The equivariant-pullback invariance of associating noncommutative vector bundles

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Abstract. For any finite-dimensional corepresentation V of a coalgebra $\mathcal{C}$ coacting principally on an algebra $\mathcal{A}$, we can form an associated finitely generated projective module $\mathcal{A}\Box V$ over the coaction-invariant subalgebra $\mathcal{B}$. The module $\mathcal{A}\Box V$ is the section module of the associated noncommutative vector bundle. Any equvariant (colinear) algebra homomorphism $\mathcal{A}\to\mathcal{A}'$, where $\mathcal{A}'$ is an algebra with a principal coaction of $\mathcal{C}$ and the coaction-invariant subalgebra $\mathcal{B}'$, restricts and corestricts to an algebra homomorphism $\mathcal{B}\to\mathcal{B}'$ making $\mathcal{B}$ a $(\mathcal{B}-\mathcal{B}')$-bimodule. Our main result is that the finitely generated left $\mathcal{B}'$-modules $\mathcal{B}'\otimes_{\mathcal{B}}(\mathcal{A}\Box V)$ and $\mathcal{A}'\Box V$ are *isomorphic*. As a corollary, we conclude that, for any equivariant *-homomorphism $f : A\to A'$ between unital C*-algebras equipped with a free action of a compact quantum group, the induced K-theory map $f_*\colon K_0(B)\to K_0(B')$, where B and B' are the respective fixed-point subalgebras, satisfies $f_*([A\Box V]) = [A'\Box V]$. As a key application, we show that any finitely-iterated equivariant noncommutative join of $SU_q(2)$ with itself is *not* trivializable as an $SU_q(2)$-compact quantum principal bundle.

Our result is motivated by the search of K_0-invariants. The main idea is to use equivariant homomorphisms to facilitate computations of such invariants by moving them from more complicated to simpler algebras. This strategy was recently successfully applied in [12] to distinguish the K_0-classes of noncommutative line bundles over two different types of quantum complex projective spaces. Herein we generalize from associated noncommutative line bundles to associated noncommutative *vector* bundles. Then we apply our general result to a vector bundle associated through the fundamental represention of $SU_q(2)$ with a finitely-iterated equivariant noncommutative join of $SU_q(2)$ with itself. Thus we prove the Borsuk-Ulam-type conjecture [1, Conjecture 2.3 type 2] in this case: there does *not* exist an equivariant *-homomorphism from the C*-algebra $C(SU_q(2))$ to the C*-algebra of the finitely-iterated equivariant noncommutative join of $C(SU_q(2))$ with itself.

A classical argument proving the non-triviality of a vector bundle associated with a principal bundle by restricting the vector bundle to an appropriate subspace uses the fact that the thus restricted vector bundle is associated with a restricted principal bundle. The latter restriction is encoded by an equivariant map of total spaces of principal bundles, which induces a natural transformation of Chern characters of the vector bundles in question. The present paper generalizes this reasoning to the noncommutative setting.

We begin by stating our main result in the standard and easily accessible Hopf-algebraic setting. Then we state and prove two slightly different coalgebraic versions of the result: one based on faithful flatness and coflatness, and one based on Chern-Galois theory [3]. Next we use the Peter-Weyl functor to make the result applicable to free actions of compact quantum groups on unital C*-algebras [2]. In the C*-algebraic setting we consider our example and main application: the finitely-iterated equivariant noncommutative join of $C(SU_q(2))$ with itself.

1 Modules associated with Galois-type coactions

Let $\mathcal{C}$ be a coalgebra, $\delta_M\colon M \to M\otimes\mathcal{C}$ a right coaction, and ${}_N\delta\colon N \to \mathcal{C}\otimes N$ a left coaction. The *cotensor* product of M with N is $M\Box^C N := \ker(\delta_M\otimes\mathrm{id} - \mathrm{id}\otimes{}_N\delta)$. In what follows, we will also use the Heyneman-Sweedler notation (with the summation sign suppressed) for comultiplications and right coactions:

$$\Delta(c) =: c_{(1)}\otimes c_{(2)}, \quad \delta_M(m) =: m_{(0)}\otimes m_{(1)}\,.$$

Next, let $\mathcal{H}$ be a Hopf algebra with bijective antipode S, comultiplication Δ, counit ε. Also, let $\delta_{\mathcal{A}}\colon \mathcal{A}\to\mathcal{A}\otimes\mathcal{H}$ be a coaction rendering $\mathcal{A}$ a right $\mathcal{H}$-comodule algebra. The subalgebra of coaction invariants $\{b\in\mathcal{A} \mid \delta_{\mathcal{A}}(b) = b\otimes 1\}$ is called the coaction- invariant (or fixed-point) subalgebra. We say that $\mathcal{A}$ is a *principal* comodule algebra iff there exists a *strong connection* [10, 7], i.e., a unital linear map $\ell:\mathcal{H}\to\mathcal{A}\otimes\mathcal{A}$ satisfying:

(1) $(\mathrm{id}\otimes\delta_{\mathcal{A}})\circ\ell = (\ell\otimes\mathrm{id})\circ\Delta$, $\big(((S^{-1}\otimes\mathrm{id})\circ\mathrm{flip}\circ\delta_{\mathcal{A}})\otimes\mathrm{id}\big)\circ\ell = (\mathrm{id}\otimes\ell)\circ\Delta$;

(2) $m\circ\ell = \varepsilon$, where $m\colon\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$ is the multiplication map.

