Combinations of structures*

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Abstract

We investigate combinations of structures by families of structures relative to families of unary predicates and equivalence relations. Conditions preserving ω -categoricity and Ehrenfeuchtness under these combinations are characterized. The notions of e-spectra are introduced and possibilities for e-spectra are described.

Key words: combination of structures, P-combination, E-combination, spectrum.

The aim of the paper is to introduce operators (similar to [1, 2, 3, 4]) on classes of structures producing structures approximating given structure, as well as to study properties of these operators.

In Section 1 we define P-operators, E-operators, and corresponding combinations of structures. In Section 2 we characterize the preservation of ω -categoricity for P-combinations and E-combinations as well as Ehrenfeuchtness for P-combinations. In Section 3 we pose and investigate questions on variations of structures under P-operators and E-operators. The notions of e-spectra for P-operators and E-operators are introduced in Section 4. Here values for e-spectra are described. In Section 5 the preservation of Ehrenfeuchtness for E-combinations is characterized.

Throughout the paper we consider structures of relational languages.

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1 P-operators, E-operators, combinations

Let $P = (P_i)_{i \in I}$, be a family of nonempty unary predicates, $(\mathcal{A}_i)_{i \in I}$ be a family of structures such that P_i is the universe of \mathcal{A}_i , $i \in I$, and the symbols P_i are disjoint with languages for the structures \mathcal{A}_j , $j \in I$. The structure $\mathcal{A}_P \Longrightarrow \bigcup_{i \in I} \mathcal{A}_i$ expanded by the predicates P_i is the P-union of the structures \mathcal{A}_i , and the operator mapping $(\mathcal{A}_i)_{i \in I}$ to \mathcal{A}_P is the P-operator. The structure \mathcal{A}_P is called the P-combination of the structures \mathcal{A}_i and denoted by $\operatorname{Comb}_P(\mathcal{A}_i)_{i \in I}$ if $\mathcal{A}_i = (\mathcal{A}_P \upharpoonright A_i) \upharpoonright \Sigma(\mathcal{A}_i)$, $i \in I$. Structures \mathcal{A}' , which are elementary equivalent to $\operatorname{Comb}_P(\mathcal{A}_i)_{i \in I}$, will be also considered as P-combinations.

By the definition, without loss of generality we can assume for $\operatorname{Comb}_P(\mathcal{A}_i)_{i\in I}$ that all languages $\Sigma(\mathcal{A}_i)$ coincide interpreting new predicate symbols for \mathcal{A}_i by empty relation.

Clearly, all structures $\mathcal{A}' \equiv \operatorname{Comb}_P(\mathcal{A}_i)_{i \in I}$ are represented as unions of their restrictions $\mathcal{A}'_i = (\mathcal{A}' \upharpoonright P_i) \upharpoonright \Sigma(\mathcal{A}_i)$ if and only if the set $p_{\infty}(x) = \{\neg P_i(x) \mid i \in I\}$ is inconsistent. If $\mathcal{A}' \neq \operatorname{Comb}_P(\mathcal{A}'_i)_{i \in I}$, we write $\mathcal{A}' = \operatorname{Comb}_P(\mathcal{A}'_i)_{i \in I \cup \{\infty\}}$, where $\mathcal{A}'_{\infty} = \mathcal{A}' \upharpoonright \bigcap_{i \in I} \overline{P_i}$, maybe applying Morleyzation.

Moreover, we write $\operatorname{Comb}_P(\mathcal{A}_i)_{i\in I\cup\{\infty\}}$ for $\operatorname{Comb}_P(\mathcal{A}_i)_{i\in I}$ with the empty structure \mathcal{A}_{∞} .

Note that if all predicates P_i are disjoint, a structure \mathcal{A}_P is a P-combination and a disjoint union of structures \mathcal{A}_i [1]. In this case the P-combination \mathcal{A}_P is called *disjoint*. Clearly, for any disjoint P-combination \mathcal{A}_P , $\operatorname{Th}(\mathcal{A}_P) = \operatorname{Th}(\mathcal{A}_P')$, where \mathcal{A}_P' is obtained from \mathcal{A}_P replacing \mathcal{A}_i by pairwise disjoint $\mathcal{A}_i' \equiv \mathcal{A}_i$, $i \in I$. Thus, in this case, similar to structures the P-operator works for the theories $T_i = \operatorname{Th}(\mathcal{A}_i)$ producing the theory $T_P = \operatorname{Th}(\mathcal{A}_P)$, which is denoted by $\operatorname{Comb}_P(T_i)_{i \in I}$.

On the opposite side, if all P_i coincide then $P_i(x) \equiv (x \approx x)$ and removing the symbols P_i we get the restriction of \mathcal{A}_P which is the combination of the structures \mathcal{A}_i [3, 4].

For an equivalence relation E replacing disjoint predicates P_i by E-classes we get the structure \mathcal{A}_E being the E-union of the structures \mathcal{A}_i . In this case the operator mapping $(\mathcal{A}_i)_{i\in I}$ to \mathcal{A}_E is the E-operator. The structure \mathcal{A}_E is also called the E-combination of the structures \mathcal{A}_i and denoted by $\operatorname{Comb}_E(\mathcal{A}_i)_{i\in I}$; here $\mathcal{A}_i = (\mathcal{A}_E \upharpoonright A_i) \upharpoonright \Sigma(\mathcal{A}_i), i \in I$. Similar above, structures \mathcal{A}' , which are elementary equivalent to \mathcal{A}_E , are denoted by $\operatorname{Comb}_E(\mathcal{A}'_j)_{j\in J}$, where \mathcal{A}'_j are restrictions of \mathcal{A}' to its E-classes.

If $\mathcal{A}_E \prec \mathcal{A}'$, the restriction $\mathcal{A}' \upharpoonright (A' \setminus A_E)$ is denoted by \mathcal{A}'_{∞} . Clearly, $\mathcal{A}' = \mathcal{A}'_E \coprod \mathcal{A}'_{\infty}$, where $\mathcal{A}'_E = \text{Comb}_E(\mathcal{A}'_i)_{i \in I}$, \mathcal{A}'_i is a restriction of \mathcal{A}' to its E-class containing the universe A_i , $i \in I$.

Considering an E-combination \mathcal{A}_E we will identify E-classes A_i with structures \mathcal{A}_i .

Clearly, the nonempty structure \mathcal{A}'_{∞} exists if and only if I is infinite.

Notice that any E-operator can be interpreted as P-operator replacing or naming E-classes for \mathcal{A}_i by unary predicates P_i . For infinite I, the difference between 'replacing' and 'naming' implies that \mathcal{A}_{∞} can have unique or unboundedly many E-classes returning to the E-operator.

Thus, for any E-combination \mathcal{A}_E , $\operatorname{Th}(\mathcal{A}_E) = \operatorname{Th}(\mathcal{A}'_E)$, where \mathcal{A}'_E is obtained from \mathcal{A}_E replacing \mathcal{A}_i by pairwise disjoint $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. In this case, similar to structures the E-operator works for the theories $T_i = \operatorname{Th}(\mathcal{A}_i)$ producing the theory $T_E = \operatorname{Th}(\mathcal{A}_E)$, which is denoted by $\operatorname{Comb}_E(T_i)_{i \in I}$, by \mathcal{T}_E , or by $\operatorname{Comb}_E \mathcal{T}$, where $\mathcal{T} = \{T_i \mid i \in I\}$.

Note that P-combinations and E-unions can be interpreted by randomizations [5] of structures.

Sometimes we admit that combinations $\operatorname{Comb}_P(\mathcal{A}_i)_{i\in I}$ and $\operatorname{Comb}_E(\mathcal{A}_i)_{i\in I}$ are expanded by new relations or old relations are extended by new tuples. In these cases the combinations will be denoted by $\operatorname{EComb}_P(\mathcal{A}_i)_{i\in I}$ and $\operatorname{EComb}_E(\mathcal{A}_i)_{i\in I}$ respectively.

