

Numerical analysis of a family of optimal distributed control problems governed by an elliptic variational inequality *

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Abstract

The numerical analysis of a family of distributed mixed optimal control problems governed by elliptic variational inequalities (with parameter $\alpha > 0$) is obtained through the finite element method when its parameter $h \rightarrow 0$. We also obtain the limit of the discrete optimal control and the associated state system solutions when $\alpha \rightarrow \infty$ (for each $h > 0$) and a commutative diagram for two continuous and two discrete optimal control and its associated state system solutions is obtained when $h \rightarrow 0$ and $\alpha \rightarrow \infty$. Moreover, the double convergence is also obtained when $(h, \alpha) \rightarrow (0, \infty)$.

1 Introduction

Following [8], we consider a bounded domain $\Omega \subset \mathbb{R}^n$ whose regular boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ consists of the union of two disjoint portions Γ_1 and Γ_2 with $meas(\Gamma_1) > 0$, and we state, for each $\alpha > 0$, the following free boundary system:

$$u \geq 0; \quad u(-\Delta u - g) = 0; \quad -\Delta u - g \geq 0 \text{ in } \Omega; \quad (1.1)$$

$$-\frac{\partial u}{\partial n} = \alpha(u - b) \text{ on } \Gamma_1; \quad -\frac{\partial u}{\partial n} = q \text{ on } \Gamma_2; \quad (1.2)$$

where the function g in (1.1) can be considered as the internal energy in Ω , $\alpha > 0$ is the heat transfer coefficient on Γ_1 , $b > 0$ is the constant environment

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temperature, and q is the heat flux on Γ_2 . The variational formulation of the above problem is given as (system (S_α)):

Find $u = u_{\alpha g} \in K_+$ such that, $\forall v \in K_+$

$$a_\alpha(u_{\alpha g}, v - u_{\alpha g}) \geq (g, v - u_{\alpha g})_H - (q, v - u_{\alpha g})_Q + \alpha(b, v - u_{\alpha g})_R, \quad (1.3)$$

where

$$V = H^1(\Omega), \quad K_+ = \{v \in V : v \geq 0 \text{ in } \Omega\},$$

$$H = L^2(\Omega), \quad Q = L^2(\Gamma_2), \quad \text{and} \quad R = L^2(\Gamma_1),$$

$$(u, v)_H = \int_{\Omega} u v \, dx, \quad (u, v)_Q = \int_{\Gamma_2} u v \, ds, \quad (u, v)_R = \int_{\Gamma_1} u v \, ds,$$

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx$$

and

$$a_\alpha(u, v) = a(u, v) + \alpha(u, v)_R. \quad (1.4)$$

We note that a_1 , and therefore a_α , is a bilinear, continuous, symmetric and coercive form on V [25, 34], that is to say: there exists a constant $\lambda_1 > 0$ and $0 < \lambda_\alpha = \lambda_1 \min\{1, \alpha\}$ such that

$$a_\alpha(v, v) \geq \lambda_\alpha \|v\|_V^2 \quad \forall v \in V. \quad (1.5)$$

In [8] the following family of continuous distributed optimal control problem associated with the system (S_α) was considered for each $\alpha > 0$:

Problem (P_α) : Find the distributed optimal control $g_{op_\alpha} \in H$ such that

$$J_\alpha(g_{op_\alpha}) = \min_{g \in H} J_\alpha(g) \quad (1.6)$$

where the quadratic cost functional $J_\alpha : H \rightarrow \mathbb{R}_0^+$ was defined by:

$$J_\alpha(g) = \frac{1}{2} \|u_{\alpha g}\|_H^2 + \frac{M}{2} \|g\|_H^2 \quad (1.7)$$

with $M > 0$ a given constant and $u_{\alpha g}$ is the corresponding solution of the elliptic variational inequality (1.3) associated to the control $g \in H$.

Several optimal control problems are governed by elliptic variational inequalities ([1],[2],[3],[5],[6], [13],[14],[27],[31],[32],[39]) and there exists an abundant literature about continuous and numerical analysis of optimal control problems governed by elliptic variational equalities or inequalities ([4],[10],[11],[15],[16], [17],[18],[19],[20], [21],[22], [23],[24],[26], [29],[30],[35], [36],[40]) and by parabolic variational equalities or inequalities ([7], [28]).

The objective of this work is to make the numerical analysis of the continuous optimal control problem (P_α) which is governed by the elliptic variational inequality (1.3) by proving the convergence of a discrete solution to the solution of the continuous optimal control problem.

In Section 2, we establish the discrete elliptic variational inequality (2.3) which is the discrete formulation of the continuous elliptic variational inequality (1.3), and we obtain that these discrete problem has unique solutions for all positive h . Moreover, we define a family $(P_{h\alpha})$ of discrete optimal control problems (2.8) and, we obtain several properties for the state system (2.3) and for the discrete cost functional $J_{h\alpha}$ defined in (2.7).