Let $\mathcal{H}$ be a Hopf algebra with bijective antipode. In [11] the principality of an $\mathcal{H}$-comodule algebra was defined by requiring the bijectivity of the canonical map (see Definition 1.1 (1)) and equivariant projectivity (see Definition 1.2 (2)). One can prove (see [5] and references therein) that an $\mathcal{H}$-comodule algebra is principal in this sense if and only if it admits a strong connection. Therefore we will treat the existence of a strong connection as a condition defining the principality of a comodule algebra and avoid the original definition of a principal comodule algebra. The latter is important when going beyond coactions that are algebra homomorphisms — then the existence of a strong connection is implied by principality but we do not have the reverse implication [3].

Theorem 1.1. *Let $\mathcal{A}$ and $\mathcal{A}'$ be right $\mathcal{H}$-comodule algebras for a Hopf algebra $\mathcal{H}$ with bijective antipode, and V be a finite-dimensional left $\mathcal{H}$-comodule. Denote by $\mathcal{B}$ and $\mathcal{B}'$ the respective coaction-invariant subalgebras. Assume that $\mathcal{A}$ is principal and that there exists an $\mathcal{H}$-equivariant algebra homomorphism $f\colon\mathcal{A}\to\mathcal{A}'$. The restriction-corestriction of f to $\mathcal{B}\to\mathcal{B}'$ makes $\mathcal{B}'$ a $(\mathcal{B}'-\mathcal{B})$-bimodule such that the associated left $\mathcal{B}'$-modules $\mathcal{B}'\otimes_B(\mathcal{A}\Box^{\mathcal{C}}V)$ and $\mathcal{A}'\Box^{\mathcal{C}}V$ are iso-*morphic. *In particular, the induced map $f_*\colon K_0(\mathcal{B})\to K_0(\mathcal{B}')$ satisfies $f_*([\mathcal{A}\Box^{\mathcal{C}}V]) = [\mathcal{A}'\Box^{\mathcal{C}}V]$.*

As will be explained later on, the above Theorem 1.1 specializes Theorem 1.3, and Theorem 2.1 is a common denominator of Theorem 1.2 and Theorem 1.1.

1.1 Faithfully flat coalgebra-Galois extensions

Definition 1.1. [5] Let $\mathcal{C}$ be a coalgebra coaugmented by a group-like element $e\in\mathcal{C}$, and $\mathcal{A}$ an algebra and a right $\mathcal{C}$-comodule via $\delta_{\mathcal{A}}:\mathcal{A}\to\mathcal{A}\otimes\mathcal{C}$. Put $\mathcal{B} := \{b\in\mathcal{A} \mid \delta_{\mathcal{A}}(b) = b\otimes e\}$ (coaction-invariant subalgebra). We say that the inclusion $\mathcal{B}\subseteq\mathcal{A}$ is an *e-coaugmented $\mathcal{C}$-Galois extension* iff

(1) the *canonical map* $can:\mathcal{A}\otimes_{\mathcal{B}}\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$, $a\otimes a'\mapsto a\delta_{\mathcal{A}}(a')$ is bijective,

(2) $\delta_{\mathcal{A}}(1) = 1\otimes e$.

Theorem 1.2. *Let $\mathcal{B}\subseteq\mathcal{A}$ and $\mathcal{B}'\subseteq\mathcal{A}'$ be e-coaugmented $\mathcal{C}$-Galois extensions, let V be a left $\mathcal{C}$-comodule. Assume that $\mathcal{A}'$ is faithfully flat as a left $\mathcal{B}'$-module and that coalgebra $\mathcal{C}$ is cosemisimple. Then every $\mathcal{C}$-equivariant algebra map $f:\mathcal{A}\to\mathcal{A}'$ restricts and corestricts to an algebra homomorphism $\mathcal{B}\to\mathcal{B}'$, and induces an isomorphism*

$$\mathcal{B}'\otimes_{\mathcal{B}}(\mathcal{A}\Box^{\mathcal{C}}V)\cong\mathcal{A}'\Box^{\mathcal{C}}V$$

of left $\mathcal{B}'$-modules that is natural in V.

