Closures and generating sets related to combinations of structures*

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Abstract

We investigate closure operators and describe their properties for E-combinations and P-combinations of structures and their theories. We prove, for E-combinations, that the existence of a minimal generating set of theories is equivalent to the existence of the least generating set, and characterize syntactically and semantically the property of the existence of the least generating set. For the class of linearly ordered language uniform theories we solve the problem of the existence of least generating set with respect to E-combinations and characterize that existence in terms of orders.

Key words: E-combination, P-combination, closure operator, generating set, language uniform theory.

1 Introduction and preliminaries

We continue to study structural properties of E-combinations and P-combinations of structures and their theories [1].

In Section 2, using the E-operators and P-operators we introduce topologies (related to topologies in [2]) and investigate their properties.

In Section 3, we prove, for E-combinations, that the existence of a minimal generating set of theories is equivalent to the existence of the least

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generating set, and characterize syntactically and semantically the property of the existence of the least generating set.

In Section 4, for the class of linearly ordered language uniform theories, we solve the problem of the existence of least generating set with respect to E-combinations and characterize that existence in terms of orders.

In Section 5 we describe some properties of e-spectra for E-combinations of linearly ordered language uniform theories.

Throughout the paper we use the following terminology in [1].

Let $P = (P_i)_{i\in I}$, be a family of nonempty unary predicates, $(\mathcal{A}_i)_{i\in I}$ be a family of structures such that P_i is the universe of $\mathcal{A}_i$, $i \in I$, and the symbols P_i are disjoint with languages for the structures $\mathcal{A}_j$, $j \in I$. The structure $\mathcal{A}_P \rightleftharpoons \bigcup\limits_{i\in I} \mathcal{A}_i$ expanded by the predicates P_i is the *P-union* of the structures $\mathcal{A}_i$, and the operator mapping $(\mathcal{A}_i)_{i\in I}$ to $\mathcal{A}_P$ is the *P-operator*. The structure $\mathcal{A}_P$ is called the *P-combination* of the structures $\mathcal{A}_i$ and denoted by $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ if $\mathcal{A}_i = (\mathcal{A}_P \upharpoonright A_i) \upharpoonright \Sigma(\mathcal{A}_i)$, $i \in I$. Structures $\mathcal{A}'$, which are elementary equivalent to $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$, will be also considered as P-combinations.

Clearly, all structures $\mathcal{A}' \equiv \mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ are represented as unions of their restrictions $\mathcal{A}'_i = (\mathcal{A}' \upharpoonright P_i) \upharpoonright \Sigma(\mathcal{A}_i)$ if and only if the set $p_\infty(x) = \{\neg P_i(x) \mid i \in I\}$ is inconsistent. If $\mathcal{A}' \neq \mathrm{Comb}_P(\mathcal{A}'_i)_{i\in I}$, we write $\mathcal{A}' = \mathrm{Comb}_P(\mathcal{A}'_i)_{i\in I\cup\{\infty\}}$, where $\mathcal{A}'_\infty = \mathcal{A}' \upharpoonright \bigcap\limits_{i\in I} \overline{P_i}$, maybe applying Morleyzation. Moreover, we write $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I\cup\{\infty\}}$ for $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ with the empty structure $\mathcal{A}_\infty$.

Note that if all predicates P_i are disjoint, a structure $\mathcal{A}_P$ is a P-combination and a disjoint union of structures $\mathcal{A}_i$. In this case the P-combination $\mathcal{A}_P$ is called *disjoint*. Clearly, for any disjoint P-combination $\mathcal{A}_P$, $\mathrm{Th}(\mathcal{A}_P) = \mathrm{Th}(\mathcal{A}'_P)$, where $\mathcal{A}'_P$ is obtained from $\mathcal{A}_P$ replacing $\mathcal{A}_i$ by pairwise disjoint $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. Thus, in this case, similar to structures the P-operator works for the theories $T_i = \mathrm{Th}(\mathcal{A}_i)$ producing the theory $T_P = \mathrm{Th}(\mathcal{A}_P)$, which is denoted by $\mathrm{Comb}_P(T_i)_{i\in I}$.

For an equivalence relation E replacing disjoint predicates P_i by E-classes we get the structure $\mathcal{A}_E$ being the *E-union* of the structures $\mathcal{A}_i$. In this case the operator mapping $(\mathcal{A}_i)_{i\in I}$ to $\mathcal{A}_E$ is the *E-operator*. The structure $\mathcal{A}_E$ is also called the *E-combination* of the structures $\mathcal{A}_i$ and denoted by $\mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$; here $\mathcal{A}_i = (\mathcal{A}_E \upharpoonright A_i) \upharpoonright \Sigma(\mathcal{A}_i)$, $i \in I$. Similar above, structures $\mathcal{A}'$, which are elementary equivalent to $\mathcal{A}_E$, are denoted by $\mathrm{Comb}_E(\mathcal{A}'_j)_{j\in J}$, where $\mathcal{A}'_j$ are restrictions of $\mathcal{A}'$ to its E-classes.

Clearly, $\mathcal{A}' \equiv \mathcal{A}_P$ realizing $p_\infty(x)$ is not elementary embeddable into $\mathcal{A}_P$ and can not be represented as a disjoint P-combination of $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. At the same time, there are E-combinations such that all $\mathcal{A}' \equiv \mathcal{A}_E$ can be represented as E-combinations of some $\mathcal{A}'_j \equiv \mathcal{A}_i$. We call this representability of $\mathcal{A}'$ to be the *E-representability*.

If there is $\mathcal{A}' \equiv \mathcal{A}_E$ which is not E-representable, we have the E'-representability replacing E by E' such that E' is obtained from E adding equivalence classes with models for all theories T, where T is a theory of a restriction $\mathcal{B}$ of a structure $\mathcal{A}' \equiv \mathcal{A}_E$ to some E-class and $\mathcal{B}$ is not elementary equivalent to the structures $\mathcal{A}_i$. The resulting structure $\mathcal{A}_{E'}$ (with the E'-representability) is a *e-completion*, or a *e-saturation*, of $\mathcal{A}_E$. The structure $\mathcal{A}_{E'}$ itself is called *e-complete*, or *e-saturated*, or *e-universal*, or *e-largest*.

For a structure $\mathcal{A}_E$ the number of *new* structures with respect to the structures $\mathcal{A}_i$, i. e., of the structures $\mathcal{B}$ which are pairwise elementary non-equivalent and elementary non-equivalent to the structures $\mathcal{A}_i$, is called the *e-spectrum* of $\mathcal{A}_E$ and denoted by e-Sp($\mathcal{A}_E$). The value $\sup\{e\text{-Sp}(\mathcal{A}')) \mid \mathcal{A}' \equiv \mathcal{A}_E\}$ is called the *e-spectrum* of the theory $\mathrm{Th}(\mathcal{A}_E)$ and denoted by e-Sp($\mathrm{Th}(\mathcal{A}_E)$).

If $\mathcal{A}_E$ does not have E-classes $\mathcal{A}_i$, which can be removed, with all E-classes $\mathcal{A}_j \equiv \mathcal{A}_i$, preserving the theory $\mathrm{Th}(\mathcal{A}_E)$, then $\mathcal{A}_E$ is called *e-prime*, or *e-minimal*.

For a structure $\mathcal{A}' \equiv \mathcal{A}_E$ we denote by $\mathrm{TH}(\mathcal{A}')$ the set of all theories $\mathrm{Th}(\mathcal{A}_i)$ of E-classes $\mathcal{A}_i$ in $\mathcal{A}'$.

By the definition, an e-minimal structure $\mathcal{A}'$ consists of E-classes with a minimal set $\mathrm{TH}(\mathcal{A}')$. If $\mathrm{TH}(\mathcal{A}')$ is the least for models of $\mathrm{Th}(\mathcal{A}')$ then $\mathcal{A}'$ is called *e-least*.

2 Closure operators

Definition. Let $\overline{\mathcal{T}}$ be the class of all complete elementary theories of relational languages. For a set $\mathcal{T} \subset \overline{\mathcal{T}}$ we denote by $\mathrm{Cl}_E(\mathcal{T})$ the set of all theories $\mathrm{Th}(\mathcal{A})$, where $\mathcal{A}$ is a structure of some E-class in $\mathcal{A}' \equiv \mathcal{A}_E$, $\mathcal{A}_E = \mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$, $\mathrm{Th}(\mathcal{A}_i) \in \mathcal{T}$. As usual, if $\mathcal{T} = \mathrm{Cl}_E(\mathcal{T})$ then $\mathcal{T}$ is said to be *E-closed*.

By the definition,
$$\mathrm{Cl}_E(\mathcal{T}) = \mathrm{TH}(\mathcal{A}'_{E'}), \qquad (1)$$

where $\mathcal{A}'_{E'}$ is an e-largest model of $\mathrm{Th}(\mathcal{A}_E)$, $\mathcal{A}_E$ consists of E-classes representing models of all theories in $\mathcal{T}$.

Note that the equality (1) does not depend on the choice of e-largest model of $\mathrm{Th}(\mathcal{A}_E)$.

The following proposition is obvious.