In Section 3, on adequate functional spaces, we obtain a result of global strong convergence when the parameter $h \rightarrow 0$ (for each $\alpha > 0$) and when $\alpha \rightarrow \infty$ (for each $h > 0$) for the discrete state systems and for the discrete optimal optimal problem corresponding to (P_α) . We end this work proving the double convergence of the discrete optimal solutions of $(P_{h\alpha})$ when $(h, \alpha) \rightarrow (0, \infty)$ obtaining a complete commutative diagram among two discrete and two continuous optimal control problems given in Fig. 1. We generalize recent results obtained for optimal control problems governed by elliptic variational equalities given in [37, 38].

2 Properties of the discretization of the problem (P_α)

Let $\Omega \subset \mathbb{R}^n$ be a bounded polygonal domain; b a positive constant and τ_h a regular triangulation with Lagrange triangles of type 1, constituted by affine-equivalent finite elements of class C^0 over Ω being h the parameter of the finite element approximation which goes to zero ([9], [12]). We take h equal to the longest side of the triangles $T \in \tau_h$ and we can approximate the sets V and K_+ by:

$$V_h = \{v_h \in C^0(\overline{\Omega}) : v_h|_T \in P_1(T), \forall T \in \tau_h\},$$

$$K_{+h} = \{v_h \in V_h : v_h \geq 0 \text{ in } \Omega\}$$

where $P_1(T)$ is the set of the polynomials of degree less than or equal to 1 in the triangle T . Let $\Pi_h : C^0(\overline{\Omega}) \rightarrow V_h$ be the corresponding linear interpolation operator and $c_0 > 0$ a constant (independent of the parameter h) such that, if $1 < r \leq 2$ ([9]):

$$\|v - \Pi_h(v)\|_H \leq c_0 h^r \|v\|_r \quad \forall v \in H^r(\Omega), \quad (2.1)$$

$$\|v - \Pi_h(v)\|_V \leq c_0 h^{r-1} \|v\|_r \quad \forall v \in H^r(\Omega). \quad (2.2)$$

The discrete formulation $(S_{h\alpha})$ of the continuous system (S_α) is, for each $\alpha > 0$, defined as: Find $u_{h\alpha g} \in K_{+h}$ such that, for all $v_h \in K_{+h}$

$$a_\alpha(u_{h\alpha g}, v_h - u_{h\alpha g}) \geq (g, v_h - u_{h\alpha g})_H - (q, v_h - u_{h\alpha g})_Q + \alpha(b, v_h - u_{h\alpha g})_R. \quad (2.3)$$

Theorem 2.1. *Let $g \in H$ and $q \in Q$ be, then there exists unique solution of the elliptic variational inequality (2.3).*

Proof. It follows from the application of Lax-Milgram Theorem ([25], [27]). \square

Lemma 2.1. *A) Let g_n and $g \in H$, and $u_{h\alpha g_n}$ and $u_{h\alpha g} \in K_{+h}$ be the associated solutions of the system $(S_{h\alpha})$ for each $\alpha > 0$. If $g_n \rightharpoonup g$ in H weak, then we have that:*

i) $\exists C > 0$ (independent of h, α and of n) such that:

$$\|u_{h\alpha g_n}\|_V \leq C; \quad (2.4)$$

ii) $\forall h > 0$,

$$\lim_{n \rightarrow \infty} \|u_{h\alpha g_n} - u_{h\alpha g}\|_V = 0. \quad (2.5)$$

B) We have that

$$\|u_{h\alpha g_2} - u_{h\alpha g_1}\|_V \leq \frac{1}{\lambda_\alpha} \|g_2 - g_1\|_H$$

where $u_{h\alpha g_i}$ is the associated solution of the system $(S_{h\alpha})$ for $g_i, i = 1, 2$.

Proof. We follow a similar methodology as in ([16], [33]). \square

Lemma 2.2. *Let $u_{\alpha g} \in K_+ \cap H^r(\Omega)$, ($1 < r \leq 2$) and $u_{h\alpha g} \in K_{+h}$ be the solutions of the elliptic variational inequalities (1.3) and (2.3) respectively for the control $g \in H$, there exists a positive constant C such that*

$$\|u_{h\alpha g} - u_{\alpha g}\|_V \leq C(\alpha)h^{(r-1)/2}. \quad (2.6)$$

Proof. If we consider $v = u_{h\alpha g} \in K_{+h} \subset K_+$ in the elliptic variational inequality (1.3) and $v_h = \Pi_h(u_{\alpha g}) \in K_{+h}$ in (2.3), and calling $w = \Pi_h(u_{\alpha g}) - u_{\alpha g}$, we have that:

$$a_\alpha(u_{h\alpha g} - u_{\alpha g}, u_{h\alpha g} - u_{\alpha g}) \leq a_\alpha(u_{h\alpha g}, w) - (g, w)_H + (q, w)_Q - \alpha(b, w)_R$$

By using the coerciveness of a_α , the estimation (2.2) and by some mathematical computation, we obtain that:

$$\|u_{h\alpha g} - u_{\alpha g}\|_V^2 \leq \frac{C}{\lambda_\alpha} \|\Pi_h(u_{\alpha g}) - u_{\alpha g}\|_V \leq \frac{C}{\lambda_\alpha} h^{r-1} \|u_{\alpha g}\|_r$$