2 ω -categoricity and Ehrenfeuchtness for combinations

Proposition 2.1. If predicates P_i are pairwise disjoint, the languages $\Sigma(A_i)$ are at most countable, $i \in I$, $|I| \leq \omega$, and the structure A_P is infinite then the theory $\text{Th}(A_P)$ is ω -categorical if and only if I is finite and each structure A_i is either finite or ω -categorical.

Proof. If I is infinite or there is an infinite structure \mathcal{A}_i which is not ω -categorical then $T = \operatorname{Th}(\mathcal{A}_P)$ has infinitely many n-types, where n = 1 if $|I| \geq \omega$ and $n = n_0$ for $\operatorname{Th}(\mathcal{A}_i)$ with infinitely many n_0 -types. Hence by Ryll-Nardzewski Theorem $\operatorname{Th}(\mathcal{A}_P)$ is not ω -categorical.

If $\operatorname{Th}(\mathcal{A}_P)$ is ω -categorical then by Ryll-Nardzewski Theorem having finitely many n-types for each $n \in \omega$, we have both finitely many predicates P_i and finitely many n-types for each P_i -restriction, i. e., for $\operatorname{Th}(\mathcal{A}_i)$. \square

Notice that Proposition 2.1 is not true if a P-combination is not disjoint: taking, for instance, a graph \mathcal{A}_1 with a set P_1 of vertices and with infinitely many R_1 -edges such that all vertices have degree 1, as well as taking a graph \mathcal{A}_2 with the same set P_1 of vertices and with infinitely many R_2 -edges such that all vertices have degree 1, we can choose edges such that $R_1 \cap R_2 = \emptyset$, each vertex in P_1 has $(R_1 \cup R_2)$ -degree 2, and alternating R_1 - and R_2 -edges there is an infinite sequence of $(R_1 \cup R_2)$ -edges. Thus, \mathcal{A}_1 and \mathcal{A}_2 are ω -categorical whereas $\operatorname{Comb}(\mathcal{A}_1, \mathcal{A}_2)$ is not.

Note also that Proposition 2.1 does not hold replacing \mathcal{A}_P by \mathcal{A}_E . Indeed, taking infinitely many infinite E-classes with structures of the empty languages we get an ω -categorical structure of the equivalence relation E. At the same time, Proposition 2.1 is preserved if there are finitely many E-classes. In general case \mathcal{A}_E does not preserve the ω -categoricity if and only if E_i -classes approximate infinitely many n-types for some $n \in \omega$, i. e., there are infinitely many n-types $q_m(\bar{x})$, $m \in \omega$, such that for any $m \in \omega$, there are infinitely many n-types $q_m(\bar{x})$, $m \in \omega$, such that for any $m \in \omega$, $\varphi_j(\bar{x}) \in q_j(\bar{x})$, $j \leq m$, and classes E_{k_1}, \ldots, E_{k_m} , all formulas $\varphi_j(\bar{x})$ have realizations in $A_E \setminus \bigcup_{r=1}^\infty E_{k_r}$. Indeed, assuming that all \mathcal{A}_i are ω -categorical we can lose the ω -categoricity for $Th(\mathcal{A}_E)$ only having infinitely many n-types (for some n) inside \mathcal{A}_{∞} . Since all n-types in \mathcal{A}_{∞} are locally (for any formulas in these types) realized in infinitely many \mathcal{A}_i , E_i -classes approximate infinitely many n-types and $Th(\mathcal{A}_E)$ is not ω -categorical. Thus, we have the following

Proposition 2.2. If the languages $\Sigma(A_i)$ are at most countable, $i \in I$, $|I| \leq \omega$, and the structure A_E is infinite then the theory $\operatorname{Th}(A_E)$ is ω -categorical if and only if each structure A_i is either finite or ω -categorical, and I is either finite, or infinite and E_i -classes do not approximate infinitely many n-types for any $n \in \omega$.

As usual we denote by $I(T, \lambda)$ the number of pairwise non-isomorphic models of T having the cardinality λ .

Recall that a theory T is *Ehrenfeucht* if T has finitely many countable models $(I(T,\omega) < \omega)$ but is not ω -categorical $(I(T,\omega) > 1)$. A structure with an Ehrenfeucht theory is also *Ehrenfeucht*.

Theorem 2.3. If predicates P_i are pairwise disjoint, the languages $\Sigma(A_i)$ are at most countable, $i \in I$, and the structure A_P is infinite then the theory $Th(A_P)$ is Ehrenfeucht if and only if the following conditions hold:

- (a) I is finite;
- (b) each structure A_i is either finite, or ω -categorical, or Ehrenfeucht;

(c) some A_i is Ehrenfeucht.

Proof. If I is finite, each structure A_i is either finite, or ω -categorical, or Ehrenfeucht, and some A_i is Ehrenfeucht then $T = \text{Th}(A_P)$ is Ehrenfeucht since each model of T is composed of disjoint models with universes P_i and

$$I(T,\omega) = \prod_{i \in I} I(\operatorname{Th}(\mathcal{A}_i), \min\{|A_i|, \omega\}). \tag{1}$$

Now if I is finite and all \mathcal{A}_i are ω -categorical then by (1), $I(T, \omega) = 1$, and if some $I(\operatorname{Th}(\mathcal{A}_i), \omega) \geq \omega$ then again by (1), $I(T, \omega) \geq \omega$.

Assuming that $|I| \geq \omega$ we have to show that the non- ω -categorical theory T has infinitely many countable models. Assuming on contrary that $I(T,\omega) < \omega$, i. e., T is Ehrenfeucht, we have a nonisolated powerful type $q(\bar{x}) \in S(T)$ [6], i. e., a type such that any model of T realizing $q(\bar{x})$ realizes all types in S(T). By the construction of disjoint union, $q(\bar{x})$ should have a realization of the type $p_{\infty}(x) = \{\neg P_i(x) \mid i \in I\}$. Moreover, if some $\mathrm{Th}(\mathcal{A}_i)$ is not ω -categorical for infinite A_i then $q(\bar{x})$ should contain a powerful type of $\mathrm{Th}(\mathcal{A}_i)$ and the restriction $r(\bar{y})$ of $q(\bar{x})$ to the coordinates realized by $p_{\infty}(x)$ should be powerful for the theory $\mathrm{Th}(\mathcal{A}_{\infty})$, where \mathcal{A}_{∞} is infinite and saturated, as well as realizing $r(\bar{y})$ in a model $\mathcal{M} \models T$, all types with coordinates satisfying $p_{\infty}(x)$ should be realized in \mathcal{M} too. As shown in [4, 7], the type $r(\bar{y})$ has the local realizability property and satisfies the following conditions: for each formula $\varphi(\bar{y}) \in r(\bar{y})$, there exists a formula $\psi(\bar{y}, \bar{z})$ of T (where $l(\bar{y}) = l(\bar{z})$), satisfying the following conditions:

- (i) for each $\bar{a} \in r(M)$, the formula $\psi(\bar{a}, \bar{y})$ is equivalent to a disjunction of principal formulas $\psi_i(\bar{a}, \bar{y})$, $i \leq m$, such that $\psi_i(\bar{a}, \bar{y}) \vdash r(\bar{y})$, and $\models \psi_i(\bar{a}, \bar{b})$ implies, that \bar{b} does not semi-isolate \bar{a} ;
- (ii) for every $\bar{a}, \bar{b} \in r(M)$, there exists a tuple \bar{c} such that $\models \varphi(\bar{c}) \land \psi(\bar{c}, \bar{a}) \land \psi(\bar{c}, \bar{b})$.

