THE WEAK IDEAL PROPERTY AND TOPOLOGICAL DIMENSION ZERO

CORNEL PASNICU AND N. CHRISTOPHER PHILLIPS

Abstract. Following up on previous work, we prove a number of results for C*-algebras with the weak ideal property or topological dimension zero, and some results for C*-algebras with related properties. Some of the more important results include:

- The weak ideal property implies topological dimension zero.
- For a separable C*-algebra A, topological dimension zero is equivalent to $\mathrm{RR}(\mathcal{O}_2 \otimes A) = 0$, to $D \otimes A$ having the ideal property for some (or any) Kirchberg algebra D, and to A being residually hereditarily in the class of all C*-algebras B such that $\mathcal{O}_\infty \otimes B$ contains a nonzero projection.
- Extending the known result for $\mathbb{Z}_2$, the classes of C*-algebras with topological dimension zero, with the weak ideal property, and with residual (SP) are closed under crossed products by arbitrary actions of abelian 2-groups.
- If X is a totally disconnected locally compact Hausdorff space and A is a $C_0(X)$-algebra all of whose fibers have one of the weak ideal property, topological dimension zero, residual (SP), or the combination of pure infiniteness and the ideal property, then A also has the corresponding property.
- For a substantial class of separable C*-algebras including all separable locally AH algebras, topological dimension zero, the weak ideal property, and the ideal property are all equivalent.
- The weak ideal property does not imply the ideal property for separable Z-stable C*-algebras.

We also give counterexamples to several other statements one might hope for.

The weak ideal property (recalled in Definition 1.3 below) was introduced in [24]; it is the property for which there are good permanence results (see Section 8 of [24]) which seems to be closest to the ideal property. (The ideal property fails to pass to extensions, by Theorem 5.1 of [17], to corners, by Example 2.8 of [23], and to fixed point algebras under actions of $\mathbb{Z}_2$, by Example 2.7 of [23]. The weak ideal property does all of these.) Topological dimension zero was introduced in [3]; it is a non-Hausdorff version of total disconnectedness of the primitive ideal space of a C*-algebra. These two properties are related, although not identical, and the purpose of this paper is to study them and their connections further.

The main results are as follows. We prove that the weak ideal property implies topological dimension zero in complete generality. For separable C*-algebras which are purely infinite in the sense of [13], it is equivalent to the ideal property and

Date: 31 December 2015.

2010 *Mathematics Subject Classification.* Primary 46L05.

Key words and phrases. Ideal property, weak ideal property, topological dimension zero, $C_0(X)$-algebra, purely infinite C*-algebra.

The work of the second author was supported by the US National Science Foundation under Grants DMS-1101742 and DMS-1501144.

to topological dimension zero. A general separable C*-algebra A has topological dimension zero if and only if $\mathcal{O}_2 \otimes A$ has real rank zero; this is also equivalent to $D \otimes A$ having the ideal property for some (or any) Kirchberg algebra D. We rule out by example other results in this direction that one might hope for. Topological dimension zero, at least for separable C*-algebras, is also equivalent to a property of the sort considered in [24]. That is, there is an upwards directed class $\mathcal{C}$ such that a separable C*-algebra A has topological dimension zero if and only if A is residually hereditarily in $\mathcal{C}$. (See the end of the introduction for other examples of this kind of property.) All this is in Section 1.

In Section 2, we improve the closure properties of the class of C*-algebras residually hereditarily in a class $\mathcal{C}$ by replacing an arbitrary action of $\mathbb{Z}_2$ with an arbitrary action of a finite abelian 2-group. This refinement was overlooked in [24]. It applies to the weak ideal property as well as to residual (SP) and to the combination of pure infiniteness and the ideal property. For topological dimension zero, better results are already known (Theorem 3.17 of [23]), but, in the separable case, we remove the technical hypothesis in Theorem 3.14 of [23], and show that if a finite group acts on a C*-algebra A and the fixed point algebra has topological dimension zero, then A has topological dimension zero.

Section 3 is a brief look at minimal tensor products. For the tensor product to have the weak ideal property or topological dimension zero, it is usually necessary that both tensor factors have the corresponding property. In the separable case and with one factor exact, this is sufficient for topological dimension zero, but we don't know about the weak ideal property. We show by example that this result fails without the exactness hypothesis.

Proceeding to a $C_0(X)$-algebra A, we show that if X is totally disconnected and the fibers all have the weak ideal property, topological dimension zero, residual (SP), or the combination of pure infiniteness and the ideal property, then A also has the corresponding property. This result is the analog for these properties of Theorem 2.1 of [20] (for real rank zero) and Theorem 2.1 of [21] (for the ideal property), but we do not assume that the $C_0(X)$-algebra is continuous. If A is a separable continuous $C_0(X)$-algebra with nonzero fibers, then total disconnectedness of X is also necessary. This is in Section 4. In the short Section 5, we consider locally trivial $C_0(X)$-algebras with fibers which are strongly purely infinite in the sense of Definition 5.1 of [14], and show (slightly generalizing the known result for $C_0(X, B)$) that A is again strongly purely infinite. In particular, this applies if the fibers are separable, purely infinite, and have topological dimension zero.

Section 6 gives a substantial class of C*-algebras for which the ideal property, the weak ideal property, and topological dimension zero are all equivalent. This class includes all separable locally AH algebras (and further generalizations of AH algebras, such as separable LS-algebras). However, we show by example that there is a Z-stable C*-algebra with just one nontrivial ideal which has the weak ideal property but not the ideal property.

Ideals in C*-algebras are assumed to be closed and two sided. We write $\mathbb{Z}_n$ for $\mathbb{Z}/n\mathbb{Z}$, since the p-adic integers will not appear. If $\alpha \colon G \to \mathrm{Aut}(A)$ is a action of a group G on a C*-algebra A, then A^{α} denotes the fixed point algebra.

Because of the role they play in this paper, we recall the following definitions from [24].

Definition 0.1 (Definition 5.1 of [24]). Let $\mathcal{C}$ be a class of C*-algebras. We say that $\mathcal{C}$ is *upwards directed* if whenever A is a C*-algebra which contains a subalgebra isomorphic to an algebra in $\mathcal{C}$, then $A \in \mathcal{C}$.

Definition 0.2 (Definition 5.2 of [24]). Let $\mathcal{C}$ be an upwards directed class of C*-algebras, and let A be a C*-algebra.

(1) We say that A is *hereditarily in* $\mathcal{C}$ if every nonzero hereditary subalgebra of A is in $\mathcal{C}$.
(2) We say that A is *residually hereditarily in* $\mathcal{C}$ if A/I is hereditarily in $\mathcal{C}$ for every ideal $I \subset A$ with $I \neq A$.

Section 5 of [24] gives permanence properties for a general condition defined this way. We recall the conditions of this type considered in [24], and add one more to be proved here.

(1) Let $\mathcal{C}$ be the class of all C*-algebras which contain an infinite projection. Then $\mathcal{C}$ is upwards directed (clear) and a C*-algebra A is purely infinite and has the ideal property if and only if A is residually hereditarily in $\mathcal{C}$. See the equivalence of conditions (ii) and (iv) of Proposition 2.11 of [26] (valid, as shown there, even when A is not separable).
(2) Let $\mathcal{C}$ be the class of all C*-algebras which contain an infinite element. Then $\mathcal{C}$ is upwards directed (clear) and a C*-algebra A is (residually) hereditarily infinite (Definition 6.1 of [24]) if and only if A is residually hereditarily in $\mathcal{C}$. (See Corollary 6.5 of [24]. We should point out that, by Lemma 2.2(iii) of [13], if D is a C*-algebra, $B \subset D$ is a hereditary subalgebra, and a and b are positive elements of B such that a is Cuntz subequivalent to b relative to D, then a is Cuntz subequivalent to b relative to B.)
(3) Let $\mathcal{C}$ be the class of all C*-algebras which contain a properly infinite element. Then $\mathcal{C}$ is upwards directed (clear) and a C*-algebra A is (residually) hereditarily properly infinite (Definition 6.2 of [24]) if and only if A is residually hereditarily in $\mathcal{C}$. (Lemma 2.2(iii) of [13] plays the same role here as in (2).)
(4) Let $\mathcal{C}$ be the class of all C*-algebras which contain a nonzero projection. Then $\mathcal{C}$ is upwards directed (clear). A C*-algebra A has Property (SP) if and only if A is hereditarily in $\mathcal{C}$, and has residual (SP) (Definition 7.1 of [24]) if and only if A is residually hereditarily in $\mathcal{C}$. (Both statements are clear.)
(5) Let $\mathcal{C}$ be the class of all C*-algebras B such that $K \otimes B$ contains a nonzero projection. Then $\mathcal{C}$ is upwards directed (clear) and a C*-algebra A has the weak ideal property (Definition 8.1 of [24]; recalled in Definition 1.3 below) if and only if A is residually hereditarily in $\mathcal{C}$. (This is shown at the beginning of the proof of Theorem 8.5 of [24].)
(6) Let $\mathcal{C}$ be the class of all C*-algebras B such that $\mathcal{O}_2 \otimes B$ contains a nonzero projection. Then $\mathcal{C}$ is upwards directed. (This is clear.) A separable C*-algebra A has topological dimension zero if and only if A is residually hereditarily in $\mathcal{C}$. (This will be proved in Theorem 1.10 below.)

1. Topological dimension zero

In this section, we prove (Theorem 1.8) that the weak ideal property implies topological dimension zero for general C*-algebras. We then give characterizations of topological dimension zero for separable C*-algebras (Theorem 1.10) and purely infinite separable C*-algebras (Theorem 1.9), in terms of other properties of the algebra, in terms of properties of their tensor products with suitable Kirchberg algebras, and (for general separable C*-algebras) of the form of being residually hereditarily in suitable upwards directed classes. We also give two related counterexamples. In particular, there is a separable purely infinite unital nuclear C*-algebra A with one nontrivial ideal such that $\mathcal{O}_2 \otimes A \cong A$ and $\mathrm{RR}(A) = 0$, and an action $\alpha \colon \mathbb{Z}_2 \to \mathrm{Aut}(A)$, such that $\mathrm{RR}(C^*(\mathbb{Z}_2, A, \alpha)) \neq 0$.

We recall two definitions from [23]. We call a not necessarily Hausdorff space *locally compact* if the compact (but not necessarily closed) neighborhoods of every point $x \in X$ form a neighborhood base at x.

Definition 1.1 (Remark 2.5(vi) of [3]; Definition 3.2 of [23])**.** Let X be a locally compact but not necessarily Hausdorff topological space. We say that X has *topological dimension zero* if for every $x \in X$ and every open set $U \subset X$ such that $x \in U$, there exists a compact open (but not necessarily closed) subset $Y \subset X$ such that $x \in Y \subset U$. (Equivalently, X has a base for its topology consisting of subsets which are compact and open, but not necessarily closed.) We further say that a C*-algebra A has *topological dimension zero* if $\mathrm{Prim}(A)$ has topological dimension zero.

Definition 1.2 (Definition 3.4 of [23])**.** Let X be a not necessarily Hausdorff topological space. A *compact open exhaustion* of X is an increasing net $(Y_\lambda)_{\lambda \in \Lambda}$ of compact open subsets $Y_\lambda \subset X$ such that $X = \bigcup_{\lambda \in \Lambda} Y_\lambda$.

We further recall (Lemma 3.10 of [23]; see Definition 3.9 of [23] or page 53 of [26] for the original definition) that if A is a C*-algebra and $I \subset A$ is an ideal, then I is compact if and only if $\mathrm{Prim}(I)$ is a compact open (but not necessarily closed) subset of $\mathrm{Prim}(A)$.

Finally, we recall the definition of the weak ideal property.

Definition 1.3 (Definition 8.1 of [24])**.** Let A be a C*-algebra. We say that A has the *weak ideal property* if whenever $I \subset J \subset K \otimes A$ are ideals in $K \otimes A$ such that $I \neq J$, then J/I contains a nonzero projection.

Lemma 1.4. Let A be a C*-algebra with the weak ideal property. Let $I_1 \subset I_2 \subset A$ be ideals with $I_1 \neq I_2$. Then there exists an ideal $J \subset A$ with $I_1 \subsetneqq J$ and such that $K \otimes (J/I_1)$ is generated as an ideal by a single nonzero projection.

Proof. Since A has the weak ideal property and $K \otimes (I_2/I_1) \neq 0$, there is a nonzero projection $e \in K \otimes (I_2/I_1)$. Let $I \subset K \otimes (I_2/I_1)$ be the ideal generated by e. Then there is an ideal $J \subset A$ with $I_1 \subset J \subset I_2$ such that $I = K \otimes (J/I_1)$. Since $J/I_1 \neq 0$, it follows that $J \neq I_1$. □

Lemma 1.5. Let A be a C*-algebra, let $F \subset A$ be a finite set of projections, and let $I \subset A$ be the ideal generated by F. Then $\mathrm{Prim}(I)$ is a compact open subset of $\mathrm{Prim}(A)$.