Proof. Since $\mathcal{A}'$ is a faithfully flat right $\mathcal{B}'$-module, the map of left $\mathcal{B}'$-modules right $\mathcal{C}$-comodules

$$\widetilde{f} := m_{\mathcal{A}'} \circ (\mathrm{id}_{\mathcal{B}'} \otimes_{\mathcal{B}} f)\colon \mathcal{B}' \otimes_{\mathcal{B}} \mathcal{A} \longrightarrow \mathcal{A}', \quad m_{\mathcal{A}'}(b' \otimes a') = b'a',$$

is an isomorphism if and only if the map of left $\mathcal{A}'$-modules right $\mathcal{C}$-comodules

$$\mathrm{id}_{\mathcal{A}'} \otimes_{\mathcal{B}'} \widetilde{f}\colon \mathcal{A}' \otimes_{\mathcal{B}'} \mathcal{B}' \otimes_{\mathcal{B}} \mathcal{A} \longrightarrow \mathcal{A}' \otimes_{\mathcal{B}'} \mathcal{A}'$$

is an isomorphism. Replacing the left-hand-side $\mathcal{B}'$ by $\mathcal{A}$, the latter is an isomorphism if and only if

$$\mathrm{id}_{\mathcal{A}'} \otimes_{A} (f \otimes_{\mathcal{B}'} f)\colon \mathcal{A}' \otimes_{\mathcal{A}} \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \longrightarrow \mathcal{A}' \otimes_{\mathcal{B}'} \mathcal{A}'$$

is an isomorphism. Thus, from the commutativity of the diagram

$$\begin{array}{ccc}
\mathcal{A}' \otimes_{\mathcal{A}} \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} & \xrightarrow{\mathrm{id}_{\mathcal{A}'}\otimes_{\mathcal{A}}(f\otimes_{\mathcal{B}'} f)} & \mathcal{A}' \otimes_{\mathcal{B}'} \mathcal{A}' \\
\downarrow{\scriptstyle \mathrm{id}_{\mathcal{A}'}\otimes_{\mathcal{A}}\, can} & & \downarrow{\scriptstyle can'} \\
\mathcal{A}' \otimes_{\mathcal{A}} \mathcal{A} \otimes \mathcal{C} & \xrightarrow{\cong} & \mathcal{A}' \otimes \mathcal{C}
\end{array}$$

and the bijectivity of the canonical maps, we infer that $\widetilde{f}$ is an isomorphism.

Furthermore, as $\widetilde{f}$ is a homomorphism of left $\mathcal{B}$-modules right $\mathcal{C}$-comodules, we conclude that

$$\widetilde{f}\,\Box^{\mathcal{C}}\mathrm{id}_V : (\mathcal{B}' \otimes_{\mathcal{B}} \mathcal{A})\Box^{\mathcal{C}} V \longrightarrow \mathcal{A}'\Box^{\mathcal{C}} V$$

is an isomorphism of left $\mathcal{B}'$-modules. Finally, since $\mathcal{C}$ is cosemisimple, and any comodule over a cosemisimple coalgebra is injective [6, Theorem 3.1.5 (iii)], whence coflat [6, Theorem 2.4.17 (i)-(iii)], the balanced tensor product $\mathcal{B}' \otimes_{\mathcal{B}} (-)$ and the cotensor product $(-)\Box^{\mathcal{C}} V$ commute. Therefore, there is a natural in V isomorphism of left $\mathcal{B}'$-modules

$$\mathcal{B}'\otimes_{\mathcal{B}}(\mathcal{A}\Box^{\mathcal{C}} V) \cong (\mathcal{B}'\otimes_{\mathcal{B}}\mathcal{A})\Box^{\mathcal{C}} V \cong \mathcal{A}'\Box^{\mathcal{C}} V,$$

as claimed. ■

1.2 Principal coactions

Definition 1.2. [3] Let $\mathcal{B} \subseteq \mathcal{A}$ be an e-coaugmented $\mathcal{C}$-Galois extension. We call such an extension a *principal $\mathcal{C}$-extension* iff

(1) $\psi : \mathcal{C} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{C}$, $c \otimes a \mapsto can(can^{-1}(1 \otimes c)a)$ is bijective (invertibility of the *canonical entwining*),

(2) there exists a left $\mathcal{B}$-linear right $\mathcal{C}$-colinear splitting of the multiplication map $\mathcal{B} \otimes \mathcal{A} \to \mathcal{A}$ (*equivariant projectivity*).