Since the type $p_{\infty}(x)$ is not isolated each formula $\varphi(\bar{y}) \in r(\bar{y})$ has realizations \bar{d} in $\bigcup_{i \in I} A_i$. On the other hand, as we consider the disjoint union of A_i and there are no non-trivial links between distinct P_i and $P_{i'}$, the sets of solutions for $\psi(\bar{d}, \bar{y})$ with $\models \varphi(\bar{d})$ in $\{\neg P_i(x) \mid \models P_i(d_j) \text{ for some } d_j \in \bar{d}\}$ are either equal or empty being composed by definable sets without parameters. If these sets are nonempty the item (i) can not be satisfied: $\psi(\bar{a}, \bar{y})$ is not equivalent to a disjunction of principal formulas. Otherwise all ψ -links for realizations of $r(\bar{y})$ are situated inside the set of solutions for $\bar{p}_{\infty}(\bar{y}) = \bigcup_{y_j \in \bar{y}} p_{\infty}(y_j)$. In

this case for $\bar{a} \models r(\bar{y})$ the formula $\exists \bar{z}(\psi(\bar{z},\bar{a}) \land \psi(\bar{z},\bar{y}))$ does not cover the set r(M) since it does not cover each φ -approximation of r(M). Thus, the property (ii) fails.

Hence, (i) and (ii) can not be satisfied, there are no powerful types, and the theory T is not Ehrenfeucht. \square

3 Variations of structures related to combinations and *E*-representability

Clearly, for a disjoint P-combination \mathcal{A}_P with infinite I, there is a structure $\mathcal{A}' \equiv \mathcal{A}_P$ with a structure \mathcal{A}'_{∞} . Since the type $p_{\infty}(x)$ is nonisolated (omitted in \mathcal{A}_P), the cardinalities for \mathcal{A}'_{∞} are unbounded. Infinite structures \mathcal{A}'_{∞} are not necessary elementary equivalent and can be both elementary equivalent to some \mathcal{A}_i or not. For instance, if infinitely many structures \mathcal{A}_i contain unary predicates Q_0 , say singletons, without unary predicates Q_1 and infinitely many $\mathcal{A}_{i'}$ for $i' \neq i$ contain Q_1 , say again singletons, without Q_0 then \mathcal{A}'_{∞} can contain Q_0 without Q_1 , Q_1 without Q_0 , or both Q_0 and Q_1 . For the latter case, \mathcal{A}'_{∞} is not elementary equivalent neither \mathcal{A}_i , nor $\mathcal{A}_{i'}$.

A natural question arises:

Question 1. What can be the number of pairwise elementary non-equivalent structures \mathcal{A}'_{∞} ?

Considering an E-combination \mathcal{A}_E with infinite I, and all structures $\mathcal{A}' \equiv \mathcal{A}_E$, there are two possibilities: each non-empty E-restriction of \mathcal{A}'_{∞} , i. e. a restriction to some E-class, is elementary equivalent to some \mathcal{A}_i , $i \in I$, or some E-restriction of \mathcal{A}'_{∞} is not elementary equivalent to all structures \mathcal{A}_i , $i \in I$.

Similarly Question 1 we have:

Question 2. What can be the number of pairwise elementary non-equivalent E-restrictions of structures \mathcal{A}'_{∞} ?

Example 3.1. Let \mathcal{A}_P be a disjoint P-combination with infinite I and composed by infinite \mathcal{A}_i , $i \in I$, such that I is a disjoint union of infinite I_j , $j \in J$, where \mathcal{A}_{i_j} contains only unary predicates and unique nonempty unary predicate Q_j being a singleton. Then \mathcal{A}'_{∞} can contain any singleton Q_j and finitely or infinitely many elements in $\bigcap_{i \in I} \overline{Q}_j$. Thus, there are $2^{|J|} \cdot (\lambda + 1)$

non-isomorphic \mathcal{A}'_{∞} , where λ is a least upper bound for cardinalities $\left|\bigcap_{j\in J} \overline{Q}_j\right|$.

For $T = \text{Th}(\mathcal{A}_P)$, we denote by $I_{\infty}(T, \lambda)$ the number of pairwise non-isomorphic structures \mathcal{A}'_{∞} having the cardinality λ .

Clearly, $I_{\infty}(T,\lambda) \leq I(T,\lambda)$.

If structures \mathcal{A}'_{∞} exist and do not have links with \mathcal{A}'_P (for instance, for a disjoint P-combination) then $I_{\infty}(T,\lambda)+1\leq I(T,\lambda)$, since if models of T are isomorphic then their restrictions to $p_{\infty}(x)$ are isomorphic too, and $p_{\infty}(x)$ can be omitted producing $\mathcal{A}'_{\infty} = \emptyset$. Here $I_{\infty}(T,\lambda)+1=I(T,\lambda)$ if and only if all $I(\operatorname{Th}(\mathcal{A}_i),\lambda)=1$ and, moreover, for any $\left(\bigcup_{i\in I}P_i\right)$ -restrictions $\mathcal{B}_P,\mathcal{B}'_P$ of $\mathcal{B},\mathcal{B}' \models T$ respectively, where $|B|=|B'|=\lambda$, and their P_i -restrictions $\mathcal{B}_i,\mathcal{B}'_i$, there are isomorphisms $f_i\colon \mathcal{B}_i \cong \mathcal{B}'_i$ preserving P_i and with an isomorphism $\bigcup_{i\in I}f_i\colon \mathcal{B}_P\cong \mathcal{B}'_P$.

The following example illustrates the equality $I_{\infty}(T,\lambda)+1=I(T,\lambda)$ with some $I(\operatorname{Th}(\mathcal{A}_i),\lambda)>1$.

Example 3.2. Let P_0 be a unary predicate containing a copy of the Ehrenfeucht example [8] with a dense linear order \leq and an increasing chain of singletons coding constants c_k , $k \in \omega$; P_n , $n \geq 1$, be pairwise disjoint unary predicates disjoint to P_0 such that $P_1 = (-\infty, c'_0) P_{n+2} = [c'_n, c'_{n+1})$, $n \in \omega$, and $\bigcup_{n \geq 1} P_n$ forms a universe of prime model (over \varnothing) for another copy of the Ehrenfeucht example with a dense linear order \leq' and an increasing chain of constants c'_k , $k \in \omega$. Now we extend the language

$$\Sigma = \langle \leq, \leq', P_n, \{c_n\}, \{c_n'\} \rangle_{n \in \omega}$$

by a bijection f between $P_0 = \{a \mid a \leq c_0 \text{ or } c_0 \leq a\}$ and $\{a' \mid a' \leq c' \}$ or $c'_0 \leq a' \}$ such that $a \leq b \Leftrightarrow f(a) \leq f(a)$. The structures \mathcal{A}'_{∞} consist of realizations $p_{\infty}(x)$ which are bijective with realizations of the type $\{c_n < x \mid n \in \omega\}$.

For the theory T of the described structure $\mathrm{EComb}_P(\mathcal{A}_i)_{i\in I}$ we have $I(T,\omega)=3$ (as for the Ehrenfeucht example and the restriction of T to P_0) and $I_\infty(T,\omega)=2$ (witnessed by countable structures with least realizations of $p_\infty(x)$ and by countable structure with realizations of $p_\infty(x)$ all of which are not least).