Proposition 2.1. (1) *If $\mathcal{T}_0$, $\mathcal{T}_1$ are sets of theories, $\mathcal{T}_0 \subseteq \mathcal{T}_1 \subset \overline{\mathcal{T}}$, then $\mathcal{T}_0 \subseteq \mathrm{Cl}_E(\mathcal{T}_0) \subseteq \mathrm{Cl}_E(\mathcal{T}_1)$.*

(2) *For any set $\mathcal{T} \subset \overline{\mathcal{T}}$, $\mathcal{T} \subset \mathrm{Cl}_E(\mathcal{T})$ if and only if the structure composed by E-classes of models of theories in $\mathcal{T}$ is not e-largest.*

(3) *Every finite set $\mathcal{T} \subset \overline{\mathcal{T}}$ is E-closed.*

(4) (Negation of finite character) *For any $T \in \mathrm{Cl}_E(\mathcal{T}) \setminus \mathcal{T}$ there are no finite $\mathcal{T}_0 \subset \mathcal{T}$ such that $T \in \mathrm{Cl}_E(\mathcal{T}_0)$.*

(5) *Any intersection of E-closed sets is E-closed.*

For a set $\mathcal{T} \subset \overline{\mathcal{T}}$ of theories in a language Σ and for a sentence φ with $\Sigma(\varphi) \subseteq \Sigma$ we denote by $\mathcal{T}_\varphi$ the set $\{T \in \mathcal{T} \mid \varphi \in T\}$. Denote by $\mathcal{T}_F$ the family of all sets $\mathcal{T}_\varphi$.

Clearly, the partially ordered set $\langle \mathcal{T}_F; \subseteq \rangle$ forms a Boolean algebra with the least element $\varnothing = \mathcal{T}_{\neg(x\approx x)}$, the greatest element $\mathcal{T} = \mathcal{T}_{(x\approx x)}$, and operations $\wedge$, $\vee$, $^-$ satisfying the following equalities: $\mathcal{T}_\varphi \wedge \mathcal{T}_\psi = \mathcal{T}_{(\varphi\wedge\psi)}$, $\mathcal{T}_\varphi \vee \mathcal{T}_\psi = \mathcal{T}_{(\varphi\vee\psi)}$, $\overline{\mathcal{T}_\varphi} = \mathcal{T}_{\neg\varphi}$.

By the definition, $\mathcal{T}_\varphi \subseteq \mathcal{T}_\psi$ if and only if for any model $\mathcal{M}$ of a theory in $\mathcal{T}$ satisfying φ we have $\mathcal{M} \models \psi$.

Proposition 2.2. *If $\mathcal{T} \subset \overline{\mathcal{T}}$ is an infinite set and $T \in \overline{\mathcal{T}} \setminus \mathcal{T}$ then $T \in \mathrm{Cl}_E(\mathcal{T})$ (i.e., T is an accumulation point for $\mathcal{T}$ with respect to E-closure Cl_E) if and only if for any formula $\varphi \in T$ the set $\mathcal{T}_\varphi$ is infinite.*

Proof. Assume that there is a formula $\varphi \in T$ such that only finitely many theories in $\mathcal{T}$, say $T_1, \ldots, T_n$, satisfy φ. Since $T \notin \mathcal{T}$ then there is $\psi \in T$ such that $\psi \notin T_1 \cup \ldots \cup T_n$. Then $(\varphi \wedge \psi) \in T$ does not belong to all theories in $\mathcal{T}$. Since $(\varphi \wedge \psi)$ does not satisfy E-classes in models of $T_E = \mathrm{Comb}_E(T_i)_{i\in I}$, where $\mathcal{T} = \{T_i \mid i \in I\}$, we have $T \notin \mathrm{Cl}_E(\mathcal{T})$.

If for any formula $\varphi \in T$, $\mathcal{T}_\varphi$ is infinite then $\{\varphi^E \mid \varphi \in T\} \cup T_E$ (where φ^E are E-relativizations of the formulas φ) is locally satisfied and so satisfied. Since T_E is a complete theory then $\{\varphi^E \mid \varphi \in T\} \subset T_E$ and hence $T \in \mathrm{Cl}_E(\mathcal{T})$. $\square$

Proposition 2.2 shows that the closure Cl_E corresponds to the closure with respect to the ultraproduct operator [3, 4, 5, 6].

Theorem 2.3. *For any sets* $\mathcal{T}_0, \mathcal{T}_1 \subset \overline{\mathcal{T}}$, $\mathrm{Cl}_E(\mathcal{T}_0 \cup \mathcal{T}_1) = \mathrm{Cl}_E(\mathcal{T}_0) \cup \mathrm{Cl}_E(\mathcal{T}_1)$.

Proof. We have $\mathrm{Cl}_E(\mathcal{T}_0) \cup \mathrm{Cl}_E(\mathcal{T}_1) \subseteq \mathrm{Cl}_E(\mathcal{T}_0 \cup \mathcal{T}_1)$ by Proposition 2.1 (1).

Let $T \in \mathrm{Cl}_E(\mathcal{T}_0 \cup \mathcal{T}_1)$ and we argue to show that $T \in \mathrm{Cl}_E(\mathcal{T}_0) \cup \mathrm{Cl}_E(\mathcal{T}_1)$. Without loss of generality we assume that $T \notin \mathcal{T}_0 \cup \mathcal{T}_1$ and by Proposition 2.1 (3), $\mathcal{T}_0 \cup \mathcal{T}_1$ is infinite. Define a function $f\colon T \to \mathcal{P}(\{0,1\})$ by the following rule: $f(\varphi)$ is the set of indexes $k \in \{0,1\}$ such that φ belongs to infinitely many theories in $\mathcal{T}_k$. Note that $f(\varphi)$ is always nonempty since by Proposition 2.2, φ belong to infinitely many theories in $\mathcal{T}_0 \cup \mathcal{T}_1$ and so to infinitely many theories in $\mathcal{T}_0$ or to infinitely many theories in $\mathcal{T}_1$. Again by Proposition 2.2 we have to prove that $0 \in f(\varphi)$ for each formula $\varphi \in T$ or $1 \in f(\varphi)$ for each formula $\varphi \in T$. Assuming on contrary, there are formulas $\varphi, \psi \in T$ such that $f(\varphi) = \{0\}$ and $f(\psi) = \{1\}$. Since $(\varphi \wedge \psi) \in T$ and $f(\varphi \wedge \psi)$ is nonempty we have $0 \in f(\varphi \wedge \psi)$ or $1 \in f(\varphi \wedge \psi)$. In the first case, since $\mathcal{T}_{\varphi\wedge\psi} \subseteq \mathcal{T}_\psi$ we get $0 \in f(\psi)$. In the second case, since $\mathcal{T}_{\varphi\wedge\psi} \subseteq \mathcal{T}_\varphi$ we get $1 \in f(\varphi)$. Both cases contradict the assumption. Thus, $T \in \mathrm{Cl}_E(\mathcal{T}_0) \cup \mathrm{Cl}_E(\mathcal{T}_1)$. □

Corollary 2.4. (Exchange property) *If* $T_1 \in \mathrm{Cl}_E(\mathcal{T} \cup \{T_2\}) \setminus \mathrm{Cl}_E(\mathcal{T})$ *then* $T_2 \in \mathrm{Cl}_E(\mathcal{T} \cup \{T_1\})$.

Proof. Since $T_1 \in \mathrm{Cl}_E(\mathcal{T} \cup \{T_2\}) = \mathrm{Cl}_E(\mathcal{T}) \cup \{T_2\}$ by Proposition 2.1 (3) and Theorem 2.3, and $T_1 \notin \mathrm{Cl}_E(\mathcal{T})$, then $T_1 = T_2$ and $T_2 \in \mathrm{Cl}_E(\mathcal{T} \cup \{T_1\})$ in view of Proposition 2.1 (1). □

Definition [7]. A *topological space* is a pair $(X, \mathcal{O})$ consisting of a set X and a family $\mathcal{O}$ of *open* subsets of X satisfying the following conditions:

(O1) $\varnothing \in \mathcal{O}$ and $X \in \mathcal{O}$;

(O2) If $U_1 \in \mathcal{O}$ and $U_2 \in \mathcal{O}$ then $U_1 \cap U_2 \in \mathcal{O}$;

(O3) If $\mathcal{O}' \subseteq \mathcal{O}$ then $\cup\mathcal{O}' \in \mathcal{O}$.

Definition [7]. A topological space $(X, \mathcal{O})$ is a T_0-*space* if for any pair of distinct elements $x_1, x_2 \in X$ there is an open set $U \in \mathcal{O}$ containing exactly one of these elements.

Definition [7]. A topological space $(X, \mathcal{O})$ is *Hausdorff* if for any pair of distinct points $x_1, x_2 \in X$ there are open sets $U_1, U_2 \in \mathcal{O}$ such that $x_1 \in U_1$, $x_2 \in U_2$, and $U_1 \cap U_2 = \varnothing$.