Using the invertibility of the canonical entwining ψ, one can define a left coaction

$${}_{\mathcal{A}}\delta : \mathcal{A} \to \mathcal{C} \otimes \mathcal{A}, \quad {}_{\mathcal{A}}\delta(a) = \psi^{-1}(a \otimes e). \tag{1}$$

Lemma 1.1. *Let $\mathcal{A}$ and $\mathcal{A}'$ be e-coaugmented $\mathcal{C}$-Galois extensions with invertible canonical entwinings. Then, if an algebra map $f\colon \mathcal{A} \to \mathcal{A}'$ is right $\mathcal{C}$-colinear, it is also left $\mathcal{C}$-colinear.*

Proof. If $f\colon \mathcal{A} \to \mathcal{A}'$ is a $\mathcal{C}$-colinear algebra homomorphism, then it intertwines the canonical maps can and can' of $\mathcal{A}$ and $\mathcal{A}'$ respectively in the following way:

$$(f \otimes \mathrm{id}_{\mathcal{C}}) \circ can = can' \circ (f \otimes_{\mathcal{B}} f). \tag{2}$$

Since both can and can' are invertible, this implies that

$$(can')^{-1}\circ(f\otimes \mathrm{id}_{\mathcal{C}})=(f\otimes_{\mathcal{B}} f)\circ can^{-1}. \tag{3}$$

Therefore, canonical entwinings ψ and ψ' are related as follows:

$$\begin{aligned}
\big((f\otimes \mathrm{id}_{\mathcal{C}})\circ\psi\big)(c\otimes a) &= (f\otimes \mathrm{id}_{\mathcal{C}})\Big(can\big((can^{-1}(1\otimes c)a)\big)\Big) && (4)\\
&= \big(can'\circ(f\otimes_{\mathcal{B}} f)\big)\big(can^{-1}(1\otimes c)a\big) && (5)\\
&= can'\big((f\otimes_{\mathcal{B}} f)\big(can^{-1}(1\otimes c)a\big)\big) && (6)\\
&= can'\big((f\otimes_{\mathcal{B}} f)\big(can^{-1}(1\otimes c)\big)f(a)\big) && (7)\\
&= can'\Big(\big((can')^{-1}(f(1)\otimes c)\big)f(a)\Big) && (8)\\
&= can'\Big(\big((can')^{-1}(1\otimes c)\big)f(a)\Big) && (9)\\
&= \psi'(c\otimes f(a)) && (10)\\
&= \big(\psi'\circ(\mathrm{id}_{\mathcal{C}}\otimes f)\big)(c\otimes a). && (11)
\end{aligned}$$

As c and a are arbitrary, we conclude that

$$(f\otimes \mathrm{id}_{\mathcal{C}})\circ\psi=\psi'\circ(\mathrm{id}_{\mathcal{C}}\otimes f). \tag{12}$$

Now it follows from the invertibility of the entwinings that

$$(\psi')^{-1}\circ(f\otimes \mathrm{id}_{\mathcal{C}})=(\mathrm{id}_{\mathcal{C}}\otimes f)\circ\psi^{-1}. \tag{13}$$

Finally, evaluating the above equation on $a\otimes e$, we get

$$\big((\psi')^{-1}\circ(f\otimes \mathrm{id}_{\mathcal{C}})\big)(a\otimes e)=\big((\mathrm{id}_{\mathcal{C}}\otimes f)\circ\psi^{-1}\big)(a\otimes e), \tag{14}$$

which reads

$$\big({}_{\mathcal{A}'}\delta\circ f\big)(a)=\big((\mathrm{id}_{\mathcal{C}}\otimes f)\circ{}_{\mathcal{A}}\delta\big)(a). \tag{15}$$

Since a is arbitrary, we infer the left $\mathcal{C}$-colinearity of f as claimed. ■

Note that in the Hopf-algebraic setting of comodule algebras, the invertibility of the canonical entwining ψ is equivalent to the bijectivity of the antipode S. Then the left-coaction formula (1) reads ${}_{\mathcal{A}}\delta=(S^{-1}\otimes\mathrm{id})\circ\mathrm{flip}\circ\delta_{\mathcal{A}}$, and the above lemma is trivially true.

Next, let us consider $\mathcal{A}\otimes\mathcal{A}$ as a $\mathcal{C}$-bicomodule via the right coaction $\mathrm{id}\otimes\delta_{\mathcal{A}}$ and the left coaction ${}_{\mathcal{A}}\delta\otimes\mathrm{id}$, and $\mathcal{C}$ as a $\mathcal{C}$-bicomodule via its comultiplication.

Definition 1.3. A *strong connection* is a $\mathcal{C}$-bicolinear map $\ell:\mathcal{C}\to\mathcal{A}\otimes\mathcal{A}$ such that $\ell(e)=1\otimes 1$ and $m\circ\ell=\varepsilon$, where m and ε stand for the multiplication in $\mathcal{A}$ and the counit of $\mathcal{C}$, respectively.

It is clear that the above definition of a strong connection coincides with its Hopf-algebraic counterpart by choosing $e=1$ (see the beginning of Section 1).