For Example 3.1 of a theory T with singletons Q_j in \mathcal{A}_i and for a cardinality $\lambda \geq 1$, we have

$$I_{\infty}(T,\lambda) = \left\{ \begin{array}{ll} \sum\limits_{i=0}^{\min\{|J|,\lambda\}} C^i_{|J|}, & \text{if J and λ are finite;} \\ |J|, & \text{if J is infinite and } |J| > \lambda; \\ 2^{|J|}, & \text{if J is infinite and } |J| \leq \lambda. \end{array} \right.$$

Clearly, $\mathcal{A}' \equiv \mathcal{A}_P$ realizing $p_{\infty}(x)$ is not elementary embeddable into \mathcal{A}_P and can not be represented as a disjoint P-combination of $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. At the same time, there are E-combinations such that all $\mathcal{A}' \equiv \mathcal{A}_E$ can be represented as E-combinations of some $\mathcal{A}'_j \equiv \mathcal{A}_i$. We call this representability of \mathcal{A}' to be the E-representability. If, for instance, all \mathcal{A}_i are infinite structures of the empty language then any $\mathcal{A}' \equiv \mathcal{A}_E$ is an E-combination of some infinite structures \mathcal{A}'_j of the empty language too.

Thus we have:

Question 3. What is a characterization of E-representability for all $A' \equiv A_E$?

Definition (cf. [9]). For a first-order formula $\varphi(x_1, \ldots, x_n)$, an equivalence relation E and a formula $\sigma(x)$ we define a (E, σ) -relativized formula $\varphi^{E,\sigma}$ by induction:

- (i) if φ is an atomic formula then $\varphi^{E,\sigma} = \varphi(x_1,\ldots,x_n) \wedge \bigwedge_{i,j=1}^n E(x_i,x_j) \wedge \exists y (E(x_1,y) \wedge \sigma(y));$
- (ii) if $\varphi = \psi \tau \chi$, where $\tau \in \{\land, \lor, \rightarrow\}$, and $\psi^{E,\sigma}$ and $\chi^{E,\sigma}$ are defined then $\varphi^{E,\sigma} = \psi^{E,\sigma} \tau \chi^{E,\sigma}$;
- (iii) if $\varphi(x_1, \ldots, x_n) = \neg \psi(x_1, \ldots, x_n)$ and $\psi^{E,\sigma}(x_1, \ldots, x_n)$ is defined then $\varphi^{E,\sigma}(x_1, \ldots, x_n) = \neg \psi^{E,\sigma}(x_1, \ldots, x_n) \wedge \bigwedge_{i,j=1}^n (E(x_i, x_j) \wedge \exists y (E(x_1, y) \wedge \sigma(y));$
- (iv) if $\varphi(x_1,\ldots,x_n)=\exists x\psi(x,x_1,\ldots,x_n)$ and $\psi^{E,\sigma}(x,x_1,\ldots,x_n)$ is defined then

$$\varphi^{E,\sigma}(x_1,\ldots,x_n) = \exists x \left(\bigwedge_{i=1}^n (E(x,x_i) \wedge \exists y (E(x,y) \wedge \sigma(y)) \wedge \psi^{E,\sigma}(x,x_1,\ldots,x_n) \right);$$

(v) if
$$\varphi(x_1,\ldots,x_n) = \forall x \psi(x,x_1,\ldots,x_n)$$
 and $\psi^{E,\sigma}(x,x_1,\ldots,x_n)$ is defined

then

$$\varphi^{E,\sigma}(x_1,\ldots,x_n) = \forall x \left(\bigwedge_{i=1}^n E(x,x_i) \wedge \exists y (E(x,y) \wedge \sigma(y)) \to \psi^{E,\sigma}(x,x_1,\ldots,x_n) \right).$$

We write E instead of (E, σ) if $\sigma = (x \approx x)$.

Note that two E-classes E_i and E_j with structures \mathcal{A}_i and \mathcal{A}_j (of a language Σ), respectively, are not elementary equivalent if and only if there is a Σ -sentence φ such that $\mathcal{A}_E \upharpoonright E_i \models \varphi^E$ (with $\mathcal{A}_i \models \varphi$) and $\mathcal{A}_E \upharpoonright E_j \models (\neg \varphi)^E$ (with $\mathcal{A}_j \models \neg \varphi$). In this case, the formula φ is called (i, j)-separating.

The following properties are obvious:

- (1) If φ is (i, j)-separating then $\neg \varphi$ is (j, i)-separating.
- (2) If φ is (i, j)-separating and ψ is (i, k)-separating then $\varphi \wedge \psi$ is both (i, j)-separating and (i, k)-separating.
- (3) There is a set Φ_i of (i, j)-separating sentences, for j in some $J \subseteq I \setminus \{i\}$, which separates A_i from all structures $A_j \not\equiv A_i$.

The set Φ_i is called *e-separating* (for \mathcal{A}_i) and \mathcal{A}_i is *e-separable* (witnessed by Φ_i).

Assuming that some $\mathcal{A}' \equiv \mathcal{A}_E$ is not E-representable, we get an E'-class with a structure \mathcal{B} in \mathcal{A}' which is e-separable from all \mathcal{A}_i , $i \in I$, by a set Φ . It means that for some sentences φ_i with $\mathcal{A}_E \upharpoonright E_i \models \varphi_i^E$, i. e., $\mathcal{A}_i \models \varphi_i$, the sentences $\left(\bigwedge_{i \in I_0} \neg \varphi_i\right)^E$, where $I_0 \subseteq_{\text{fin}} I$, form a consistent set, satisfying the restriction of \mathcal{A}' to the class E'_B with the universe B of \mathcal{B} .

Thus, answering Question 3 we have

Proposition 3.3. For any E-combination A_E the following conditions are equivalent:

- (1) there is $A' \equiv A_E$ which is not E-representable;
- (2) there are sentences φ_i such that $\mathcal{A}_i \models \varphi_i$, $i \in I$, and the set of sentences $\left(\bigwedge_{i \in I_0} \neg \varphi_i\right)^E$, where $I_0 \subseteq_{\text{fin}} I$, is consistent with $\text{Th}(\mathcal{A}_E)$.

Proposition 3.3 implies

Corollary 3.4. If A_E has only finitely many pairwise elementary non-equivalent E-classes then each $A' \equiv A_E$ is E-representable.

4 e-spectra

If there is $\mathcal{A}' \equiv \mathcal{A}_E$ which is not E-representable, we have the E'-representability replacing E by E' such that E' is obtained from E adding equivalence classes with models for all theories T, where T is a theory of a restriction \mathcal{B} of a structure $\mathcal{A}' \equiv \mathcal{A}_E$ to some E-class and \mathcal{B} is not elementary equivalent to the structures \mathcal{A}_i . The resulting structure $\mathcal{A}_{E'}$ (with the E'-representability) is a e-completion, or a e-saturation, of \mathcal{A}_E . The structure $\mathcal{A}_{E'}$ itself is called e-complete, or e-saturated, or e-universal, or e-largest.

For a structure \mathcal{A}_E the number of *new* structures with respect to the structures \mathcal{A}_i , i. e., of the structures \mathcal{B} which are pairwise elementary non-equivalent and elementary non-equivalent to the structures \mathcal{A}_i , is called the e-spectrum of \mathcal{A}_E and denoted by e-Sp(\mathcal{A}_E). The value sup{e-Sp(\mathcal{A}')) | $\mathcal{A}' \equiv \mathcal{A}_E$ } is called the e-spectrum of the theory Th(\mathcal{A}_E) and denoted by e-Sp(Th(\mathcal{A}_E)).

If \mathcal{A}_E does not have *E*-classes \mathcal{A}_i , which can be removed, with all *E*-classes $\mathcal{A}_j \equiv \mathcal{A}_i$, preserving the theory $\text{Th}(\mathcal{A}_E)$, then \mathcal{A}_E is called *e*-prime, or *e*-minimal.

For a structure $\mathcal{A}' \equiv \mathcal{A}_E$ we denote by $TH(\mathcal{A}')$ the set of all theories $Th(\mathcal{A}_i)$ of E-classes \mathcal{A}_i in \mathcal{A}' .

By the definition, an e-minimal structure \mathcal{A}' consists of E-classes with a minimal set $\mathrm{TH}(\mathcal{A}')$. If $\mathrm{TH}(\mathcal{A}')$ is the least for models of $\mathrm{Th}(\mathcal{A}')$ then \mathcal{A}' is called e-least.

