Integrable orbit equivalence rigidity for free groups

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Abstract

It is shown that every accessible group which is integrable orbit equivalent to a free group is virtually free. Moreover, we also show that any integrable orbit-equivalence between finitely generated groups extends to their end compactifications.

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1 Introduction

Measure equivalence (ME) is an equivalence relation on groups, introduced by M. Gromov [Gro93] as a measure-theoretic counterpart to quasi-isometry. Most of the research in this area has focussed on rigidity phenomena. For example, Furman proved [Fur99a, Fur99b] that any group ME to a lattice in a higher rank simple Lie group has a finite normal subgroup whose quotient is commensurable to a lattice in the same Lie group. See [Fur11] for a survey of further results.

Here we consider the class of free groups. This class is far from rigid: there is a large variety of groups measure-equivalent to a free group [Gab05] and we do not even have a conjectural classification of such groups. So it makes sense to consider the more restrictive

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notion of measure-equivalence known as **integrable orbit equivalence** (IOE) or L^1-orbit-equivalence (L^1-OE) which takes into account the metric in addition to measure-theoretic structure. This notion is stricter than integrable measure equivalence (IME) (also called L^1-measurable equivalence) that first appeared in [BFS13] where it was shown that any group that is L^1-ME to a lattice in $SO(n,1)$ ($n \geq 3$) has a finite normal subgroup whose quotient is a lattice in $SO(n,1)$. Another milestone is T. Austin's work proving L^1-ME rigidity for nilpotent groups [Aus13] (see also M. Cantrell's recent strengthening of Austin's results [Can15]).

The main result of this paper is that any finitely generated accessible group that admits a strict integrable embedding into a free group is virtually free. In particular, any finitely generated accessible group that is IOE to a free group is virtually free. These terms are defined next.

1.1 Accessible groups

According to Stallings' Ends Theorem [Sta68], if a finitely generated group Γ has more than one end then it splits as either a nontrivial free product with amalgamation or as an HNN extension over a finite subgroup. If such splittings cannot occur indefinitely, then the group is called *accessible*. C.T.C. Wall conjectured [Wal71] that all finitely generated groups are accessible. A counterexample was obtained by Dunwoody [Dun93]. However, Dunwoody showed that all finitely presented groups are accessible [Dun85].

1.2 Strict integrable embeddings

Let Γ, G be finitely generated groups. Intuitively, a strict integrable embedding of Γ into G is a random map from Γ into G that is 'Lipschitz on average' and has a bounded number of preimages. To be precise, fix a finite symmetric generating set S_G of G and define the **word length** of any $g \in G$ by $|g|_G := n$ where n is the smallest natural number such that there exist $s_1, \ldots, s_n \in S_G$ with $g = s_1 \cdots s_n$.

Given an action of Γ on a set X, a **cocycle** into G is a map $\alpha : \Gamma \times X \to G$ such that

$$\alpha(g_1 g_2, x) = \alpha(g_1, g_2 x)\alpha(g_2, x) \quad \forall g_1, g_2 \in \Gamma, x \in X.$$

In the case of concern, X is endowed with a probability measure μ, the action $\Gamma \curvearrowright (X, \mu)$ is measure-preserving and α is measurable. Then we say that α is **integrable** if

$$\int |\alpha(g,x)|_G \, d\mu(x) < \infty$$

for every $g \in \Gamma$. While the precise value of $\int |\alpha(g,x)|_G \, d\mu(x)$ depends on the generating set S_G, its finiteness does not and therefore integrability of α does not depend on S_G.

We say that α is a **strict integrable embedding** if in addition to being integrable there is a constant $C > 0$ such that for every $h \in G$,

$$\#\{g \in \Gamma : \ \alpha(g,x) = h\} \leq C$$

for a.e. x. This notion is more restrictive than the notion of integrable embedding defined in the appendix of [Aus13].

Our main result is:

Theorem 1.1. *Let Γ be a finitely generated accessible group. If Γ admits a strict integrable embedding into a free group then Γ is virtually free.*

Definition 1. Two groups Γ, Λ are **integrably orbit equivalent** (IOE) if there exist probability measure-preserving essentially free ergodic actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (X, \mu)$ such that for a.e. $x \in X$, $\Gamma x = \Lambda x$ and the orbit cycles, defined by

$$\alpha : \Gamma \times X \to \Lambda, \quad \alpha(g, x)x = gx,$$

$$\beta : \Lambda \times X \to \Gamma, \quad \beta(h, x)x = hx$$

are integrable.

Clearly an IOE cocycle is a strict integrable embedding. So Theorem 1.1 implies that any finitely generated accessible group IOE to a free group is virtually free.