Let $\mathcal{T} \subset \overline{\mathcal{T}}$ be a set, $\mathcal{O}_E(\mathcal{T}) = \{\mathcal{T} \setminus \mathrm{Cl}_E(\mathcal{T}') \mid \mathcal{T}' \subseteq \mathcal{T}\}$. Proposition 2.1 and Theorem 2.3 imply that the axioms (O1)–(O3) are satisfied. Moreover, since for any theory $T \in \overline{\mathcal{T}}$, $\mathrm{Cl}_E(\{T\}) = \{T\}$ and hence, $\mathcal{T} \setminus \mathrm{Cl}_E(\{T\}) = \mathcal{T}\{T\}$ is an open set containing all theories in $\mathcal{T}$, which are not equal to T, then $(\mathcal{T}, \mathcal{O}_E(\mathcal{T}))$ is a T_0-space. Moreover, it is Hausdorff. Indeed, taking two distinct theories $T_1, T_2 \in \mathcal{T}$ we have a formula φ such that $\varphi \in T_1$ and $\neg\varphi \in T_2$. By Proposition 2.2 we have that $\mathcal{T}_\varphi$ and $\mathcal{T}_{\neg\varphi}$ are closed containing T_1 and T_2 respectively; at the same time $\mathcal{T}_\varphi$ and $\mathcal{T}_{\neg\varphi}$ form a partition of $\mathcal{T}$, so $\mathcal{T}_\varphi$ and $\mathcal{T}_{\neg\varphi}$ are disjoint open sets. Thus we have

Theorem 2.5. *For any set $\mathcal{T} \subset \overline{\mathcal{T}}$ the pair $(\mathcal{T}, \mathcal{O}_E(\mathcal{T}))$ is a Hausdorff topological space.*

Similarly to the operator $\mathrm{Cl}_E(\mathcal{T})$ we define the operator $\mathrm{Cl}_P(\mathcal{T})$ for families P of predicates P_i as follows.

Definition. For a set $\mathcal{T} \subset \overline{\mathcal{T}}$ we denote by $\mathrm{Cl}_P(\mathcal{T})$ the set of all theories $\mathrm{Th}(\mathcal{A})$ such that $\mathrm{Th}(\mathcal{A}) \in \mathcal{T}$ or $\mathcal{A}$ is a structure of type $p_\infty(x)$ in $\mathcal{A}' \equiv \mathcal{A}_P$, where $\mathcal{A}_P = \mathrm{Comb}_P(\mathcal{A}_i)_{i \in I}$ and $\mathrm{Th}(\mathcal{A}_i) \in \mathcal{T}$ are pairwise distinct. As above, if $\mathcal{T} = \mathrm{Cl}_P(\mathcal{T})$ then $\mathcal{T}$ is said to be *P-closed*.

Using above only disjoint P-combinations $\mathcal{A}_P$ we get the closure $\mathrm{Cl}^d_P(\mathcal{T})$ being a subset of $\mathrm{Cl}_P(\mathcal{T})$.

The following example illustrates the difference between $\mathrm{Cl}_P(\mathcal{T})$ and $\mathrm{Cl}^d_P(\mathcal{T})$.

Example 2.7. Taking disjoint copies of predicates $P_i = \{a \in M_0 \mid a < c_i\}$ with their $<$-structures as in [1, Example 4.8], $\mathrm{Cl}^d_P(\mathcal{T}) \setminus \mathcal{T}$ produces models of the Ehrenfeucht example and unboundedly many connected components each of which is a copy of a model of the Ehrenfeucht example. At the same time $\mathrm{Cl}_P(\mathcal{T})$ produces two new structures: densely ordered structures with and without the least element.

The following proposition is obvious.

Proposition 2.8. (1) *If $\mathcal{T}_0$, $\mathcal{T}_1$ are sets of theories, $\mathcal{T}_0 \subseteq \mathcal{T}_1 \subset \overline{\mathcal{T}}$, then $\mathcal{T}_0 \subseteq \mathrm{Cl}_P(\mathcal{T}_0) \subseteq \mathrm{Cl}_P(\mathcal{T}_1)$.*

(2) *Every finite set $\mathcal{T} \subset \overline{\mathcal{T}}$ is P-closed.*

(3) (Negation of finite character) *For any $T \in \mathrm{Cl}_P(\mathcal{T}) \setminus \mathcal{T}$ there are no finite $\mathcal{T}_0 \subset \mathcal{T}$ such that $T \in \mathrm{Cl}_P(\mathcal{T}_0)$.*

(4) *Any intersection of P-closed sets is P-closed.*

Remark 2.9. Note that an analogue of Proposition 2.8 for P-combinations fails. Indeed, taking disjoint predicates P_i, $i \in \omega$, with $2i+1$ elements and with structures $\mathcal{A}_i$ of the empty language, we get, for the set $\mathcal{T}$ of theories $T_i = \mathrm{Th}(\mathcal{A}_i)$, that $\mathrm{Cl}_P(\mathcal{T})$ consists of the theories whose models have cardinalities witnessing all ordinals in $\omega+1$. Thus, for instance, theories in $\mathcal{T}$ do not contain the formula

$$\exists x, y(\neg(x \approx y) \wedge \forall z((z \approx x) \vee (z \approx y))) \tag{2}$$

whereas $\mathrm{Cl}_P(\mathcal{T})$ (which is equal to $\mathrm{Cl}^d_P(\mathcal{T})$) contains a theory with the formula (2).

More generally, for $\mathrm{Cl}^d_P(\mathcal{T})$ with infinite $\mathcal{T}$, we have the following.

Since there are no links between distinct P_i, the structures of $p_\infty(x)$ are defined as disjoint unions of connected components $C(a)$, for a realizing $p_\infty(x)$, where each $C(a)$ consists of a set of realizations of p_∞-preserving formulas $\psi(a,x)$ (i.e., of formulas $\varphi(a,x)$ with $\psi(a,x) \vdash p_\infty(x)$). Similar to Proposition 2.2 theories $T_{\infty,C(a)}$ of $C(a)$-restrictions of $\mathcal{A}_\infty$ coincide and are characterized by the following property: $T_{\infty,C(a)} \in \mathrm{Cl}^d_P(\mathcal{T})$ if and only if $T_{\infty,C(a)} \in \mathcal{T}$ or for any formula $\varphi \in T_{\infty,C(a)}$, there are infinitely many theories T in $\mathcal{T}$ such that φ satisfies all structures approximating $C(a)$-restrictions of models of T.

Thus similarly to 2.3–2.5 we get the following three assertions for disjoint P-combinations.

Theorem 2.10. *For any sets* $\mathcal{T}_0, \mathcal{T}_1 \subset \overline{\mathcal{T}}$, $\mathrm{Cl}^d_P(\mathcal{T}_0 \cup \mathcal{T}_1) = \mathrm{Cl}^d_P(\mathcal{T}_0) \cup \mathrm{Cl}^d_P(\mathcal{T}_1)$.

Corollary 2.11. (Exchange property) *If* $T_1 \in \mathrm{Cl}^d_P(\mathcal{T} \cup \{T_2\}) \setminus \mathrm{Cl}^d_P(\mathcal{T})$ *then* $T_2 \in \mathrm{Cl}^d_P(\mathcal{T} \cup \{T_1\})$.

Let $\mathcal{T} \subset \overline{\mathcal{T}}$ be a set, $\mathcal{O}^d_P(\mathcal{T}) = \{\mathcal{T} \setminus \mathrm{Cl}^d_P(\mathcal{T}') \mid \mathcal{T}' \subseteq \mathcal{T}\}$.

Theorem 2.12. *For any set* $\mathcal{T} \subset \overline{\mathcal{T}}$ *the pair* $(\mathcal{T}, \mathcal{O}^d_P(\mathcal{T}))$ *is a topological* T_0*-space.*

Remark 2.13. By Proposition 2.8 (2), for any finite $\mathcal{T}$ the spaces $(\mathcal{T}, \mathcal{O}_P(\mathcal{T}))$ and $(\mathcal{T}, \mathcal{O}^d_P(\mathcal{T}))$ are Hausdorff, moreover, here $\mathcal{O}_P(\mathcal{T}) = \mathcal{O}^d_P(\mathcal{T})$ consisting of all subsets of $\mathcal{T}$. However, in general, the spaces $(\mathcal{T}, \mathcal{O}_P(\mathcal{T}))$ and $(\mathcal{T}, \mathcal{O}^d_P(\mathcal{T}))$ are not Hausdorff.

Indeed, consider structures $\mathcal{A}_i$, $i \in I$, where $I = (\omega+1) \setminus \{0\}$, of the empty language and such that $|\mathcal{A}_i| = i$. Let $T_i = \mathrm{Th}(\mathcal{A}_i)$, $i \in I$, $\mathcal{T} = \{T_i \mid i \in I\}$. Coding the theories T_i by their indexes we have the following. For any finite set $F \subset I$, $\mathrm{Cl}_P(F) = \mathrm{Cl}^d_P(F) = F$, and for any infinite set $\mathrm{INF} \subseteq I$, $\mathrm{Cl}_P(\mathrm{INF}) = \mathrm{Cl}^d_P(\mathrm{INF}) = I$. So any open set U is either cofinite or empty. Thus any two nonempty open sets are not disjoint.

Notice that we get a similar effect replacing elements in $\mathcal{A}_i$ by equivalence classes with pairwise isomorphic finite structures, may be with additional classes having arbitrary structures.