The following proposition is obvious:

Proposition 4.1. 1. For a given language Σ , $0 \leq e \operatorname{-Sp}(\operatorname{Th}(\mathcal{A}_E)) \leq 2^{\max\{|\Sigma|,\omega\}}$

- 2. A structure A_E is e-largest if and only if e-Sp(A_E) = 0. In particular, an e-minimal structure A_E is e-largest is and only if e-Sp(Th(A_E)) = 0.
- 3. Any weakly saturated structure A_E , i. e., a structure realizing all types of $Th(A_E)$ is e-largest.
- 4. For any E-combination \mathcal{A}_E , if $\lambda \leq e\text{-Sp}(\operatorname{Th}(\mathcal{A}_E))$ then there is a structure $\mathcal{A}' \equiv \mathcal{A}_E$ with $e\text{-Sp}(\mathcal{A}') = \lambda$; in particular, any theory $\operatorname{Th}(\mathcal{A}_E)$ has an e-largest model.
- 5. For any structure \mathcal{A}_E , $e\text{-Sp}(\mathcal{A}_E) = |\mathrm{TH}(\mathcal{A}'_{E'}) \setminus \mathrm{TH}(\mathcal{A}_E)|$, where $\mathcal{A}'_{E'}$ is an e-largest model of $\mathrm{Th}(\mathcal{A}_E)$.

6. Any prime structure A_E is e-minimal (but not vice versa as the e-minimality is preserved, for instance, extending an infinite E-class of given structure to a greater cardinality). Any small theory $\operatorname{Th}(A_E)$ has an e-minimal model (being prime), and in this case, the structure A_E is e-minimal if and only if

$$\mathrm{TH}(\mathcal{A}_E) = \bigcap_{\mathcal{A}' \equiv \mathcal{A}_E} \mathrm{TH}(\mathcal{A}'),$$

i. e., A_E is e-least.

- 7. If A_E is e-least then e-Sp(A_E) = e-Sp(Th(A_E)).
- 8. If $e\operatorname{-Sp}(\operatorname{Th}(\mathcal{A}_E))$ finite and $\operatorname{Th}(\mathcal{A}_E)$ has $e\operatorname{-least}$ model then \mathcal{A}_E is $e\operatorname{-minimal}$ if and only if \mathcal{A}_E is $e\operatorname{-least}$ and if and only if $e\operatorname{-Sp}(\mathcal{A}_E)=e\operatorname{-Sp}(\operatorname{Th}(\mathcal{A}_E))$.
- 9. If e-Sp(Th(\mathcal{A}_E)) is infinite then there are $\mathcal{A}' \equiv \mathcal{A}_E$ such that e-Sp(\mathcal{A}') = e-Sp(Th(\mathcal{A}_E)) but \mathcal{A}' is not e-minimal.
- 10. A countable e-minimal structure A_E is prime if and only if each E-class A_i is a prime structure.

Reformulating Proposition 2.2 we have

Proposition 4.2. For E-combinations which are not EComb, a countable theory $Th(\mathcal{A}_E)$ without finite models is ω -categorical if and only if e- $Sp(Th(\mathcal{A}_E)) = 0$ and each E-class \mathcal{A}_i is either finite or ω -categorical.

Note that if there are no links between E-classes (i. e., the Comb is considered, not EComb) and there is $\mathcal{A}' \equiv \mathcal{A}_E$ which is not E-representable, then by Compactness the e-completion can vary adding arbitrary (finitely or infinitely) many new E-classes with a fixed structure which is not elementary equivalent to structures in old E-classes.

Proposition 4.3. For any cardinality λ there is a theory $T = \text{Th}(A_E)$ of a language Σ such that $|\Sigma| = |\lambda + 1|$ and $e\text{-Sp}(T) = \lambda$.

Proof. Clearly, for structures \mathcal{A}_i of fixed cardinality and with empty language we have e-Sp(Th(\mathcal{A}_E)) = 0. For $\lambda > 0$ we take a language Σ consisting of unary predicate symbols P_i , $i < \lambda$. Let $\mathcal{A}_{i,n+1}$ be a structure having a universe $A_{i,n}$ with n elements and $P_i = A_{i,n}$, $P_j = \emptyset$, $i, j < \lambda$, $i \neq j$, $n \in \omega \setminus \{0\}$. Clearly, the structure \mathcal{A}_E , formed by all $\mathcal{A}_{i,n}$, is e-minimal. It produces structures $\mathcal{A}' \equiv \mathcal{A}_E$ containing E-classes with infinite predicates P_i ,

and structures of these classes are not elementary equivalent to the structures $\mathcal{A}_{i,n}$. Thus, for the theory $T = \text{Th}(\mathcal{A}_E)$ we have $e\text{-Sp}(T) = \lambda$. \square

In Proposition 4.3, we have e-Sp $(T) = |\Sigma(T)|$. At the same time the following proposition holds.

Proposition 4.4. For any infinite cardinality λ there is a theory $T = \text{Th}(\mathcal{A}_E)$ of a language Σ such that $|\Sigma| = \lambda$ and $e\text{-Sp}(T) = 2^{\lambda}$.

Proof. Let P_j be unary predicate symbols, $j < \lambda$, forming the language Σ , and \mathcal{A}_i be structures consisting of only finitely many nonempty predicates P_{j_1}, \ldots, P_{j_k} and such that these predicates are independent. Taking for the structures \mathcal{A}_i all possibilities for cardinalities of sets of solutions for formulas $P_{j_1}^{\delta_{j_1}}(x) \wedge \ldots \wedge P_{j_k}^{\delta_{j_k}}(x)$, $\delta_{j_l} \in \{0,1\}$, we get an e-minimal structure \mathcal{A}_E such that for the theory $T = \text{Th}(\mathcal{A}_E)$ we have e-Sp $(T) = 2^{\lambda}$.

Another approach for e-Sp $(T) = 2^{\lambda}$ was suggested by E.A. Palyutin. Taking infinitely many \mathcal{A}_i with arbitrarily finitely many disjoint singletons R_{j_1}, \ldots, R_{j_k} , where Σ consists of R_j , $j < \lambda$, we get $\mathcal{A}' \equiv \mathcal{A}_E$ with arbitrarily many singletons for any subset of λ producing 2^{λ} E-classes which are pairwise elementary non-equivalent. \square

If e-Sp(T) = 0 the theory T is called e-non-abnormalized or (e,0)-abnormalized. Otherwise, i. e., if e-Sp(T) > 0, T is e-abnormalized. An e-abnormalized theory T with e-Sp(T) = λ is called (e,λ) -abnormalized. In particular, an (e,1)-abnormalized theory is e-categorical, an (e,n)-abnormalized theory with $n \in \omega \setminus \{0,1\}$ is e-Ehrenfeucht, an (e,ω) -abnormalized theory is e-countable, and an $(e,2^{\lambda})$ -abnormalized theory is (e,λ) -maximal.

If e-Sp(T) = λ and T has a model \mathcal{A}_E with e-Sp(\mathcal{A}_E) = μ then \mathcal{A}_E is called (e, \varkappa) -abnormalized, where \varkappa is the least cardinality with $\mu + \varkappa = \lambda$. By proofs of Propositions 4.3 and 4.4 we have

Corollary 4.5. For any cardinalities $\mu \leq \lambda$ and the least cardinality \varkappa with $\mu + \varkappa = \lambda$ there is an (e, λ) -abnormalized theory T with an (e, \varkappa) -abnormalized model A_E .

