ON A POISSON STRUCTURE ON BOTT-SAMELSON VARIETIES: COMPUTATIONS IN COORDINATES

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Abstract. For a connected complex semi-simple Lie group G, we consider the Poisson structure π_n on an n-dimensional Bott-Samelson variety $Z_{\mathbf{u}}$ of G defined by a standard multiplicative Poisson structure $\pi_{\rm st}$ on G, where $\mathbf{u}$ is any sequence of length n of simple reflections in the Weyl group of G. We explicitly express π_n on each of the 2^n affine coordinate charts, one for every subexpression of $\mathbf{u}$, in terms of the root strings and the structure constants of the Lie algebra of G. We show that the restriction of π_n to each affine coordinate chart gives rise to a Poisson structure on the polynomial algebra $\mathbb{C}[z_1, \ldots, z_n]$ which is a Poisson-Ore extension of $\mathbb{C}$ compatible with a rational action by a maximal torus of G. For canonically chosen $\pi_{\rm st}$, we show that the induced Poisson structure on $\mathbb{C}[z_1, \ldots, z_n]$ for every affine coordinate chart is in fact defined over $\mathbb{Z}$, thus giving rise to a Poisson-Ore extension of any field $\mathbf{k}$ of arbitrary characteristic. The special case of π_n on the affine chart corresponding to the full subexpression of $\mathbf{u}$ yields an explicit formula for the standard Poisson structures on extended Bruhat cells in Bott-Samelson coordinates.

1. Introduction

1.1. **Introduction.** Let G be a connected complex semi-simple Lie group with a fixed Borel subgroup B and a maximal torus $T \subset B$, and let $\mathfrak{g}$, $\mathfrak{b}$, and $\mathfrak{h}$ be the respective Lie algebras of G, B, and T. Let $\Delta \subset \mathfrak{h}^*$ be the set of roots and $\Gamma \subset \Delta$ the set of simple roots determined by $\mathfrak{b}$. Let $W = N_G(T)/T$ be the Weyl group, where $N_G(T)$ is the normalizer of T in G.

Let $\mathbf{u} = (s_1, s_2, \cdots, s_n)$ be any sequence of simple reflections in W, and for $1 \leq i \leq n$, let $P_{s_i} = B \cup Bs_iB$ be the parabolic subgroup of G associated s_i that contains B. Consider the product manifold $P_{s_1} \times \ldots \times P_{s_n}$ with the right action of B^n (the n-fold product of B) by

$$(p_1, p_2 \ldots p_n) \cdot (b_1, b_2, \ldots, b_n) = (p_1 b_1,\, b_1^{-1} p_2 b_2,\, \ldots,\, b_{n-1}^{-1} p_n b_n), \qquad p_i \in P_{s_i},\, b_i \in B,\, 1 \leq i \leq n.$$

The quotient space, denoted by $Z_{\mathbf{u}} = P_{s_1} \times_B \ldots \times_B P_{s_n}/B$, is the Bott-Samelson variety associated to $\mathbf{u}$. For $(p_1, \ldots, p_n) \in P_{s_1} \times \ldots \times P_{s_n}$, let $[p_1, \ldots, p_n] \in Z_{\mathbf{u}}$ denote the image of $(p_1, \ldots, p_n)$ in $Z_{\mathbf{u}}$. Multiplication in the group G gives a well-defined map

$$\mu : Z_{\mathbf{u}} \longrightarrow G/B : \ \mu([p_1, p_2, \ldots, p_n]) = p_1 p_2 \cdots p_n/B.$$

When $\mathbf{u}$ is a reduced word, μ is a resolution of singularities of the Schubert variety $\overline{BuB/B}$ in G/B, where $u = s_1 s_2 \cdots s_n \in W$. Bott-Samelson varieties have been studied extensively in the literature and play an important role in geometric representation theory. See, for example, [1, 2] and the references therein.