Proof. This can be shown by using the same argument as in (iii) implies (i) in the proof of Proposition 2.7 of [26]. However, we can give a more direct proof (not involving the Pedersen ideal). As there, we prove that I is compact (as recalled after Definition 1.2). So let $(I_\lambda)_{\lambda \in \Lambda}$ be an increasing net of ideals in A such that $\overline{\bigcup_{\lambda \in \Lambda} I_\lambda} = I$. Standard functional calculus arguments produce $\varepsilon > 0$ such that whenever B is a C*-algebra, $C \subset B$ is a subalgebra, and $p \in B$ is a projection such that $\operatorname{dist}(p, C) < \varepsilon$, then there is a projection $q \in C$ such that $\|q - p\| < 1$, and in particular q is Murray-von Neumann equivalent q. Write $F = \{p_1, p_2, \ldots, p_n\}$. Choose $\lambda \in \Lambda$ such that $\operatorname{dist}(p_j, I_\lambda) < \varepsilon$ for $j = 1, 2, \ldots, n$. Let $q_1, q_2, \ldots, q_n \in I_\lambda$ be projections obtained from the choice of ε. Then there are partial isometries $s_1, s_2, \ldots, s_n \in A$ such that $p_j = s_j q_j s_j^*$ for $j = 1, 2, \ldots, n$. So $p_1, p_2, \ldots, p_n \in I_\lambda$, whence $I_\lambda = I$. This completes the proof. □

Lemma 1.6. Let A be a C*-algebra, and let $I \subset A$ be an ideal. Suppose that there is a collection $(I_\lambda)_{\lambda \in \Lambda}$ (not necessarily a net) of ideals in A such that I is the ideal generated by $\bigcup_{\lambda \in \Lambda} I_\lambda$ and such that $\operatorname{Prim}(I_\lambda)$ has a compact open exhaustion (as in Definition 1.2) for every $\lambda \in \Lambda$. Then $\operatorname{Prim}(I)$ has a compact open exhaustion.

Proof. It is easily checked that a union of open sets with compact open exhaustions also has a compact open exhaustion. □

Proposition 1.7. Let A be a C*-algebra. Then there is a largest ideal $I \subset A$ such that $\operatorname{Prim}(I)$ has a compact open exhaustion.

Proof. Let I be the closure of the union of all ideals $J \subset A$ such that $\operatorname{Prim}(J)$ has a compact open exhaustion. Then $\operatorname{Prim}(I)$ has a compact open exhaustion by Lemma 1.6. □

Theorem 1.8. Let A be a C*-algebra with the weak ideal property. Then A has topological dimension zero.

Proof. We will show that for every ideal $I \subset A$, the subset $\operatorname{Prim}(I)$ has a compact open exhaustion. The desired conclusion will then follow from Lemma 3.6 of [23].

So let $I \subset A$ be an ideal. By Proposition 1.7, there is a largest ideal $J \subset I$ such that $\operatorname{Prim}(J)$ has a compact open exhaustion. We prove that $J = I$. Suppose not. Use Lemma 1.4 to find an ideal $N \subset I$ with $J \subsetneqq N$ and such that $K \otimes (N/J)$ is generated by one nonzero projection. Then $\operatorname{Prim}(K \otimes (N/J))$ is a compact open subset of $\operatorname{Prim}(K \otimes (I/J))$ by Lemma 1.5. So $\operatorname{Prim}(N/J)$ is a compact open subset of $\operatorname{Prim}(I/J)$. Since $\operatorname{Prim}(J)$ has a compact open exhaustion, we can apply Lemma 3.7 of [23] (taking $U = \operatorname{Prim}(J)$) to deduce that $\operatorname{Prim}(N)$ has a compact open exhaustion. Since $J \subsetneqq N$, we have a contradiction. Thus $J = I$, and $\operatorname{Prim}(I)$ has a compact open exhaustion. □

The list of equivalent conditions in the next theorem extends the list in Corollary 4.3 of [26], by adding condition (5). As discussed in the introduction, this condition is better behaved than the related condition (4).

Theorem 1.9. Let A be a separable C*-algebra which is purely infinite in the sense of Definition 4.1 of [13]. Then the following are equivalent:

(1) $\mathcal{O}_2 \otimes A$ has real rank zero.
(2) $\mathcal{O}_2 \otimes A$ has the ideal property.
(3) A has topological dimension zero.
(4) A has the ideal property.
(5) A has the weak ideal property.

Proof. The equivalence of conditions (1), (2), (3), and (4) is Corollary 4.3 of [26]. That (4) implies (5) is trivial. That (5) implies (3) is Theorem 1.8. □

We presume that Theorem 1.9 holds without separability. However, some of the results used in the proof of Corollary 4.3 of [26] are only known in the separable case, and it seems likely to require some work to generalize them.

Recall that a Kirchberg algebra is a simple separable nuclear purely infinite C*-algebra.

Theorem 1.10. Let A be a separable C*-algebra. Then the following are equivalent:

(1) A has topological dimension zero.
(2) $\mathcal{O}_2 \otimes A$ has real rank zero.
(3) $\mathcal{O}_2 \otimes A$ has the ideal property.
(4) $\mathcal{O}_2 \otimes A$ has the weak ideal property.
(5) $\mathcal{O}_\infty \otimes A$ has the ideal property.
(6) $\mathcal{O}_\infty \otimes A$ has the weak ideal property.
(7) There exists a Kirchberg algebra D such that $D \otimes A$ has the weak ideal property.
(8) For every Kirchberg algebra D, the algebra $D \otimes A$ has the ideal property.
(9) A is residually hereditarily in the class of all C*-algebras B such that $\mathcal{O}_2 \otimes B$ contains a nonzero projection.
(10) A is residually hereditarily in the class of all C*-algebras B such that $K \otimes \mathcal{O}_2 \otimes B$ contains a nonzero projection.
(11) A is residually hereditarily in the class of all C*-algebras B such that $\mathcal{O}_\infty \otimes B$ contains a nonzero projection.

We presume that Theorem 1.10 also holds without separability.

To put conditions (9), (10), and (11) in context, we point out that it is clear that the classes used in them are upwards directed in the sense of Definition 0.1. However, applying the results of Section 5 of [24] does not give any closure properties for the collection of C*-algebras with topological dimension zero which are not already known.

The conditions in Theorem 1.10 are not equivalent to A having the weak ideal property, since there are nonzero simple separable C*-algebras A, such as those classified in [29], for which $K \otimes A$ has no nonzero projections. They are also not equivalent to $\mathrm{RR}(\mathcal{O}_\infty \otimes A) = 0$. See Example 1.13 below.

Proof of Theorem 1.10. Since A has topological dimension zero if and only if $\mathcal{O}_2 \otimes A$ has topological dimension zero, and since $\mathcal{O}_2 \otimes A$ is purely infinite (by Proposition 4.5 of [13]), the equivalence of (1), (2), (3), and (4) follows by applying Theorem 1.9 to $\mathcal{O}_2 \otimes A$. Since $\mathcal{O}_\infty \otimes A$ is purely infinite (by Proposition 4.5 of [13]) and $\mathcal{O}_2 \otimes \mathcal{O}_\infty \cong \mathcal{O}_2$, the equivalence of (3), (5), and (6) follows by applying Theorem 1.9 to $\mathcal{O}_\infty \otimes A$.

We prove the equivalence of (1) and (9). Let $\mathcal{C}$ be the class of all C*-algebras B such that $\mathcal{O}_2 \otimes B$ contains a nonzero projection.

Assume that A has topological dimension zero; we prove that A is residually hereditarily in $\mathcal{C}$. Let $I \subset A$ be an ideal, and let $B \subset A/I$ be a nonzero hereditary subalgebra. Then A/I has topological dimension zero by Proposition 2.6 of [4] and Lemma 3.6 of [23]. It follows from Lemma 3.3 of [23] that B has topological dimension zero. Use (3) implies (1) in Theorem 1.9 to conclude that $\mathcal{O}_2 \otimes B$ contains a nonzero projection.

Conversely, assume that A is residually hereditarily in $\mathcal{C}$. We actually prove that $\mathcal{O}_2 \otimes A$ has the weak ideal property. By (5) implies (3) in Theorem 1.9, and since $\mathcal{O}_2 \otimes A$ is purely infinite (by Proposition 4.5 of [13]), it will follow that $\mathcal{O}_2 \otimes A$ has topological dimension zero. Since $\mathrm{Prim}(\mathcal{O}_2 \otimes A) \cong \mathrm{Prim}(A)$, it will follow that A has topological dimension zero.

Thus, let $I \subset J \subset \mathcal{O}_2 \otimes A$ be ideals such that $J \neq I$; we must show that $K \otimes (J/I)$ contains a nonzero projection. Since $\mathcal{O}_2$ is simple and nuclear, there are ideals $I_0 \subset J_0 \subset A$ such that $I = \mathcal{O}_2 \otimes I_0$ and $J = \mathcal{O}_2 \otimes J_0$; moreover, $J/I \cong \mathcal{O}_2 \otimes (J_0/I_0)$. Since J_0/I_0 is a nonzero hereditary subalgebra of A/I_0, the definition of being hereditarily in $\mathcal{C}$ implies that $\mathcal{O}_2 \otimes (J_0/I_0)$ contains a nonzero projection, so $K \otimes (J/I) \cong K \otimes \mathcal{O}_2 \otimes (J_0/I_0)$ does also. This completes the proof of the equivalence of (1) and (9).

We prove equivalence of (9) and (11) by showing that the two classes involved are equal, that is, by showing that if B is any C*-algebra, then $\mathcal{O}_2 \otimes B$ contains a nonzero projection if and only if $\mathcal{O}_\infty \otimes B$ contains a nonzero projection. If $\mathcal{O}_2 \otimes B$ contains a nonzero projection, use an injective (nonunital) homomorphism $\mathcal{O}_2 \to \mathcal{O}_\infty$ to produce an injective homomorphism of the minimal tensor products $\mathcal{O}_2 \otimes_{\min} B \to \mathcal{O}_\infty \otimes_{\min} B$. Since $\mathcal{O}_2$ and $\mathcal{O}_\infty$ are nuclear, we have an injective homomorphism $\mathcal{O}_2 \otimes B \to \mathcal{O}_\infty \otimes B$, and hence a nonzero projection in $\mathcal{O}_\infty \otimes B$. Using an injective (unital) homomorphism from $\mathcal{O}_\infty$ to $\mathcal{O}_2$, the same argument also shows that if $\mathcal{O}_\infty \otimes B$ contains a nonzero projection then so does $\mathcal{O}_2 \otimes B$.

The proof of the equivalence of (9) and (10) is essentially the same as in the previous paragraph, using injective homomorphisms

$$\mathcal{O}_2 \longrightarrow K \otimes \mathcal{O}_2 \quad \text{and} \quad K \otimes \mathcal{O}_2 \longrightarrow \mathcal{O}_2 \otimes \mathcal{O}_2 \xrightarrow{\cong} \mathcal{O}_2.$$

We have now proved the equivalence of all the conditions except (7) and (8).

It is trivial that (6) implies (7) and that (8) implies (5).

Assume (7), so that there is a Kirchberg algebra D_0 such that $D_0 \otimes A$ has the weak ideal property. We prove (8). Let D be any Kirchberg algebra. By Theorem 1.8, the algebra $D_0 \otimes A$ has topological dimension zero. Since

$$\mathrm{Prim}(D_0 \otimes A) \cong \mathrm{Prim}(A) \cong \mathrm{Prim}(D \otimes A),$$

$D \otimes A$ has topological dimension zero. Apply the already proved implication from (1) to (5) with $D \otimes A$ in place of A, concluding that $\mathcal{O}_\infty \otimes D \otimes A$ has the ideal property. Since $\mathcal{O}_\infty \otimes D \cong D$ (Theorem 3.15 of [12]), we see that $D \otimes A$ has the ideal property. □

A naive look at condition (1) of Theorem 1.9 and the permanence properties for C*-algebras which are residually hereditarily in some class $\mathcal{C}$ (see Corollary 5.6 and Theorem 5.3 of [24]) might suggest that if $\mathcal{O}_\infty \otimes A$ has real rank zero and one has an arbitrary action of $\mathbb{Z}_2$ on $\mathcal{O}_\infty \otimes A$, or a spectrally free (Definition 1.3 of [24]) action of any discrete group on $\mathcal{O}_\infty \otimes A$, then the crossed product should also have real rank zero. This is false. We give an example of a nonsimple purely infinite unital nuclear C*-algebra A satisfying the Universal Coefficient Theorem (in fact, with $\mathcal{O}_2 \otimes A \cong A$), with exactly one nontrivial ideal, and such that $\mathrm{RR}(A) = 0$, and an action $\alpha \colon \mathbb{Z}_2 \to \mathrm{Aut}(A)$, such that $C^*(\mathbb{Z}_2, A, \alpha)$ does not have real rank zero.