Let \mathcal{A}_E and $\mathcal{B}_{E'}$ be structures and $\mathcal{C}_{E''} = \mathcal{A}_E \coprod \mathcal{B}_{E'}$ be their disjoint union, where $E'' = E \coprod E'$. We denote by ComLim $(\mathcal{A}_E, \mathcal{B}_{E'})$ the number of elementary pairwise non-equivalent structures \mathcal{D} which are both a restriction of $\mathcal{A}' \equiv \mathcal{A}_E$ to some E-class and a restriction of $\mathcal{B}' \equiv \mathcal{B}_{E'}$ to some E'-class as well as \mathcal{D} is not elementary equivalent to the structures \mathcal{A}_i and \mathcal{B}_j .

We have:

$$\operatorname{ComLim}(\mathcal{A}_{E}, \mathcal{B}_{E'}) \leq \min\{e\operatorname{-Sp}(\operatorname{Th}(\mathcal{A}_{E})), e\operatorname{-Sp}(\operatorname{Th}(\mathcal{B}_{E'}))\},$$

$$\max\{e\operatorname{-Sp}(\operatorname{Th}(\mathcal{A}_{E})), e\operatorname{-Sp}(\operatorname{Th}(\mathcal{B}_{E'}))\} \leq e\operatorname{-Sp}(\operatorname{Th}(\mathcal{C}_{E''})),$$

$$e\operatorname{-Sp}(\operatorname{Th}(\mathcal{A}_{E})) + e\operatorname{-Sp}(\operatorname{Th}(\mathcal{B}_{E'})) = e\operatorname{-Sp}(\operatorname{Th}(\mathcal{C}_{E''})) + \operatorname{ComLim}(\mathcal{A}_{E}, \mathcal{B}_{E'}).$$

Indeed, all structures witnessing the value e-Sp(Th($\mathcal{C}_{E''}$)) can be obtained by Th(\mathcal{A}_E) or Th($\mathcal{B}_{E'}$) and common structures are counted for ComLim(\mathcal{A}_E , $\mathcal{B}_{E'}$).

If $\mathcal{A}_E = \mathcal{B}_{E'}$ then $\operatorname{ComLim}(\mathcal{A}_E, \mathcal{B}_{E'}) = e\operatorname{-Sp}(\operatorname{Th}(\mathcal{A}_E))$. Assuming that \mathcal{A}_E and $\mathcal{B}_{E'}$ do not have elementary equivalent classes \mathcal{A}_i and \mathcal{B}_j , the number $\operatorname{ComLim}(\mathcal{A}_E, \mathcal{B}_{E'})$ can vary from 0 to $2^{|\Sigma| + \omega}$.

Indeed, if $\operatorname{Th}(\mathcal{A}_E)$ or $\operatorname{Th}(\mathcal{B}_{E'})$ does not produce new, elementary non-equivalent classes then $\operatorname{ComLim}(\mathcal{A}_E, \mathcal{B}_{E'}) = 0$. Otherwise we can take structures \mathcal{A}_i and \mathcal{B}_i with one unary predicate symbol P such that P has 2i elements for \mathcal{A}_i and 2i+1 elements for \mathcal{B}_i , $i \in \omega$. In this case we have $\operatorname{Sp}(\operatorname{Th}(\mathcal{A}_E)) = 1$, $\operatorname{Sp}(\operatorname{Th}(\mathcal{B}_{E'})) = 1$, $\operatorname{ComLim}(\mathcal{A}_E, \mathcal{B}_{E'}) = 1$, and $\mathcal{C}_{E''}$ witnessed by structures with infinite interpretations for P. Extending the language by unary predicates P_i , $i < \lambda$, and interpreting P_i in disjoint structures as for P above, we get $\operatorname{Sp}(\operatorname{Th}(\mathcal{A}_E)) = \lambda$, $\operatorname{Sp}(\operatorname{Th}(\mathcal{B}_{E'})) = \lambda$, $\operatorname{ComLim}(\mathcal{A}_E, \mathcal{B}_{E'}) = \lambda$. Thus we have

Proposition 4.6. For any cardinality λ there are structures \mathcal{A}_E and $\mathcal{B}_{E'}$ of a language Σ such that $|\Sigma| = |\lambda + 1|$ and $\operatorname{ComLim}(\mathcal{A}_E, \mathcal{B}_{E'}) = \lambda$.

Applying proof of Proposition 4.4 with even and odd cardinalities for intersections of predicates in \mathcal{A}_i and \mathcal{B}_j respectively, we have $\operatorname{Sp}(\operatorname{Th}(\mathcal{A}_E)) = 2^{\lambda}$, $\operatorname{Sp}(\operatorname{Th}(\mathcal{B}_{E'})) = 2^{\lambda}$, $\operatorname{ComLim}(\mathcal{A}_E, \mathcal{B}_{E'}) = 2^{\lambda}$. In particular, we get

Proposition 4.7. For any infinite cardinality λ are structures \mathcal{A}_E and $\mathcal{B}_{E'}$ of a language Σ such that $|\Sigma| = \lambda$ and $\operatorname{ComLim}(\mathcal{A}_E, \mathcal{B}_{E'}) = 2^{\lambda}$.

Replacing E-classes by unary predicates P_i (not necessary disjoint) being universes for structures \mathcal{A}_i and restricting models of $\operatorname{Th}(\mathcal{A}_P)$ to the set of realizations of $p_{\infty}(x)$ we get the e-spectrum e-Sp(Th(\mathcal{A}_P)), i. e., the number of pairwise elementary non-equivalent restrictions of $\mathcal{M} \models \operatorname{Th}(\mathcal{A}_P)$ to $p_{\infty}(x)$. We also get the notions of (e, λ) -abnormalized theory $\operatorname{Th}(\mathcal{A}_P)$, of (e, λ) -abnormalized model of $\operatorname{Th}(\mathcal{A}_P)$, and related notions.

Note that for any countable theory $T = \text{Th}(A_P)$, $e\text{-Sp}(T) \leq I(T, \omega)$. In particular, if $I(T, \omega)$ is finite then e-Sp(T) is finite too. Moreover, if T is ω -categorical then e-Sp(T)=0, and if T is an Ehrenfeucht theory, then e-Sp $(T) < I(T, \omega)$. Illustrating the finiteness for Ehrenfeucht theories we consider

Example 4.8. Similar to Example 3.2, let T_0 be the Ehrenfeucht theory of a structure \mathcal{M}_0 , formed from the structure $\langle \mathbb{Q}; \langle \rangle$ by adding singletons R_k for elements c_k , $c_k < c_{k+1}$, $k \in \omega$, such that $\lim_{k \to \infty} c_k = \infty$. It is well known that the theory T_3 has exactly 3 pairwise non-isomorphic models:

- (a) a prime model \mathcal{M}_0 ($\lim_{k\to\infty} c_k = \infty$);
- (b) a prime model \mathcal{M}_1 over a realization of powerful type $p_{\infty}(x) \in S^1(\varnothing)$, isolated by sets of formulas $\{c_k < x \mid k \in \omega\};$

(c) a saturated model \mathcal{M}_2 (the limit $\lim_{k\to\infty} c_k$ is irrational). Now we introduce unary predicates $P_i = \{a \in M_0 \mid a < c_i\}, i < \omega$, on \mathcal{M}_0 . The structures $\mathcal{A}_i = \mathcal{M}_0 \upharpoonright P_i$ form the P-combination \mathcal{A}_P with the universe M_0 . Realizations of the type $p_{\infty}(x)$ in \mathcal{M}_1 and in \mathcal{M}_2 form two elementary non-equivalent structures \mathcal{A}_{∞} and \mathcal{A}'_{∞} respectively, where \mathcal{A}_{∞} has a dense linear order with a least element and \mathcal{A}'_{∞} has a dense linear order without endpoints. Thus, e-Sp $(T_0) = 2$ and T_0 is e-Ehrenfeucht.