It is well known (see, for example, [5, §1.5] or [8, §4.4]) that the choice of the pair (B, T), together with that of a symmetric non-degenerate invariant bilinear form $\langle\, , \,\rangle$ on $\mathfrak{g}$, give rise to a multiplicative holomorphic Poisson structure $\pi_{\rm st}$ on G (see §2.1), and the Poisson-Lie group $(G, \pi_{\rm st})$, referred to as a *standard complex semi-simple Poisson Lie group*, is the semi-classical limit of the much studied quantum group associated to G (see [5, 6, 8]). Every parabolic subgroup of G containing B is a Poisson submanifold of $(G, \pi_{\rm st})$. Consequently, for any sequence $\mathbf{u} = (s_1, \ldots, s_n)$ of simple reflections in W, the restriction to $P_{s_1} \times \ldots \times P_{s_n} \subset G^n$ of the n-fold product Poisson structure $\pi_{\rm st}^n = \pi_{\rm st} \times \ldots \times \pi_{\rm st}$ on G^n projects to a well-defined Poisson structure, denoted by π_n, on the Bott-Samelson variety $Z_{\mathbf{u}}$ (see §2.2 for details). We refer to π_n as *a standard Poisson structure on* $Z_{\mathbf{u}}$.

The Bott-Samelson variety $Z_{\mathbf{u}}$, where $\mathbf{u} = (s_1, \ldots, s_n)$, has 2^n affine charts $\{\mathcal{O}^\gamma : \gamma \in \Upsilon_{\mathbf{u}}\}$, where $\Upsilon_{\mathbf{u}}$ is the set of all subexpressions of $\mathbf{u}$, and the choice of a set $\{e_\alpha : \alpha \in \Gamma\}$ of root vectors for the simple roots gives rise to coordinates $(z_1, \ldots, z_n)$ on each affine chart $\mathcal{O}^\gamma$ (see §3.1). In §3.2, we give our first formula (Lemma 3.1) of the Poisson structure π_n in each coordinate chart in terms of certain vector fields on Bott-Samelson subvarieties of $Z_{\mathbf{u}}$. It is also shown in §3.3 that π_n is log-canonical in some affine charts. The first main result of the paper is Theorem 4.14, which further expresses the vector fields in Lemma 3.1 in terms of root strings and the structure constants of $\mathfrak{g}$. Identify the algebra of regular functions on $\mathcal{O}^\gamma$ with the polynomial algebra $\mathbb{C}[z_1, \ldots, z_n]$ using the coordinates $(z_1, \ldots, z_n)$ and denote by π_γ the Poisson structure on $\mathbb{C}[z_1, \ldots, z_n]$ defined by π_n. As consequences of Theorem 4.14, we prove the following prominent features of the Poisson polynomial algebras $(\mathbb{C}[z_1, \ldots, z_n], \pi_\gamma)$, where $\gamma \in \Upsilon_{\mathbf{u}}$:

1) For each $\gamma \in \Upsilon_{\mathbf{u}}$, the Poisson structure π_γ on $\mathbb{C}[z_1, \ldots, z_n]$ is independent of the re-scalings of the coordinates $(z_1, \ldots, z_n)$ resulted from different choices of root vectors for the simple roots (Proposition 5.2), and the Poisson polynomial algebra $(\mathbb{C}[z_1, \ldots, z_n], \pi_\gamma)$ is a *Poisson-Ore extension of* $\mathbb{C}$ compatible with a natural action by the maximal torus T (Proposition 5.12). When $\gamma = \mathbf{u}$ is the full subexpression of $\mathbf{u}$, the Poisson-Ore extension $(\mathbb{C}[z_1, \ldots, z_n], \pi_\gamma)$ of $\mathbb{C}$ is *symmetric and nilpotent* in the sense of [12, Definition 4] (Remark 5.9 and Proposition 5.12). The pairs of derivations at each step of the Poisson-Ore extension are expressed in terms of the action of Borel subgroups on Bott-Samelson varieties (see, in particular, §5.3).