Remark 2.14. If the closure operator $\mathrm{Cl}^{d,r}_P$ is obtained from Cl^d_P permitting repetitions of structures for predicates P_i, we can lose both the property of T_0-space and the identical closure for finite sets of theories. Indeed, for the example in Remark 2.13, $\mathrm{Cl}^{d,r}_P(\mathcal{T})$ is equal to the $\mathrm{Cl}^{d,r}_P$-closure of any singleton $\{T\} \in \mathrm{Cl}^{d,r}_P(\mathcal{T})$ since the type $p_\infty(x)$ has arbitrarily many realizations producing models for each element in $\mathcal{T}$. Thus there are only two possibilities for open sets U: either $U = \varnothing$ or $U = \mathcal{T}$.

Remark 2.15. Let $\mathcal{T}_{\rm fin}$ be the class of all theories for finite structures. By compactness, for a set $\mathcal{T} \subset \mathcal{T}_{\rm fin}$, $\mathrm{Cl}_E(\mathcal{T})$ is a subset of $\mathcal{T}_{\rm fin}$ if and only if models of $\mathcal{T}$ have bounded cardinalities, whereas $\mathrm{Cl}_P(\mathcal{T})$ is a subset of $\mathcal{T}_{\rm fin}$ if and only if $\mathcal{T}$ is finite. Proposition 2.2 and its P-analogue allows to describe both $\mathrm{Cl}_E(\mathcal{T})$ and $\mathrm{Cl}_P(\mathcal{T})$, in particular, the sets $\mathrm{Cl}_E(\mathcal{T}) \setminus \mathcal{T}_{\rm fin}$ and $\mathrm{Cl}_P(\mathcal{T}) \setminus \mathcal{T}_{\rm fin}$. Clearly, there is a broad class of theories in $\overline{\mathcal{T}}$ which do not lay in

$$\bigcup_{\mathcal{T} \subset \mathcal{T}_{\rm fin}} \mathrm{Cl}_E(\mathcal{T}) \cup \bigcup_{\mathcal{T} \subset \mathcal{T}_{\rm fin}} \mathrm{Cl}^d_P(\mathcal{T}).$$

For instance, finitely axiomatizable theories with infinite models can not be approximated by theories in $\mathcal{T}_{\rm fin}$ in such way.

3 Generating subsets of E-closed sets

Definition. Let $\mathcal{T}_0$ be a closed set in a topological space $(\mathcal{T}, \mathcal{O}_E(\mathcal{T}))$. A subset $\mathcal{T}'_0 \subseteq \mathcal{T}_0$ is said to be *generating* if $\mathcal{T}_0 = \mathrm{Cl}_E(\mathcal{T}'_0)$. The generating set $\mathcal{T}'_0$ (for $\mathcal{T}_0$) is *minimal* if $\mathcal{T}'_0$ does not contain proper generating subsets. A minimal generating set $\mathcal{T}'_0$ is the *least* is $\mathcal{T}'_0$ is contained in each generating set for $\mathcal{T}_0$.

Remark 3.1. Each set $\mathcal{T}_0$ has a generating subset $\mathcal{T}'_0$ with a cardinality $\leq \max\{|\Sigma|, \omega\}$, where Σ is the union of the languages for the theories in $\mathcal{T}_0$. Indeed, the theory $T = \mathrm{Th}(\mathcal{A}_E)$, whose E-classes are models for theories in $\mathrm{Cl}_E(\mathcal{T}_0)$, has a model $\mathcal{M}$ with $|M| \leq \max\{|\Sigma|, \omega\}$. The E-classes of $\mathcal{M}$ are models of theories in $\mathrm{Cl}_E(\mathcal{T}_0)$ and the set of these theories is the required generating set.

Theorem 3.2. *If $\mathcal{T}_0'$ is a generating set for a E-closed set $\mathcal{T}_0$ then the following conditions are equivalent:*

(1) *$\mathcal{T}_0'$ is the least generating set for $\mathcal{T}_0$;*

(2) *$\mathcal{T}_0'$ is a minimal generating set for $\mathcal{T}_0$;*

(3) *any theory in $\mathcal{T}_0'$ is isolated by some set $(\mathcal{T}_0')_\varphi$, i.e., for any $T \in \mathcal{T}_0'$ there is $\varphi \in T$ such that $(\mathcal{T}_0')_\varphi = \{T\}$;*

(4) *any theory in $\mathcal{T}_0'$ is isolated by some set $(\mathcal{T}_0)_\varphi$, i.e., for any $T \in \mathcal{T}_0'$ there is $\varphi \in T$ such that $(\mathcal{T}_0)_\varphi = \{T\}$.*

Proof. (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (3) are obvious.

(2) $\Rightarrow$ (1). Assume that $\mathcal{T}_0'$ is minimal but not least. Then there is a generating set $\mathcal{T}_0''$ such that $\mathcal{T}_0' \setminus \mathcal{T}_0'' \neq \varnothing$ and $\mathcal{T}_0'' \setminus \mathcal{T}_0' \neq \varnothing$. Take $T \in \mathcal{T}_0' \setminus \mathcal{T}_0''$.

We assert that $T \in \mathrm{Cl}_E(\mathcal{T}_0' \setminus \{T\})$, i.e., T is an accumulation point of $\mathcal{T}_0' \setminus \{T\}$. Indeed, since $\mathcal{T}_0'' \setminus \mathcal{T}_0' \neq \varnothing$ and $\mathcal{T}_0'' \subset \mathrm{Cl}_E(\mathcal{T}_0')$, then by Proposition 2.1, (3), $\mathcal{T}_0'$ is infinite and by Proposition 2.2 it suffices to prove that for any $\varphi \in T$, $(\mathcal{T}_0' \setminus \{T\})_\varphi$ is infinite. Assume on contrary that for some $\varphi \in T$, $(\mathcal{T}_0' \setminus \{T\})_\varphi$ is finite. Then $(\mathcal{T}_0')_\varphi$ is finite and, moreover, as $\mathcal{T}_0'$ is generating for $\mathcal{T}_0$, by Proposition 2.2, $(\mathcal{T}_0)_\varphi$ is finite, too. So $(\mathcal{T}_0'')_\varphi$ is finite and, again by Proposition 2.2, T does not belong to $\mathrm{Cl}_E(\mathcal{T}_0'')$ contradicting to $\mathrm{Cl}_E(\mathcal{T}_0'') = \mathcal{T}_0$.

Since $T \in \mathrm{Cl}_E(\mathcal{T}_0' \setminus \{T\})$ and $\mathcal{T}_0'$ is generating for $\mathcal{T}_0$, then $\mathcal{T}_0' \setminus \{T\}$ is also generating for $\mathcal{T}_0$ contradicting the minimality of $\mathcal{T}_0'$.

(2) $\Rightarrow$ (3). If $\mathcal{T}_0'$ is finite then by Proposition 2.1 (3), $\mathcal{T}_0' = \mathcal{T}_0$. Since $\mathcal{T}_0$ is finite then for any $T \in \mathcal{T}_0$ there is a formula $\varphi \in T$ negating all theories in $\mathcal{T}_0 \setminus \{T\}$. Therefore, $(\mathcal{T}_0)_\varphi = (\mathcal{T}_0')_\varphi$ is a singleton containing T and thus, $(\mathcal{T}_0')_\varphi$ isolates T.

Now let $\mathcal{T}_0'$ be infinite. Assume that some $T \in \mathcal{T}_0'$ is not isolated by the sets $(\mathcal{T}_0')_\varphi$. It implies that for any $\varphi \in T$, $(\mathcal{T}_0' \setminus \{T\})_\varphi$ is infinite. Using Proposition 2.2 we obtain $T \in \mathrm{Cl}_E(\mathcal{T}_0' \setminus \{T\})$ contradicting the minimality of $\mathcal{T}_0'$.

(3) $\Rightarrow$ (2). Assume that any theory T in $\mathcal{T}_0'$ is isolated by some set $(\mathcal{T}_0')_\varphi$. By Proposition 2.2 it implies that $T \notin \mathrm{Cl}_E(\mathcal{T}_0' \setminus \{T\})$. Thus, $\mathcal{T}_0'$ is a minimal generating set for $\mathcal{T}_0$.

(3) $\Rightarrow$ (4) is obvious for finite $\mathcal{T}_0'$. If $\mathcal{T}_0'$ is infinite and any theory T in $\mathcal{T}_0'$ is isolated by some set $(\mathcal{T}_0')_\varphi$ then T is isolated by the set $(\mathcal{T}_0)_\varphi$, since otherwise using Proposition 2.2 and the property that $\mathcal{T}_0'$ generates $\mathcal{T}_0$, there are infinitely many theories in $\mathcal{T}_0'$ containing φ contradicting $|(\mathcal{T}_0')_\varphi| = 1$. □

The equivalences (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) in Theorem 3.2 were noticed by E.A. Palyutin.

Theorem 3.2 immediately implies

Corollary 3.3. *For any structure $\mathcal{A}_E$, $\mathcal{A}_E$ is e-minimal if and only if $\mathcal{A}_E$ is e-least.*

Definition. Let T be the theory $\mathrm{Th}(\mathcal{A}_E)$, where $\mathcal{A}_E = \mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$, $\{\mathrm{Th}(\mathcal{A}_i) \mid i \in I\} = \mathcal{T}_0$. We say that T has a *minimal/least generating set* if $\mathrm{Cl}_E(\mathcal{T}_0)$ has a minimal/least generating set.