To put our example in context, we recall the following. First, Example 9 of [7] gives an example of a pointwise outer action α of $\mathbb{Z}_2$ on a simple unital AF algebra A such that $C^*(\mathbb{Z}_2, A, \alpha)$ does not have real rank zero. Second, by Corollary 4.4 of [9], if A is purely infinite and simple, then for any action $\alpha \colon \mathbb{Z}_2 \to \mathrm{Aut}(A)$ the crossed product is again purely infinite. If α is pointwise outer, then $C^*(\mathbb{Z}_2, A, \alpha)$ is again simple, so automatically has real rank zero. Otherwise, α must be an inner action. (See Lemma 1.11 below.) Then $C^*(\mathbb{Z}_2, A, \alpha) \cong A \oplus A$, so has real rank zero. Thus, no such example is possible when A is purely infinite and simple. Third, it is possible for A to satisfy $\mathcal{O}_2 \otimes A \cong A$ but to have $\mathcal{O}_2 \otimes C^*(\mathbb{Z}_2, A, \alpha) \not\cong C^*(\mathbb{Z}_2, A, \alpha)$. See Lemma 4.7 of [8], where this happens with $A = \mathcal{O}_2$.

The following lemma is well known, but we don't know a reference.

Lemma 1.11. Let A be a simple C*-algebra, let G be a finite cyclic group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of G on A. Let $g_0 \in G$ be a generator of G. If α_{g_0} is inner, then α is an inner action, that is, there is a homomorphism $g \mapsto u_g$ from G to the unitary group of $M(A)$ such that $\alpha_g(a) = u_g a u_g^*$ for all $g \in G$ and $a \in A$.

Proof. Let n be the order of G.

By hypothesis, there is a unitary $v \in M(A)$ such that $\alpha_{g_0}(a) = vav^*$ for all $a \in A$. Then $a = \alpha_{g_0}^n(a) = v^n a v^{-n}$ for all $a \in A$. Simplicity of A implies that the center of $M(A)$ contains only scalars, so there is $\lambda \in S^1$ such that $v^n = \lambda \cdot 1$. Now choose $\omega \in S^1$ such that $\omega^n = \lambda^{-1}$, giving $(\omega v)^n = 1$. Define $u_{g_0^k} = \omega^k v^k$ for $k = 0, 1, \ldots, n-1$. □

Example 1.12. There is a separable purely infinite unital nuclear C*-algebra A with exactly one nontrivial ideal I, satisfying the Universal Coefficient Theorem, satisfying $\mathcal{O}_2 \otimes A \cong A$, and such that $\mathrm{RR}(A) = 0$, and there is an action $\alpha \colon \mathbb{Z}_2 \to \mathrm{Aut}(A)$ such that $\mathrm{RR}(C^*(\mathbb{Z}_2, A, \alpha)) \neq 0$. Moreover, α is strongly pointwise outer in the sense of Definition 4.11 of [28] (Definition 1.1 of [24]) and spectrally free in the sense of Definition 1.3 of [24].

To start the construction, let $\nu \colon \mathbb{Z}_2 \to \mathrm{Aut}(\mathcal{O}_2)$ be the action considered in Lemma 4.7 of [8]. Define $B = C^*(\mathbb{Z}_2, \mathcal{O}_2, \nu)$. Lemma 4.7 of [8] implies that B is a Kirchberg algebra (simple, separable, nuclear, and purely infinite) which is unital and satisfies the Universal Coefficient Theorem, and moreover that $K_0(B) \cong \mathbb{Z}\big[\tfrac{1}{2}\big]$ and $K_1(B) = 0$.

Let P be the unital Kirchberg algebra satisfying the Universal Coefficient Theorem, $K_0(P) = 0$, and $K_1(P) \cong \mathbb{Z}$. The Künneth formula (Theorem 4.1 of [31]) implies that $K_0(P \otimes \mathcal{O}_4) = 0$ and $K_1(P \otimes \mathcal{O}_4) \cong \mathbb{Z}_3$.

Since $\mathcal{O}_4$ satisfies the Universal Coefficient Theorem, and since $K \otimes P \otimes \mathcal{O}_4$ and $\mathcal{O}_4$ are separable and nuclear, every possible six term exact sequence

$$\begin{array}{ccccc} K_0(P \otimes \mathcal{O}_4) & \longrightarrow & M_0 & \longrightarrow & K_0(\mathcal{O}_4) \\ {\scriptstyle \exp}\uparrow & & & & \downarrow{\scriptstyle \partial} \\ K_1(\mathcal{O}_4) & \longleftarrow & M_1 & \longleftarrow & K_1(P \otimes \mathcal{O}_4) \end{array}$$

(for any possible choice of abelian groups M_0 and M_1 and homomorphisms exp and ∂) is realized as the K-theory of an exact sequence

$$0 \longrightarrow K \otimes P \otimes \mathcal{O}_4 \longrightarrow D \longrightarrow \mathcal{O}_4 \longrightarrow 0, \tag{1.1}$$

in which D is unital, $K_0(D) \cong M_0$, and $K_1(D) \cong M_1$. Choose the exact sequence (1.1) such that the connecting map

$$\exp \colon K_0(\mathcal{O}_4) \to K_1\big(P \otimes \mathcal{O}_4\big) \tag{1.2}$$

is an isomorphism. Define $A = \mathcal{O}_2 \otimes D$. Let $\iota\colon \mathbb{Z}_2 \to \mathrm{Aut}(D)$ be the trivial action, and let $\alpha = \nu \otimes \iota\colon \mathbb{Z}_2 \to \mathrm{Aut}(A)$ be the obvious action on the tensor product.

It follows from the isomorphisms $\mathcal{O}_2 \otimes \mathcal{O}_4 \cong \mathcal{O}_2$ and $\mathcal{O}_2 \otimes P \otimes \mathcal{O}_4 \cong \mathcal{O}_2$ that A fits into an exact sequence

$$0 \longrightarrow K \otimes \mathcal{O}_2 \longrightarrow A \longrightarrow \mathcal{O}_2 \longrightarrow 0.$$

Theorem 3.14 and Corollary 3.16 of [3] therefore imply that $\mathrm{RR}(A) = 0$. There is also an exact sequence of crossed products

$$0 \longrightarrow C^*\big(\mathbb{Z}_2,\, \mathcal{O}_2 \otimes K \otimes P \otimes \mathcal{O}_4\big) \longrightarrow C^*(\mathbb{Z}_2, A, \alpha) \longrightarrow C^*\big(\mathbb{Z}_2,\, \mathcal{O}_2 \otimes \mathcal{O}_4\big) \longrightarrow 0,$$

in which the actions on the ideal and quotient are the tensor product of ν and the trivial action. This sequence reduces to

$$0 \longrightarrow B \otimes K \otimes P \otimes \mathcal{O}_4 \longrightarrow B \otimes D \longrightarrow B \otimes \mathcal{O}_4 \longrightarrow 0,$$

in which the maps are gotten from those of (1.1) by tensoring them with id_B. It follows from Künneth formula (Theorem 4.1 of [31]) that

$$K_0(B \otimes \mathcal{O}_4) \cong K_1\big(B \otimes K \otimes P \otimes \mathcal{O}_4\big) \cong \mathbb{Z}\big[\tfrac{1}{2}\big] \otimes \mathbb{Z}_3 \cong \mathbb{Z}_3.$$

By naturality, the connecting map

$$K_0(B \otimes \mathcal{O}_4) \to K_1\big(B \otimes K \otimes P \otimes \mathcal{O}_4\big)$$

is the tensor product of the isomorphism (1.2) with $\mathrm{id}_{\mathbb{Z}[\frac{1}{2}]}$, and is hence nonzero. Since every class in $K_0(B \otimes \mathcal{O}_4)$ is represented by a projection in $B \otimes \mathcal{O}_4$, it follows from the six term exact sequence in K-theory that projections in $B \otimes \mathcal{O}_4$ need not lift to projections in $B \otimes D$. Theorem 3.14 of [3] therefore implies that $\mathrm{RR}(B \otimes D) \neq 0$. Thus $\mathrm{RR}(C^*(\mathbb{Z}_2, A, \alpha)) \neq 0$.

It remains to prove strong pointwise outerness and spectral freeness. In our case, these are equivalent by Theorem 1.16 of [24], so we prove strong pointwise outerness. This reduces to proving that automorphisms of $\mathcal{O}_2 \otimes K \otimes P \otimes \mathcal{O}_4$ and $\mathcal{O}_2 \otimes \mathcal{O}_4$ coming from the nontrivial element of $\mathbb{Z}_2$ are outer. The automorphism of $\mathcal{O}_2$ coming from the action ν and the nontrivial element of $\mathbb{Z}_2$ is outer, since otherwise the action would be inner by Lemma 1.11, so crossed product would be $\mathcal{O}_2 \oplus \mathcal{O}_2$. We can now apply Proposition 1.19 of [24] twice, both times using $\nu\colon \mathbb{Z}_2 \to \mathrm{Aut}(\mathcal{O}_2)$ in place of $\alpha\colon G \to \mathrm{Aut}(A)$, and in one case using $K \otimes P \otimes \mathcal{O}_4$ in place of B and in the other case using $\mathcal{O}_4$.

We would like to get outerness from Theorem 1 of [32], but that theorem is only stated for unital C*-algebras.

Example 1.13. There is a separable purely infinite unital nuclear C*-algebra A with exactly one nontrivial ideal and which has the ideal property but such that $\mathcal{O}_\infty \otimes A$ does not have real rank zero.

Let D be as in (1.1) in Example 1.12, with the property that the connecting map in (1.2) is nonzero. Set $A = \mathcal{O}_\infty \otimes D$. Since $\mathcal{O}_\infty \otimes K \otimes P \otimes \mathcal{O}_4$ and $\mathcal{O}_\infty \otimes \mathcal{O}_4$ have the weak ideal property (for trivial reasons), it follows from Theorem 8.5(5) of [24] that A has the weak ideal property, and from Theorem 1.9 that A has the ideal property. However, A is by construction not K_0-liftable in the sense of Definition 3.1 of [26], so Corollary 4.3 of [26] implies that $\mathcal{O}_\infty \otimes A$ (which is of course isomorphic to A) does not have real rank zero.

2. Permanence properties for crossed products

In [24] we proved that if $\mathcal{C}$ is an upwards directed class of C*-algebras, α is a completely arbitrary action of $\mathbb{Z}_2$ on a C*-algebra A, and A^{α} is (residually) hereditarily in $\mathcal{C}$, then A is (residually) hereditarily in $\mathcal{C}$. (See Theorem 5.5 of [24].) In particular, by considering dual actions, it follows (Corollary 5.6 of [24]) that crossed products by arbitrary actions of $\mathbb{Z}_2$ preserve the class of C*-algebras which are (residually) hereditarily in $\mathcal{C}$. Here, we show how one can easily extend the first result to arbitrary groups of order a power of 2 and the second result to arbitrary abelian groups of order a power of 2. This should have been done in [24], but was overlooked there. We believe these results should be true for any finite group in place of $\mathbb{Z}_2$, or at least any finite abelian group, but we don't know how to prove them in this generality.

The following lemma is surely well known.

Lemma 2.1. Let G be a topological group, let A be a C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G on A. Let $N \subset G$ be a closed normal subgroup. Then there is an action $\overline{\alpha}\colon G/N \to \mathrm{Aut}(A^{\alpha|_N})$ such that for $g \in G$ and $a \in A^{\alpha|_N}$ we have $\overline{\alpha}_{gN}(a) = \alpha_g(a)$. Moreover, $(A^{\alpha|_N})^{\overline{\alpha}} = A^{\alpha}$.

Proof. The only thing requiring proof is that if $g \in G$ and $a \in A^{\alpha|_N}$ then $\alpha_g(a) \in A^{\alpha|_N}$. So let $k \in N$. Since $g^{-1}kg \in N$, we get

$$\alpha_k(\alpha_g(a)) = \alpha_g\big(\alpha_{g^{-1}kg}(a)\big) = \alpha_g(a).$$

This completes the proof. □

Theorem 2.2. Let $\mathcal{C}$ be an upwards directed class of C*-algebras. Let G be a finite 2-group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an arbitrary action of G on a C*-algebra A.

(1) If A^{α} is hereditarily in $\mathcal{C}$, then A is hereditarily in $\mathcal{C}$.
(2) If A^{α} is residually hereditarily in $\mathcal{C}$, then A is residually hereditarily in $\mathcal{C}$.

Proof. We prove both parts at once.