As E.A. Palyutin noticed, varying unary predicates P_i in the following way: $P_{2i} = \{a \in M_0 \mid a < c_{2i}\}, P_{2i+1} = \{a \in M_0 \mid a \le c_{2i+1}\}, \text{ we get } e$ $\operatorname{Sp}(T_3) = 4$ since the structures \mathcal{A}'_{∞} have dense linear orders with(out) least elements and with(out) greatest elements.

Modifying Example above, let T_n be the Ehrenfeucht theory of a structure \mathcal{M}^n , formed from the structure $\langle \mathbb{Q}; \langle \rangle$ by adding constants $c_k, c_k < c_{k+1}$, $k \in \omega$, such that $\lim_{k \to \infty} c_k = \infty$, and unary predicates R_0, \ldots, R_{n-2} which form a partition of the set \mathbb{Q} of rationals, with

$$\models \forall x, y ((x < y) \rightarrow \exists z ((x < z) \land (z < y) \land R_i(z))), i = 0, \dots, n - 2.$$

The theory T_n has exactly n+1 pairwise non-isomorphic models:

- (a) a prime model \mathcal{M}^n ($\lim_{k\to\infty} c_k = \infty$);
- (b) prime models \mathcal{M}_i^n over realizations of powerful types $p_i(x) \in S^1(\varnothing)$, isolated by sets of formulas $\{c_k < x \mid k \in \omega\} \cup \{P_i(x)\}, i = 0, \dots, n-2$ ($\lim c_k \in P_i$);
 - (c) a saturated model $\mathcal{M}_i nfty^n$ (the limit $\lim_{k\to\infty} c_k$ is irrational).

Now we introduce unary predicates $P_i = \{a \in M^n \mid a < c_i\}, i < \omega, \text{ on }$ \mathcal{M}^n . The structures $\mathcal{A}_i = \mathcal{M}^n \upharpoonright P_i$ form the P-combination \mathcal{A}_P with the universe M^n . Realizations of the type $p_{\infty}(x)$ in \mathcal{M}_i^n and in \mathcal{M}_{∞}^n form n-1 elementary non-equivalent structures \mathcal{A}_j^n , $j \leq n-2$, and \mathcal{A}_{∞}^n , where \mathcal{A}_j^n has a dense linear order with a least element in R_j , and \mathcal{A}_{∞}^n has a dense linear order without endpoints. Thus, $e\text{-Sp}(T_n) = n$ and T_n is e-Ehrenfeucht.

Note that in the example above the type $p_{\infty}(x)$ has n-1 completions by formulas $R_0(x), \ldots, R_{n-2}(x)$.

Example 4.9. Taking a disjoint union \mathcal{M} of $m \in \omega \setminus \{0\}$ copies of \mathcal{M}_0 in the language $\{<_j, R_k\}_{j < m, k \in \omega}$ and unary predicates $P_i = \{a \mid \mathcal{M} \models \exists x (a < x \land R_i(x))\}$ we get the P-combination \mathcal{A}_P with the universe M for the structures $\mathcal{A}_i = \mathcal{M} \upharpoonright P_i$, $i \in \omega$. We have e-Sp(Th(\mathcal{A}_P)) = $3^m - 1$ since each connected component of \mathcal{M} produces at most two possibilities for dense linear orders or can be empty on the set of realizations of $p_{\infty}(x)$, and at least one connected component has realizations of $p_{\infty}(x)$.

Marking the relations $<_i$ by the same symbol < we get the theory T with

$$e$$
-Sp $(T) = \sum_{l=1}^{m} (l+1) = \frac{m(m+1)}{2} + m = \frac{m^2 + 3m}{2}.$

Examples 4.8 and 4.9 illustrate that having a powerful type $p_{\infty}(x)$ we get e-Sp(Th(\mathcal{A}_P)) $\neq 1$, i. e., there are no e-categorical theories Th(\mathcal{A}_P) with a powerful type $p_{\infty}(x)$. Moreover, we have

Theorem 4.10. For any theory $\operatorname{Th}(\mathcal{A}_P)$ with non-symmetric or definable semi-isolation on the complete type $p_{\infty}(x)$, $e\operatorname{-Sp}(\operatorname{Th}(\mathcal{A}_P)) \neq 1$.

Proof. Assuming the hypothesis we take a realization a of $p_{\infty}(x)$ and construct step-by-step a $(a, p_{\infty}(x))$ -thrifty $model \mathcal{N}$ of $Th(\mathcal{A}_P)$, i. e., a model satisfying the following condition: if $\varphi(x, y)$ is a formula such that $\varphi(a, y)$ is consistent and there are no consistent formulas $\psi(a, y)$ with $\psi(a, y) \vdash p_{\infty}(x)$ then $\varphi(a, \mathcal{N}) = \emptyset$.

At the same time, since $p_{\infty}(x)$ is non-isolated, for any realization a of $p_{\infty}(x)$ the set $p_{\infty}(x) \cup \{\neg \varphi(a,x) \mid \varphi(a,x) \vdash p_{\infty}(x)\}$ is consistent. Then there is a model $\mathcal{N}' \models \operatorname{Th}(\mathcal{A}_P)$ realizing $p_{\infty}(x)$ and which is not $(a', p_{\infty}(x))$ -thrifty for any realization a' of $p_{\infty}(x)$.

If semi-isolation is non-symmetric, $\mathcal{N} \upharpoonright p_{\infty}(x)$ and $\mathcal{N}' \upharpoonright p_{\infty}(x)$ are not elementary equivalent since the formula $\varphi(a, y)$ witnessing the non-symmetry of semi-isolation has solutions in $\mathcal{N}' \upharpoonright p_{\infty}(x)$ and does not have solutions in $\mathcal{N} \upharpoonright p_{\infty}(x)$.

If semi-isolation is definable and witnessed by a formula $\psi(a, y)$ then again $\mathcal{N} \upharpoonright p_{\infty}(x)$ and $\mathcal{N}' \upharpoonright p_{\infty}(x)$ are not elementary equivalent since $\neg \psi(a, y)$ is realized in $\mathcal{N}' \upharpoonright p_{\infty}(x)$ and it does not have solutions in $\mathcal{N} \upharpoonright p_{\infty}(x)$

Thus, e-Sp(Th(\mathcal{A}_P)) > 1. \square

Since non-definable semi-isolation implies that there are infinitely many 2-types, we have

Corollary 4.11. For any theory $\operatorname{Th}(\mathcal{A}_P)$ with $e\operatorname{-Sp}(\operatorname{Th}(\mathcal{A}_P))=1$ the structures \mathcal{A}'_{∞} are not ω -categorical.

Applying modifications of the Ehrenfeucht example as well as constructions in [4], the results for e-spectra of E-combinations are modified for P-combinations:

Proposition 4.12. For any cardinality λ there is a theory $T = \text{Th}(\mathcal{A}_P)$ of a language Σ such that $|\Sigma| = \max\{\lambda, \omega\}$ and $e\text{-Sp}(T) = \lambda$.

Proof. Clearly, if $p_{\infty}(x)$ is inconsistent then e-Sp(T) = 0. Thus, the assertion holds for $\lambda = 0$.

If $\lambda = 1$ we take a theory T_1 with disjoint unary predicates P_i , $i \in \omega$, and a symmetric irreflexive binary relation R such that each vertex has R-degree 2, each P_i has infinitely many connected components, and each connected component on P_i has diameter i. Now structures on $p_{\infty}(x)$ have connected components of infinite diameter, all these structures are elementary equivalent, and e-Sp $(T_1) = 1$.