Theorem 1.3. *Let $\mathcal{B}\subseteq\mathcal{A}$ and $\mathcal{B}'\subseteq\mathcal{A}'$ be principal $\mathcal{C}$-extensions, and V a finite-dimensional left $\mathcal{C}$-comodule. Then every $\mathcal{C}$-equivariant algebra map $f\colon\mathcal{A}\to\mathcal{A}'$ restricts and corestricts to an algebra homomorphism $\mathcal{B}\to\mathcal{B}'$, and induces an isomorphism of finitely generated projective left $\mathcal{B}'$-modules*

$$\mathcal{B}'\otimes_{\mathcal{B}}(\mathcal{A}\Box^{\mathcal{C}}V)\cong\mathcal{A}'\Box^{\mathcal{C}}V$$

that is natural in V. In particular, the induced map $f_*\colon K_0(\mathcal{B})\to K_0(\mathcal{B}')$ satisfies

$$f_*([\mathcal{A}\Box^{\mathcal{C}}V])=[\mathcal{A}'\Box^{\mathcal{C}}V].$$

Proof. Note first that combining [3, Lemma 2.2] with [3, Lemma 2.3] implies that a principal $\mathcal{C}$-extension always admits a strong connection:

$$\ell : \mathcal{C} \longrightarrow \mathcal{A} \otimes \mathcal{A}, \quad \sum_{\mu} a_\mu \otimes r_\mu(c) := \ell(c) =: \ell(c)^{\langle 1 \rangle} \otimes \ell(c)^{\langle 2 \rangle} \ \text{(summation suppressed)}, \tag{16}$$

where $\{a_\mu\}_\mu$ is a basis of $\mathcal{A}$. Given a unital linear functional $\varphi : \mathcal{A} \to \mathbb{C}$, one can construct [11] a left $\mathcal{B}$-linear map $\sigma : \mathcal{A} \to \mathcal{B}$,

$$\sigma(a) := a_{(0)} \ell\left(a_{(1)}\right)^{\langle 1 \rangle} \varphi\left(\ell\left(a_{(1)}\right)^{\langle 2 \rangle}\right), \tag{17}$$

such that $\sigma(b) = b$ for all $b \in \mathcal{B}$. For a finite-dimensional left $\mathcal{C}$-comodule V with a basis $\{v_i\}_i$, we define the coefficient matrix of the coaction $\varrho \colon V \to \mathcal{C} \otimes V$ with respect to $\{v_i\}_i$ by $\varrho(v_i) =: \sum_j c_{ij} \otimes v_j$. By [3, Theorem 3.1], we can now combine (16) and (17) with the c_{ij} to obtain a finite-size (say N) idempotent matrix e with entries

$$e_{(\mu,i)(\nu,j)} := \sigma(r_\mu(c_{ij})a_\nu) \in \mathcal{B}$$

such that $\mathcal{A}\Box^{\mathcal{C}}V \cong \mathcal{B}^N e$ as left $\mathcal{B}$-modules. Consequently, $\mathcal{A}\Box^{\mathcal{C}}V$ is finitely generated projective, and its class in $K_0(\mathcal{B})$ can be represented by e.

Since $f : \mathcal{A} \to \mathcal{A}'$ satisfies the assumptions of Lemma 1.1,

$$\ell' := (f \otimes f) \circ \ell : \mathcal{C} \longrightarrow \mathcal{A}' \otimes \mathcal{A}' \tag{18}$$

is a strong connection on $\mathcal{A}'$. Next, we choose bases $\{a_\mu \mid \mu \in J\}$ and $\{a'_\mu \mid \mu \in J'\}$ of $\mathcal{A}$ and $\mathcal{A}'$ respectively in such a way that

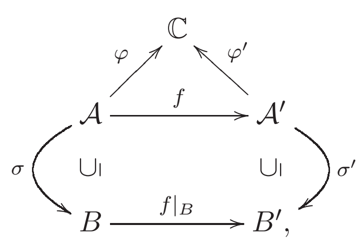
$$\{a'_\mu = f(a_\mu) \mid \mu \in I\} \text{ is a basis of } f(\mathcal{A}) \quad \text{and} \quad \{a_\mu \mid \mu \notin I\} \text{ is a basis of } \ker f.$$

Under the above choices, using (16) and (18) we compute

$$\sum_\mu a'_\mu \otimes r'_\mu(c) := \ell'(c) = \sum_\mu f(a_\mu) \otimes f(r_\mu(c)) = \sum_\mu a'_\mu \otimes f(r_\mu(c)).$$

Thus we obtain $r'_\mu(h) = f(r_\mu(h))$ for all $\mu \in I$.