□

Now, we consider the continuous optimal control problem which was established in (1.6). The associated discrete cost functional $J_{h\alpha} : H \rightarrow \mathbb{R}_0^+$ is defined by the following expression:

$$J_{h\alpha}(g) = \frac{1}{2} \|u_{h\alpha g}\|_H^2 + \frac{M}{2} \|g\|_H^2 \quad (2.7)$$

where $u_{h\alpha g}$ is the unique solution of the elliptic variational inequality (2.3) for a given control $g \in H$ and a given parameter $\alpha > 0$. Then, we establish the following discrete distributed optimal control problem ($P_{h\alpha}$): Find $g_{op_{h\alpha}} \in H$ such that

$$J_{h\alpha}(g_{op_{h\alpha}}) = \min_{g \in H} J_{h\alpha}(g). \quad (2.8)$$

We remark that the discrete (in the space) distributed optimal control problem ($P_{h\alpha}$) is still an infinite dimensional optimal control problem since the control space H is not discretized.

Theorem 2.2. *For the control $g \in H$, the parameters $\alpha > 0$ and $h > 0$, we have:*

a)

$$\lim_{\|g\|_H \rightarrow \infty} J_{h\alpha}(g) = \infty.$$

b) $J_{h\alpha}(g) \geq \frac{M}{2}\|g\|_H^2 - C\|g\|_H$ for some constant C independent of $h > 0$.

c) The functional $J_{h\alpha}$ is a lower weakly semi-continuous application in H .

d) For each $h > 0$ and $\alpha > 0$, there exists a solution of the discrete distributed optimal control problem (2.8).

Proof. From the definition of $J_{h\alpha}(g)$ we obtain a) and b).

c) Let $g_n \rightharpoonup g$ in H weak, then by using the equality $\|g_n\|_H^2 = \|g_n - g\|_H^2 - \|g\|_H^2 + 2(g_n, g)_H$ we obtain that $\|g\|_H \leq \liminf_{n \rightarrow \infty} \|g_n\|_H$. Therefore, we have

$$\liminf_{n \rightarrow \infty} J_{h\alpha}(g_n) \geq \frac{1}{2}\|u_{h\alpha g}\|_H^2 + \frac{M}{2}\|g\|_H^2 = J_{h\alpha}(g).$$

d) It follows from [27]. □

Lemma 2.3. *If the continuous state system has the regularity $u_{\alpha g} \in H^r(\Omega)$ ($1 < r \leq 2$) for $g \in H$ and $\alpha > 0$, then we have the following estimation $\forall g \in H$:*

$$|J_{h\alpha}(g) - J_\alpha(g)| \leq C(\alpha)h^{\frac{r-1}{2}} \quad (2.9)$$

where C is a positive constant independent of $h > 0$.

Proof. By definition of the discrete cost functional $J_{h\alpha}$, we have:

$$J_{h\alpha}(g) - J_\alpha(g) = \frac{1}{2}(\|u_{h\alpha g}\|_H^2 - \|u_{\alpha g}\|_H^2) = \frac{1}{2}\|u_{h\alpha g} - u_{\alpha g}\|_H^2 + (u_{\alpha g}, u_{h\alpha g} - u_{\alpha g})_H$$

and therefore, if we apply (2.6), it results:

$$|J_{h\alpha}(g) - J_\alpha(g)| \leq \left(\frac{1}{2}\|u_{h\alpha g} - u_{\alpha g}\|_H + \|u_{\alpha g}\|_H\right)\|u_{h\alpha g} - u_{\alpha g}\|_H \leq C(\alpha)h^{\frac{r-1}{2}},$$

and (2.9) holds. □

3 Results of Convergence

3.1 Convergence when $h \rightarrow 0$

Theorem 3.1. *Let $u_{\alpha g} \in K_+ \cap H^r(\Omega)$, ($1 < r \leq 2$) and $u_{h\alpha g} \in K_{+h}$ be the solutions of the elliptic variational inequalities (1.3) and (2.3) respectively for the control $g \in H$, then $u_{h\alpha g} \rightarrow u_{\alpha g}$ in V when $h \rightarrow 0^+$.*

Proof. Similarly to the part a) of the Lemma 2.1, we can show that there exist a constant $C > 0$ such that $\|u_{h\alpha g}\|_V \leq C$, $\forall h > 0$. Therefore, we conclude that there exists $\eta_\alpha \in V$ so that $u_{h\alpha g} \rightharpoonup \eta_\alpha$ in V (in H strong) as $h \rightarrow 0^+$ and $\eta \in K_+$. On the other hand, given $v \in K_+$ let be $v_h = \Pi(v) \in K_{+h}$ for each h such that $v_h \rightarrow v$ in V when h goes to zero. Now, by considering $v_h \in K_{+h}$ in the discrete elliptic variational inequality (2.3) we get:

$$a_\alpha(u_{h\alpha g}, v_h - u_{h\alpha g}) \geq (g, v_h - u_{h\alpha g})_H - (q, v_h - u_{h\alpha g})_Q + \alpha(b, v_h - u_{h\alpha g})_R \quad (3.1)$$

and when we pass to the limit as $h \rightarrow 0^+$ in (3.1) by using that the bilinear form a is lower weak semi-continuous in V , we obtain:

$$a_\alpha(\eta_\alpha, v - \eta_\alpha) \geq (g, v - \eta_\alpha)_H - (q, v - \eta_\alpha)_Q + \alpha(b, v - \eta_\alpha)_R, \quad \forall v \in K_+$$

and from the uniqueness of the solution of the discrete elliptic variational inequality (1.3), we obtain that $\eta = u_{\alpha g}$.