Since by Theorem 3.2 the notions of minimality and to be least coincide in the context, below we shall consider least generating sets as well as e-least structures in cases of minimal generating sets.

Proposition 3.4. *For any closed nonempty set $\mathcal{T}_0$ in a topological space $(\mathcal{T}, \mathcal{O}_E(\mathcal{T}))$ and for any $\mathcal{T}'_0 \subseteq \mathcal{T}_0$, the following conditions are equivalent:*

(1) *$\mathcal{T}'_0$ is the least generating set for $\mathcal{T}_0$;*

(2) *any/some structure $\mathcal{A}_E = \mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$, where $\{\mathrm{Th}(\mathcal{A}_i) \mid i \in I\} = \mathcal{T}'_0$, is an e-least model of the theory $\mathrm{Th}(\mathcal{A}_E)$ and E-classes of each/some e-largest model of $\mathrm{Th}(\mathcal{A}_E)$ form models of all theories in $\mathcal{T}_0$;*

(3) *any/some structure $\mathcal{A}_E = \mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$, where $\{\mathrm{Th}(\mathcal{A}_i) \mid i \in I\} = \mathcal{T}'_0$, $\mathcal{A}_i \not\equiv \mathcal{A}_j$ for $i \neq j$, is an e-least model of the theory $\mathrm{Th}(\mathcal{A}_E)$, where E-classes of $\mathcal{A}_E$ form models of the least set of theories and E-classes of each/some e-largest model of $\mathrm{Th}(\mathcal{A}_E)$ form models of all theories in $\mathcal{T}_0$.*

Proof. (1) $\Rightarrow$ (2). Let $\mathcal{T}'_0$ be the least generating set for $\mathcal{T}_0$. Consider the structure $\mathcal{A}_E = \mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$, where $\{\mathrm{Th}(\mathcal{A}_i) \mid i \in I\} = \mathcal{T}'_0$. Since $\mathcal{T}'_0$ is the least generating set for $\mathcal{T}_0$, then $\mathcal{A}_E$ is an e-least model of the theory $\mathrm{Th}(\mathcal{A}_E)$. Moreover, by Proposition 2.2, E-classes of models of $\mathrm{Th}(\mathcal{A}_E)$ form models of all theories in $\mathcal{T}_0$. Thus, E-classes of $\mathcal{A}_E$ form models of the least set $\mathcal{T}'_0$ of theories such that E-classes of each/some e-largest model of $\mathrm{Th}(\mathcal{A}_E)$ form models of all theories in $\mathcal{T}_0$.

Similarly, constructing $\mathcal{A}_E$ with $\mathcal{A}_i \not\equiv \mathcal{A}_j$ for $i \neq j$, we obtain (1) $\Rightarrow$ (3).

Since (3) is a particular case of (2), we have (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1). Let $\mathcal{A}_E$ be an e-least model of the theory $\mathrm{Th}(\mathcal{A}_E)$ and E-classes of each/some e-largest model of $\mathrm{Th}(\mathcal{A}_E)$ form models of all theories in $\mathcal{T}_0$. Then by the definition of Cl_E, $\mathcal{T}'_0$ is the least generating set for $\mathcal{T}_0$. $\square$

Note that any prime structure $\mathcal{A}_E$ (or a structure with finitely many E-classes, or a prime structure extended by finitely many E-classes), is e-minimal forming, by its E-classes, the least generating set $\mathcal{T}'_0$ of theories for the set $\mathcal{T}_0$ of theories corresponding to E-classes of e-largest $\mathcal{A}'_E \equiv \mathcal{A}_E$. Indeed, if a set $\mathcal{T}_0''$ is generating for $\mathcal{T}_0$ then by Proposition 2.2 there is a model $\mathcal{M}$ of T consisting of E-classes with the set of models such that their theories form the set $\mathcal{T}_0''$. Since $\mathcal{A}_E$ prime (or with finitely many E-classes, or a prime structure extended by finitely many E-classes), then $\mathcal{A}_E$ is elementary embeddable into $\mathcal{M}$ (respectively, has E-classes with theories forming $\mathcal{T}_0''$, or elementary embeddable to a restriction without finitely many E-classes), then $\mathcal{T}_0' \subseteq \mathcal{T}_0''$, and so $\mathcal{T}_0'$ is the least generating set for $\mathcal{T}_0$. Thus, Proposition 3.4 implies

Corollary 3.5. *Any theory* $\mathrm{Th}(\mathcal{A}_E)$ *with a prime model* $\mathcal{M}$*, or with a finite set* $\{\mathrm{Th}(\mathcal{A}_i) \mid i \in I\}$*, or both with* E*-classes for* $\mathcal{M}$ *and* $\mathcal{A}_i$*, has the least generating set.*

Clearly, the converse for prime models does not hold, since finite sets $\mathcal{T}_0$ are least generating whereas theories in $\mathcal{T}_0$ can be arbitrary, in particular, without prime models. Again the converse for finite sets does not hold since there are prime models with infinite $\mathcal{T}_0$. Finally the general converse is not true since we can combine a theory T having a prime model with infinite $\mathcal{T}_0$ and a theory T' with infinitely many E-classes of disjoint languages and without prime models for these classes. Denoting by $\mathcal{T}_0'$ the set of theories for these E-classes, we get the least infinite generating set $\mathcal{T}_0 \cup \mathcal{T}_0'$ for the combination of T and T', which does not have a prime model.

Replacing E-combinations by P-combinations we obtain the notions of (minimal/least) generating set for $\mathrm{Cl}_P(\mathcal{T}_0)$.

The following example shows that Corollary 3.5 does not hold even for disjoint P-combinations.

Example 3.6. Take structures $\mathcal{A}_i$, $i \in (\omega+1)\setminus\{0\}$, in Remark 2.13 and the theories $T_i = \mathrm{Th}(\mathcal{A}_i)$ forming the Cl^d_P-closed set $\mathcal{T}$. Since $\mathcal{T}$ is generated by any its infinite subset, we get that having prime models of $\mathrm{Th}(\mathcal{A}_P)$, the closure $\mathrm{Cl}^d_P(\mathcal{T})$ does not have minimal generating sets.

For the example above, with the empty language, $\mathrm{Cl}^{d,r}_P(\mathcal{T})$ is generated by any singleton $\{T\} \in \mathrm{Cl}^{d,r}_P(\mathcal{T})$ since the type $p_\infty(x)$ has arbitrarily many realizations producing models for each T_i, $i \in (\omega+1)\setminus\{0\}$. Thus, each element of $\mathrm{Cl}^{d,r}_P(\mathcal{T})$ forms a minimal generating set.

Adding to the language Σ countably many unary predicate symbols $R^{(1)}_i$, $i \in \omega\setminus\{0\}$, for constants and putting each singleton R_i into $\mathcal{A}_i$, $i \in \omega\setminus\{0\}$, we get examples of $\mathrm{Cl}^d_P(\mathcal{T})$ and $\mathrm{Cl}^{d,r}_P(\mathcal{T})$ with the least (infinite) generating sets. Thus, the property of the non-existence of minimal/least generating sets is not preserved under expansions of theories.

We again obtain the non-existence of minimal/least generating sets for Cl_P^d and $\mathrm{Cl}_P^{d,r}$, respectively, expanding theories T_i in the previous example by singletons R_j, $j \neq i$, which are equal to R_i.

Natural questions arise concerning minimal generating sets:

Question 1. *What is a characterization for the existence of least generating sets?*

Question 2. *Is there exists a theory* $\mathrm{Th}(\mathcal{A}_E)$ *without the least generating set?*

Obviously, for E-combinations, Question 1 has an answer in terms of Proposition 2.2 (clarified in Theorem 3.2) taking the least, under inclusion, set $\mathcal{T}'_0$ generating the set $\mathrm{Cl}_E(\mathcal{T}'_0)$. It means that $\mathcal{T}'_0$ does not have accumulation points inside $\mathcal{T}'_0$ (with respect to the sets $(\mathcal{T}'_0)_\varphi$), i.e., any element in $\mathcal{T}'_0$ is isolated by some formula, whereas each element T in $\mathrm{Cl}_E(\mathcal{T}'_0)\setminus\mathcal{T}'_0$ is an accumulation point of $\mathcal{T}'_0$ (again with respect to $(\mathcal{T}'_0)_\varphi$), i.e., $\mathcal{T}'_0$ is dense in its E-closure.

Note that a positive answer to Question 2 for Cl_P is obtained in Remark 2.13.

Below we will give a more precise formulation for this answer related to E-combinations and answer Question 2 for special cases with languages.

4 Language uniform theories and related E-closures

Definition. A theory T in a predicate language Σ is called *language uniform*, or a LU-*theory* if for each arity n any substitution on the set of non-empty n-ary predicates preserves T. The LU-theory T is called IILU-*theory* if it has non-empty predicates and as soon as there is a non-empty n-ary predicate then there are infinitely many non-empty n-ary predicates and there are infinitely many empty n-ary predicates.

Below we point out some basic examples of LU-theories:

• Any theory T_0 of infinitely many independent unary predicates R_k is a LU-theory; expanding T_0 by infinitely many empty predicates R_l we get a IILU-theory T_1.