Now we choose a unital functional φ' on $\mathcal{A}'$ and take $\varphi := \varphi' \circ f$. For σ' produced from φ' and ℓ' as in (17), we check that the diagram



commutes by the following calculation. First we compute

$$\begin{aligned}\sigma'(a') &= a'_{(0)} \ell'(a'_{(1)})^{\langle 1 \rangle} \varphi'\big(\ell'(a'_{(1)})^{\langle 2 \rangle}\big) \\ &= a'_{(0)} f\big(\ell(a'_{(1)})^{\langle 1 \rangle}\big) \varphi'\big(f(\ell(a'_{(1)})^{\langle 2 \rangle})\big) \\ &= a'_{(0)} f\big(\ell(a'_{(1)})^{\langle 1 \rangle}\big) \varphi\big(\ell(a'_{(1)})^{\langle 2 \rangle}\big).\end{aligned}$$

Next we plug in $a' = f(a)$ to get

$$\begin{aligned}\sigma'(f(a)) &= f(a)_{(0)} f\left(\ell\left(f(a)_{(1)}\right)^{\langle 1\rangle}\right)\varphi\left(\ell(f(a)_{(1)})^{\langle 2\rangle}\right)\\ &= f\left(a_{(0)}\right) f\left(\ell\left(a_{(1)}\right)^{\langle 1\rangle}\right)\varphi\left(\ell(a_{(1)})^{\langle 2\rangle}\right)\\ &= f\left(a_{(0)}\ell\left(a_{(1)}\right)^{\langle 1\rangle}\varphi\left(\ell(a_{(1)})^{\langle 2\rangle}\right)\right)\\ &= f(\sigma(a)).\end{aligned}$$

Hence

$$\begin{aligned}f\left(e_{(\mu,i)(\nu,j)}\right) &= f\left(\sigma\left(r_\mu(c_{ij})a_\nu\right)\right) = \sigma'\left(f\left(r_\mu(c_{ij})a_\nu\right)\right)\\ &= \sigma'\left(f\left(r_\mu(c_{ij})\right)f\left(a_\nu\right)\right) = \sigma'\left(r'_\mu(c_{ij})a'_\nu\right).\end{aligned}$$

Note that $f\left(e_{(\mu,i)(\nu,j)}\right)$ is zero for $\nu \notin I$ because then $f(a_\nu) = 0$.

Furthermore, applying [3, Theorem 3.1] to the strong connection ℓ', the basis $\{a'_\mu\}_\mu$, and the matrix coefficients c_{ij}, for all $\mu, \nu \in I$, $i, j \in \{1, \ldots, \dim V\}$, we obtain

$$\sigma'\left(r'_\mu(c_{ij})a'_\nu\right) =: e'_{(\mu,i)(\nu,j)}\,,$$

where the $e'_{(\mu,i)(\nu,j)}$ are the entries of an idempotent matrix e' such that $\mathcal{B}'^{N'}e' \cong \mathcal{A}'\Box^{\mathcal{C}}V$ as left $\mathcal{B}'$-modules. Thus, in the block matrix notation, we arrive at the following crucial equality

$$f(e) = \begin{pmatrix} e' & 0\\ d & 0\end{pmatrix},$$

where d is unspecified. Now, taking into account that $f(e)$ is an idempotent matrix, we derive the equality $de' = d$, which allows us to verify that

$$\begin{pmatrix} e' & 0\\ d & 0\end{pmatrix} = \begin{pmatrix} 1 & 0\\ d & 1\end{pmatrix}\begin{pmatrix} e' & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} 1 & 0\\ d & 1\end{pmatrix}^{-1}.$$

Hence the corresponding finitely generated projective left $\mathcal{B}'$-modules are isomorphic:

$$\mathcal{B}'\otimes_{\mathcal{B}}\left(A\Box^{\mathcal{C}}V\right) \cong \mathcal{B}'\otimes_{\mathcal{B}}\left(B^N e\right) \cong \mathcal{B}'^N f(e) \cong \mathcal{B}'^{N'}e' \cong \mathcal{A}'\Box^{\mathcal{C}}V.$$

In particular, $(f|_{\mathcal{B}})_*[e] := [f(e)] = [e'] \in K_0(\mathcal{B}')$. ■

The importance of the above theorem relies on a possibility to apply an explicit formula for the Chern-Galois character [3] to show that a given principal extension is not cleft [13, §8.2].

1.3 The Hopf-algebraic case revisited

We end this section by arguing that Theorem 1.3 specializes to Theorem 1.1 in the Hopf-algebraic setting. First, observe that the lacking assumption of the principality of $\mathcal{A}'$ in Theorem 1.1 is redundant. Indeed, if ℓ is a strong connection on $\mathcal{A}$ and $f\colon \mathcal{A}\to\mathcal{A}'$ is an $\mathcal{H}$-equivariant algebra homomorphism, then $(f\otimes f)\circ\ell$ is immediately a strong connection on $\mathcal{A}'$. Furthermore, the bijectivity of the antipode S is equivalent to the invertibility of the canonical entwining, and the coaugmentation is readily provided by $1\in\mathcal{H}$. Finally, as is explained at the beginning of Section 1, the existence of a strong connection implies both the bijectivity of the canonical map and equivariant projectivity.