We do not know whether accessibility is a necessary condition nor whether 'strict integrable embedding' can be weakened to 'integrable embedding' or IOE weakened to IME.

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2 Preliminaries

Definition 2. If X is a connected locally connected σ-compact topological space then let

$$\mathrm{End}(X) := \varprojlim_{K} \pi_0(X \setminus K).$$

To be precise, the inverse limit is over all compact subsets $K \subset X$ and $\pi_0(X \setminus K)$ denotes the set of noncompact connected components of $X \setminus K$. We give $\pi_0(X \setminus K)$ the discrete topology. If $K \subset L$ are compact subsets of X then there is a natural map from $\pi_0(X \setminus L)$ to $\pi_0(X \setminus K)$ and $\mathrm{End}(X)$ is the inverse limit of this system (where the collection of compacts of X is ordered by inclusion). In particular, for every compact K, there is a natural map $\pi_K : \mathrm{End}(X) \to \pi_0(X \setminus K)$.

The **end compactification of** X, denoted $\overline{X}$ is the space $\overline{X} := X \cup \mathrm{End}(X)$ with the following topology: every open subset of X is $\overline{X}$. Also, for every compact $K \subset X$ and $C \in \pi_0(X \setminus K)$, the set

$$C \cup \{\xi \in \mathrm{End}(X) : \ \pi_K(\xi) \in C\} \subset \overline{X}$$

is open. These sets form a basis for the topology of $\overline{X}$.

Let Γ be a group with a finite generating set S. The **Cayley graph of** (Γ, S), denoted $\mathrm{Cay}(\Gamma, S)$, has vertex set Γ and edge set $\{(g, gs):\ g \in G, s \in S\}$. We usually let $\bar{\Gamma}$ denote the end compactification $\overline{\mathrm{Cay}(\Gamma, S)}$, leaving the generating set implicit.

It is well-known that any finitely generated group quasi-isometric to a free group is itself virtually free. This is usually attributed to Gromov via Stallings' Ends Theorem. Alternatively, it follows from Thomassen-Woess [TW93] that accessibility is a quasi-isometric-invariant and from Papasoglu-Whyte [PW02] that any accessible group quasi-isometric to a free group must be virtually free. Indeed, this implies more:

Theorem 2.1. *If Γ is the fundamental group of a finite graph of groups in which all vertex and edge groups are finite then Γ is virtually free.*

Proof. This follows from [PW02] although it may have been known earlier. □

We also note:

Lemma 2.2. *If Γ is the fundamental group of a finite graph of groups in which all edge groups are finite and $\Gamma_v \leq \Gamma$ is a vertex subgroup then Γ_v is quasi-isometrically embedded in Γ.*

Finally, we introduce a notion of L^1-embedding:

Definition 3. Let $\Gamma \curvearrowright (X, \mu)$ a probability measure-preserving action and $\alpha : \Gamma \times X \to G$ a measurable cocycle. We say α is an *L^1-embedding* if

- α is L^1: for every $g \in \Gamma$, $\int |\alpha(g, x)|_G \, d\mu(x) < \infty$ where $|\cdot|_G$ denotes length with respect to a fixed word metric on G;
- there is a constant $C > 0$ such that for any $h \in G$,

$$\#\{g \in \Gamma :\ \alpha(g, x) = h\} \leq C$$

  for a.e. x.

Remark 1. It is straightforward to check that a composition of L^1-embeddings is an L^1-embedding and that any cocycle arising from an L^1-OE is an L^1-embedding.

In order to prove Theorem 1.1, it now suffices to show the following. Let $\Gamma \curvearrowright (X, \mu)$ be a probability measure-preserving action and $\alpha : \Gamma \times X \to G$ an L^1-embedding. Then Γ is not 1-ended. We will show this in the next section.

3 The Space of Ends

Theorem 3.1. *Let Γ, G be finitely generated groups, $\Gamma \curvearrowright (X, \mu)$ a probability measure-preserving action, $\alpha : \Gamma \times X \to G$ an L^1-embedding. Define $\alpha' : \Gamma \times X \to G$ by*

$$\alpha'(g, x) = \alpha(g^{-1}, x)^{-1}.$$

Let $\bar{\Gamma}, \bar{G}$ denote the end-compactifications of Γ, G respectively with respect to fixed finite generating subsets. Then α' extends to a map, also denoted by α' from $\bar{\Gamma} \times X \to \bar{G}$ such that

- $\alpha'_{gx}(g\xi) = \alpha(g,x)\alpha'_x(\xi)$ *(for* $g \in G, \xi \in \bar{\Gamma}$ *and a.e.* $x \in X$*);*

- $\alpha'_x : \bar{\Gamma} \to \bar{G}$ *is continuous for a.e.* x*.*

Here, $\alpha'_x(g) := \alpha'(g,x)$*.*

Remark 2. The Theorem above implies that every finitely generated group is 1-taut relative to its space of ends, in the terminology of [BFS13]. We will not need this fact.