We use induction on the number $n \in \mathbb{Z}_{\geq 0}$ such that the order of G is 2^n. When $n = 0$, the statement is trivial. So assume $n \in \mathbb{Z}_{\geq 0}$, the statement is known for all groups of order 2^n, G is a group with $\mathrm{card}(G) = 2^{n+1}$, A is a C*-algebra, $\alpha\colon G \to \mathrm{Aut}(A)$ is an action, and A^{α} is (residually) hereditarily in $\mathcal{C}$. The Sylow Theorems provide a subgroup $N \subset G$ such that $\mathrm{card}(N) = 2^n$. Since N has index 2, N must be normal. Let $\overline{\alpha}\colon G/N \to \mathrm{Aut}(A^{\alpha|_N})$ be as in Lemma 2.1. Then $(A^{\alpha|_N})^{\overline{\alpha}} = A^{\alpha}$ is (residually) hereditarily in $\mathcal{C}$. Since $G/N \cong \mathbb{Z}_2$, it follows from Theorem 5.5 of [24] that $A^{\alpha|_N}$ is (residually) hereditarily in $\mathcal{C}$. The induction hypothesis now implies that A is (residually) hereditarily in $\mathcal{C}$. □

Corollary 2.3. Let $\mathcal{C}$ be an upwards directed class of C*-algebras. Let G be a finite abelian 2-group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an arbitrary action of G on a C*-algebra A.

(1) If A is hereditarily in $\mathcal{C}$, then $C^*(G, A, \alpha)$ and A^{α} are hereditarily in $\mathcal{C}$.
(2) If A is residually hereditarily in $\mathcal{C}$, then $C^*(G, A, \alpha)$ and A^{α} are residually hereditarily in $\mathcal{C}$.

Proof. For $C^*(G, A, \alpha)$, apply Theorem 2.2 with $C^*(G, A, \alpha)$ in place of A and the dual action $\widehat{\alpha}$ in place of α.

For A^α, use the Proposition in [30] to see that A^α is isomorphic to a corner of $C^*(G, A, \alpha)$, and apply Proposition 5.10 of [24]. □

Presumably Corollary 2.3 is valid for crossed products by coactions of not necessarily abelian 2-groups. Indeed, possibly the appropriate context is that of actions of finite dimensional Hopf C*-algebras. We will not pursue this direction here.

Corollary 2.4. Let G be a finite 2-group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an arbitrary action of G on a C*-algebra A. Suppose A^α has one of the following properties: residual hereditary infiniteness, residual hereditary proper infiniteness, residual (SP), or the combination of the ideal property and pure infiniteness. Then A has the same property.

Proof. As discussed in the introduction, for each of these properties there is an upwards directed class $\mathcal{C}$ such that a C*-algebra has the property if and only if it is residually hereditarily in the class $\mathcal{C}$. Apply Theorem 2.2. □

Corollary 2.5. Let G be a finite abelian 2-group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an arbitrary action of G on a C*-algebra A. Suppose A has one of the following properties: residual hereditary infiniteness, residual hereditary proper infiniteness, residual (SP), or the combination of the ideal property and pure infiniteness. Then $C^*(G, A, \alpha)$ and A^α have the same property.

Proof. The proof is the same as that of Corollary 2.4, using Corollary 2.3 instead of Theorem 2.2. □

We omit the weak ideal property in Corollary 2.4 and Corollary 2.5, because better results are already known (Theorem 8.9 and Corollary 8.10 of [24]). We also already know (Theorem 3.17 of [23]) that topological dimension zero is preserved by crossed products by actions of arbitrary finite abelian groups, not just abelian 2-groups. The result analogous to Corollary 2.4 is Theorem 3.14 of [23], but it has an extra technical hypothesis. In the separable case, we remove this hypothesis.

Theorem 2.6. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a separable C*-algebra A. Suppose that A^α has topological dimension zero. Then A has topological dimension zero.

Proof. Define an action $\beta\colon G \to \mathrm{Aut}(\mathcal{O}_2 \otimes A)$ by $\beta_g = \mathrm{id}_{\mathcal{O}_2} \otimes \alpha_g$ for $g \in G$. The implication from (1) to (4) in Theorem 1.10 shows that $(\mathcal{O}_2 \otimes A)^\beta = \mathcal{O}_2 \otimes A^\alpha$ has the weak ideal property. Theorem 8.9 of [24] now implies that $\mathcal{O}_2 \otimes A$ has the weak ideal property. So A has topological dimension zero by the implication from (4) to (1) in Theorem 1.10. □

3. Permanence properties for tensor products

In this section, we consider permanence properties for tensor products. Its main purpose is to serve as motivation for Section 4. The new positive result is Theorem 3.6: if A and B are separable C*-algebras and A is exact, then $A \otimes_{\mathrm{min}} B$ has topological dimension zero if and only if A and B have topological dimension zero. The exactness hypothesis is necessary (Example 3.3). We also give a partial result for the weak ideal property (Proposition 3.9).

The following two examples show that the properties we are considering are certainly not preserved by taking tensor products with arbitrary C*-algebras.

Example 3.1. The algebra $\mathbb{C}$ has all of topological dimension zero, the ideal property, the weak ideal property, and residual (SP), but $C([0,1]) \otimes \mathbb{C}$ has none of these.

Example 3.2. The algebra $\mathcal{O}_2$ is purely infinite and has the ideal property but $C([0,1]) \otimes \mathcal{O}_2$ does not have the ideal property.

In particular, there is no hope of any general theorem about tensor products for properties of the form "residually hereditarily in $\mathcal{C}$" when only one tensor factor has the property. Permanence theorems will therefore have to assume that both factors have the property in question. The following example shows that we will also need to assume that at least one tensor factor is exact.

Example 3.3. We show that there are separable unital C*-algebras A and C (neither of which is exact) which have topological dimension zero and such that $A \otimes_{\min} C$ does not have topological dimension zero. In fact, A and C even have real rank zero, and C is simple. We also show that there are separable unital C*-algebras B and D which are purely infinite and have the ideal property, but such that $B \otimes_{\min} D$ does not have the ideal property. In fact, B and D even tensorially absorb $\mathcal{O}_2$, and D is simple.

Since topological dimension zero and the weak ideal property are preserved by passing to quotients, it follows that no other tensor product of A and C has topological dimension zero, and that no other tensor product of B and D even has the weak ideal property.

Let A and C be as in Theorem 2.6 of [25]. As there, A and C are separable unital C*-algebras with real rank zero, C is simple, and $A \otimes_{\min} C$ does not have the ideal property. These are the same algebras A and C as used in the proof of Proposition 4.5 of [26]. Thus, $A \otimes_{\min} C$ does not have topological dimension zero by Proposition 4.5(1) of [26]. Also, $\mathcal{O}_2 \otimes A \otimes_{\min} C$ does not have the ideal property by Proposition 4.5(2) of [26]. Thus taking $B = \mathcal{O}_2 \otimes A$ and $D = \mathcal{O}_2 \otimes C$ gives algebras B and D with the required properties.

We have several positive results, but no answers for several obvious questions. We recall known results, then give the new result we can prove, and conclude with open questions and a partial result.

We will assume one of the algebras is exact, but this assumption can be replaced by any of the other conditions in Proposition 2.17 of [1] which ensure that $\mathrm{Prim}(A \otimes_{\min} B) \cong \mathrm{Prim}(A) \times \mathrm{Prim}(B)$.

Theorem 3.4 (Corollary 1.3 of [25])**.** Let A and B be C*-algebras with the ideal property. Assume that A is exact. Then $A \otimes_{\min} B$ has the ideal property.

Theorem 3.5 (Proposition 4.6 of [26])**.** Let A and B be C*-algebras with the ideal property. Assume that B is purely infinite and A is exact. Then $A \otimes_{\min} B$ is purely infinite and has the ideal property.

Theorem 3.6. Let A and B be separable C*-algebras. Assume that A is exact. Then $A \otimes_{\min} B$ has topological dimension zero if and only if both A and B have topological dimension zero.

Proof. By Proposition 2.17 of [1] (see Remark 2.11 of [1] for the notation in Proposition 2.16 of [1], to which it refers), the spaces of closed prime ideals satisfy

$$\mathrm{prime}(A \otimes_{\min} B) \cong \mathrm{prime}(A) \times \mathrm{prime}(B),$$

with the homeomorphism being implemented in the obvious way. (See Proposition 2.16(iii) of [1].) Since A, B, and $A \otimes_{\min} B$ are all separable, Proposition 4.3.6 of [27] implies that prime ideals are primitive; the reverse is well known. So

$$\operatorname{Prim}(A \otimes_{\min} B) \cong \operatorname{Prim}(A) \times \operatorname{Prim}(B). \tag{3.1}$$

Assume A and B have topological dimension zero. Then (see Definition 1.1) we need to prove that if X and Y are locally compact but not necessarily Hausdorff spaces which have topological dimension zero, then $X \times Y$ has topological dimension zero. So let $(x, y) \in X \times Y$, and let $W \subset X \times Y$ be an open set with $(x, y) \in W$. By the definition of the product topology, there are open subsets $U_0 \subset X$ and $V_0 \subset Y$ such that $x \in U_0$ and $y \in V_0$. By the definition of topological dimension zero, there are compact open (but not necessarily closed) subsets $U_0 \subset X$ and $V_0 \subset Y$ such that $x \in U \subset U_0$ and $y \in V \subset V_0$. Then $U \times V$ is a compact open subset of $X \times Y$ such that $(x, y) \in U \times V \subset W$.

Now assume $A \otimes_{\min} B$ has topological dimension zero. We prove that B has topological dimension zero; the proof that A has topological dimension zero is the same, except that we don't need to know that exactness passes to quotients. Choose a maximal ideal $I \subset A$. Then A/I is also exact, by Proposition 7.1(ii) of [10]. Apply (3.1) as is and also with A/I in place of A, use the formula for the homeomorphism from Proposition 2.16(iii) of [1], and use the quotient map $A \to A/I$ and the map $\operatorname{Prim}((A/I) \otimes_{\min} B) \to \operatorname{Prim}(A \otimes_{\min} B)$ it induces. The outcome is that

$$\begin{aligned} \operatorname{Prim}(B) &\cong \operatorname{Prim}((A/I) \otimes_{\min} B) \cong \{I\} \times \operatorname{Prim}(B) \\ &\subset \operatorname{Prim}(A) \times \operatorname{Prim}(B) \cong \operatorname{Prim}(A \otimes_{\min} B), \end{aligned}$$

and is a closed subset. Combining Lemmas 3.6 and 3.8 of [23], we see that closed subsets of spaces with topological dimension zero also have topological dimension zero. □

Question 3.7. Let A and B be C*-algebras, with A exact. If A and B have residual (SP), does $A \otimes_{\min} B$ have residual (SP)?

Question 3.8. Let A and B be C*-algebras, with A exact. If A and B have the weak ideal property, does $A \otimes_{\min} B$ have the weak ideal property?

Using results from Section 6 below, we can get a partial result towards Question 3.8. But its proof depends on relating the weak ideal property to topological dimension zero and the ideal property, and doesn't seem to help with the general case.

Proposition 3.9. Let A and B be C*-algebras in the class $\mathcal{W}$ of Theorem 6.11. If A and B have the weak ideal property and A is exact, then $A \otimes_{\min} B$ has the weak ideal property.

The class $\mathcal{W}$ is the smallest class of separable C*-algebras which contains the separable locally AH algebras, the separable LS algebras, the separable type I C*-algebras, and the separable purely infinite C*-algebras, and is closed under finite and countable direct sums and under minimal tensor products when one tensor factor is exact.

Proof of Proposition 3.9. By Theorem 1.8, the algebras A and B have topological dimension zero. Combining Lemma 6.3, Lemma 6.4, Lemma 6.8, Lemma 6.7(2), Proposition 6.9, and Lemma 6.10, we see that A and B are in the class $\mathcal{P}$ of Notation 6.1. Therefore A and B have the ideal property. So $A \otimes_{\mathrm{min}} B$ has the ideal property by Theorem 3.4. It clearly follows that $A \otimes_{\mathrm{min}} B$ has the weak ideal property. $\square$

4. Permanence properties for bundles over totally disconnected spaces

We now turn to section algebras of continuous fields over totally disconnected base spaces. We prove that if A is the section algebra of a bundle over a totally disconnected space, and the fibers all have one of the properties residual (SP), topological dimension zero, the weak ideal property, or the combination of the ideal property and pure infiniteness, then A also has the same property. Moreover, if A has one of these properties, so do all the fibers.

The section algebra of a continuous field over a space which is not totally disconnected will not have the weak ideal property except in trivial cases, and the same is true of the other properties involving the existence of projections in ideals. See Example 3.1 and Example 3.2, showing that this fails even for trivial continuous fields. Accordingly, we can't drop the requirement that the base space be totally disconnected. Indeed, we prove that for a continuous field with nonzero fibers, and assuming everything is separable, if the section algebra has one of the four properties above then the base space must be totally disconnected.