2) Choose the bilinear form $\langle \, , \, \rangle$ on $\mathfrak{g}$ such that $\frac{\langle \alpha, \alpha \rangle}{2} \in \mathbb{Z}$ for each root α. Then for any $\gamma \in \Upsilon_{\mathbf{u}}$ and $1 \leq i < k \leq n$, the polynomial $\{z_i, z_k\} \in \mathbb{C}[z_1, \ldots, z_n]$ have integer coefficients, so π_γ defines a Poisson structure on $\mathbb{Z}[z_1, \ldots, z_n]$. Consequently, each $\gamma \in \Upsilon_{\mathbf{u}}$ gives rise to a Poisson-Ore extension $(\mathbf{k}[z_1, \ldots, z_n], \pi_\gamma)$ of any field $\mathbf{k}$ of arbitrary characteristic. See §6.1.

Parts of the paper, notably §4 and §6, are from the first author's Mphil thesis. Based on Theorem 4.14, the first author has also written a computer program in the GAP language [9] to compute the Poisson structures π_γ on polynomial algebras for any simple Lie algebra $\mathfrak{g}$, any sequence $\mathbf{u}$ of simple reflections, and any $\gamma \in \Upsilon_{\mathbf{u}}$. Some examples are given in §6.2.

Poisson-Ore extensions of fields with compatible torus actions are the semi-classical analogs of quantum CGL extensions (see [12, 13] and references therein) and have been studied in the context of Dixmier-Moeglin equivalences of Poisson algebras and quantum cluster algebras (see [3, 10, 11, 12, 18]). We have shown in this paper that the Bott-Samelson Poisson manifolds $(Z_{\mathbf{u}}, \pi_n)$ provide a rich source of systematic and concrete examples for such algebras. In light of their geometrical setting, it is thus interesting to work out for these Bott-Samelson examples some of the general theories on Poisson-Ore extensions, such as their T-Poisson prime ideals [3, 11] and higher Poisson derivations in prime characteristics [18]. These will be done in separate papers.

Our motivation for studying the Bott-Samelson Poisson manifolds $(Z_{\mathbf{u}}, \pi_n)$ also comes from their relations to naturally defined Poisson structures on the so-called *extended Bruhat cells*, which we explain in §1.2.

1.2. **The standard Poisson structures on extended Bruhat cells.** With the notation as in §1.1, for any integer $n \geq 1$, let the product group B^n act on the product manifold G^n by

$$(g_1, g_2 \ldots g_n) \cdot (b_1, b_2, \ldots, b_n) = (g_1 b_1,\, b_1^{-1} g_2 b_2,\, \ldots,\, b_{n-1}^{-1} g_n b_n), \qquad g_i \in G,\, b_i \in B,\, 1 \leq i \leq n,$$

and denote the quotient manifold of G^n by B^n by

$$F_n = G \times_B \cdots \times_B G/B. \tag{1}$$

It is shown in [19, §7] (see also §2.2) that the n-fold product Poisson structure π_{st}^n on G^n projects to a well-defined Poisson structure on F_n, which will also be denoted by π_n. Note that for any sequence $\mathbf{u} = (s_1, \ldots, s_n)$ of simple reflections in W, the Bott-Samelson variety $Z_{\mathbf{u}}$ can be regarded as a closed submanifold of F_n under the embedding $P_{s_1} \times \cdots \times P_{s_n} \subset G^n$. As

$P_{s_1} \times \cdots \times P_{s_n}$ is a Poisson submanifold of G^n with respect to $\pi_{\rm st}^n$, it follows from the definitions that $Z_{\mathbf{u}}$, with the Poisson structure π_n defined in §1.1, is a Poisson submanifold of (F_n, π_n).

For a sequence $\mathbf{u} = (u_1, \ldots, u_n)$ of elements in the Weyl group W, where the u_i's are not necessarily simple reflections, the image of $Bu_1B \times \cdots \times Bu_nB \subset G^n$ in F_n, denoted by $B\mathbf{u}B/B$, is called an *extended Bruhat cell* in [20]. The Bruhat decomposition $G = \bigsqcup_{u \in W} BuB$ of G then gives rise to the decomposition

$$F_n = \bigsqcup_{\mathbf{u} \in W^n} B\mathbf{u}B/B \tag{2}$$

of F_n into the disjoint union of extended Bruhat cells. As each BuB, where $u \in W$, is a Poisson submanifold of G with respect to $\pi_{\rm st}$, the decomposition in (2) is a decomposition of the Poisson manifold (F_n, π_n) into Poisson submanifolds.