Theorem 3.1 follows immediately from the next two lemmas.

Lemma 3.2. *Let* X, Y *be connected locally connected* σ*-compact topological spaces. Let* $\alpha : X \to Y$ *be a continuous map. Assume that for every compact* $K \subset Y$ *there exists a compact* $F \subset X$ *such that* $\alpha(X \setminus F) \subset Y \setminus K$ *and* α *descends to a well defined map* $\pi_0(X \setminus F) \to \pi_0(Y \setminus K)$*. Then* α *extends continuously to* $\bar{\alpha} : \overline{X} \to \overline{Y}$*.*

Proof. For every $K \subset Y$ we have a map

$$\mathrm{End}(X) \to \pi_0(X \setminus D) \to \pi_0(Y \setminus K),$$

so the lemma follows by the definition of the inverse limit $\varprojlim_D \pi_0(Y \setminus K)$. □

Lemma 3.3. *For a.e.* $x \in X$ *and every finite set* $K \subset G$ *there exists a finite set* $F \subset \Gamma$ *(depending on* x *and* K*) such that* $\alpha'_x(\Gamma \setminus F) \subset G \setminus K$ *and* α'_x *descends to a map* $\pi_0(\Gamma \setminus F) \to \pi_0(G \setminus K)$*.*

Proof. Let S_Γ, S_G be finite generating sets for Γ, G respectively. Let $|\cdot|_\Gamma, |\cdot|_G$ denote word length on Γ, G respectively.

For each $h_1, h_2 \in G$, choose a geodesic segment $\gamma[h_1,h_2]$ from h_1 to h_2. More precisely, for every integer $0 \le n \le |h_1^{-1}h_2|_G$, there is an element $\gamma[h_1,h_2](n) \in G$ so that

$$\gamma[h_1,h_2](n)^{-1}\gamma[h_1,h_2](n+1) \in S_G$$

if $n < |h_1^{-2}h_2|_G$ and $\gamma[h_1,h_2](0) = h_1, \gamma[h_1,h_2](|h^{-1}h_2|_G) = h_2$. Let us also require that this choice is left-invariant so that $h\gamma[h_1,h_2] = \gamma[hh_1,hh_2]$ for any $h, h_1, h_2 \in G$.

For each $x \in X$, $g \in \Gamma$, $s \in S_\Gamma$, we imagine an airplane flying from $\alpha'_x(g)$ to $\alpha'_x(gs)$. The path of the flight is the geodesic $\gamma[\alpha'_x(g), \alpha'_x(gs)]$. We call this an s-**flight**. For $k \in G$, we let $F_{s,k}(x)$ denote the set of elements $g \in \Gamma$ such that the s-flight from $\alpha'_x(g)$ to $\alpha'_x(gs)$ contains k. That is:

$$F_{s,k}(x) := \{g \in \Gamma : \ k \in \gamma[\alpha'_x(g), \alpha'_x(gs)]\}.$$

Claim 1. $F_{s,k}(x)$ is finite for a.e. x. In fact,

$$\int \#F_{s,k}(x)\ d\mu(x) \le C \int |\alpha(s^{-1},x)^{-1}|_G\ d\mu(x) < \infty$$

where $C > 0$ is the constant in the definition of L^1-embedding.

Proof of Claim 1. It suffices to show that $\int \#F_{s,k}(x)\ d\mu(x) < \infty$. In order to prove this, let

$$L_{s,k} = \{(g,x) \in \Gamma \times X :\ g^{-1} \in F_{s,k}(x)\}.$$

Let c_Γ denote the counting measure on Γ. Then $\int \#F_{s,k}\ d\mu = c_\Gamma \times \mu(L_{s,k})$. Because the action $\Gamma \curvearrowright (X,\mu)$ is invariant,

$$c_\Gamma \times \mu(L_{s,k}) = c_\Gamma \times \mu(R_{s,k})$$

where $R_{s,k} = \{(g^{-1}, gx) :\ (g,x) \in L_{s,k}\}$. By definition

$$c_\Gamma \times \mu(R_{s,k}) \ = \ \int \#\{g \in \Gamma :\ (g,x) \in R_{s,k}\}\ d\mu(x).$$