The fact that the properties we consider are equivalent to being residually hereditarily in a suitable class $\mathcal{C}$ underlies some of our reasoning, but knowing that a property has this form does not seem to be sufficient for our results.

Following standard notation, if A is a C*-algebra then $M(A)$ is its multiplier algebra and $Z(A)$ is its center.

Definition 4.1. Let X be a locally compact Hausdorff space. Then a $C_0(X)$-*algebra* is a C*-algebra A together with a nondegenerate (see below) homomorphism $\iota \colon C_0(X) \to Z(M(A))$. Here ι is *nondegenerate* if $\overline{\iota(C_0(X))A} = A$.

Unlike in Definition 2.1 of [16], we do not assume that ι is injective. This permits a hereditary subalgebra of A to also be a $C_0(X)$-algebra, without having to replace X by a closed subspace.

Notation 4.2. Let the notation be as in Definition 4.1. For an open set $U \subset X$ we identify $C_0(U)$ with the obvious ideal of $C_0(X)$. Then $\overline{\iota(C_0(U))A}$ is an ideal in A. For $x \in X$, we define

$$A_x = A / \overline{\iota(C_0(X \setminus \{x\}))A},$$

and we let $\mathrm{ev}_x \colon A \to A_x$ be the quotient map. For a closed subset $L \subset X$, we define $A|_L = A / \overline{\iota(C_0(X \setminus L))A}$. We equip it with the $C_0(L)$-algebra structure which comes from the fact that $C_0(X \setminus L)$ is contained in the kernel of the composition

$$C_0(X) \longrightarrow Z(M(A)) \longrightarrow Z(M(A|_L)).$$

Thus $A_x = A|_{\{x\}}$. Strictly speaking, A is the section algebra of a bundle and $A|_L$ is the section algebra of the restriction of this bundle to L, but the abuse of notation is convenient.

Lemma 4.3. Let the notation be as in Definition 4.1 and Notation 4.2. Let $a \in A$. Then:

(1) $\|a\| = \sup_{x \in X} \|\mathrm{ev}_x(a)\|$.
(2) For every $\varepsilon > 0$, the set $\{x \in X : \|\mathrm{ev}_x(a)\| \geq \varepsilon\} \subset X$ is compact.
(3) The function $x \mapsto \|\mathrm{ev}_x(a)\|$ is upper semicontinuous.
(4) For $f \in C_0(X)$ and $x \in X$, we have $\mathrm{ev}_x(\iota(f)a) = f(x)\mathrm{ev}_x(a)$.

Proof. When ι is injective, the first three parts are Corollary 2.2 of [16], and the last part is contained in the proof of Theorem 2.3 of [16]. In the general case, let $Y \subset X$ be the closed subset such that

$$\mathrm{Ker}(\iota) = \{f \in C_0(X) : f|_Y = 0\}.$$

Then A is a $C_0(Y)$-algebra in the obvious way. We have $A_x = 0$ for $x \notin Y$, and the function $x \mapsto \|\mathrm{ev}_x(a)\|$ associated with the $C_0(X)$-algebra structure is gotten by extending the one associated with the $C_0(Y)$-algebra structure to be zero on $X \setminus Y$. The first three parts then follow from those for the $C_0(Y)$-algebra structure, as does the last when $x \in Y$. The last part is trivial for $x \in X \setminus Y$. □

Definition 4.4. Let X be a locally compact Hausdorff space, and let A be a $C_0(X)$-algebra. We say that A is a *continuous* $C_0(X)$*-algebra* if for all $a \in A$, the map $x \mapsto \|\mathrm{ev}_x(a)\|$ of Lemma 4.3(3) is continuous.

Proposition 4.5. Let X be a locally compact Hausdorff space and let A be a C*-algebra. Then homomorphisms $\iota : C_0(X) \to Z(M(A))$ which make A is a continuous $C_0(X)$-algebra correspond bijectively to isomorphisms of A with the algebra of continuous sections vanishing at infinity of a continuous field of C*-algebras over X, as in 10.4.1 of [6].

Proof. This is essentially contained in Theorem 2.3 of [16], referring to the definitions at the end of Section 1 of [16]. □

We will also need to use results from [15], so we compare definitions.

Proposition 4.6. Let X be a locally compact Hausdorff space.

(1) Let $\big(X, (\pi_x : A \to A_x)_{x \in X}, A\big)$ be a C*-bundle in the sense of Definition 1.1 of [15]. Then A is a $C_0(X)$-algebra, with structure map $\iota : C_0(X) \to M(A)$ determined by the product in Definition 1.1(ii) of [15], if and only if for every $a \in A$ the function $x \mapsto \|\pi_x(a)\|$ is upper semicontinuous and vanishes at infinity.
(2) Let A be a $C_0(X)$-algebra. Then $\big(X, (\mathrm{ev}_x : A \to A_x)_{x \in X}, A\big)$ is a C*-bundle in the sense of Definition 1.1 of [15] which satisfies the condition in (1).

Proof. Theorem 2.3 of [16] and the preceding discussion gives a one to one correspondence between $C_0(X)$-algebras and upper semicontinuous bundles over X in the sense of the definitions at the end of Section 1 of [16]. (The version stated there is for a special case: the structure map of the $C_0(X)$-algebra is required to be injective and the fibers of the bundle are required to be nonzero on a dense subset of X. But the argument in [16] also proves the general case. Some of the argument is also contained in Lemma 2.1 of [15].)

The difference between Definition 1.1 of [15] and the definition of [16] is that [15] omits the requirement (condition (ii) in [16]) that for $a \in A$ and $r > 0$, the set $\{x \in X : \|a(x)\| \geq r\}$ be compact. It is easy to check that a function $f : X \to [0, \infty)$ is upper semicontinuous and vanishes at infinity if and only if for every $r > 0$ the set $\{x \in X : f(x) \geq r\}$ is compact. □

We prove results stating that if X is totally disconnected and the fibers of a $C_0(X)$-algebra A have a particular property, then so does A. These don't require continuity. We will return to continuity later in this section, when we want to prove that if a continuous $C_0(X)$-algebra with nonzero fibers has one of our properties, then X is totally disconnected.

Lemma 4.7. Let the notation be as in Definition 4.1 and Notation 4.2. Let $B \subset A$ be a hereditary subalgebra. Let $a \in A$. Then $a \in B$ if and only if $\mathrm{ev}_x(a) \in \mathrm{ev}_x(B)$ for all $x \in X$.

Proof. The forward implication is immediate.

For the reverse implication, we first claim that if $f \in C_0(X)$ satisfies $0 \leq f \leq 1$ and if $b \in B$, then $\iota(f)b \in B$. To prove the claim, it suffices to consider the case $b \geq 0$. In this case, $\iota(f)b = b^{1/2}\iota(f)b^{1/2}$, and the claim follows from the fact that B is also a hereditary subalgebra in $M(A)$.

To prove the result, it is enough to prove that for every $\varepsilon > 0$ there is $b \in B$ such that $\|a - b\| < \varepsilon$. So let $\varepsilon > 0$. Define $K \subset X$ by

$$K = \left\{ x \in X \colon \|\mathrm{ev}_x(a)\| \geq \frac{\varepsilon}{2} \right\}.$$

For $x \in K$ choose $c_x \in B$ such that $\mathrm{ev}_x(c_x) = \mathrm{ev}_x(a)$, and define $U_x \subset X$ by

$$U_x = \left\{ y \in X \colon \|\mathrm{ev}_y(c_x - a)\| < \frac{\varepsilon}{2} \right\}.$$

It follows from Lemma 4.3(2) that K is compact and from Lemma 4.3(3) that U_x is open for all $x \in K$. Choose $x_1, x_2, \ldots, x_n \in K$ such that the sets $U_{x_1}, U_{x_2}, \ldots, U_{x_n}$ cover K. Choose continuous functions $f_k \colon X \to [0,1]$ with compact support contained in U_{x_k} for $k = 1, 2, \ldots, n$, and such that for $x \in K$ we have $\sum_{k=1}^{n} f_k(x) = 1$ and for $x \in X \setminus K$ we have $\sum_{k=1}^{n} f_k(x) \leq 1$. Define $b \in A$ by

$$b = \sum_{k=1}^{n} \iota(f_k) c_{x_k}.$$

Then $b \in B$ by the claim. Moreover, if $x \in K$ then, using Lemma 4.3(4) at the first step, and $\|\mathrm{ev}_x(c_{x_k} - a)\| < \frac{\varepsilon}{2}$ whenever $f_k(x) \neq 0$ at the second step, we have

$$\|\mathrm{ev}_x(b - a)\| \leq \sum_{k=1}^{n} f_k(x) \|\mathrm{ev}_x(c_{x_k} - a)\| < \frac{\varepsilon}{2}.$$

Define $f(x) = 1 - \sum_{k=1}^{n} f_k(x)$ for $x \in X$. For $x \in X \setminus K$, similar reasoning gives

$$\begin{aligned}
\|\mathrm{ev}_x(b - a)\| &\leq \big\|\mathrm{ev}_x(b - [1 - f(x)]a)\big\| + \|f(x)\mathrm{ev}_x(a)\| \\
&\leq \sum_{k=1}^{n} f_k(x) \|\mathrm{ev}_x(c_{x_k} - a)\| + f(x)\|\mathrm{ev}_x(a)\| \\
&\leq [1 - f(x)]\frac{\varepsilon}{2} + f(x)\|\mathrm{ev}_x(a)\| \leq \frac{\varepsilon}{2}.
\end{aligned}$$

It now follows from Lemma 4.3(1) that $\|b - a\| < \varepsilon$. This completes the proof. ☐

Corollary 4.8. Let X be a locally compact Hausdorff space, let A be a $C_0(X)$-algebra with structure map $\iota \colon C_0(X) \to Z(M(A))$, and let $B \subset A$ be a hereditary subalgebra. Then there is a homomorphism $\mu \colon C_0(X) \to Z(M(B))$ which makes B a $C_0(X)$-algebra and such that for all $b \in B$ and $f \in C_0(X)$ we have $\mu(f)b = \iota(f)b$. Moreover, $B_x = \mathrm{ev}_x(B)$ for all $x \in X$.

Proof. It follows from Lemma 4.7 that if $f \in C_0(X)$ and $b \in B$ then $\iota(f)b \in B$. For $f \in C_0(X)$ we define $T_f \colon B \to B$ by $T_f(b) = \iota(f)b$ for $b \in B$. It is easy to check that (T_f, T_f) is a double centralizer of B, and that $f \mapsto (T_f, T_f)$ defines a homomorphism $\mu \colon C_0(X) \to Z(M(B))$. Nondegeneracy of μ follows from nondegeneracy of ι. The relations $\mu(f)b = \iota(f)b$ and $B_x = \mathrm{ev}_x(B)$ hold by construction. □

Lemma 4.9. Let X be a locally compact Hausdorff space, let A be a $C_0(X)$-algebra with structure map $\iota \colon C_0(X) \to Z(M(A))$, let $F \subset A$ be a finite set, and let $\varepsilon > 0$. Then there is $f \in C_{\mathrm{c}}(X)$ such that $0 \leq f \leq 1$ and $\|\iota(f)a - a\| < \varepsilon$ for all $a \in F$.

Proof. Define $K \subset X$ by

$$K = \left\{x \in X \colon \text{there is } a \in F \text{ such that } \|\mathrm{ev}_x(a)\| \geq \tfrac{\varepsilon}{3}\right\}.$$

It follows from Lemma 4.3(2) that K is compact. Choose $f \in C_{\mathrm{c}}(X)$ such that $0 \leq f \leq 1$ and $f(x) = 1$ for all $x \in K$.

Fix $a \in F$. Let $x \in X$. If $x \in K$ then, using Lemma 4.3(4), $\|\mathrm{ev}_x(\iota(f)a - a)\| = 0$. Otherwise, again using Lemma 4.3(4),

$$\|\mathrm{ev}_x(\iota(f)a - a)\| \leq f(x)\|\mathrm{ev}_x(a)\| + \|\mathrm{ev}_x(a)\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}.$$

Clearly $\sup_{x \in X} \|\mathrm{ev}_x(\iota(f)a - a)\| \leq \frac{2\varepsilon}{3} < \varepsilon$. So $\|\iota(f)a - a\| < \varepsilon$ by Lemma 4.3(1). □

Lemma 4.10. Let X be a locally compact Hausdorff space, let A be a $C_0(X)$-algebra with structure map $\iota \colon C_0(X) \to Z(M(A))$, let $z \in X$, let $F \subset \mathrm{Ker}(\mathrm{ev}_z)$ be a finite set, and let $\varepsilon > 0$. Then there is $f \in C_{\mathrm{c}}(X \setminus \{z\})$ such that $0 \leq f \leq 1$ and $\|\iota(f)a - a\| < \varepsilon$ for all $a \in F$.