2 Noncommutative vector bundles associated with free actions of compact quantum groups

Let (H, Δ) be a compact quantum group [15]. Let A be a unital C*-algebra and $\delta_A : A \to A \otimes_{\min} H$ an injective unital $*$-homomorphism. We call δ_A a *right coaction* of H on A (or a *right action of the compact quantum group on a compact quantum space*) iff

(1) $(\delta_A \otimes \mathrm{id}_H) \circ \delta_A = (\mathrm{id}_H \otimes \Delta) \circ \delta_A$ (coassociativity),

(2) $\{\delta_A(a)h \mid a \in A,\, h \in H\}^{\mathrm{cls}} = A \underset{\min}{\otimes} H$ (counitality).

Furthermore, a coaction δ_A is called *free* [9] iff

$$\{a\delta_A(\tilde{a}) \mid a, \tilde{a} \in A\}^{\mathrm{cls}} = A \underset{\min}{\otimes} H, \quad \text{where “cls” stands for “closed linear span”.}$$

Given a compact quantum group (H, Δ), we denote by $\mathcal{O}(H)$ its dense cosemisimple Hopf $*$-subalgebra spanned by the matrix coefficients of irreducible (or finite dimensional) unitary corepresentations [15]. In the same spirit, we define the *Peter-Weyl subalgebra* [2] of A as

$$\mathcal{P}_H(A) := \{\, a \in A \,|\, \delta_A(a) \in A \otimes \mathcal{O}(H)\,\}\,.$$

It follows from Woronowicz's definition of a compact quantum group that the left and right coactions of H on itself by the comultiplication are free. Also, it is easy to check that $\mathcal{P}_H(H) = \mathcal{O}(H)$, and $\mathcal{P}_H(H)$ becomes a right $\mathcal{O}(H)$-comodule via the restriction-corestriction of δ_A [2]. Its coaction-invariant subalegbra coincides with the fixed-point subalgebra

$$B := \{b \in A \mid \delta_A(b) = b \otimes 1\}.$$

A fundamental result concerning Peter-Weyl comodule algebras is that the freeness of an action of a compact quantum group (H, Δ) on a unital C*-algebra A is *equivalent* to the principality of the Peter-Weyl $\mathcal{O}(H)$-comodule algebra $\mathcal{P}_H(A)$ [2]. The result bridges algebra and analysis allowing us to conclude from Theorem 1.1 the following crucial claim.

Theorem 2.1. *Let (H, Δ) be a compact quantum group, let A and A' be (H, Δ)-C*-algebras, B and B' the corresponding fixed-point subalgebras, and $f : A \to A'$ an equivariant *-homomorphism. Then, if the coaction of (H, Δ) on A is free and V is a representation of (H, Δ), the following left B'-modules are isomorphic*

$$B' \otimes_B \left(\mathcal{P}_H(A) \Box^{\mathcal{O}(H)} V\right) \cong \mathcal{P}_H(A') \Box^{\mathcal{O}(H)} V.$$

In particular, if V is finite dimensional, then the induced map $f_* \colon K_0(B) \to K_0(B')$ satisfies

$$f_*([\mathcal{P}_H(A) \Box^{\mathcal{O}(H)} V]) = [\mathcal{P}_H(A') \Box^{\mathcal{O}(H)} V].$$

Proof. Note that, since $\mathcal{O}(H)$ is cosemisimple, any comodule is a direct sum of finite-dimensional comodules, so that it suffices to prove the theorem for finite-dimesional representations of (H, Δ). By [2], the freeness of the (H, Δ)-action is equivalent to principality of the Peter-Weyl comodule algebra $\mathcal{P}_H(A)$, which (as explained at the beginning of Section 1) is tantamount to the existence of a strong connection: $\ell : \mathcal{O}(H) \to \mathcal{P}_H(A) \otimes \mathcal{P}_H(A)$. Now the claim follows from Theorem 1.1 applied to the case $\mathcal{A} = \mathcal{P}_H(A)$, $\mathcal{H} = \mathcal{O}(H)$, and $\mathcal{B} = B$. ■

Recall that the existence of a strong connection implies equivariant projectivity, which (by [14]) is equivalent to faithful flatness. Combining this with the cosemisimplicity of $\mathcal{O}(H)$, we can view the above Theorem 2.1 as a specialization of Theorem 1.2.

