# HILBERT-KUNZ MULTIPLICITY OF PRODUCTS OF IDEALS

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ABSTRACT. We give bounds for Hilbert-Kunz multiplicity of product of two ideals and we characterize the equality in terms of the tight closures of the ideals. Connections are drawn with $*$-spread and with ordinary length calculations.

## 1. Introduction

The Hilbert-Kunz multiplicity [Mon83] $e_{\mathrm{HK}}(I)$ of a finite colength ideal $I$ of a local Noetherian ring $(R, \mathfrak{m})$ of prime characteristic $p > 0$ is an important invariant in prime characteristic commutative algebra. It has been extensively studied when the ideal in question is the *maximal* ideal $\mathfrak{m}$, in which case it characterizes the regularity of the ring [WY00, Theorem 1.5] and has been used to explore other properties as well (e.g. finiteness of projective dimension in [Mil00] and strong semistability of vector bundles in [Tri05]). One of the main applications of the invariant applied to arbitrary $\mathfrak{m}$-primary ideals is the fact that it governs their tight closures [HH90]. However, it has always been clear that in order to understand the Hilbert-Kunz multiplicity of the maximal ideal, one must understand the Hilbert-Kunz multiplicity of arbitrary $\mathfrak{m}$-primary ideals, even if one does not care about tight closure *per se* [Mon83, Cha97]. It is then natural to ask, given a pair of ideals $I, J$, what can one say about the Hilbert-Kunz multiplicity of their product? With the exception of some work on asymptotic properties of $e_{\mathrm{HK}}(I^n)$ for $n \gg 0$ [WY01, Han03, Tri15], it seems that the Hilbert-Kunz multiplicity of products of ideals has not been widely explored, even in well-behaved rings. Hence, the current work serves as a first foray into this interesting area.

In §2, we start in a general setting, proving a certain length inequality (Proposition 2.1) involving colengths of ideals $I$, $J$, their product $IJ$, and the number of generators of $J$. We show that if this inequality is an equality, then $J \subseteq I$, and when $J$ is of the principal class, we obtain a converse (Theorem 2.4). In §3, we specialize to the prime characteristic case and give an analogous inequality involving the Hilbert-Kunz multiplicities of $I$, $J$, and $IJ$, along with the $*$-*spread* of $J$ (Proposition 3.1) which, if it is an equality, we show that $J$ is contained in the *tight closure* of $I$ (Theorem 3.4). Again we obtain a converse, this time when $J$ is a parameter ideal up to tight closure (Theorem 3.5). In Examples 2.6 and 3.8,

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we show that the assumptions on the second ideal are necessary in the converse statements. In §4, we revisit some interesting old results with our new perspective. As indicated above, our computations in the prime characteristic case use the invariant of $*$-*spread* [Eps05] in an essential way, which in turn allows us to recover a special case of a result of Epstein and Vraciu, but with a better bound (Proposition 4.2). We also recover a Lech-like inequality of Huneke and Yao [HY02] in the $F$-finite case (Proposition 4.3) using our methods.

## 2. Length inequalities in the general case

Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let $\mathbf{a} = a_1, \dots, a_\ell$ be a system of generators for a finite $R$-module $M$. Then we have an exact sequence

$$R^s \xrightarrow{u_{\mathbf{a}}} R^\ell \xrightarrow{\pi_{\mathbf{a}}} M \to 0 \tag{1}$$

where $\pi_{\mathbf{a}}$ is the map sending each $e_j \mapsto a_j$, where $e_1, \dots, e_\ell$ is the canonical standard basis for the free module $R^\ell$, and $u_{\mathbf{a}}$ is a matrix with cokernel $M$. Let $K_{\mathbf{a}}$ denote the kernel of $\pi_{\mathbf{a}}$. This gives rise to a short exact sequence

$$0 \to K_{\mathbf{a}} \xrightarrow{j_{\mathbf{a}}} R^\ell \xrightarrow{\pi_{\mathbf{a}}} M \to 0.$$

Now let $I$ be an $\mathfrak{m}$-primary ideal. Taking the tensor product of (1) with $R/I$, we have

$$R^s/IR^s \xrightarrow{u_{\mathbf{a},I}} (R/I)^\ell \xrightarrow{\pi_{\mathbf{a},I}} M/IM \to 0.$$