Proof. The proof is essentially the same as that of Lemma 4.9. We define K as there, observe that $z \notin K$, and require that $\mathrm{supp}(f)$, in addition to being compact, be contained in $X \setminus \{z\}$. □

Lemma 4.11. Let X be a locally compact Hausdorff space, let A be a $C_0(X)$-algebra with structure map $\iota \colon C_0(X) \to Z(M(A))$, and let $I \subset A$ be an ideal. Let $\pi \colon A \to A/I$ be the quotient map. Then there is a homomorphism $\mu \colon C_0(X) \to Z(M(A/I))$ which makes A/I a $C_0(X)$-algebra and such that for all $a \in A$ and $f \in C_0(X)$ we have $\mu(f)\pi(a) = \pi(\iota(f)a)$. Moreover, giving I the $C_0(X)$-algebra from Corollary 4.8, for every $x \in X$ we have $(A/I)_x \cong A_x/I_x$.

Proof. Let $\overline{\pi} \colon M(A) \to M(A/I)$ be the map on multiplier algebras induced by $\pi \colon A \to A/I$. Define $\mu = \overline{\pi} \circ \iota$. All required properties of μ are obvious except for nondegeneracy.

To prove nondegeneracy, let $b \in A/I$ and let $\varepsilon > 0$. Choose $a \in A$ such that $\pi(a) = b$. Use Lemma 4.9 to choose $f \in C_{\mathrm{c}}(X)$ such that $0 \leq f \leq 1$ and $\|\iota(f)a - a\| < \varepsilon$. Then

$$\|\mu(f)b - b\| = \big\|\pi\big(\iota(f)a - a\big)\big\| < \varepsilon.$$

This completes the proof of nondegeneracy.

It remains to prove the last statement. Let $x \in X$. Let $\mathrm{ev}_x \colon A \to A_x$ be as in Notation 4.2, and let $\overline{\mathrm{ev}}_x \colon A/I \to (A/I)_x$ be the corresponding map with A/I in place of A. Also let $\pi_x \colon A_x \to A_x/I_x$ be the quotient map. Then $\pi_x \circ \mathrm{ev}_x$ and $\overline{\mathrm{ev}}_x \circ \pi$ are surjective, so it suffices to show that they have the same kernel.

Let $a \in A$. Suppose first $(\pi_x \circ \mathrm{ev}_x)(a) = 0$. Let $\varepsilon > 0$; we prove that $\|(\overline{\mathrm{ev}}_x \circ \pi)(a)\| < \varepsilon$. We have $\mathrm{ev}_x(a) \in I_x$. So there is $b \in I$ such that $\mathrm{ev}_x(b) = \mathrm{ev}_x(a)$.

Then $\mathrm{ev}_x(a-b) = 0$. So Lemma 4.10 provides $f \in C_{\mathrm{c}}(X \setminus \{x\})$ such that $0 \leq f \leq 1$ and $\|\iota(f)(a-b) - (a-b)\| < \varepsilon$. By Corollary 4.8, we have $\iota(f)b \in I$. So $\pi(b) = \pi(\iota(f)b) = 0$. Thus

$$\|\mu(f)\pi(a) - \pi(a)\| = \big\|\pi\big(\iota(f)(a-b) - (a-b)\big)\big\| < \varepsilon.$$

Since $\overline{\mathrm{ev}}_x(\mu(f)\pi(a)) = 0$, it follows that $\|(\overline{\mathrm{ev}}_x \circ \pi)(a)\| < \varepsilon$.

Now assume that $(\overline{\mathrm{ev}}_x \circ \pi)(a) = 0$. Let $\varepsilon > 0$; we prove that $\|(\pi_x \circ \mathrm{ev}_x)(a)\| < \varepsilon$. Apply Lemma 4.10 to the $C_0(X)$-algebra A/I, getting $f \in C_{\mathrm{c}}(X \setminus \{x\})$ such that $0 \leq f \leq 1$ and $\|\mu(f)\pi(a) - \pi(a)\| < \varepsilon$. Thus $\|\pi(\iota(f)a - a)\| < \varepsilon$. Choose $b \in I$ such that $\big\|[\iota(f)a - a] - b\big\| < \varepsilon$. It follows that

$$\big\|(\pi_x \circ \mathrm{ev}_x)\big(\iota(f)a - a - b\big)\big\| < \varepsilon.$$

Since $\mathrm{ev}_x(\iota(f)a) = 0$ and $(\pi_x \circ \mathrm{ev}_x)(b) = 0$, it follows that $\|(\pi_x \circ \mathrm{ev}_x)(a)\| < \varepsilon$, as desired. □

Lemma 4.12. Let X be a locally compact Hausdorff space, let A be a $C_0(X)$-algebra with structure map $\iota\colon C_0(X) \to Z(M(A))$, and let D be a C*-algebra. Then there is a homomorphism $\mu\colon C_0(X) \to Z(M(D \otimes_{\max} A))$ which makes $D \otimes_{\max} A$ a $C_0(X)$-algebra and such that for all $a \in A$, $d \in D$, and $f \in C_0(X)$ we have $\mu(f)(d \otimes a) = d \otimes \iota(f)a$. Moreover, for every $x \in X$ we have $(D \otimes_{\max} A)_x \cong D \otimes_{\max} A_x$.

Proof. The family

$$\big(X, (\mathrm{id}_D \otimes_{\max} \pi_x\colon A \to A_x)_{x \in X}, D \otimes_{\max} A\big)$$

is a C*-bundle in the sense of Definition 1.1 of [15]. (See (2) on page 678 of [15].)

Using exactness of the maximal tensor product and Lemma 4.10, one verifies the hypothesis of Lemma 2.3 of [15]. This lemma therefore implies that for $b \in D \otimes_{\max} A$ the function $x \mapsto \|\mathrm{ev}_x(b)\|$ is upper semicontinuous. It is clear that for $d \in D$ and $a \in A$ the function $x \mapsto \|\mathrm{ev}_x(d \otimes a)\|$ vanishes at infinity, and it then follows from density that for all $b \in D \otimes_{\max} A$ the function $x \mapsto \|\mathrm{ev}_x(b)\|$ vanishes at infinity. Now apply Proposition 4.6. □

Lemma 4.13. Let X be a totally disconnected locally compact Hausdorff space, let A be a $C_0(X)$-algebra with structure map $\iota\colon C_0(X) \to Z(M(A))$, and let $x \in X$.

(1) Let $p \in A_x$ be a projection. Then there is a projection $e \in A$ such that $\mathrm{ev}_x(e) = p$.
(2) Let $p \in A_x$ be an infinite projection. Then there is an infinite projection $e \in A$ such that $\mathrm{ev}_x(e) = p$.

The use of semiprojectivity is slightly indirect, because we don't know that there is a countable neighborhood base at x.

Proof of Lemma 4.13. We prove (1). Since $\mathbb{C}$ is semiprojective, there is $\varepsilon > 0$ such that whenever B and C are C*-algebras, $\varphi\colon B \to C$ is a homomorphism, and $b \in B$ satisfies $\|b^* - b\| < \varepsilon$, $\|b^2 - b\| < \varepsilon$, and $\varphi(b)$ is a projection, then there exists a projection $e \in B$ such that $\varphi(e) = \varphi(b)$. Since ev_x is surjective, there is $a \in A$ such that $\mathrm{ev}_x(a) = p$. By Lemma 4.3(3), there is an open set $U \subset X$ with $x \in U$ such that for all $y \in U$ we have $\|\mathrm{ev}_y(a^* - a)\| < \frac{\varepsilon}{2}$ and $\|\mathrm{ev}_y(a^2 - a)\| < \frac{\varepsilon}{2}$. Since X is totally disconnected, there is a compact open set $K \subset X$ such that $x \in K \subset U$. Define $b = \iota(\chi_K)a$. Using Lemma 4.3(4), we get

$$\|\mathrm{ev}_y(b^* - b)\| < \frac{\varepsilon}{2} \quad \text{and} \quad \|\mathrm{ev}_y(b^2 - b)\| < \frac{\varepsilon}{2}$$

when $y \in K$, and $\mathrm{ev}_y(b^* - b) = \mathrm{ev}_y(b^2 - b) = 0$ when $y \in X \setminus K$. It follows from Lemma 4.3(1) that

$$\|b^* - b\| \leq \frac{\varepsilon}{2} < \varepsilon \quad \text{and} \quad \|b^2 - b\| \leq \frac{\varepsilon}{2} < \varepsilon.$$

Now obtain e by using the choice of ε with $B = A$ and $C = A_x$.

We describe the changes needed for the proof of (2). Let T be the Toeplitz algebra, generated by an isometry s (so $s^*s = 1$ but $ss^* \neq 1$). By hypothesis, there is a homomorphism $\varphi_0 \colon T \to A_x$ such that $\varphi_0(1) = p$ and $\varphi_0(1 - ss^*) \neq 0$. Since T is semiprojective, an argument similar to that in the proof of (2) shows that there is a homomorphism $\varphi \colon T \to A$ such that $\mathrm{ev}_x \circ \varphi = \varphi_0$. Set $e = \varphi(1)$. Then $\varphi(s)^*\varphi(s) = e$ and $\varphi(s)\varphi(s)^* \leq e$ We have $e - \varphi(s)\varphi(s)^* \neq 0$ because $\mathrm{ev}_x(e - \varphi(s)\varphi(s)^*) \neq 0$. So e is an infinite projection. □

Theorem 4.14. Let X be a totally disconnected locally compact Hausdorff space and let A be a $C_0(X)$-algebra.

(1) Assume that A_x has residual (SP) for all $x \in X$. Then A has residual (SP).
(2) Assume that A_x is purely infinite and has the ideal property for all $x \in X$. Then A is purely infinite and has the ideal property.
(3) Assume that A_x has the weak ideal property for all $x \in X$. Then A has the weak ideal property.
(4) Assume that A is separable and A_x has topological dimension zero for all $x \in X$. Then A has topological dimension zero.

Proof. We prove (1). Recall (Definition 7.1 of [24]) that a C*-algebra D has residual (SP) if and only if D is residually hereditarily in the class $\mathcal{C}$ of all C*-algebras which contain a nonzero projection. (See (4) in the introduction.)

We verify the definition directly. So let $I \subset A$ be an ideal such that $A/I \neq 0$, and let $B \subset A/I$ be a nonzero hereditary subalgebra. Combining Lemma 4.11 and Corollary 4.8, we see that B is a $C_0(X)$-algebra. Since $B \neq 0$, Lemma 4.3(1) provides $x \in X$ such that $B_x \neq 0$. Let $\overline{\mathrm{ev}}_x \colon A/I \to (A/I)_x$ be the map of Notation 4.2 for the $C_0(X)$-algebra A/I. Then $B_x = \overline{\mathrm{ev}}_x(B)$ by Corollary 4.8 and $(A/I)_x \cong A_x/I_x$ by Lemma 4.11. Thus B_x is isomorphic to a nonzero hereditary subalgebra of A_x/I_x. Since A_x has residual (SP), it follows that there is a nonzero projection $p \in B_x$. Lemma 4.13(1) provides a projection $e \in B$ such that $\overline{\mathrm{ev}}_x(e) = p$. Then $e \neq 0$ since $\overline{\mathrm{ev}}_x(e) \neq 0$. We have thus verified that A has residual (SP).

We next prove (2). Let $\mathcal{C}$ be the class of all C*-algebras which contain an infinite projection. By the equivalence of conditions (ii) and (iv) of Proposition 2.11 of [26] (valid, as shown there, even when A is not separable), a C*-algebra D is purely infinite and has the ideal property if and only if D is residually hereditarily in $\mathcal{C}$. (See (1) in the introduction.) The argument is now the same as for (1), except using Lemma 4.13(2) in place of Lemma 4.13(1).

Now we prove (3). Let $\mathcal{C}$ be the class of all C*-algebras B such that $K \otimes B$ contains a nonzero projection. It is shown at the beginning of the proof of

Theorem 8.5 of [24] that a C*-algebra D has the weak ideal property if and only if D is residually hereditarily in $\mathcal{C}$. (See (5) in the introduction.)

We verify that A satisfies this condition. So let $I \subset A$ be an ideal such that $A/I \neq 0$, and let $B \subset A/I$ be a nonzero hereditary subalgebra. As in the proof of (1), B is a $C_0(X)$-algebra and there is $x \in X$ such that B_x is isomorphic to a nonzero hereditary subalgebra of A_x/I_x. Therefore $K \otimes B_x$ contains a nonzero projection p. Since K is nuclear, Lemma 4.12 implies that $K \otimes B$ is a $C_0(X)$-algebra with $(K \otimes B)_x \cong K \otimes B_x$. So Lemma 4.13(1) provides a projection $e \in K \otimes B$ such that $\overline{\mathrm{ev}}_x(e) = p$. Then $e \neq 0$ since $\overline{\mathrm{ev}}_x(e) \neq 0$. This shows that A is residually hereditarily in $\mathcal{C}$, as desired.