An extended Bruhat cell $B(s_1, \ldots, s_n)B/B \subset F_n$, where each s_i is a simple reflection, is said to be of *Bott-Samelson type* [20]. In the notation of the current paper, an extended Bruhat cell $B(s_1, \ldots, s_n)B/B$ in F_n of Bott-Samelson type is nothing but the open affine chart $\mathcal{O}^{(s_1, \ldots, s_n)}$ in the Bott-Samelson variety $Z_{(s_1, \ldots, s_n)} \subset F_n$. Given an arbitrary $\mathbf{u} = (u_1, \ldots, u_n) \in W^n$, choose any reduced decomposition $u_i = s_{i,1}s_{i,2}\cdots s_{i,l(u_i)}$ for each u_i, where $l : W \to \mathbb{N}$ is the length function of W, and consider the sequence

$$\tilde{\mathbf{u}} = (s_{1,1}, \ \ldots, \ s_{1,l(u_1)}, \ s_{2,1}, \ \ldots, \ s_{2,l(u_2)}, \ \ldots, s_{n,1}, \ \ldots, \ s_{n,l(u_n)})$$

of simple reflections of length $l(\mathbf{u}) = l(u_1) + \cdots + l(u_n)$. Then the multiplication map on G induces a Poisson isomorphism

$$(Z_{\tilde{\mathbf{u}}}, \, \pi_{l(\mathbf{u})}) \supset (\mathcal{O}^{\tilde{\mathbf{u}}}, \, \pi_{l(\mathbf{u})}) = (B\tilde{\mathbf{u}}B/B, \, \pi_{l(\mathbf{u})}) \ \stackrel{\sim}{\longrightarrow} \ (B\mathbf{u}B/B, \, \pi_n) \subset (F_n, \, \pi_n) \tag{3}$$

(see [20, §1.3]). Referring to the coordinates $(z_1, \ldots, z_{l(\mathbf{u})})$ on $\mathcal{O}^{\tilde{\mathbf{u}}} = B\tilde{\mathbf{u}}B/B$ introduced in §3.1 of the present paper as *Bott-Samelson coordinates* on $B\mathbf{u}B/B$ via the isomorphism in (3), Theorem 4.14 then computes explicitly the Poisson structure π_n on $B\mathbf{u}B/B$ in these coordinates, and the discussions in §5 show that the corresponding Poisson polynomial algebra $\mathbb{C}[z_1, \ldots, z_{l(\mathbf{u})}]$ is a symmetric nilpotent semi-quadratic Poisson-Ore extension of $\mathbb{C}$ in the sense of [12].

The study of the Poisson manifolds (F_n, π_n), $n \geq 1$, together with that of some other holomorphic Poisson structures on spaces related to flag varieties of G, was initiated in [19], where they were identified as *mixed product Poisson structures defined by quasitriangular r-matrices.* The study was continued in [20], where the orbits of their symplectic leaves under the action of the maximal torus T, also called T-leaves, were described explicitly in terms of extended Bruhat cells and extended double Bruhat cells associated to conjugacy classes. Together with [19] and [20], the current paper belongs to a series of papers devoted to a detailed study of the Poisson manifolds (F_n, π_n) and related Poisson structures, as the results in the current paper show that the Poisson manifold (F_n, π_n) is paved (in the sense of being a disjoint union) by affine Poisson spaces with the corresponding Poisson polynomial algebras being symmetric nilpotent semi-quadratic Poisson-Ore extensions of $\mathbb{C}$.

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1.4. **Notation.** Continuing with the notation from §1.1, let $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$ be the root decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$. For $\alpha \in \Delta$, let h_α be the unique element in $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that $\alpha(h_\alpha) = 2$, and let $\alpha^\vee : \mathbb{C}^\times \to T$ be the co-character of T defined by h_α. Let $\Delta_+ \subset \Delta$ be the set of positive roots determined by $\mathfrak{b}$, and let $\mathfrak{b}_- = \mathfrak{h} + \sum_{\alpha \in \Delta_+} \mathfrak{g}_{-\alpha}$. The Borel subgroup of G with Lie algebra $\mathfrak{b}_-$ is denoted by B_-.