Note that $K_{\mathbf{a},I} := \operatorname{im} u_{\mathbf{a},I} = (K_{\mathbf{a}} + IR^\ell)/IR^\ell$, giving us the short exact sequence

$$0 \to K_{\mathbf{a},I} \xrightarrow{j_{\mathbf{a},I}} (R/I)^\ell \xrightarrow{\pi_{\mathbf{a},I}} M/IM \to 0. \tag{2}$$

Since $I$ has finite colength, and since we could have taken $\mathbf{a}$ to be a minimal generating set for $M$, we could assume $\ell$ is the minimal number of generators of $M$, denoted by $\mu(M)$. Thus Sequence 2 yields the length equality

$$\mu(M) \cdot \lambda_R(R/I) = \lambda_R(K_{\mathbf{a},I}) + \lambda_R(M/IM). \tag{3}$$

In particular, *if $M$ is also an $\mathfrak{m}$-primary ideal $J$*, then we have

$$\mu(J) \cdot \lambda_R(R/I) + \lambda_R(R/J) = \lambda_R(K_{\mathbf{a},I}) + \lambda_R(R/IJ). \tag{4}$$

As $K_{\mathbf{a},I}$ has non-negative length, Equations 3 and 4 give rise to the inequalities in the following result.

**Proposition 2.1.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let $M$ be a finite $R$-module and $I$ an $\mathfrak{m}$-primary ideal. Then*

$$\lambda_R(M/IM) \le \mu(M) \cdot \lambda_R(R/I). \tag{5}$$

*If $M$ is also an $\mathfrak{m}$-primary ideal $J$, then*

$$\lambda_R(R/IJ) \le \mu(J) \cdot \lambda_R(R/I) + \lambda_R(R/J). \tag{6}$$

One obtains immediately the following two corollaries:

**Corollary 2.2.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then for $\mathfrak{m}$-primary ideals $I, J$*

$$\lambda_R(R/IJ) \leq \min\{\mu(J) \cdot \lambda_R(R/I) + \lambda_R(R/J), \lambda_R(R/I) + \mu(I) \cdot \lambda_R(R/J)\}.$$

**Corollary 2.3.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and $I$ an $\mathfrak{m}$-primary ideal minimally generated by $\ell$ elements. Then*

$$\lambda_R(R/I^n) \leq \left(1 + \ell + \cdots + \ell^{n-1}\right) \cdot \lambda_R(R/I).$$

*Proof.* Induct on $n$. □

Now, *assume $M$ is an ideal $J$*, and let $(\mathbb{K}_\bullet(\mathbf{a}), \partial_\bullet)$ be the Koszul complex on the sequence $\mathbf{a} = a_1, \ldots, a_\ell$. Note that $\partial_1 = i \circ \pi_\mathbf{a}$, where $i : J \hookrightarrow R$ is the natural inclusion. Since $\mathbb{K}_\bullet(\mathbf{a})$ is a complex, we have $\operatorname{im} \partial_2 \subseteq \ker \partial_1 = \ker \pi_\mathbf{a} = K_\mathbf{a}$, with *equality* if and only if $\mathbf{a}$ is a regular sequence [BH97, Corollary 1.6.19]. Therefore, equality holds in (6) if and only if $K_{\mathbf{a},I} = 0$, i.e. $K_\mathbf{a} \subseteq IR^\ell$, which implies that $\operatorname{im} \partial_2 \subseteq IR^\ell$, and all these conditions are equivalent if $\mathbf{a}$ is a regular sequence. Now *assume further that* $\ell \geq 2$ (that is, $J$ is not principal). Then $\operatorname{im} \partial_2$ is a sum of cyclic modules of the form $C_{ij} := R \cdot v_{ij}(\mathbf{a})$ for every pair $(i, j)$ with $1 \leq i < j \leq \ell$, where

$$v_{ij}(\mathbf{a}) := -a_j e_i + a_i e_j,$$

recalling that $e_1, \ldots, e_\ell$ is the canonical basis for $R^\ell$ as a free $R$-module. So if equality holds in (6), we have

$$-a_j e_i + a_i e_j \in C_{ij} \subseteq \operatorname{im} \partial_2 \subseteq IR^\ell = \oplus_{h=1}^{\ell} I e_h,$$

so that $a_i, a_j \in I$, which in turn implies that all $a_i \in I$, so that $J \subseteq I$. Conversely, if $\mathbf{a}$ is a regular sequence and $J \subseteq I$, then each $C_{ij} \subseteq IR^\ell$, so that

$$K_\mathbf{a} = \ker \partial_1 = \operatorname{im} \partial_2 = \sum_{i<j} C_{ij} \subseteq IR^\ell,$$

whence $K_{\mathbf{a},I} = 0$, which means that equality holds in (6). Combining all this together, we have proved the following result.