Finally, we prove (4). Since A is separable, by Theorem 1.10 it suffices to show that A is residually hereditarily in the class $\mathcal{C}$ of all C*-algebras D such that $\mathcal{O}_2 \otimes D$ contains a nonzero projection. Also, for every $x \in X$, the algebra A_x is separable. So Theorem 1.10 implies that A_x is residually hereditarily in $\mathcal{C}$. The proof is now the same as for (3), except using $\mathcal{O}_2$ in place of K. □

We will next show that when the $C_0(X)$-algebra is continuous, the fibers are all nonzero, and the algebra is separable, then the algebra has one of our properties if and only if all the fibers have this property and X is totally disconnected.

Separability should not be necessary.

Having nonzero fibers is necessary. The zero C*-algebra is a $C_0(X)$-algebra for any X, and it certainly has all our properties. For a less trivial example, let X_0 be the Cantor set, take $X = X_0 \amalg [0,1]$, and make $C(X_0, \mathcal{O}_2)$ a $C(X)$-algebra via restriction of functions in $C(X)$ to X_0.

Continuity is necessary, at least without separability. Let X be any compact Hausdorff space, and for $x \in X$ let B_x be any nonzero unital C*-algebra. Let A be the C*-algebra product $\prod_{x \in X} B_x$, consisting of elements a in the set theoretic product such that $\sup_{x \in X} \|a_x\|$ is finite. Define a homomorphism $\iota \colon C(X) \to A$ by $\iota(f) = (f(x) \cdot 1_{A_x})_{x \in X}$ for $f \in C(X)$. This homomorphism makes A a $C(X)$-algebra. If, say, $B_x = \mathcal{O}_2$ for all $x \in X$, then A has all our properties, but X need not be totally disconnected. (The construction is easily adapted to spaces X which are only locally compact and possibly nonunital fibers.) It is possible that requiring separability and that all fibers be nonzero will force X to be totally disconnected.

Lemma 4.15. Let X be a second countable locally compact Hausdorff space, and let A be a separable continuous $C_0(X)$-algebra such that $A_x \neq 0$ for all $x \in X$. If A has topological dimension zero then X is totally disconnected.

Proof. By Theorem 3.2 of [15] (see condition (v) there), $\mathcal{O}_2 \otimes A$ is a continuous $C_0(X)$-algebra. It follows from Theorem 1.9 that $\mathcal{O}_2 \otimes A$ has the ideal property, and then from Theorem 2.1 of [21] that X is totally disconnected. □

Theorem 4.16. Let X be a second countable locally compact Hausdorff space, and let A be a separable continuous $C_0(X)$-algebra such that $A_x \neq 0$ for all $x \in X$.

(1) A has residual (SP) if and only if X is totally disconnected and A_x has residual (SP) for all $x \in X$.
(2) A is purely infinite and has the ideal property if and only if X is totally disconnected and A_x is purely infinite and has the ideal property for all $x \in X$.
(3) A has the weak ideal property if and only if X is totally disconnected and A_x has the weak ideal property for all $x \in X$.

(4) A has topological dimension zero if and only if X is totally disconnected and A_x has topological dimension zero for all $x \in X$.

Proof. In all four parts, the reverse implications follow from Theorem 4.14. Also, in all four parts, the fact that A_x has the appropriate property for all $x \in X$ follows from the general fact that the property passes to arbitrary quotients. See Theorem 7.4(7) of [24] for residual (SP), Theorem 6.8(7) of [24] for the combination of purely infiniteness and the ideal property, Theorem 8.5(5) of [24] for the weak ideal property, and combine Proposition 5.8 of [24] with Theorem 1.10(9) for the weak ideal property.

It remains to show that all four properties imply that X is totally disconnected. All four properties imply topological dimension zero (using Theorem 1.8 as necessary), so this follows from Lemma 4.15. □

The proofs in this section depend on properties of projections, and so do not work for a general property defined by being residually hereditarily in an upwards directed class of C*-algebras. However, we know of no counterexamples to either version of the following question.

Question 4.17. Let $\mathcal{C}$ be an upwards directed class of C*-algebras, let X be a totally disconnected locally compact space, and let A be a $C_0(X)$-algebra such that A_x is residually hereditarily in $\mathcal{C}$ for all $x \in X$ Does it follow that A is residually hereditarily in $\mathcal{C}$? What if we assume that A is a continuous $C_0(X)$-algebra?

5. Strong pure infiniteness for bundles

It seems to be unknown whether $C_0(X) \otimes A$ is purely infinite when X is a locally compact Hausdorff space and A is a general purely infinite C*-algebra, even when A is additionally assumed to be simple. (To apply Theorem 5.11 of [13], one also needs to know that A is approximately divisible.) Efforts to prove this by working locally on X seem to fail. Even in cases in which they work, such methods are messy. It therefore seems worthwhile to give the following result, which, given what is known already, has a simple proof.

Theorem 5.1. Let X be a locally compact Hausdorff space, and let A be a locally trivial $C_0(X)$-algebra whose fibers A_x are strongly purely infinite in the sense of Definition 5.1 of [14]. Then A is strongly purely infinite.

Since X is locally compact, local triviality is equivalent to the requirement that every point $x \in X$ have a compact neighborhood L such that, using the $C(L)$-algebra structure on $A|_L$ from Notation 4.2 and the obvious $C(L)$-algebra structure on $C(L, A_x)$, these two algebras are isomorphic as $C(L)$-algebras. We say in this case that $A|_L$ is trivial.

Proof of Theorem 5.1. Let $\iota\colon C_0(X) \to Z(M(A))$ be the structure map.

We first prove the result when X is compact, by induction on the least $n \in \mathbb{Z}_{>0}$ for which there are open sets $U_1, U_2, \ldots, U_n \subset X$ which cover X and such that $A|_{\overline{U_j}}$ is trivial for $j = 1, 2, \ldots, n$. If $n = 1$, there is a strongly purely infinite C*-algebra B such that $A \cong C(X, B)$, and A is strongly purely infinite by Corollary 5.3 of [11]. Assume the result is known for some $n \in \mathbb{Z}_{>0}$, and suppose that there are open sets $U_1, U_2, \ldots, U_{n+1} \subset X$ which cover X and such that $A|_{\overline{U_j}}$ is trivial for $j = 1, 2, \ldots, n+1$. Define $U = \bigcup_{j=1}^{n} U_j$. If $X \setminus U = \varnothing$ then the induction hypothesis applies directly. Otherwise, use $X \setminus U \subset U_{n+1}$ to choose an open set $W \subset X$ such that $X \setminus U \subset W \subset \overline{W} \subset U_{n+1}$. Define $Y = X \setminus W$ and $L = \overline{W}$. Then

$$L \cup Y = X, \quad X \setminus L \subset Y, \quad Y \subset U, \quad \text{and} \quad L \subset U_{n+1}.$$

Since $L \subset \overline{U_{n+1}}$, there is a strongly purely infinite C*-algebra B such that $A|_L \cong C(L, B)$. By definition (see Notation 4.2), there is a short exact sequence

$$0 \longrightarrow \overline{\iota(C_0(X \setminus L))A} \longrightarrow A \longrightarrow A|_L \longrightarrow 0.$$

We can identify the algebra $\overline{\iota(C_0(X \setminus L))A}$ with an ideal in $A|_Y$. Consideration of the sets $U_1 \cap Y$, $U_2 \cap Y$, $\ldots$, $U_n \cap Y$ shows that the induction hypothesis applies to $A|_Y$, which is therefore strongly purely infinite. So $\overline{\iota(C_0(X \setminus L))A}$ is strongly purely infinite by Proposition 5.11(ii) of [14]. Also $A|_L$ is strongly purely infinite by Corollary 5.3 of [11], so A is strongly purely infinite by Theorem 1.3 of [11]. This completes the induction step, and the proof of the theorem when X is compact.

We now prove the general case. Let $(U_\lambda)_{\lambda \in \Lambda}$ be an increasing net of open subsets of X such that $\overline{U_\lambda}$ is compact for all $\lambda \in \Lambda$ and $\bigcup_{\lambda \in \Lambda} U_\lambda = X$. For $\lambda \in \Lambda$, the algebra $A|_{\overline{U_\lambda}}$ is strongly purely infinite by the case already done. So its ideal $\overline{\iota(C_0(U_\lambda))A}$ is strongly purely infinite by Proposition 5.11(ii) of [14]. Using Lemma 4.9, one checks that $A \cong \varinjlim_{\lambda \in \Lambda} \overline{\iota(C_0(U_\lambda))A}$, so A is strongly purely infinite by Proposition 5.11(iv) of [14]. □

Lemma 5.2. Let A be a separable C*-algebra. Then the following are equivalent:

(1) A is purely infinite and has topological dimension zero.
(2) A is strongly purely infinite and has the ideal property.

Proof. Condition (2) implies condition (1) because strong pure infiniteness implies pure infiniteness (Proposition 5.4 of [14]), the ideal property implies the weak ideal property, and the weak ideal property implies topological dimension zero (Theorem 1.8).

Now assume (1). Then A has the ideal property by Theorem 1.9. Apply Proposition 2.14 of [26]. □

Corollary 5.3. Let X be a locally compact Hausdorff space, and let A be a locally trivial $C_0(X)$-algebra whose fibers A_x are all strongly purely infinite, separable, and have topological dimension zero. Then A is strongly purely infinite.

Proof. Lemma 5.2 implies that the fibers are all strongly purely infinite, so that Theorem 5.1 applies. □

6. When does the weak ideal property imply the ideal property?

The weak ideal property seems to be the property most closely related to the ideal property which has good behavior on passing to hereditary subalgebras, fixed point algebras, and extensions. (Example 2.7 of [23] gives a separable unital C*-algebra A with the ideal property and an action of $\mathbb{Z}_2$ on A such that the fixed point algebra does not have the ideal property. Example 2.8 of [23] gives a separable unital C*-algebra A such that $M_2(A)$ has the ideal property but A does not have the ideal property. Theorem 5.1 of [17] gives an extension of separable C*-algebras with the ideal property such that the extension does not have the ideal property.) On the other hand, the ideal property came first, and in some ways seems more natural. Accordingly, it seems interesting to find conditions under which the weak ideal property implies the ideal property. Our main result in this direction is Theorem 6.11. We also give an example to show that this implication can fail for Z-stable C*-algebras.

It is convenient to work with the following class of C*-algebras.

Notation 6.1. We denote by $\mathcal{P}$ the class of all separable C*-algebras for which topological dimension zero, the ideal property, and the weak ideal property are all equivalent.

That is, a separable C*-algebra A is in $\mathcal{P}$ exactly when either A has all of the properties topological dimension zero, the ideal property, and the weak ideal property, or none of them.

The class $\mathcal{P}$ is not particularly interesting in itself. (For example, all cones over nonzero C*-algebras are in it, because they have none of the three properties.) However, proving results about it will make possible a result to the effect that these properties are all equivalent for the smallest class of separable C*-algebras which contains the AH algebras (as well as some others) and is closed under certain operations.

The following lemma isolates, for convenient reference, what we actually need to prove to show that a separable C*-algebra is in $\mathcal{P}$.

Lemma 6.2. Let A be a separable C*-algebra for which topological dimension zero implies the ideal property. Then $A \in \mathcal{P}$.

Proof. The ideal property implies the weak ideal property by Proposition 8.2 of [24]. The weak ideal property implies topological dimension zero by Theorem 1.8. □

We prove two closure property for the class $\mathcal{P}$. What can be done here is limited by the failure of other closure properties for the class of C*-algebras with the ideal property. See the introduction to this section. (It is hopeless to try to prove results for $\mathcal{P}$ for quotients, since the cone over every C*-algebra is in $\mathcal{P}$).

Lemma 6.3. Let $(A_\lambda)_{\lambda \in \Lambda}$ be a countable family of C*-algebras in $\mathcal{P}$. Then $\bigoplus_{\lambda \in \Lambda} A_\lambda \in \mathcal{P}$.

Proof. Set $A = \bigoplus_{\lambda \in \Lambda} A_\lambda$. Then A is separable, since Λ is countable and A_λ is separable for all $\lambda \in \Lambda$. By Lemma 6.2, we need to show that if A has topological dimension zero then A has the ideal property. For $\lambda \in \Lambda$, the algebra A_λ is a quotient of A, so has topological dimension zero by Proposition 2.6 of [4] and Lemma 3.6 of [23]. Therefore A_λ has the ideal property by hypothesis.