Now, we will prove the strong convergence. As consequence of Lemma 2.2, by passing to the limit when $h \rightarrow 0^+$ in the inequality (2.6), it results:

$$\lim_{h \rightarrow 0^+} \|u_{h\alpha g} - u_{\alpha g}\|_V = 0.$$

□

Henceforth we will consider the following:

Definition Given $\mu \in [0, 1]$ and $g_1, g_2 \in H$, we define:

- a) the convex combinations of two data g_1 and g_2 as

$$g_3(\mu) = \mu g_1 + (1 - \mu)g_2 \in H, \quad (3.2)$$

- b) the convex combination of two discrete solutions

$$u_{h\alpha 3}(\mu) = \mu u_{h\alpha g_1} + (1 - \mu)u_{h\alpha g_2} \in K_{+h} \quad (3.3)$$

- c) and $u_{h\alpha 4}(\mu)$ as the associated discrete state system which is the solution of the discrete elliptic variational inequality (2.3) for the control $g_3(\mu)$.

Following the idea given in [8, 33] we define two open problems. Given the controls $g_1, g_2 \in H$,

a)

$$0 \leq u_{h\alpha 4}(\mu) \leq u_{h\alpha 3}(\mu) \text{ in } \Omega, \quad \forall \mu \in [0, 1], \forall h > 0, \quad (3.4)$$

b)

$$\|u_{h\alpha 4}(\mu)\|_H \leq \|u_{h\alpha 3}(\mu)\|_H \quad \forall \mu \in [0, 1], \forall h > 0. \quad (3.5)$$

Remark 1: We have that (3.4) \Rightarrow (3.5).

Remark 2: If (3.4) (or (3.5)) is true, then the functional $J_{h\alpha}$ is H-elliptic and a strictly convex application because we have:

i)

$$\|g_{3\mu}\|_H^2 = \mu \|g_1\|_H^2 + (1 - \mu) \|g_2\|_H^2 - \mu(1 - \mu) \|g_2 - g_1\|_H^2 \quad \forall g_1, g_2 \in H, \forall \mu \in [0, 1]$$

ii)

$$\|u_{h\alpha 3}(\mu)\|_H^2 = \mu \|u_{h\alpha 1}\|_H^2 + (1 - \mu) \|u_{h\alpha 2}\|_H^2 - \mu(1 - \mu) \|u_{h\alpha 2} - u_{h\alpha 1}\|_H^2$$

$$\forall g_1, g_2 \in H, \forall \mu \in [0, 1], \forall \alpha > 0.$$

Then we get:

$$\begin{aligned} & \mu J_{h\alpha}(g_1) + (1 - \mu) J_{h\alpha}(g_2) - J_{h\alpha}(g_3(\mu)) \\ &= \frac{\mu(1 - \mu)}{2} \|u_{h\alpha g_2} - u_{h\alpha g_1}\|_H^2 + \frac{M}{2} \mu(1 - \mu) \|g_2 - g_1\|_H^2 + \frac{1}{2} [\|u_{h\alpha 3}\|_H^2 - \|u_{h\alpha 4}\|_H^2] \\ &\geq \frac{\mu(1 - \mu)}{2} \|u_{h\alpha g_2} - u_{h\alpha g_1}\|_H^2 + \frac{M}{2} \mu(1 - \mu) \|g_2 - g_1\|_H^2 \geq \\ &\quad \frac{M}{2} \mu(1 - \mu) \|g_2 - g_1\|_H^2 > 0 \quad \forall \mu \in (0, 1), \quad g_1 \neq g_2 \in H \end{aligned}$$

and therefore, the uniqueness for the discrete optimal control problem $(P_{h\alpha})$, defined in (2.8), holds.

Theorem 3.2. Let $u_{\alpha g_{op}} \in K_+$ be the continuous state system associated to the optimal control $g_{op\alpha} \in H$ which is the solution of the continuous distributed optimal control problem (1.6). If, for each $h > 0$, we choose an discrete optimal control $g_{op_{h\alpha}} \in H$ which is a solution of the discrete distributed optimal control problem (2.8) and its corresponding discrete state system $u_{h\alpha g_{op_{h\alpha}}} \in K_{+h}$, we obtain that:

$$u_{h\alpha g_{op_{h\alpha}}} \rightarrow u_{\alpha g_{op\alpha}} \text{ in } V \text{ strong when } h \rightarrow 0^+, \quad (3.6)$$

and

$$g_{op_{h\alpha}} \rightarrow g_{op\alpha} \text{ in } H \text{ strong when } h \rightarrow 0^+. \quad (3.7)$$