**Theorem 2.4.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let $J$ be a non-principal proper ideal and $I$ an $\mathfrak{m}$-primary ideal. Then the equality*

$$\lambda_R(R/IJ) = \mu(J) \cdot \lambda_R(R/I) + \lambda_R(R/J)$$

*implies that $J \subseteq I$. The converse holds if $J$ is generated by a regular sequence.*

We also have the following characterization of the equality in (5).

**Proposition 2.5.** $\lambda_R(M/IM) = \mu(M) \cdot \lambda_R(R/I)$ *if and only if $M/IM$ is $(R/I)$-free.*

*Proof.* We may assume (1) is part of a minimal free resolution of $M$. Then we have $K_\mathbf{a} \subseteq \mathfrak{m}R^\ell$, and hence $K_{\mathbf{a},I} \subseteq \mathfrak{m} \cdot (R/I)^\ell$. If $M/IM$ is a free $(R/I)$-module, then in particular it is a projective $(R/I)$-module, so the short exact sequence (2) splits. In particular, there is an $(R/I)$-linear map $p : (R/I)^\ell \to K_{\mathbf{a},I}$ such that $p \circ j_{\mathbf{a},I}$ is the identity map on $K_{\mathbf{a},I}$. Hence,

$$K_{\mathbf{a},I} = p(j_{\mathbf{a},I}(K_{\mathbf{a},I})) \subseteq p(\mathfrak{m} \cdot (R/I)^\ell) = \mathfrak{m} p((R/I)^\ell) = \mathfrak{m} \cdot \operatorname{im} p \subseteq \mathfrak{m} K_{\mathbf{a},I}.$$

Then $K_{\mathbf{a},I} = 0$ by the Nakayama's lemma, which means that equality holds in (5). □

Moreover, the regular sequence condition is necessary to get the converse in Theorem 2.4, as the following example shows.

**Example 2.6.** Let $(R, \mathfrak{m})$ be a regular local ring, and let $J$ be any $\mathfrak{m}$-primary ideal which is not generated by an $R$-sequence. Since the projective dimension of $R/J$ over $R$ is finite, it follows from a result of Vasconcelos [Vas67, Corollary 1] that $J/J^2$ is not free over $R/J$. Then by Proposition 2.5, equality does not hold in (6) when we let $I = J$. For instance, let $R = k[\![x, y]\!]$, where $k$ is any field and $x, y$ are indeterminates, $\mathfrak{m} = (x, y)R$, and $I = J = \mathfrak{m}^2$. Then the left hand side of (6) equals 12, while the right hand side equals 10.

**Remark 2.7.** Note that the condition $\mu(J) \geq 2$ is essential in Theorem 2.4. Indeed, suppose $J = (a)$ is principal. We have $K_{a,I} = ((0 : a) + I)/I$, so that equality holds in (5) if and only if $(0 : a) \subseteq I$. In particular, if $a$ is *any* $R$-regular element, whether $a$ belongs to $I$ or not, we have equality in (5). Of course, if $\mu(J) = 0$, then equality holds in (5) independently of $I$, as both sides vanish.

We do, however, get the following corollary to Theorem 2.4, which may be well-known, but we could not find it in the literature.

**Corollary 2.8.** *Let $R$ be a Cohen-Macaulay local ring of dimension $d \geq 2$, and let $J$ be an $\mathfrak{m}$-primary parameter ideal. Then*

$$\lambda_R(R/J^2) = (d + 1) \cdot \lambda_R(R/J).$$

*Proof.* Note that $J$ is generated by a regular sequence of length $d$. Then letting $I = J$ and $\ell = d$ in Theorem 2.4 gives the result. □