• Replacing independent predicates R_k for T_0 and T_1 by disjoint unary predicates R'_k with a cardinality $\lambda \in (\omega+1)\setminus\{0\}$ such that each R'_k has λ elements; the obtained theories are denoted by T_0^λ and T_1^λ respectively; here, T_0^λ and T_1^λ are LU-theories, and, moreover, T_1^λ is a IILU-theory; we denote T_0^1 and T_1^1 by T_0^c and T_1^c; in this case nonempty predicates R'_k are singletons symbolizing constants which are replaced by the predicate languages.

• Any theory T of equal nonempty unary predicates R_k is a LU-theory;

• Similarly, LU-theories and IILU-theories can be constructed using n-ary predicate symbols of arbitrary arity n.

• The notion of language uniform theory can be extended for an arbitrary language taking graphs for language functions; for instance, theories of free algebras can be considered as LU-theories.

• Acyclic graphs with colored edges (arcs), for which all vertices have same degree with respect to each color, has LU-theories. If there are infinitely many colors and infinitely many empty binary relations then the colored graph has a IILU-theory.

• Generic arc-colored graphs without colors for vertices [8, 9], free polygonometries of free groups [10], and cube graphs with coordinated colorings of edges [11] have LU-theories.

The simplest example of a theory, which is not language uniform, can be constructed taking two nonempty unary predicates R_1 and R_2, where $R_1 \subset R_2$. More generally, if a theory T, with nonempty predicates R_i, $i \in I$, of a fixed arity, is language uniform then cardinalities of $R_{i_1}^{\delta_1}(\bar{x}) \wedge \ldots \wedge R_{i_j}^{\delta_1}(\bar{x})$ do not depend on pairwise distinct $i_1, \ldots, i_j$.

Remark 4.1. Any countable theory T of a predicate language Σ can be transformed to a LU-theory T'. Indeed, since without loss of generality Σ is countable consisting of predicate symbols $R_n^{(k_n)}$, $n \in \omega$, then we can step-by-step replace predicates R_n by predicates R'_n in the following way. We put $R'_0 \rightleftharpoons R_0$. If predicates $R'_0, \ldots, R'_n$ of arities $r_0 < \ldots < r_n$, respectively, are already defined, we take for R'_{n+1} a predicate of an arity $r_{n+1} > \max\{r_n, k_{n+1}\}$, which is obtained from R'_{n+1} adding $r_{n+1} - k_{n+1}$ fictitious variables corresponding to the formula

$$R'(x_1, \ldots, x_{k_{n+1}}) \wedge (x_{k_{n+2}} \approx x_{k_{n+2}}) \wedge (x_{r_{n+1}} \approx x_{r_{n+1}}).$$

If the resulted LU-theory T' has non-empty predicates, it can be transformed to a countable IILU-theory T'' copying these non-empty predicated with same domains countably many times and adding countably many empty predicates for each arity r_n.

Clearly, the process of the transformation of T to T' do not hold for for uncountable languages, whereas any LU-theory can be transformed to an IILU-theory as above.

Definition. Recall that theories T_0 and T_1 of languages Σ_0 and Σ_1 respectively are said to be *similar* if for any models $\mathcal{M}_i \models T_i$, $i = 0, 1$, there are formulas of T_i, defining in $\mathcal{M}_i$ predicates, functions and constants of language Σ_{1-i} such that the corresponding structure of Σ_{1-i} is a model of T_{1-i}.

Theories T_0 and T_1 of languages Σ_0 and Σ_1 respectively are said to be *language similar* if T_0 can be obtained from T_1 by some bijective replacement of language symbols in Σ_1 by language symbols in Σ_0 (and vice versa).

Clearly, any language similar theories are similar, but not vice versa. Note also that, by the definition, any LU-theory T is language similar to any theory T^σ which is obtained from T replacing predicate symbols R by $\sigma(R)$, where σ is a substitution on the set of predicate symbols in $\Sigma(T)$ corresponding to nonempty predicates for T as well as a substitution on the set of predicate symbols in $\Sigma(T)$ corresponding to empty predicates for T. Thus we have

Proposition 4.2. *Let T_1 and T_2 be* LU*-theories of same language such that T_2 is obtained from T_1 by a bijection f_1 (respectively f_2) mapping (non)empty predicates for T_1 to (non)empty predicates for T_2. Then T_1 and T_2 are language similar.*

Corollary 4.3. *Let T_1 and T_2 be countable* IILU*-theories of same language such that the restriction T_1' of T_1 to non-empty predicates is language similar to the restriction T_2' of T_2 to non-empty predicates. Then T_1 and T_2 are language similar.*

Proof. By the hypothesis, there is a bijection f_2 for non-empty predicates of T_1 and T_2. Since T_1 and T_2 be countable IILU-theories then T_1 and T_2 have countably many empty predicates of each arity with non-empty predicates, there is a bijection f_1 for empty predicates of T_1 and T_2. Now Corollary is implied by Proposition 4.2. $\Box$

Definition. For a theory T in a predicate language Σ, we denote by

$\mathrm{Supp}_\Sigma(T)$ the *support* of Σ for T, i. e., the set of all arities n such that some n-ary predicate R for T is not empty.

Clearly, if T_1 and T_2 are language similar theories, in predicate languages Σ_1 and Σ_2 respectively, then $\mathrm{Supp}_{\Sigma_1}(T_1) = \mathrm{Supp}_{\Sigma_2}(T_2)$.

Definition. Let T_1 and T_2 be language similar theories of same language Σ. We say that T_2 *language dominates* T_1 and write $T_1 \sqsubseteq^L T_2$ if for any symbol $R \in \Sigma$, if $T_1 \vdash \exists\bar{x}R(\bar{x})$ then $T_2 \vdash \exists\bar{x}R(\bar{x})$, i. e., all predicates, which are non-empty for T_1, are nonempty for T_2. If $T_1 \sqsubseteq^L T_2$ and $T_2 \sqsubseteq^L T_1$, we say that T_1 and T_2 are *language domination-equivalent* and write $T_1 \sim^L T_2$.

Proposition 4.4. *The relation $\sqsubseteq^L$ is a partial order on any set of* LU-*theories.*

Proof. Since $\sqsubseteq^L$ is always reflexive and transitive, it suffices to note that if $T_1 \sqsubseteq^L T_2$ and $T_2 \sqsubseteq^L T_1$ then $T_1 = T_2$. It follows as language similar LU-theories coincide having the same set of nonempty predicates. □

Definition. We say that T_2 *infinitely language dominates* T_1 and write $T_1 \sqsubset^L_\infty T_2$ if $T_1 \sqsubseteq^L T_2$ and for some n, there are infinitely many new nonempty predicates for T_2 with respect to T_1.

Since there are infinitely many elements between any distinct comparable elements in a dense order, we have

Proposition 4.5. *If a class of theories $\mathcal{T}$ has a dense order $\sqsubseteq^L$ then $T_1 \sqsubset^L_\infty T_2$ for any distinct $T_1, T_2 \in \mathcal{T}$ with $T_1 \sqsubseteq^L T_2$.*

Clearly, if $T_1 \sqsubseteq^L T_2$ then $\mathrm{Supp}_\Sigma(T_1) \subseteq \mathrm{Supp}_\Sigma(T_2)$ but not vice versa. In particular, there are theories T_1 and T_2 with $T_1 \sqsubset^L_\infty T_2$ and $\mathrm{Supp}_\Sigma(T_1) = \mathrm{Supp}_\Sigma(T_2)$.

Let T_0 be a LU-theory with infinitely many nonempty predicate of some arity n, and I_0 be the set of indexes for the symbols of these predicates.

Now for each infinite $I \subseteq I_0$ with $|I| = |I_0|$, we denote by T_I the theory which is obtained from the complete subtheory of T_0 in the language $\{R_k \mid k \in I\}$ united with symbols of all arities $m \neq n$ and expanded by empty predicates R_l for $l \in I_0 \setminus I$, where $|I_0 \setminus I|$ is equal to the cardinality of the set empty predicates for T_0, of the arity n.

By the definition, each T_I is language similar to T_0: it suffices to take a bijection f between languages of T_I and T_0 such that (non)empty predicates of T_I in the arity n correspond to (non)empty predicates of T_0 in the arity n, and f is identical for predicate symbols of the arities $m \neq n$. In particular, Let $\mathcal{T}$ be an infinite family of theories T_I, and T_J be a theory of the form above (with infinite $J \subseteq I_0$ such that $|J| = |I_0|$). The following proposition modifies Proposition 2.2 for the E-closure $\mathrm{Cl}_E(\mathcal{T})$.

Proposition 4.6. *If $T_J \notin \mathcal{T}$ then $T_J \in \mathrm{Cl}_E(\mathcal{T})$ if and only if for any finite set $J_0 \subset I_0$ there are infinitely many T_I with $J \cap J_0 = I \cap J_0$.*

Proof. By the definition each theory T_J is defined by formulas describing $P_k \neq \varnothing \Leftrightarrow k \in J$. Each such a formula φ asserts for a finite set $J_0 \subset I_0$ that if $k \in J_0$ then $R_k \neq \varnothing \Leftrightarrow k \in J$. It means that $\{k \in J_0 \mid P_k \neq \varnothing\} = J \cap J_0$. On the other hand, by Proposition 2.2, $T_J \in \mathrm{Cl}_E(\mathcal{T})$ if and only if each such formula φ belongs to infinitely many theories T_I in $\mathcal{T}$, i.e., for infinitely many indexes I we have $I \cap J_0 = J \cap J_0$. $\square$

Now we take an infinite family F of infinite indexes I such that F is linearly ordered by $\subseteq$ and if $I_1 \subset I_2$ then $I_2 \setminus I_1$ is infinite. The set $\{T_I \mid I \in F\}$ is denoted by $\mathcal{T}_F$.