If $\lambda = n > 1$ is finite, we take the theory T_n in Example 4.8 with e- $\operatorname{Sp}(T_n) = n$, as well as we can take a generic Ehrenfeucht theory T'_{λ} with $\operatorname{RK}(T'_{\lambda}) = 2$ and with $\lambda - 1$ limit model \mathcal{M}_i over the type $p_{\infty}(x)$, $i < \lambda - 1$, such that each \mathcal{M}_i has a Q_j -chains, $j \leq i$, and does not have Q_k -chains for k > i. Restricting the limit models to $p_{\infty}(x)$ we get λ elementary non-equivalent structures including the prime structure \mathcal{N}^0 without Q_i -chains and structures $\mathcal{M}_i \upharpoonright p_{\infty}(x)$, $i < \lambda - 1$, which are elementary non-equivalent by distinct (non)existence of Q_j -chains.

Similarly, taking $\lambda \geq \omega$ disjoint binary predicates R_j for the Ehrenfeucht example in 4.8 we have λ structures with least elements in R_j which are not elementary equivalent each other. Producing the theory T_{λ} we have e-Sp $(T_{\lambda}) = \lambda$.

Modifying the generic Ehrenfeucht example taking λ binary predicates Q_j with Q_j -chains which do not imply Q_k -chains for k > i we get λ elementary non-equivalent restrictions to $p_{\infty}(x)$. \square

Note that as in Example 4.8 the type $p_{\infty}(x)$ for the Ehrenfeucht-like example T_{λ} has λ completions by the formulas $R_{j}(x)$ whereas the type $p_{\infty}(x)$ for the generic Ehrenfeucht theory is complete. At the same time having λ completions for the $p_{\infty}(x)$ -restrictions related to T_{λ} , the $p_{\infty}(x)$ -restrictions the generic Ehrenfeucht examples with complete $p_{\infty}(x)$ can violet the uniqueness of the complete 1-type like the Ehrenfeucht example T_{0} , where \mathcal{A}_{∞} realizes two complete 1-types: the type of the least element and the type of elements which are not least.

Proposition 4.13. For any infinite cardinality λ there is a theory $T = \text{Th}(\mathcal{A}_P)$ of a language Σ such that $|\Sigma| = \lambda$ and $e\text{-Sp}(T) = 2^{\lambda}$.

Proof. Let T be the theory of independent unary predicates R_j , $j < \lambda$, (defined by the set of axioms $\exists x \, (R_{k_1}(x) \land \ldots \land R_{k_m}(x) \land \neg R_{l_1}(x) \land \ldots \land \neg R_{l_n}(x))$, where $\{k_1, \ldots, k_m\} \cap \{l_1, \ldots, l_n\} = \varnothing$) such that countably many of them form predicates P_i , $i < \omega$, and infinitely many of them are independent with P_i . Thus, T can be considered as $\text{Th}(\mathcal{A}_P)$. Restrictions of models of T to sets of realizations of the type $p_{\infty}(x)$ witness that predicates R_j distinct with all P_i are independent. Denote indexes of these predicates R_j by J. Since $p_{\infty}(x)$ is non-isolated, for any family $\Delta = (\delta_j)_{j \in J}$, where $\delta_j \in \{0,1\}$, the types $q_{\Delta}(x) = \{R_j^{\delta_j} \mid j \in J\}$ can be pairwise independently realized and omitted in structures $\mathcal{M} \upharpoonright p_{\infty}(x)$ for $\mathcal{M} \models T$. Then any predicate R_j can be independently realized and omitted in these restrictions. Thus there are 2^{λ} restrictions with distinct theories, i. e., e-Sp $(T) = 2^{\lambda}$. \square

Since for E-combinations \mathcal{A}_E and P-combinations \mathcal{A}_P and their limit structures \mathcal{A}_{∞} , being respectively structures on E-classes and $p_{\infty}(x)$, the theories $\text{Th}(\mathcal{A}_{\infty})$ are defined by types restricted to E(x,y) and $p_{\infty}(x)$, and for any countable theory there are either countably many types or continuum many types, Propositions 4.3, 4.4, 4.12, and 4.13 implies the following

Theorem 4.14. If $T = \operatorname{Th}(\mathcal{A}_E)$ (respectively, $T = \operatorname{Th}(\mathcal{A}_P)$) is a countable theory then $e\text{-Sp}(T) \in \omega \cup \{\omega, 2^{\omega}\}$. All values in $\omega \cup \{\omega, 2^{\omega}\}$ have realizations in the class of countable theories of E-combinations (of P-combinations).

5 Ehrenfeuchtness for *E*-combinations

Theorem 5.1. If the language $\bigcup_{i \in I} \Sigma(A_i)$ is at most countable and the struc-

ture A_E is infinite then the theory $T = \text{Th}(A_E)$ is Ehrenfeucht if and only if $e\text{-Sp}(T) < \omega$ (which is equivalent here to e-Sp(T) = 0) and for an e-largest model $A_{E'} \models T$ consisting of E'-classes A_j , $j \in J$, the following conditions hold:

- (a) for any $j \in J$, $I(\operatorname{Th}(\mathcal{A}_i), \omega) < \omega$;
- (b) there are positively and finitely many $j \in J$ such that $I(\operatorname{Th}(\mathcal{A}_j), \omega) > 1$;
- (c) if $I(\operatorname{Th}(A_j), \omega) \leq 1$ then there are always finitely many $A_{j'} \equiv A_j$ or always infinitely many $A_{j'} \equiv A_j$ independent of $A_{E'} \models T$.

Proof. If e-Sp(T) < ω and the conditions (a)–(c) hold then the theory T is Ehrenfeucht since each countable model $\mathcal{A}_{E''} \models T$ is composed of disjoint models with universes $E''_k = A_k$, $k \in K$, and $I(T,\omega)$ is a sum $\sum_{l=0}^{e-\text{Sp}(T)}$ of finitely many possibilities for models with l representatives with respect to the elementary equivalence of E''-classes that are not presented in a prime (i. e., e-minimal) model of T. These possibilities are composed by finitely many possibilities of $I(\text{Th}(\mathcal{A}_k),\omega) > 1$ for $\mathcal{A}_{k'} \equiv \mathcal{A}_k$ and finitely many of $\mathcal{A}_{k''} \not\equiv \mathcal{A}_k$ with $I(\text{Th}(\mathcal{A}_{k''}),\omega) > 1$. Moreover, there are $\hat{C}(I(\text{Th}(\mathcal{A}_k),\omega),m_i)$ possibilities for substructures consisting of $\mathcal{A}_{k'} \equiv \mathcal{A}_k$ where m_i is the number of E-classes having the theory $\text{Th}(\mathcal{A}_k)$, $\hat{C}(n,m) = C^m_{n+m-1}$ is the number of combinations with repetitions for n-element sets with m places. The formula for $I(T,\omega)$ is based on the property that each E''-class with the structure \mathcal{A}_k can be replaced, preserving the elementary equivalence of $\mathcal{A}_{E''}$, by arbitrary $\mathcal{B} \equiv \mathcal{A}_k$.

Now we assume that the theory T is Ehrenfeucht. Since models of T with distinct theories of E-classes are not isomorphic, we have e-Sp $(T) < \omega$. Applying the formula for $I(T, \omega)$ we have the conditions (a), (b). The condition (c) holds since varying unboundedly many $\mathcal{A}_{j'} \equiv \mathcal{A}_j$ we get $I(T, \omega) \geq \omega$.

The conditions $e\text{-Sp}(T) < \omega$ and e-Sp(T) = 0 are equivalent. Indeed, if e-Sp(T) > 0 then taking an e-minimal model \mathcal{M} we get, by Compactness, unboundedly many E-classes, which are elementary non-equivalent to E-classes in \mathcal{M} . It implies that $I(T, \omega) \geq \omega$. \square

Since any prime structure is e-minimal (but not vice versa as the e-minimality is preserved, for instance, extending an infinite E-class of given structure to a greater cardinality preserving the elementary equivalence) and any Ehrenfeucht theory T, being small, has a prime model, any Ehrenfeucht theory $Th(\mathcal{A}_E)$ has an e-minimal model.

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