However, $(g,x) \in R_{s,k}$ if and only if $(g^{-1}, gx) \in L_{s,k}$ if and only if $g \in F_{s,k}(gx)$ if and only if $k \in \gamma[\alpha'_{gx}(g), \alpha'_{gx}(gs)]$ if and only if

$$\alpha'_{gx}(g)^{-1}k \in \gamma[e, \alpha'_{gx}(g)^{-1}\alpha'_{gx}(gs)].$$

Let us now compute

$$\alpha'_{gx}(g)^{-1}\alpha'_{gx}(gs) = \alpha(g^{-1}, gx)\alpha(s^{-1}g^{-1}, gx)^{-1} = \alpha(s^{-1}, x)^{-1}$$

by the cocycle equation. So

$$\begin{aligned} \int \#F_{s,k}\ d\mu \ &= \ c_\Gamma \times \mu(R_{s,k}) = \int \#\{g \in \Gamma :\ \alpha'_{gx}(g)^{-1}k \in \gamma[e, \alpha(s^{-1},x)^{-1}]\}\ d\mu(x) \\ &\leq \ C \int |\alpha(s^{-1},x)^{-1}|_G\ d\mu(x) < \infty. \end{aligned}$$

□

Now let $K \subset G$ be finite and define

$$F_K(x) := \bigcup \{F_{s,k} :\ s \in S_\Gamma, k \in K\}.$$

To finish the proof of the lemma, it suffices to show that if $g_1, g_2 \in \Gamma$ are in the same connected component of $\Gamma \setminus F_K(x)$ then $\alpha'_x(g_1), \alpha'_x(g_2)$ are in the same connected component of $G \setminus K$. Because S_Γ is a generating set, we may assume that $g_2 = g_1 s$ for some $s \in S_\Gamma$. Because $g_1 \notin F_K(x)$, it follows that

$$K \cap \gamma[\alpha'_x(g_1), \alpha'_x(g_1 s)] = \emptyset.$$

So $\alpha'_x(g_1), \alpha'_x(g_1 s)$ are in the same connected component of $G \setminus K$ as required. □

Definition 4. Suppose H is a finitely generated group and $S_H \subset H$ is a finite symmetric generating set. Let $\mathrm{Cay}(H, S_H)$ be the associated Cayley graph. Given a subset $F \subset H$, let ∂F be the set of all edges e of $\mathrm{Cay}(H, S_H)$ with one endpoint in F and one endpoint in $H \setminus F$.

Lemma 3.4. *Suppose H is a finitely generated group and $S_H \subset H$ is a finite symmetric generating set. Suppose there exists a constant $C > 0$ and finite subsets $F_n \subset H$ such that*

- $|\partial F_n| \le C$ *for all* $n \in \mathbb{N}$,
- $\lim_{n\to\infty} |F_n| = \infty$.

Then H has at least 2 ends.

Proof. We identify each F_n with its induced subgraph in $\mathrm{Cay}(H, S_H)$. We may assume without loss of generality that every connected component of the complement $\mathrm{Cay}(H, S_H)\setminus F_n$ is infinite. This is because we may add all finite components of $\mathrm{Cay}(H, S_H)\setminus F_n$ to F_n without increasing the size of its boundary.

Choose elements $g_n \in F_n$, $s_n \in S_H$ so that

- $(g_n, g_n s_n) \in \partial F_n$
- if $F_n^\circ \subset F_n$ is the connected component of F_n containing g_n then $\lim_{n\to\infty} |F_n^\circ| = +\infty$
- there exists an infinite path $p_n \subset \mathrm{Cay}(H, S_H) \setminus F_n$ starting from $g_n s_n$.

Let $F_n' = g_n^{-1} F_n^\circ$ and $p_n' = g_n^{-1} p_n$. After passing to a subsequence if necessary, we may assume that F_n' converges to a limit F_∞' and p_n' converges to a limit p_∞' (in the topology of uniform convergence on compact subsets). We observe that F_∞' is infinite, $p_\infty' \subset \mathrm{Cay}(H, S_H) \setminus F_\infty'$ is an infinite path and $|\partial F_\infty'| \le C$. Thus the compact set $K := \partial F_\infty'$ is such that there are at least two infinite components of $\mathrm{Cay}(H, S_H) \setminus K$ (namely, the component containing p_∞' and the component containing F_∞'). This proves that H has at least two ends.

□

Proposition 3.5. *Suppose Γ is an infinite finitely generated group, $G = \mathbb{F}_r$ be a nonabelian free group, $\Gamma \curvearrowright (X, \mu)$ a probability measure-preserving action and $\alpha : \Gamma \times X \to G$ an L^1-embedding. Then Γ has more than one end.*

Proof. We fix a free generating set of G from which we obtain a word metric and a Cayley graph (which is a regular tree since G is a free group). We also fix a finite generating set S_Γ for Γ.