Proof. Now, we consider a fixed value of the heat transfer coefficient $\alpha > 0$. Let be $h > 0$ and $g_{op_{h\alpha}}$ a solution of (2.8) and $u_{h\alpha g_{op_{h\alpha}}}$ its associated discrete optimal state system which is the solution of the problem defined in (2.3) for each $h > 0$. From (2.7) and (2.8), we have that for all $g \in H$

$$J_{h\alpha}(g_{op_{h\alpha}}) = \frac{1}{2} \|u_{h\alpha g_{op_{h\alpha}}}\|_H^2 + \frac{M}{2} \|g_{op_{h\alpha}}\|_H^2 \leq \frac{1}{2} \|u_{h\alpha g}\|_H^2 + \frac{M}{2} \|g\|_H^2.$$

Then, if we consider $g = 0$ and $u_{h\alpha 0}$ his corresponding associated state system, it results that:

$$J_{h\alpha}(g_{op_{h\alpha}}) = \frac{1}{2} \|u_{h\alpha g_{op_{h\alpha}}}\|_H^2 + \frac{M}{2} \|g_{op_{h\alpha}}\|_H^2 \leq \frac{1}{2} \|u_{h\alpha 0}\|_H^2.$$

Since $\|u_{h\alpha 0}\|_H \leq C \quad \forall h$, then we can obtain:

$$\|u_{h\alpha g_{op_{h\alpha}}}\|_H \leq C \quad \forall h \quad (3.8)$$

and

$$\|g_{op_{h\alpha}}\|_H \leq \frac{1}{\sqrt{M}} \|u_{h\alpha 0}\|_H \leq \frac{1}{\sqrt{M}} C \quad \forall h. \quad (3.9)$$

If we consider $v_h = b \in K_{+h}$ in the inequality (2.3) for $g_{op_{h\alpha}}$ we obtain, because the coerciveness of the application a_α :

$$\|u_{h\alpha g_{op_{h\alpha}}}\|_V \leq C \quad (3.10)$$

where the constant C is independent of the parameter h y $\alpha > 0$. Now we can say that there exist $\eta_\alpha \in V$ and $f_\alpha \in H$ such that $u_{h\alpha g_{op_{h\alpha}}} \rightharpoonup \eta_\alpha$ in

V weak (in H strong), and $g_{op_{h\alpha}} \rightharpoonup f_\alpha$ in H weak when $h \rightarrow 0^+$. Moreover, $\eta_\alpha \in K_+$.

Let given $v \in K_+$, by taking $v_h = \Pi(v) \in K_{+h}$ we know that $v_h \rightarrow v$ in V strong when $h \rightarrow 0^+$. Then, if we consider the variational elliptic inequality (2.3) for $g = g_{op_{h\alpha}}$ we have, taking into account that the application a_α is a lower weak semi-continuous application in V , that:

$$a_\alpha(\eta_\alpha, v - \eta_\alpha) \geq (f_\alpha, v - \eta_\alpha) - (q, v - \eta_\alpha)_Q + \alpha(b, v - \eta_\alpha)_R, \quad \forall v \in K_+$$

and by the uniqueness of the solution of the problem given by the elliptic variational inequality (1.3), we deduce that $\eta_\alpha = u_\alpha f_\alpha$.

By using that the functional cost J_α is semi-continuous in H weak (see [8]) and Theorem 3.1, it results that $f = u_{\alpha g_{op_\alpha}}$ and $\eta_\alpha = u_{g_{op_\alpha}}$.

Now, we consider $v = u_{h\alpha g_{op_{h\alpha}}} \in K_{+h} \subset K_+$ in the system (S_α) with control g_{op_α} , and $v_h = \Pi_h(u_{\alpha g_{op_\alpha}})$ in the discrete system $(S_{h\alpha})$ for the control $g_{op_{h\alpha}}$ and define $w_h = u_{h\alpha g_{op_{h\alpha}}} - u_{\alpha g_{op_\alpha}}$. After some mathematical work, we obtain that:

$$a_\alpha(w_h, w_h) \leq -a_\alpha(u_{h\alpha g_{op_{h\alpha}}}, \Pi_h(u_{\alpha g_{op_\alpha}}) - u_{\alpha g_{op_\alpha}}) + (q, \Pi_h(u_{\alpha g_{op_\alpha}}) - u_{\alpha g_{op_\alpha}})_Q$$

$$\begin{aligned} & -\alpha(b, \Pi_h(u_{\alpha g_{op_\alpha}}) - u_{\alpha g_{op_\alpha}})_R + (g_{op_{h\alpha}}, \Pi_h(u_{\alpha g_{op_\alpha}}) - u_{h\alpha g_{op_{h\alpha}}})_H \\ & - (g_{op_\alpha}, w_h)_H. \end{aligned}$$

From the coerciveness of the application a_α , and $u_{h\alpha g_{op_{h\alpha}}} \rightarrow u_{\alpha g_{op_\alpha}}$ in H and $\Pi_h(u_{\alpha g_{op_\alpha}}) \rightarrow u_{\alpha g_{op_\alpha}}$ in H , we obtain that $\|w_h\|_V \rightarrow 0$ if $h \rightarrow 0$ and then (3.6) it holds. Its easy to see that (3.7) holds too. \square