## 3. Tight closure and Hilbert-Kunz multiplicity

In this section, we find analogues of the results of the previous section in prime characteristic, which allows us to look at ideals "up to tight closure," replacing colength with Hilbert-Kunz multiplicity, and replacing minimal number of generators with $*$-spread. For background and unexplained terminology on tight closure theory, see the monograph [Hun96]. In order that tight closure and Hilbert-Kunz multiplicity are well behaved, we make the following blanket assumptions: $(R, \mathfrak{m})$ is an excellent $d$-dimensional Noetherian local ring of prime characteristic $p > 0$, and the $\mathfrak{m}$-adic completion of $R$ is reduced and equidimensional – *i.e.* $R$ is *quasi-unmixed*. Let $q = p^e$ be a varying power of $p$. Recall that for an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ in such a ring, the $q$th *bracket power* $\mathfrak{a}^{[q]}$ is defined as the ideal generated by the $q$th powers of all the elements of $\mathfrak{a}$. It may also be defined by choosing a generating set for $\mathfrak{a}$ and raising these generators to $q$th powers. The *Hilbert-Kunz* multiplicity of such an ideal (which exists by [Mon83]) is then given by

$$e_{\mathrm{HK}}(\mathfrak{a}) := \lim_{q\to\infty} \frac{\lambda_R(R/\mathfrak{a}^{[q]})}{q^d}.$$

Now let both $I$ and $J$ be $\mathfrak{m}$-primary ideals, where $J$ is generated by a sequence $\mathbf{a} = a_1, \ldots, a_\ell$. Replacing $I$ by $I^{[q]}$, the $a_j$ by $a_j^q$, and $M$ by $J^{[q]}$, and plugging into (4), we get

$$\ell \cdot \lambda_R(R/I^{[q]}) + \lambda_R(R/J^{[q]}) = \lambda_R(K_{\mathbf{a}^q, I^{[q]}}) + \lambda_R(R/(IJ)^{[q]}).$$

Dividing by $q^d$ and taking the limit as $q \to \infty$, we have

$$\ell \cdot e_{\mathrm{HK}}(I) + e_{\mathrm{HK}}(J) = \lim_{q\to\infty} \frac{\lambda_R(K_{\mathbf{a}^q, I^{[q]}})}{q^d} + e_{\mathrm{HK}}(IJ). \tag{7}$$

Moreover, we can replace $J$ by a *minimal* $*$*-reduction* of $J$ – that is, an ideal contained in $J$, which has the same tight closure as $J$, and which is minimal with respect to this property. The first named author proved in [Eps05, Proposition 2.1 and Lemma 2.2] that under the given conditions on $R$, such an ideal always exists, and that its minimal number of generators is between $\operatorname{ht} J$ and $\mu(J)$. Diverging a bit from the terminology of [Eps05], we define the $*$*-spread* $\ell^*(\mathfrak{a})$ of an ideal $\mathfrak{a}$ to be the minimum among the minimal numbers of generators of all minimal $*$-reductions of $\mathfrak{a}$. To see that we can replace $J$ by an arbitrary minimal $*$-reduction (and hence one with $\ell^*(J)$ generators), use the fact that Hilbert-Kunz multiplicity is invariant up to tight closure (by [HH90, Theorem 8.17]) and since for any $*$-reduction $K$ of $J$, we have $(IK)^* = (IJ)^*$. Therefore, we get the following result.

**Proposition 3.1.** *Let $(R, \mathfrak{m})$ be a quasi-unmixed excellent Noetherian local ring of characteristic $p > 0$, and let $I$, $J$ be $\mathfrak{m}$-primary ideals. Then*

$$e_{\mathrm{HK}}(IJ) \le \ell^*(J) \cdot e_{\mathrm{HK}}(I) + e_{\mathrm{HK}}(J). \tag{8}$$

We obtain immediately the following two corollaries.

**Corollary 3.2.** *Let $(R, \mathfrak{m})$ be a quasi-unmixed excellent Noetherian local ring of characteristic $p > 0$, and let $I$, $J$ be $\mathfrak{m}$-primary ideals. Then*

$$e_{\mathrm{HK}}(IJ) \le \min\{\ell^*(J) \cdot e_{\mathrm{HK}}(I) + e_{\mathrm{HK}}(J),\ e_{\mathrm{HK}}(I) + \ell^*(I) \cdot e_{\mathrm{HK}}(J)\}.$$

**Corollary 3.3.** *Let $(R, \mathfrak{m})$ be a quasi-unmixed excellent Noetherian local ring of characteristic $p > 0$, let $I$ be an $\mathfrak{m}$-primary ideal, and let $\ell = \ell^*(I)$. Then*

$$e_{\mathrm{HK}}(I^n) \le (1 + \ell + \cdots + \ell^{n-1}) \cdot e_{\mathrm{HK}}(I).$$

We next give a necessary condition for equality in (8). Note that $\ell^*(J) \ge 2$ whenever the dimension is at least 2.