For any infinite $F' \subseteq F$ we denote by $\overline{\lim} F'$ the union-set $\bigcup F'$ and by $\underline{\lim} F'$ intersection-set $\bigcap F'$. If $\overline{\lim} F'$ (respectively $\underline{\lim} F'$) does not belong to F' then it is called the *upper* (*lower*) *accumulation point* (for F'). If J is an upper or lower accumulation point we simply say that J is an *accumulation point*.

Corollary 4.7. *If $T_J \notin \mathcal{T}_F$ then $T_J \in \mathrm{Cl}_E(\mathcal{T}_F)$ if and only if J is an* (*upper or lower*) *accumulation point for some infinite $F' \subseteq F$.*

Proof. If $J = \overline{\lim} F'$ or $J = \underline{\lim} F'$ then for any finite set $J_0 \subset I_0$ there are infinitely many T_I with $J \cap J_0 = I \cap J_0$. Indeed, if $J = \bigcup F'$ then for any finite $J_0 \subset I_0$ there are infinitely many $I \in F'$ such that $I \cap J_0$ contains exactly same elements as $J \cap J_0$ since otherwise we have $J \subset \bigcup F'$. Similarly the assertion holds for $J = \bigcap F'$. By Proposition 4.6 we have $T_J \in \mathrm{Cl}_E(\mathcal{T}_F)$.

Now let $J \neq \overline{\lim} F'$ and $J \neq \underline{\lim} F'$ for any infinite $F' \subseteq F$. In this case for each $F' \subseteq F$, either J contains new index j for a nonempty predicate with respect to $\bigcup F'$ for each $F' \subseteq F$ with $\bigcup F' \subseteq J$ or $\bigcap F'$ contains new index j' for a nonempty predicate with respect to J for each $F' \subseteq F$ with $\bigcap F' \supseteq J$. In the first case, for $J_0 = \{j\}$ there are no $I \in F'$ such that $I \cap J_0 = J \cap J_0$. In the second case, for $J_0 = \{j'\}$ there are no $I \in F'$ such that $I \cap J_0 = J \cap J_0$. By Proposition 4.6 we get $T_J \notin \mathrm{Cl}_E(\mathcal{T}_F)$. $\square$

By Corollary 4.7 the action of the operator Cl_E for the families $\mathcal{T}_F$ is reduced to unions and intersections of *index* subsets of F.

Now we consider possibilities for the linearly ordered sets $\mathcal{F} = \langle F; \subseteq\rangle$ and their closures $\overline{\mathcal{F}} = \langle \overline{F}; \subseteq\rangle$ related to Cl_E.

The structure $\mathcal{F}$ is called *discrete* if F does not contain accumulation points.

By Corollary 4.7, if $\mathcal{F}$ is discrete then for any $J \in F$, $T_J \notin \mathrm{Cl}_E(\mathcal{T}_{F\setminus\{J\}})$. Thus we get

Proposition 4.8. *For any discrete $\mathcal{F}$, $\mathcal{T}_F$ is the least generating set for $\mathrm{Cl}_E(\mathcal{T}_F)$.*

By Proposition 4.8, for any discrete $\mathcal{F}$, $\mathcal{T}_F$ can be reconstructed from $\mathrm{Cl}_E(\mathcal{T}_F)$ removing accumulation points, which always exist. For instance, if $\langle F; \subseteq\rangle$ is isomorphic to $\langle \omega; \leq\rangle$ or $\langle \omega^*; \leq\rangle$ (respectively, isomorphic to $\langle \mathbb{Z}; \leq\rangle$) then $\mathrm{Cl}_E(\mathcal{T}_F)$ has exactly one (two) new element(s) $\overline{\lim}\, F$ or $\underline{\lim}\, F$ (both $\overline{\lim}\, F$ and $\underline{\lim}\, F$).

Consider an opposite case: with dense $\mathcal{F}$. Here, if $\mathcal{F}$ is countable then, similarly to $\langle \mathbb{Q}; \leq\rangle$, taking cuts for $\mathcal{F}$, i. e., partitions (F^-, F^+) of F with $F^- < F^+$, we get the closure $\overline{F}$ with continuum many elements. Thus, the following proposition holds.

Proposition 4.9. *For any dense $\mathcal{F}$, $|\overline{F}| \geq 2^\omega$.*

Clearly, there are dense $\mathcal{F}$ with dense and non-dense $\overline{\mathcal{F}}$. If $\overline{\mathcal{F}}$ is dense then, since $\overline{\overline{F}} = \overline{F}$, there are dense $\mathcal{F}_1$ with $|F_1| = |\overline{F_1}|$. In particular, it is followed by Dedekind theorem on completeness of $\mathbb{R}$.

Answering Question 4 we have

Proposition 4.10. *If $\overline{\mathcal{F}}$ is dense then $\mathrm{Cl}_E(\mathcal{T}_F)$ does not contain the least generating set.*

Proof. Assume on contrary that $\mathrm{Cl}_E(\mathcal{T}_F)$ contains the least generating set with a set $F_0 \subseteq F$ of indexes. By the minimality F_0 does not contain both the least element and the greatest element. Thus taking an arbitrary $J \in F_0$ we have that for the cut $(F^-_{0,J}, F^+_{0,J})$, where $F^-_{0,J} = \{J^- \in F_0 \mid J^- \subset J\}$ and $F^+_{0,J} = \{J^+ \in F_0 \mid J^+ \supset J\}$, $J = \overline{\lim}\, F^-_{0,J}$ and $J = \underline{\lim}\, F^+_{0,J}$. Thus, $F_0 \setminus \{J\}$ is again a set of indexes for a generating set for $\mathrm{Cl}_E(\mathcal{T}_F)$. Having a contradiction we obtain the required assertion. □

Combining Proposition 3.4 and Proposition 4.10 we obtain

Corollary 4.11. *If $\overline{\mathcal{F}}$ is dense then $\mathrm{Th}(\mathcal{A}_E)$ does not have e-least models and, in particular, it is not small.*

Remark 4.12. The condition of the density of $\overline{\mathcal{F}}$ for Proposition 4.10 is essential. Indeed, we can construct step-by step a countable dense structure $\mathcal{F}$ without endpoints such that for each $J \in F$ and for its cut (F_J^-, F_J^+), where $F_J^- = \{J^- \in F \mid J^- \subset J\}$ and $F_J^+ = \{J^+ \in F \mid J^+ \supset J\}$, $J \supset \overline{\lim} F_J^-$ and $J \subset \underline{\lim} F_J^+$. In this case $\mathrm{Cl}_E(\mathcal{T}_F)$ contains the least generating set $\{T_J \mid J \in F\}$.

In general case, if an element J of F has a successor J' or a predecessor J^{-1} then J defines a connected component with respect to the operations $\cdot'$ and $\cdot^{-1}$. Indeed, taking closures of elements in F with respect to $\cdot'$ and $\cdot^{-1}$ we get a partition of F defining an equivalence relation such that two elements J_1 and J_2 are equivalent if and only if J_2 is obtained from J_1 applying $\cdot'$ or $\cdot^{-1}$ several (maybe zero) times.

Now for any connected component C we have one of the following possibilities:

(i) C is a singleton consisting of an element J such that $J \neq \overline{\lim} F_J^-$ and $J \neq \underline{\lim} F_J^+$; in this case J is not an accumulation point for $F \setminus \{J\}$ and T_J belongs to any generating set for $\mathrm{Cl}_E(\mathcal{T}_F)$;

(ii) C is a singleton consisting of an element J such that $J = \overline{\lim} F_J^-$ or $J = \underline{\lim} F_J^+$, and $\overline{\lim} F_J^- \neq \underline{\lim} F_J^+$; in this case J is an accumulation point for exactly one of F_J^- and F_J^+, J separates F_J^- and F_J^+, and T_J can be removed from any generating set for $\mathrm{Cl}_E(\mathcal{T}_F)$ preserving the generation of $\mathrm{Cl}_E(\mathcal{T}_F)$; thus T_J does not belong to minimal generating sets;

(iii) C is a singleton consisting of an element J such that $J = \overline{\lim} F_J^- = \underline{\lim} F_J^+$; in this case J is a (unique) accumulation point for both F_J^- and F_J^+, moreover, again T_J can be removed from any generating set for $\mathrm{Cl}_E(\mathcal{T}_F)$ preserving the generation of $\mathrm{Cl}_E(\mathcal{T}_F)$, and T_J does not belong to minimal generating sets;

(iv) $|C| > 1$ (in this case, for any intermediate element J of C, T_J belongs to any generating set for $\mathrm{Cl}_E(\mathcal{T}_F)$), $\underline{\lim}\, C \supset \overline{\lim} F^-_{\underline{\lim}\, C}$ and $\overline{\lim}\, C \subset \underline{\lim} F^+_{\overline{\lim}\, C}$; in this case, for the endpoint(s) J^* of C, if it (they) exists, T_{J^*} belongs to any generating set for $\mathrm{Cl}_E(\mathcal{T}_F)$;

(v) $|C| > 1$, and $\underline{\lim}\, C = \overline{\lim} F^-_{\underline{\lim}\, C}$ or $\overline{\lim}\, C = \underline{\lim} F^+_{\overline{\lim}\, C}$; in this case, for the endpoint $J^* = \underline{\lim}\, C$ of C, if it exists, T_{J^*} does not belong to minimal generating sets of $\mathrm{Cl}_E(\mathcal{T}_F)$, and for the endpoint $J^{**} = \overline{\lim}\, C$ of C, if it exists, $T_{J^{**}}$ does not belong to minimal generating sets of $\mathrm{Cl}_E(\mathcal{T}_F)$.