3.2 Convergence when $\alpha \rightarrow \infty$

Now, under the same hypothesis in §1, we consider the following free boundary system [8]:

$$u \geq 0; \quad u(-\Delta u - g) = 0; \quad -\Delta u - g \geq 0 \text{ in } \Omega; \quad (3.11)$$

$$u = b \text{ on } \Gamma_1; \quad -\frac{\partial u}{\partial n} = q \text{ on } \Gamma_2; \quad (3.12)$$

where the function g in (3.11) can be considered as the internal energy in Ω , b is the positive constant temperature on Γ_1 and q is the heat flux on Γ_2 . The variational formulation of the above problem is given as (S): Find $u_g \in K$ such that

$$a(u, v - u_g) \geq (g, v - u_g)_H - (q, v - u_g)_Q, \quad \forall v \in K \quad (3.13)$$

where

$$K = \{v \in V : v \geq 0 \text{ in } \Omega, v/\Gamma_1 = b\}.$$

In [8], the following continuous distributed optimal control problem (P) associated with the elliptic variational inequality (3.13) was considered: Find the continuous distributed optimal control $g_{op} \in H$ such that

$$J(g_{op}) = \min_{g \in H} J(g) \quad (3.14)$$

where the quadratic cost functional $J : H \rightarrow \mathbb{R}_0^+$ is defined by:

$$J(g) = \frac{1}{2} \|u_g\|_H^2 + \frac{M}{2} \|g\|_H^2 \quad (3.15)$$

with $M > 0$ a given constant and u_g is the corresponding solution of the elliptic variational inequality (3.13) associated to the control $g \in H$.

Therefore, as in §2, we define the discrete variational inequality formulation (S_h) of the system (S) as follows: Find $u_{hg} \in K_h$ such that

$$a(u_{hg}, v_h - u_{hg}) \geq (g, v_h - u_{hg})_H - (q, v_h - u_{hg})_Q \quad \forall v_h \in K_h. \quad (3.16)$$

where

$$K_h = \{v_h \in V_h : v_h \geq 0 \text{ in } \Omega, v_h/\Gamma_1 = b\}$$

The corresponding discrete distributed optimal control problem (P_h) of the continuous distributed optimal control problem (P) is defined as: Find the discrete distributed optimal control $g_{op_h} \in H$ such that

$$J_h(g_{op_h}) = \min_{g \in H} J_h(g) = \frac{1}{2} \|u_{hg}\|_H^2 + \frac{M}{2} \|g\|_H^2, \quad (3.17)$$

where u_{hg} is the solution of the elliptic variational inequality (3.16).

Theorem 3.3. *i) Let $g \in H$, and $q \in Q$ be, then there exists unique solution of elliptic variational inequality (3.16).*

ii) There exists a solution of the discrete optimal control problem (3.17)

Proof. *i)* It follows from the application of Lax-Milgram Theorem [25], [27].

ii) It follows from [33]. \square

Theorem 3.4. *Let $g \in H$, $q \in Q$ and $h > 0$ be, then we have*

$$\lim_{\alpha \rightarrow \infty} \|u_{h\alpha g} - u_{hg}\|_V = 0.$$

Proof. Without loss of generality, we consider $\alpha > 1$ and we define $w = u_{h\alpha g} - u_{hg} \in V$. By definition of a_α , we have:

$$a_\alpha(w, w) - a_1(w, w) = (\alpha - 1)\|w\|_R^2.$$

After mathematical work, we obtain that:

$$a_1(w, w) \leq a_1(w, w) + (\alpha - 1)\|w\|_R^2 \leq (g, w)_H - (q, w)_Q - a(u_{hg}, w) \quad (3.18)$$

and by coerciveness of a_1 it results that:

$$\|u_{h\alpha g} - u_{hg}\|_R^2 \leq \frac{C}{\alpha - 1}$$

and $u_{h\alpha g} \rightarrow u_{hg}$ in Γ_1 , when $\alpha \rightarrow \infty$.

Moreover, as a consequence of (3.18), we obtain that $\|u_{h\alpha g}\|_V \leq C$ (C constant independent of α and h). Then, there exist $\eta \in V$ such that

$$u_{h\alpha g} \rightharpoonup \eta \text{ in } V \text{ (in } H \text{ strong)}.$$

Then, the strong convergence in V is obtained similarly to the one in Theorem 3.1 \square

Theorem 3.5. *If, for each $h > 0$ we choose $g_{op_{h\alpha}} \in H$ a solution of the optimal control problem $(P_{h\alpha})$ and consider its respective discrete state system $u_{h\alpha g_{op_{h\alpha}}} \in K_{+h}$ the solution of (2.3), we obtain that:*

$$u_{h\alpha g_{op_{h\alpha}}} \rightarrow u_{hf_h} \text{ in } V \text{ when } \alpha \rightarrow \infty, \quad (3.19)$$

and

$$g_{op_{h\alpha}} \rightarrow f_h \text{ in } H \text{ when } \alpha \rightarrow \infty. \quad (3.20)$$

where $f_h \in H$ is a solution of the discrete optimal control problem (P_h) and u_{hf_h} is its corresponding discrete state system solution of the variational inequality (3.16).