To obtain a contradiction, we assume that $\mathrm{End}(\Gamma) = \{\xi\}$ is a singleton. Define $\phi : X \to \mathrm{End}(G)$ by $\phi(x) = \alpha'(\xi, x)$ where α' is as in Theorem 3.1. By Theorem 3.1,

$$\phi(hx) = \alpha(h, x)\phi(x).$$

For $n \in \mathbb{N}, x \in X$, let $G(n, x)$ be the set of all $g \in G$ such that $(g|\phi(x))_e \le n$ where $(\cdot|\cdot)_e$ is the Gromov product. To be precise $(g|\phi(x))_e = d(e, m)$ where, if $g \ne e$, $m \in G$ is the 'midpoint' of the geodesic triangle with vertices $\{g, \phi(x), e\}$. That is, m is the unique element contained in all three geodesic sides of the triangle with vertices $\{g, \phi(x), e\}$. If $g = e$ then by definition $m = e$. Thus $G(n, x)$ is the set of all elements $g \in G$ such that the geodesic from g to $\phi(x)$ contains a point of distance no more than n from e. Let

$F(n,x) = \{h \in \Gamma : \ \alpha'(h,x) \in G(n,x)\}$.
Claim 1. $\int |\partial F(n,x)| \ d\mu(x) \le C \sum_{s \in S_\Gamma} \int |\alpha(s,x)|_G \ d\mu(x) =: M$. Note M is independent of n.

Proof of Claim 1. Let $r_n(x) \in G$ be the unique element satisfying

$$d(e, r_n(x)) = n = (r_n(x)|\phi(x))_e.$$

In other words, $n \mapsto r_n(x)$ is the geodesic from e to $\phi(x)$. Observe that $\partial G(n,x)$ is the unique edge from $r_n(x)$ to $r_{n+1}(x)$.

By definition $\partial F(n,x)$ consists of all edges of the form (g, gs) such that $g \in F(n,x)$ and $gs \notin F(n,x)$ $(s \in S)$. Equivalently, $\alpha'_x(g) \in G(n,x)$ and $\alpha'_x(gs) \notin G(n,x)$. Equivalently, the s-flight from $\alpha'_x(g)$ to $\alpha'_x(gs)$ flies over $r_n(x)$. The claim now follows as in the proof of Lemma 3.3, Claim 1. □

Claim 2. For every $n \in \mathbb{N}$ and a.e. $x \in X$, $|F(n,x)| < \infty$.

Proof. The previous claim implies $\partial F(n,x)$ is finite for a.e. x. Because Γ is 1-ended, for a.e. $x \in X$ either $F(n,x)$ or $\Gamma \setminus F(n,x)$ is finite. Because $\alpha' : \bar{\Gamma} \times X \to \bar{G}$ is continuous and $\alpha'(\xi, x) \notin \overline{G(n,x)} = \overline{\alpha'(F(n,x),x)}$, it follows that $\Gamma \setminus F(n,x)$ must be infinite and therefore $F(n,x)$ is finite. □

Observe that $G(n,x) \subset G(n+1,x)$ and $\cup_{n \ge 0} G(n,x) = G$. Therefore $F(n,x) \subset F(n+1,x)$ for all n and $\cup_{n \ge 0} F(n,x) = \Gamma$ which in particular implies that $\lim_{n \to \infty} |F(n,x)| = +\infty$.

Because

$$\lim_{n \to \infty} \int |F(n,x)| \ d\mu(x) = +\infty, \qquad \int |\partial F(n,x)| \ d\mu(x) \le M,$$

we can choose $x_n \in X$ so that $|F(n,x_n)| \to \infty$ while $|\partial F(n,x_n)|$ stays bounded. Lemma 3.4 now implies that Γ has at least 2 ends, a contradiction. □

Proof of Theorem 1.1. By assumption there exists an L^1-embedding $\alpha : \Gamma \times X \to G$ and G is a free group. Since Γ is accessible, we may write it as the fundamental group of a finite graph of groups in which each edge group is finite and each vertex group has ≤ 1 end. By Lemma 2.2 each vertex group H quasi-isometrically embeds into Γ. So if we restrict α to $H \times X$, it is still an L^1-embedding. So Proposition 3.5 implies that H is not 1-ended. So every vertex group and edge group in the graph of groups decomposition of Γ is finite. This implies that Γ is virtually free by Theorem 2.1.

□

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