Summarizing (i)–(v) we obtain the following assertions.

Proposition 4.13. *A partition of F by the connected components forms* discrete intervals *or, in particular, singletons of $\mathcal{F}$, where only endpoints J of these intervals can be among elements J^{**} such that $T_{J^{**}}$ does not belong to minimal generating sets of* $\mathrm{Cl}_E(\mathcal{T}_F)$.

Proposition 4.14. *If (F^-,F^+) is a cut of F with $\overline{\lim}\,F^- = \underline{\lim}\,F^+$ (respectively $\overline{\lim}\,F^- \subset \underline{\lim}\,F^+$) then any generating set $\mathcal{T}^0$ for $\mathrm{Cl}_E(\mathcal{T}_F)$ is represented as a (disjoint) union of generating set $\mathcal{T}^0_{F^-}$ for $\mathrm{Cl}_E(\mathcal{T}_{F^-})$ and of generating set $\mathcal{T}^0_{F^+}$ for $\mathrm{Cl}_E(\mathcal{T}_{F^+})$, moreover, any (disjoint) union of a generating set for $\mathrm{Cl}_E(\mathcal{T}_{F^-})$ and of a generating set for $\mathrm{Cl}_E(\mathcal{T}_{F^+})$ is a generating set $\mathcal{T}^0$ for $\mathrm{Cl}_E(\mathcal{T}_F)$.*

Proposition 4.14 implies

Corollary 4.15. *If (F^-,F^+) is a cut of F then $\mathrm{Cl}_E(\mathcal{T}_F)$ has the least generating set if and only if $\mathrm{Cl}_E(\mathcal{T}_{F^-})$ and $\mathrm{Cl}_E(\mathcal{T}_{F^+})$ have the least generating sets.*

Considering $\subset$-ordered connected components we have that *discretely ordered* intervals in $\overline{\mathcal{F}}$, consisting of discrete connected components and their limits $\underline{\lim}$ and $\overline{\lim}$, are alternated with densely ordered intervals including their limits. If $\overline{\mathcal{F}}$ contains an (infinite) dense interval, then by Proposition 4.10, $\mathrm{Cl}_E(\mathcal{T}_F)$ does not have the least generating set. Conversely, if $\overline{\mathcal{F}}$ does not contain dense intervals then $\mathrm{Cl}_E(\mathcal{T}_F)$ contains the least generating set. Thus, answering Questions 1 and 2 for $\mathrm{Cl}_E(\mathcal{T}_F)$, we have

Theorem 4.16. *For any linearly ordered set $\mathcal{F}$, the following conditions are equivalent:*
(1) $\mathrm{Cl}_E(\mathcal{T}_F)$ *has the least generating set;*
(2) $\overline{\mathcal{F}}$ *does not have dense intervals.*

Remark 4.17. Theorem 4.16 does not hold for some non-linearly ordered $\mathcal{F}$. Indeed, taking countably many disjoint copies $\mathcal{F}_q$, $q \in \mathbb{Q}$, of linearly ordered sets isomorphic to $\langle\omega,\leq\rangle$ and ordering limits $J_q = \overline{\lim}\,F_q$ by the ordinary dense order on $\mathbb{Q}$ such that $\{J_q \mid q \in \mathbb{Q}\}$ is densely ordered, we obtain a dense interval $\{J_q \mid q \in \mathbb{Q}\}$ whereas the set $\cup\{F_q \mid q \in \mathbb{Q}\}$ forms the least generating set $\mathcal{T}_0$ of theories for $\mathrm{Cl}_E(\mathcal{T}_0)$.

The above operation of extensions of theories for $\{J_q \mid q \in \mathbb{Q}\}$ by theories for $\mathcal{F}_q$ as well as expansions of theories of the empty language to theories for $\{J_q \mid q \in \mathbb{Q}\}$ confirm that the (non)existence of a least/minimal generating set for $\mathrm{Cl}_E(\mathcal{T}_0)$ is not preserved under restrictions and expansions of theories.

Remark 4.18. Taking an arbitrary theory T with a non-empty predicate R of an arity n, we can modify Theorem 4.16 in the following way. Extending the language $\Sigma(T)$ by infinitely many n-ary predicates interpreted exactly as R and by infinitely many empty n-ary predicates we get a class $\mathcal{T}_{T,R}$ of theories R-*generated* by T. The class $\mathcal{T}_{T,R}$ satisfies the following: any linearly ordered $\mathcal{F}$ as above is isomorphic to some family $\mathcal{F}'$, under inclusion, sets of indexes of non-empty predicates for theories in $\mathcal{T}_{T,R}$ such that strict inclusions $J_1 \subset J_2$ for elements in $\mathcal{F}'$ imply that cardinalities $J_2 \setminus J_1$ are infinite and do not depend on choice of J_1 and J_2. Theorem 4.16 holds for linearly ordered $\mathcal{F}'$ involving the given theory T.

5 On e-spectra for families of language uniform theories

Remark 5.1. Remind [1, Proposition 4.1, (7)] that if $T = \mathrm{Th}(\mathcal{A}_E)$ has an e-least model $\mathcal{M}$ then e-$\mathrm{Sp}(T) = e$-$\mathrm{Sp}(\mathcal{M})$. Then, following [1, Proposition 4.1, (5)], e-$\mathrm{Sp}(T) = |\mathcal{T}_0 \setminus \mathcal{T}_0'|$, where $\mathcal{T}_0'$ is the (least) generating set of theories for E-classes of $\mathcal{M}$, and $\mathcal{T}_0$ is the closed set of theories for E-classes of an e-largest model of T. Note also that e-$\mathrm{Sp}(T)$ is infinite if $\mathcal{T}_0$ does not have the least generating set.

Remind that, as shown in [1, Propositions 4.3], for any cardinality λ there is a theory $T = \mathrm{Th}(\mathcal{A}_E)$ of a language Σ such that $|\Sigma| = |\lambda + 1|$ and e-$\mathrm{Sp}(T) = \lambda$. Modifying this proposition for the class of LU-theories we obtain

Proposition 5.2. (1) *For any $\mu \leq \omega$ there is an E-combination $T = \mathrm{Th}(\mathcal{A}_E)$ of* IILU-*theories in a language Σ of the cardinality ω such that T has an e-least model and e-$\mathrm{Sp}(T) = \mu$.*

(2) *For any uncountable cardinality λ there is an E-combination $T = \mathrm{Th}(\mathcal{A}_E)$ of* IILU-*theories in a language Σ of the cardinality λ such that T has an e-least model and e-$\mathrm{Sp}(T) = \lambda$.*

Proof. In view of Propositions 3.4, 4.8, and Remark 5.1, it suffices to take an E-combination of IILU-theories of a language Σ of the cardinality λ and with a discrete linearly ordered set $\mathcal{F}$ having:

1) $\mu \leq \omega$ accumulation points if $\lambda = \omega$;

2) λ accumulation points if $\lambda > \omega$.

We get the required $\mathcal{F}$ for (1) taking:

(i) finite F for $\mu = 0$;

(ii) $\mu/2$ discrete connected components, forming $\mathcal{F}$, with the ordering type $\langle\mathbb{Z};\leq\rangle$ and having pairwise distinct accumulation points, if $\mu > 0$ is even natural;

(iii) $(\mu-1)/2$ discrete connected components, forming $\mathcal{F}$, with the ordering type $\langle\mathbb{Z};\leq\rangle$ and one connected components with the ordering type $\langle\omega;\leq\rangle$ such that all accumulation points are distinct, if $\mu > 0$ is odd natural;

(iv) ω discrete connected components, forming $\mathcal{F}$, with the ordering type $\langle\mathbb{Z};\leq\rangle$, if $\mu = \omega$.

The required $\mathcal{F}$ for (2) is formed by (uncountably many) λ discrete connected components, forming $\mathcal{F}$, with the ordering type $\langle\mathbb{Z};\leq\rangle$. $\square$

Combining Propositions 3.4, 4.10, Theorem 4.16, and Remark 5.1 with $\overline{\mathcal{F}}$ having dense intervals, we get

Proposition 5.3. *For any infinite cardinality λ there is an E-combination $T = \mathrm{Th}(\mathcal{A}_E)$ of* IILU*-theories in a language Σ of cardinality λ such that T does not have e-least models and e-*$\mathrm{Sp}(T) \geq \max\{2^\omega, \lambda\}$.

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