**Theorem 3.4.** *Let $(R, \mathfrak{m})$ be a quasi-unmixed excellent Noetherian local ring of characteristic $p > 0$, and let $I$, $J$ be $\mathfrak{m}$-primary ideals such that $\ell^*(J) \geq 2$. Then the equality*

$$e_{\mathrm{HK}}(IJ) = \ell^*(J) \cdot e_{\mathrm{HK}}(I) + e_{\mathrm{HK}}(J)$$

*implies that $J \subseteq I^*$.*

*Proof.* If the statement holds for some minimal $*$-reduction of $J$, then it will hold for $J$ itself. Hence, we may pass to a minimal $*$-reduction of $J$, in which case we may let $\ell = \ell^*(J) = \mu(J)$. Let $(\mathbb{K}_\bullet(\mathbf{a}^q), \partial_{\bullet,q})$ be the Koszul complex on the sequence $\mathbf{a}^q := a_1^q, \dots, a_\ell^q$, where $\mathbf{a}$ is a minimal generating set for $J$. We have $\partial_{1,q} = i \circ \pi_{\mathbf{a}^q}$, where $i : J^{[q]} \hookrightarrow R$ is the natural inclusion. Of course we have $\operatorname{im} \partial_{2,q} \subseteq \ker \partial_{1,q} = K_{\mathbf{a}^q}$. So suppose we have equality in (8). Then by Equation 7, we have

$$\lim_{q \to \infty} \frac{\lambda_R(K_{\mathbf{a}^q, I^{[q]}})}{q^d} = 0.$$

Recall from the discussion preceding Theorem 2.4 that $\operatorname{im} \partial_{2,q}$ is a sum of cyclic modules of the form $C_{ijq} := R \cdot v_{ij}(\mathbf{a}^q)$ for every pair $i, j$ with $1 \leq i < j \leq \ell$, where $v_{ij}(\mathbf{a}^q) := -a_j^q e_i + a_i^q e_j$, where $e_1, \dots, e_\ell$ is the canonical free basis for $R^\ell$. Recall also that $C_{ijq} \subseteq K_{\mathbf{a}^q}$. Thus, we have

$$K_{\mathbf{a}^q, I^{[q]}} = \frac{K_{\mathbf{a}^q} + I^{[q]} R^\ell}{I^{[q]} R^\ell} \supseteq \frac{C_{ijq} + I^{[q]} R^\ell}{I^{[q]} R^\ell} \cong \frac{R}{I^{[q]} R^\ell :_R v_{ij}(\mathbf{a}^q)} = \frac{R}{(I^{[q]} : a_i^q) \cap (I^{[q]} : a_j^q)},$$

which maps onto $R/(I^{[q]} : a_i^q)$. Hence,

$$e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(I + (a_i)) = \lim_{q \to \infty} \frac{\lambda_R\left(\frac{I^{[q]} + (a_i^q)}{I^{[q]}}\right)}{q^d} = \lim_{q \to \infty} \frac{\lambda_R(R/(I^{[q]} : a_i^q))}{q^d} \leq \lim_{q \to \infty} \frac{\lambda_R(K_{\mathbf{a}^q, I^{[q]}})}{q^d} = 0.$$

By [HH90, Theorem 8.17] then, $a_i \in I^*$. Since this holds for all $i$, we have $J \subseteq I^*$. ☐

Next, we provide an analogue of the converse statement from Theorem 2.4. For this, *assume $J$ is generated by a system of parameters* $\mathbf{a} = a_1, \dots, a_d$, and recall that in this case, the Koszul complex $\mathbb{K}_\bullet(\mathbf{a})$ is *stably phantom acyclic* [AHH93, Proposition 5.4(d) and Proposition 5.19]. In particular, this means that $K_{\mathbf{a}^q} = \ker \partial_{1,q} \subseteq (\operatorname{im} \partial_{2,q})^*_{R^d}$, where for an inclusion $L \subseteq M$ of $R$-modules, $L^*_M$ denotes the tight closure of $L$ in $M$ [HH90]. But since the entries of the matrix corresponding to $\partial_{2,q}$ are in $J^{[q]}$, we have $(\operatorname{im} \partial_{2,q})^*_{R^d} \subseteq (J^{[q]} R^d)^*_{R^d} = (J^{[q]})^* R^d$. So we have $K_{\mathbf{a}^q} \subseteq (J^{[q]})^* R^d$. Therefore,

$$K_{\mathbf{a}^q, I^{[q]}} = \frac{K_{\mathbf{a}^q} + I^{[q]} R^d}{I^{[q]} R^d} \subseteq \frac{((J^{[q]})^* + I^{[q]}) R^d}{I^{[q]} R^d} \subseteq \frac{((I + J)^{[q]})^* R^d}{I^{[q]} R^d}.$$