It is clear that arbitrary direct sums of C*-algebras with the ideal property also have the ideal property, so it follows that A has the ideal property. □

Lemma 6.4. Let A and B be C*-algebras in $\mathcal{P}$. Assume that A is exact. Then $A \otimes_{\min} B \in \mathcal{P}$.

Proof. The algebra $A \otimes_{\min} B$ is separable because A and B are. By Lemma 6.2, we need to show that if $A \otimes_{\min} B$ has topological dimension zero then $A \otimes_{\min} B$ has the ideal property. Now A and B have topological dimension zero by Theorem 3.6, so have the ideal property by hypothesis. It now follows from Corollary 1.3 of [25] that $A \otimes_{\min} B$ has the ideal property. □

We now identify a basic collection of C*-algebras in $\mathcal{P}$. The main point of the first class we consider is that it contains the AH algebras (as described below), but in fact it is much larger.

Since there are conflicting definitions of AH algebras in the literature, we include a definition. We don't assume that the projections involved have constant rank. We thus do not need to include finite direct sums of corners of algebras of the form $C(X, M_k)$, because such finite direct sums are again of the same form—use the disjoint union of the spaces.

Definition 6.5. Let A be a C*-algebra.

(1) We say that A is an *AH algebra* if A is a direct limit of a sequence $(A_n)_{n \in \mathbb{Z}_{\geq 0}}$ of C*-algebras of the form $pC(X, M_k)p$ for a compact Hausdorff space X, $k \in \mathbb{Z}_{>0}$, and a projection $p \in C(X, M_k)$, all depending on n.
(2) We say that A is a *locally AH algebra* if for every finite set $F \subset A$ and every $\varepsilon > 0$, there exist a compact Hausdorff space X, $k \in \mathbb{Z}_{>0}$, a projection $p \in C(X, M_k)$, and a unital homomorphism $\varphi \colon pC(X, M_k)p \to A$ such that for all $a \in F$ there is $b \in pC(X, M_k)p$ with $\|\varphi(b) - a\| < \varepsilon$.

In particular, AH algebras are locally AH algebras.

Definition 6.6. Let A be a C*-algebra.

(1) We say that A is *standard* (Definition 2.7 of [5]) if A is unital and whenever B is a simple unital C*-algebra and $J \subset A \otimes_{\min} B$ is an ideal which is generated as an ideal by its projections, then there is an ideal $I \subset A$ which is generated as an ideal by its projections and such that $J = I \otimes_{\min} B$.
(2) We say that A is an *LS algebra* (Definition 2.13 of [5]) if for every finite set $F \subset A$ and every $\varepsilon > 0$, there exist a standard C*-algebra D and an injective homomorphism $\varphi \colon D \to A$ such that for all $a \in F$ there is $b \in D$ with $\|\varphi(b) - a\| < \varepsilon$.

Lemma 6.7. Let A be a C*-algebra.

(1) If $A \cong pC(X, M_k)p$ for a compact Hausdorff space X, $k \in \mathbb{Z}_{>0}$, and a projection $p \in C(X, M_k)$, then A is standard.
(2) If A is a locally AH algebra, then A is an LS algebra.

Proof. Part (1) is a special case of Remark 2.9(2) of [5]. Part (2) is immediate from part (1). □

There are many more standard C*-algebras than in Lemma 6.7(1), and therefore many more LS algebras than in Lemma 6.7(2). For example, in Definition 6.5(2) replace $pC(X, M_k)p$ by a finite direct sum of C*-algebras of the form $pC(X, D)p$ for connected compact Hausdorff spaces X, simple unital C*-algebras D, and projections $p \in C(X, D)$. Such a C*-algebra is standard by Remark 2.9(2) of [5], so a direct limit of such algebras is an LS algebra. (When all the algebras D which occur are exact, such a direct limit is called an exceptional GAH algebra in [19]. See Definitions 2.9 and 2.7 there.)

Lemma 6.8. Let A be a separable LS algebra (Definition 6.6(2)). Then $A \in \mathcal{P}$.

Proof. As usual, we use Lemma 6.2. Assume A has topological dimension zero. By the implication from (1) to (3) in Theorem 1.10, the algebra $\mathcal{O}_2 \otimes A$ has the ideal property. Apply Lemma 2.11 of [5] with $B = \mathcal{O}_2$ to conclude that A has the ideal property. □

Extending the list of properties in the discussion of type I C*-algebras in Remark 2.12 of [21] (and using essentially the same proof as there), we get the following longer list of equivalent conditions on a separable type I C*-algebra.

Proposition 6.9. Let A be a separable type I C*-algebra. Then the following are equivalent:

(1) A has topological dimension zero.
(2) A has the weak ideal property.
(3) A has the ideal property.
(4) A has the projection property (every ideal in A has an increasing approximate identity consisting of projections; Definition 1 of [18]).
(5) A has real rank zero.
(6) A is an AF algebra.

Proof. It is clear that every condition on the list implies the previous one. So we need only show that (1) implies (6). Use Lemma 3.6 of [23] to see that $\mathrm{Prim}(A)$ has a base for its topology consisting of compact open sets. Then the theorem in Section 7 of [2] implies that A is AF. □

Lemma 6.10. Every separable purely infinite C*-algebra is in $\mathcal{P}$.

Proof. By Lemma 6.2, we need to show that if A is separable, purely infinite, and has topological dimension zero, then A has the ideal property. Use Lemma 3.6 of [23] to see that $\mathrm{Prim}(A)$ has a base for its topology consisting of compact open sets. Then apply Proposition 2.11 of [26]. □

Theorem 6.11. Let $\mathcal{W}$ the smallest class of separable C*-algebras which contains the separable LS algebras (including the separable locally AH algebras), the separable type I C*-algebras, and the separable purely infinite C*-algebras, and is closed under finite and countable direct sums and under minimal tensor products when one tensor factor is exact. Then for any C*-algebra in $\mathcal{W}$, topological dimension zero, the weak ideal property, and the ideal property are all equivalent.

Proof. Combine Lemma 6.3, Lemma 6.4, Lemma 6.8, Lemma 6.7(2), Proposition 6.9, and Lemma 6.10. □

Let Z be the Jiang-Su algebra. It is unfortunately not true that the weak ideal property implies the ideal property for Z-stable C*-algebras.

Example 6.12. We give a separable C*-algebra A such that A and $Z \otimes A$ have the weak ideal property but such that neither A nor $Z \otimes A$ has the ideal property.

Let D be a Bunce-Deddens algebra, and let the extension

$$0 \longrightarrow K \otimes D \longrightarrow A \longrightarrow \mathbb{C} \longrightarrow 0$$

be as in the proof of Theorem 5.1 of [17]. (The extension is as in the first paragraph of that proof, using the choices suggested in the second paragraph.) In particular, A does not have the ideal property, and the connecting homomorphism $\exp\colon K_0(\mathbb{C}) \to K_1(K \otimes D)$ is injective. Since $K \otimes D$ and $\mathbb{C}$ have the weak ideal property (for trivial reasons), it follows from Theorem 8.5(5) of [24] that A has the weak ideal property. Clearly $Z \otimes K \otimes D$ and $Z \otimes \mathbb{C}$ have the ideal property. However, it is shown in the proof of Theorem 2.9 of [22] that $Z \otimes A$ does not have the ideal property.

Question 6.13. Let A be a separable C*-algebra which is a direct limit of recursive subhomogeneous C*-algebras. If A has the weak ideal property, does A have the ideal property?

We suspect that the answer is no, but we don't have a counterexample.

References

[1] E. Blanchard and E. Kirchberg, *Non-simple purely infinite C*-algebras: The Hausdorff case*, J. Funct. Anal. **207**(2004), 461–513.
[2] O. Bratteli and G. A. Elliott, *Structure spaces of approximately finite-dimensional C*-algebras, II*, J. Funct. Anal. **30**(1978), 74–82.
[3] L. G. Brown and G. K. Pedersen, *C*-algebras of real rank zero*, J. Funct. Anal. **99**(1991), 131–149.
[4] L. G. Brown and G. K. Pedersen, *Limits and C*-algebras of low rank or dimension*, J. Operator Theory **61**(2009), 381–417.
[5] J. R. Carrión and C. Pasnicu, *Approximations of C*-algebras and the ideal property*, J. Math. Anal. Appl. **338**(2008), 925–945.
[6] J. Dixmier, *C*-Algebras*, North-Holland, Amsterdam, New York, Oxford, 1977.
[7] G. A. Elliott, *A classification of certain simple C*-algebras*, pages 373–385 in: *Quantum and Non-Commutative Analysis*, H. Araki etc. (eds.), Kluwer, Dordrecht, 1993.
[8] M. Izumi, *Finite group actions on C*-algebras with the Rohlin property. I*, Duke Math. J. **122**(2004), 233–280.
[9] J. A. Jeong and H. Osaka, *Extremally rich C*-crossed products and the cancellation property*, J. Austral. Math. Soc. (Series A) **64**(1998), 285–301.
[10] E. Kirchberg, *Commutants of unitaries in UHF algebras and functorial properties of exactness*, J. reine angew. Math. **452**(1994), 39–77.
[11] E. Kirchberg, *On permanence properties of strongly purely infinite C*-algebras*, preprint 2003 (Preprintreihe SFB 478–Geometrische Strukturen in der Mathematik, Heft 284; ISSN 1435-1188).
[12] E. Kirchberg and N. C. Phillips, *Embedding of exact C*-algebras in the Cuntz algebra $\mathcal{O}_2$*, J. reine angew. Math. **525**(2000), 17–53.
[13] E. Kirchberg and M. Rørdam, *Non-simple purely infinite C*-algebras*, Amer. J. Math. **122**(2000), 637–666.
[14] E. Kirchberg and M. Rørdam, *Infinite non-simple C*-algebras: absorbing the Cuntz algebra $\mathcal{O}_\infty$*, Adv. Math. **167**(2002), 195–264.
[15] E. Kirchberg and S. Wassermann, *Operations on continuous bundles of C*-algebras*, Math. Annalen **303**(1995), 677–697.
[16] M. Nilsen, *C*-bundles and $C_0(X)$-algebras*, Indiana Univ. Math. J. **45**(1966), 463–477.
[17] C. Pasnicu, *On the AH algebras with the ideal property*, J. Operator Theory **43**(2000), 389–407.
[18] C. Pasnicu, *The projection property*, Glasgow Math. J. **44**(2002), 293–300.
[19] C. Pasnicu, *The ideal property and tensor products of C*-algebras*, Rev. Roumaine Math. Pures Appl. **49**(2004), 153–162.
[20] C. Pasnicu, *Real rank zero and continuous fields of C*-algebras*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **48(96)**(2005), 319–325.
[21] C. Pasnicu, *The ideal property, the projection property, continuous fields and crossed products*, J. Math. Anal. Appl. **323**(2006), 1213–1224.
[22] C. Pasnicu, *$\mathcal{D}$-stable C*-algebras, the ideal property and real rank zero*, Bull. Math. Soc. Sci. Math. Roumanie **52**(2009), 177–192.
[23] C. Pasnicu and N. C. Phillips, *Permanence properties for crossed products and fixed point algebras of finite groups*, Trans. Amer. Math. Soc. **366**(2014), 4625–4648.
[24] C. Pasnicu and N. C. Phillips, *Crossed products by spectrally free actions*, J. Funct. Anal. **269**(2015), 915–967.
[25] C. Pasnicu and M. Rørdam, *Tensor products of C*-algebras with the ideal property*, J. Funct. Anal. **177**(2000), 130–137.
[26] C. Pasnicu and M. Rørdam, *Purely infinite C*-algebras of real rank zero*, J. reine angew. Math. **613**(2007), 51–73.

[27] G. K. Pedersen, *C*-Algebras and their Automorphism Groups*, Academic Press, London, New York, San Francisco, 1979.
[28] N. C. Phillips, *Freeness of actions of finite groups on C*-algebras*, pages 217–257 in: *Operator structures and dynamical systems*, M. de Jeu, S. Silvestrov, C. Skau, and J. Tomiyama (eds.), Contemporary Mathematics vol. 503, Amer. Math. Soc., Providence RI, 2009.
[29] S. Razak *On the classification of simple stably projectionless C*-algebras*, Canad. J. Math. **54**(2002), 138-224.
[30] J. Rosenberg, *Appendix to O. Bratteli's paper on "Crossed products of UHF algebras"*, Duke Math. J. **46**(1979), 25–26.
[31] C. Schochet, *Topological methods for C*-algebras II: geometric resolutions and the Künneth formula*, Pacific J. Math. **98**(1982), 443–458.
[32] S. Wassermann, *Tensor products of *-automorphisms of C*-algebras*, Bull. London Math. Soc. **7**(1975), 65–70.

Department of Mathematics, The University of Texas at San Antonio, San Antonio TX 78249, USA.

Department of Mathematics, University of Oregon, Eugene OR 97403-1222, USA.