2.1 Equivariant noncommutative join construction

Definition 2.1. [8] For any compact quantum group (H,Δ) acting freely on a unital C*-algebra A via $\delta_A\colon A\to A\otimes_{\min}H$, we define its *equivariant join* with H to be the unital C*-algebra

$$A\circledast^{\delta_A} H := \left\{ f\in C([0,1],A)C \;\middle|\; f(0)\in\mathbb{C}\otimes H,\ f(1)\in\delta_A(A)\right\}.$$

Theorem 2.2. *[2] The *-homomorphism*

$$\mathrm{id}\otimes\Delta\colon\ C([0,1],A)\underset{\min}{\otimes}H\ \longrightarrow\ C([0,1],A)\underset{\min}{\otimes}H\underset{\min}{\otimes}H$$

*restricts and corestricts to *-homomorphism*

$$\delta_\Delta\colon A\circledast^{\delta_A}H\longrightarrow (A\circledast^{\delta_A}H)\underset{\min}{\otimes}H$$

defining a free *action of the compact quantum group (H,Δ) on the equivariant join C*-algebra $A\circledast^{\delta_A}H$.*

The following proposition relates associated noncommutative vector bundles and the Borsuk-Ulam type 2 conjecture of non-existence of equivariant maps from an equivariant join with a compact quantum group to the compact quantum group group itself [1].

Proposition 2.1. *Let (H,Δ) be a compact quantum group acting freely on a unital C*-algebra A. Assume that there exist an equivariant *-homomorphism $F: A\to A'$ of (H,Δ)-C*-algebras and a finite-dimensional representation V of (H,Δ) such that the finitely generated projective module $\mathcal{P}_H(A'\circledast^{\delta_{A'}}H)\Box^{\mathcal{O}(H)}V$ is not stably free. Then there does not exist an equivariant *-homomorphism $H\to A\circledast^{\delta_A}H$.*

Proof. Since the *-homomorphism F is equivariant, the *-homomorphism

$$\mathrm{id}\otimes F\otimes\mathrm{id}: C([0,1])\underset{\min}{\otimes}A\underset{\min}{\otimes}H\longrightarrow C([0,1])\underset{\min}{\otimes}A'\underset{\min}{\otimes}H$$

restricts and corestricts to an equivariant *-homomorphism $f: A\circledast^{\delta_A}H\to A'\circledast^{\delta_{A'}}H$. Hence, by Theorem 2.2 and Theorem 2.1,

$$\mathcal{P}_H(A'\circledast^{\delta_{A'}}H)\Box^{\mathcal{O}(H)}V\cong B'\otimes_B\left(\mathcal{P}_H(A\circledast^{\delta_A}H)\Box^{\mathcal{O}(H)}V\right)\tag{19}$$

as left B'-modules. (Here B and B' are the respective fixed-point subalgebras.)

If there would exist a *-homomorphism $H\to A\circledast^{\delta_A}H$, then, by [1, Proposition 3.2], the associated module $\mathcal{P}_H(A\circledast^{\delta_A}H)\Box^{\mathcal{O}(H)}V$ would be free. Through (19), this would contradict our assumption that $\mathcal{P}_H(A'\circledast^{\delta_{A'}}H)\Box^{\mathcal{O}(H)}V$ is not stably free. ■

2.2 Iterated equivariant noncommutative join of $SU_q(2)$

Theorem 2.3. *Let A be any finitely iterated equivariant join of $C(SU_q(2))$ with itself. Then there does not exist an equivariant *-homomorphism $C(SU_q(2))\to A\circledast^{\delta_\Delta}C(SU_q(2))$.*

Proof. Let (H,Δ) be a compact quantum group such that the C*-algebra H admits a character χ. Then

$$\mathrm{ev}_{\frac{1}{2}}\otimes\chi\otimes\mathrm{id}: H\circledast^{\Delta}H\longrightarrow H$$

is an equivariant *-homomorphism. More generally, applying character χ to the leftmost factor in an arbitrary finitely iterated equivariant join of H with itself, we obtain an equivariant map to the iterated join consisting of one less number of copies of H. Composing all these maps, we obtain an equivariant map $A \to H$. Now, by the preceding Proposition 2.1 and Theorem 2.2, it suffices to show that there exists a finite-dimensional representation V of (H, Δ) such that the module $\mathcal{P}_H(H \circledast^{\Delta} H) \Box^{\mathcal{O}(H)} V$ is not stably free.

For $H = C(SU_q(2)$ we have a circle of characters. Moreover, if V is a fundamental representation of $SU_q(2)$, then applying to it [1, Theorem 3.3] combined with an index computation in [8] shows that the module $\mathcal{P}_H(H \circledast^{\Delta} H) \Box^{\mathcal{O}(H)} V$ is not stably free, as desired. (See [1, Section 3.2] for details.) ■

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