Thus,

$$\lim_{q\to\infty} \frac{\lambda_R(K_{\mathbf{a}^q, I^{[q]}})}{q^d} \le \lim_{q\to\infty} \frac{\lambda_R\left(\dfrac{((I+J)^{[q]})^* R^d}{I^{[q]} R^d}\right)}{q^d}$$
$$= d \cdot \left( \lim_{q\to\infty} \frac{\lambda_R(R/I^{[q]})}{q^d} - \lim_{q\to\infty} \frac{\lambda_R(R/((I+J)^{[q]})^*)}{q^d} \right).$$

Since the latter of the two limits given is the Hilbert-Kunz multiplicity of $I+J$ when $R$ has a test element [CE04, Remark 2.6], which in turn follows from our blanket assumptions when $R$ is reduced [HH94, Theorem 6.1(a)], we get

$$\lim_{q\to\infty} \frac{\lambda_R(K_{\mathbf{a}^q, I^{[q]}})}{q^d} \le d \cdot (e_{\mathrm{HK}}(I) - e_{\mathrm{HK}}(I+J)).$$

Combining this with Equation 7, we obtain the displayed inequality in the following result. Note that if $J$ has the same tight closure as a parameter ideal, then $\ell^*(J) = d$.

**Theorem 3.5.** *Let $(R, \mathfrak{m})$ be a quasi-unmixed excellent reduced Noetherian local ring of characteristic $p > 0$ of dimension $d \ge 2$, and let $I$, $J$ be $\mathfrak{m}$-primary ideals such that $J$ has the same tight closure as a parameter ideal. Then*

$$e_{\mathrm{HK}}(IJ) \ge d \cdot e_{\mathrm{HK}}(I+J) + e_{\mathrm{HK}}(J). \tag{9}$$

*If $J \subseteq I^*$, then equality holds in (9), in particular equality holds in (8).*

*Proof.* All that remains is to prove the last statement. Since we are assuming $J \subseteq I^*$, we have $(I+J)^* = I^*$, so that $e_{\mathrm{HK}}(I) = e_{\mathrm{HK}}(I+J)$. Then combine Inequality 9 with Inequality 8 to obtain equality. □

**Corollary 3.6.** *Let $(R, \mathfrak{m})$ be a quasi-unmixed excellent reduced Noetherian local ring of characteristic $p > 0$ of dimension $d \ge 2$, and let $I$, $J$ be $\mathfrak{m}$-primary ideals such that $J$ has the same tight closure as a parameter ideal. Then equality holds in (8) if and only if $J \subseteq I^*$.*

**Corollary 3.7.** *Let $(R, \mathfrak{m})$ be a quasi-unmixed reduced excellent Noetherian local ring of characteristic $p > 0$ of dimension $d \ge 2$, and let $J$ be an $\mathfrak{m}$-primary parameter ideal. Then*

$$e_{\mathrm{HK}}(J^2) = (d+1) \cdot e_{\mathrm{HK}}(J) = (d+1) \cdot e(J).$$

**Example 3.8.** Let $(R, \mathfrak{m})$ be a regular local ring. Since (6) is equivalent to (8), Example 2.6 shows that if $J$ is an $\mathfrak{m}$-primary ideal, equality does not hold in (8) with $I = J$ unless $J$ is generated by a regular sequence (*i.e.* unless it is a parameter ideal).