Proof. As in Theorem 3.2, the inequalities (3.8) and (3.9) hold. Now, considering $v_h = b$ in (2.3) (and we take $\alpha > 1$ without loss of generality) for the control $g_{op_{h\alpha}}$ and $w_h = b - u_{h\alpha} g_{op_{h\alpha}}$, we obtain:

$$a_\alpha(u_{h\alpha} g_{op_{h\alpha}}, w_h) \geq (g_{op_{h\alpha}}, w_h)_H - (q, w_h)_Q + \alpha(b, w_h)_R$$

that is to say:

$$a_1(-w_h, w_h) + a_1(b, w_h) \geq (g_{op_{h\alpha}}, w_h)_H - (q, w_h)_Q + (\alpha - 1)\|w_h\|_R. \quad (3.21)$$

By the coerciveness of the application a_1 , it results that:

$$\|u_{h\alpha} g_{op_{h\alpha}}\|_V \leq C \quad \forall \alpha > 0. \quad (3.22)$$

Moreover,

$$\|u_{h\alpha} g_{op_{h\alpha}}\|_R \leq \frac{C}{\alpha - 1} \quad \forall \alpha > 0. \quad (3.23)$$

Then, there exists $f_h \in H$ and $\eta_h \in V$ (we can see that $\eta_h \in K_h$) such that

$$g_{op_{h\alpha}} \rightharpoonup f_h \text{ in } H \quad (3.24)$$

and

$$u_{h\alpha} g_{op_{h\alpha}} \rightharpoonup \eta_h \text{ in } V \text{ (in } H \text{ strong)} \quad (3.25)$$

Let be $v_h \in K_h \subset K_{+h}$ and given $w_h = v_h - u_{h\alpha} g_{op_{h\alpha}}$, we have:

$$a_\alpha(u_{h\alpha} g_{op_{h\alpha}}, w_h) \geq (g_{op_{h\alpha}}, w_h)_H - (q, w_h)_Q + \alpha(b, w_h)_R$$

$$a(u_{h\alpha} g_{op_{h\alpha}}, w_h) \geq (g_{op_{h\alpha}}, w_h)_H - (q, w_h)_Q + \alpha(b - u_{h\alpha} g_{op_{h\alpha}}, w_h)_R$$

and because of (3.24), (3.25) and similar arguments given in Theorem 3.2, and the fact that the application a is semi-continuous in V weak, we obtain that η_h is a solution of (3.16) for the control f_h . Then (by item (i) in Theorem 3.3), $\eta_h = u_h f_h$.

If we consider $v_h = u_h f_h \in K_h \subset K_{+h}$ in (2.3) for the control $g_{op_{h\alpha}} \in H$ and $w_h = u_h f_h - u_{h\alpha} g_{op_{h\alpha}}$, then:

$$a_\alpha(u_{h\alpha} g_{op_{h\alpha}}, w_h) \geq (g_{op_{h\alpha}}, w_h)_H - (q, w_h)_Q + \alpha(b, w_h)_R$$

$$a_1(w_h, w_h) \leq a_1(u_h f_h, w_h) - (g_{op_{h\alpha}}, w_h)_H + (q, w_h)_Q - \alpha(b, w_h)_R + (\alpha - 1)(u_{h\alpha g_{op_{h\alpha}}}, w_h)_R.$$

Again, as consequence of the coerciveness of the application a_1 and by (3.24), (3.25), it results (3.19).

Now we see that f_h is a solution of (2.8): because of Theorem 2.2 (c), and by the definition of optimum:

$$J_h(f_h) \leq \lim_{\alpha \rightarrow \infty} J_{h\alpha}(g_{op_{h\alpha}}) \leq \lim_{\alpha \rightarrow \infty} J_{h\alpha}(g) \quad \forall g \in H$$

and by Theorem 3.4 we conclude that

$$J_h(f_h) \leq J_h(g) \quad \forall g \in H.$$

Finally, we see that:

$$J_h(f_h) \leq \lim_{\alpha \rightarrow \infty} J_{h\alpha}(g_{op_{h\alpha}}) \leq J_h(g) \quad \forall g \in H$$

then, if we consider $g = f_h$:

$$\lim_{\alpha \rightarrow \infty} J_{h\alpha}(g_{op_{h\alpha}}) = J_h(f_h)$$

and, because (3.19),

$$\lim_{\alpha \rightarrow \infty} \|g_{op_{h\alpha}}\|_H = \|f\|_H. \quad (3.26)$$

Then, by using (3.24), (3.25), we obtain (3.20). □

Now, following the idea given in [38] we have this final theorem:

3.3 Double convergence when $(h, \alpha) \rightarrow (0^+, \infty)$

Theorem 3.6. *If, for each $h > 0$ we choose $g_{op_{h\alpha}} \in H$ a solution of the optimal control problem $(P_{h\alpha})$ and we consider its respective discrete state system $u_{h\alpha g_{op_{h\alpha}}} \in K_{+h}$, which is the unique solution of (2.3), we obtain that:*

$$u_{h\alpha g_{op_{h\alpha}}} \rightarrow u_{g_{op}} \text{ in } V \text{ when } (h, \alpha) \rightarrow (0^+, \infty), \quad (3.27)$$

and

$$g_{op_{h\alpha}} \rightarrow g_{op} \text{ in } H \text{ when } (h, \alpha) \rightarrow (0^+, \infty). \quad (3.28)$$

where $g_{op} \in H$ is the solution of the optimal control problem (P) and $u_{g_{op}}$ is its corresponding state system solution of the variational inequality (3.13).

Proof. As in Theorem 3.2, we have (3.8), (3.9) and (3.10) and, in consequence, there exist $u^* \in V$ with $u^*/\Gamma_1 = b$ and $g^* \in H$ such that:

$$u_{h\alpha g_{op_{h\alpha}}} \rightharpoonup u^* \quad (\text{strong in } H) \quad (3.29)$$

and

$$g_{op_{h\alpha}} \rightharpoonup g^* \quad (3.30)$$

when $(h, \alpha) \rightarrow (0^+, \infty)$ in both cases. Let be $v \in K$ such that $v/\Gamma_1 = b$. We consider $v_h = \Pi_h(v) \in K_{+h}$ in the state system (2.3) and we define $w_h = v_h - u_{h\alpha g_{op_{h\alpha}}}$. Then we obtain

$$a(u_{h\alpha g_{op_{h\alpha}}}, w_h) \geq (g_{op_{h\alpha}}, w_h)_H - (q, w_h)_Q.$$

Because the application a is semi-continuous weak in V and $w_h \rightarrow v - u^*$ in H when $(h, \alpha) \rightarrow (0, \infty)$, it results that u^* is solution of (3.13). But this problem has unique solution, then we conclude that $u^* = u_{g^*}$. Moreover, we have that:

$$\begin{aligned} & a_\alpha(u_{h\alpha g_{op_{h\alpha}}} - u_{g^*}, u_{h\alpha g_{op_{h\alpha}}} - u_{g^*}) \leq (g_{op_{h\alpha}}, u_{h\alpha g_{op_{h\alpha}}} - \Pi(u_{g^*}))_H \\ & + (q, u_{h\alpha g_{op_{h\alpha}}} - \Pi(u_{g^*}))_Q + \alpha(b, u_{h\alpha g_{op_{h\alpha}}} - \Pi(u_{g^*}))_R - a_\alpha(u_{g^*}, u_{h\alpha g_{op_{h\alpha}}} - \Pi(u_{g^*})) \\ & + a_\alpha(u_{h\alpha g_{op_{h\alpha}}}, \Pi(u_{g^*}) - u_{g^*}) - a_\alpha(u_{g^*}, \Pi(u_{g^*}) - u_{g^*}). \end{aligned}$$

Beacause the coerciveness of the application a_α in V and by (3.29) and (3.30), we obtain (3.27) when $(h, \alpha) \rightarrow (0^+, \infty)$.

As the functional $J_{h\alpha}$ is lower weakly semi-continuous in H (Theorem 2.2) and (3.30) we obtain that $g_{op_{h\alpha}} \rightharpoonup g_{op}$.

We also have that $\lim_{(h,\alpha) \rightarrow (0,\infty)} J_{h\alpha}(g_{op_{h\alpha}}) = J(g_{op})$, and then

$\lim_{(h,\alpha) \rightarrow (0,\infty)} \|g_{op_{h\alpha}}\|_H = \|g_{op}\|_H$, and by (3.30), (3.28) the thesis holds. \square

4 Conclusion

In conclusion, by using the previous results given in [8], and [33] we obtain the following commutative diagram among the two continuous optimal control problems (P) and (P_α), and two discrete optimal control problems (P_h) and ($P_{h\alpha}$) when $h \rightarrow 0$, $\alpha \rightarrow \infty$ and $(h, \alpha) \rightarrow (0^+, \infty)$, which can be summarized by the following figure (Fig. 1):

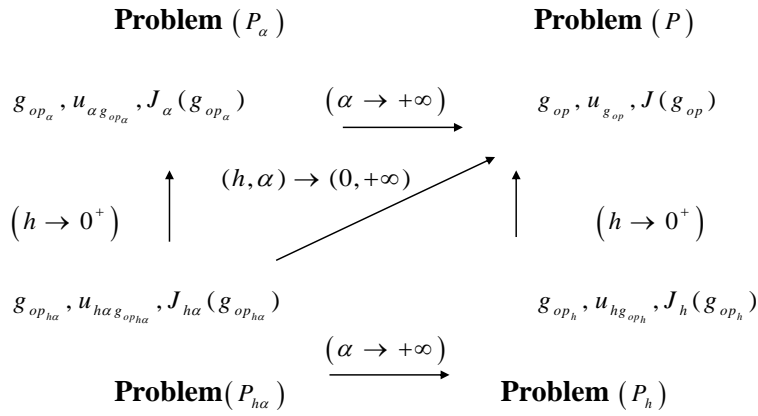


Fig. 1: Complete diagram for two continuous and two discrete optimal control and the associated state system solutions

1

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