## 4. Revisiting some old results

Recall the following theorem, stated slightly differently here than in the original paper:

**Theorem 4.1** (Special case of [EV08, Theorem 1])**.** *Let $(R, \mathfrak{m}, k)$ be an analytically irreducible excellent local ring of characteristic $p > 0$ such that $k = \kappa(\bar{R})$, where $\bar{R}$ is the normalization of $R$. Let $I$, $J$ be $\mathfrak{m}$-primary ideals. Then there exists $q_0$ such that for all $q \geq q_0$,*

$$e_{\mathrm{HK}}(IJ^{[q]}) = \ell^*(J) \cdot e_{\mathrm{HK}}(I) + e_{\mathrm{HK}}(J^{[q]}).$$

In other words (since the equality $\ell^*(J) = \ell^*(J^{[q]})$ always holds), given such a ring $R$, the inequality (8) becomes an equality *when $J$ is replaced by a sufficiently high bracket power!* We don't know how to prove this for an arbitrary quasi-unmixed reduced excellent local ring $R$, except in the case where $J$ is a parameter ideal. However, we can recover the result from [EV08] for such a general ring in case $J$ is a parameter ideal. In this case, every $J^{[q]}$ is also, of course, a parameter ideal, and there exists some $q_0$ such that $J^{[q_0]} \subseteq I$, since both $I$ and $J$ are $\mathfrak{m}$-primary. Thus, by Theorem 3.5, we get

**Proposition 4.2.** *Let $(R, \mathfrak{m}, k)$ be a quasi-unmixed reduced excellent Noetherian local ring of characteristic $p > 0$ and dimension $d$, let $I, J$ be $\mathfrak{m}$-primary ideals such that $J$ has the same tight closure as a system of parameters. Let $q_0$ be a power of $p$ such that $J^{[q_0]} \subseteq I$. Then for all $q \geq q_0$, we have*

$$e_{\mathrm{HK}}(IJ^{[q]}) = d \cdot e_{\mathrm{HK}}(I) + e_{\mathrm{HK}}(J^{[q]}).$$

Finally, we re-prove the following result of Huneke and Yao, which was used in service of a proof [HY02, Theorem 3.1] that excellent quasi-unmixed local rings with Hilbert-Kunz multiplicity 1 must be regular, a theorem originally due to Watanabe and Yoshida [WY00, Theorem 1.5]. It can also be considered as a characteristic $p > 0$ analogue of Lech's inequality [Lec60, Theorem 3] which states that for an $\mathfrak{m}$-primary ideal $I$ in a Noetherian local ring of dimension $d$, the Hilbert-Samuel multiplicity $e(I)$ is bounded above by $d! \cdot e(R) \cdot \lambda_R(R/I)$. Unlike our proof, Huneke and Yao use a filtration argument. Recall that a prime characteristic $p > 0$ reduced ring $R$ is said to be *$F$-finite* if the ring $R^{1/p}$ is finitely generated as a module over $R$ via the obvious inclusion map. In this case it also follows that $R^{1/q}$ is module-finite over $R$, for all powers $q$ of $p$.

**Proposition 4.3** (Special case of [HY02, Corollary 2.2(b)])**.** *Let $(R, \mathfrak{m}, k)$ be a reduced Noetherian local ring of dimension $d$ and positive characteristic $p > 0$. Suppose in addition that $R$ is $F$-finite. Then for any $\mathfrak{m}$-primary ideal $I$, we have*

$$e_{\mathrm{HK}}(I) \leq e_{\mathrm{HK}}(R) \cdot \lambda_R(R/I).$$

*Proof.* In Proposition 2.1, let $M = R^{1/q}$ where $q$ is a power of $p$. Therefore,

$$\lambda_R(M/IM) = \lambda_R(R^{1/q}/IR^{1/q}) = [k^{1/q} : k] \cdot \lambda_{R^{1/q}}(R^{1/q}/IR^{1/q}) = [k : k^{1/q}] \cdot \lambda_R(R/I^{[q]}),$$

where the notation $[k^{1/q}:k]$ denotes the field extension degree. Also note that

$$\begin{aligned}\mu(M) &= \lambda_R(M/\mathfrak{m}M) = \lambda_R(R^{1/q}/\mathfrak{m}R^{1/q}) \\ &= [k^{1/q}:k]\cdot\lambda_{R^{1/q}}(R^{1/q}/\mathfrak{m}R^{1/q}) = [k^{1/q}:k]\cdot\lambda_R(R/\mathfrak{m}^{[q]}).\end{aligned}$$

Thus by Proposition 2.1, after dividing both sides by $[k^{1/q}:k]$, we obtain

$$\lambda_R(R/I^{[q]}) \le \lambda_R(R/\mathfrak{m}^{[q]})\cdot\lambda_R(R/I).$$

Now the result follows by dividing both sides of the above inequality by $q^d$ and taking limits as $q\to\infty$. □

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