Let $\alpha \in \Delta_+$. If $e_\alpha \in \mathfrak{g}_\alpha$ and $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ are such that $[e_\alpha, e_{-\alpha}] = h_\alpha$, we call $\{h_\alpha, e_\alpha, e_{-\alpha}\}$ an $\mathfrak{sl}(2,\mathbb{C})$-triple for α. Clearly, any non-zero $e_\alpha \in \mathfrak{g}_\alpha$ uniquely determines an $\mathfrak{sl}(2,\mathbb{C})$-triple $\{h_\alpha, e_\alpha, e_{-\alpha}\}$, and every other $\mathfrak{sl}(2,\mathbb{C})$-triple for α is of the form $\{h_\alpha, \lambda e_\alpha, \lambda^{-1} e_{-\alpha}\}$ for a unique $\lambda \in \mathbb{C}$. Given an $\mathfrak{sl}(2,\mathbb{C})$-triple $\{h_\alpha, e_\alpha, e_{-\alpha}\}$, denote by $\theta_\alpha : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}$ the Lie algebra homomorphism defined by

$$\theta_\alpha : \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longmapsto h_\alpha, \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longmapsto e_\alpha, \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \longmapsto e_{-\alpha},$$

and denote also by $\theta_\alpha : \mathrm{SL}(2,\mathbb{C}) \to G$ the corresponding Lie group homomorphism, so that

$$\alpha^\vee(t) = \theta_\alpha\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right), \qquad t \in \mathbb{C}^\times.$$

An $\mathfrak{sl}(2,\mathbb{C})$-triple $\{h_\alpha, e_\alpha, e_{-\alpha}\}$ for $\alpha \in \Delta_+$ also gives rise to the one-parameter subgroups $u_{\pm\alpha} : \mathbb{C} \to G$ via

$$u_\alpha(z) = \theta_\alpha\left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right) = \exp(ze_\alpha), \quad u_{-\alpha}(z) = \theta_\alpha\left(\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\right) = \exp(ze_{-\alpha}), \qquad z \in \mathbb{C}.$$

Let $W = N_G(T)/T$ be again the Weyl group of (G,T). For $\alpha \in \Delta_+$, let $s_\alpha \in W$ be the reflection in W determined by α, and if $\{h_\alpha, e_\alpha, e_{-\alpha}\}$ is an $\mathfrak{sl}(2,\mathbb{C})$-triple for α, let $\dot{s}_\alpha$ be the representative of s_α in $N_G(T)$ given by

$$\dot{s}_\alpha = u_\alpha(-1) u_{-\alpha}(1)\, u_\alpha(-1) \in N_G(T). \tag{4}$$

For a complex algebraic torus $\mathbb{T}$ with Lie algebra $\mathfrak{t}$, we use the same notation for an element $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^\times)$ and its differential at the identity element of $\mathbb{T}$, which is an element in $\mathfrak{t}^*$. The values of λ on $t \in T$ and on $x \in \mathfrak{t}$ are respectively denoted as $t^\lambda \in \mathbb{C}^\times$ and $\lambda(x) \in \mathbb{C}$.

2. Definition of the Poisson structure π_n on $Z_{\mathbf{u}}$

2.1. The standard semi-simple Poisson Lie group (G, π_{st}). Recall from [5, 8] that if L is a Lie group, a Poisson bivector field π_L on L is said to be multiplicative if the map

$$(L \times L,\ \pi_L \times \pi_L) \longrightarrow (L,\ \pi_L): \quad (l_1,\ l_2) \longmapsto l_1 l_2, \qquad l_1, l_2 \in L,$$

is Poisson, where $\pi_L \times \pi_L$ is the product Poisson structure on $L \times L$. A Poisson Lie group is a pair (L, π_L), where L is a Lie group and π_L is a multiplicative Poisson bivector field on L. A Poisson Lie subgroup of a Poisson Lie group (L, π_L) is a Lie subgroup L_1 of L which is also a Poisson submanifold with respect to π_L, and in this case $(L_1, \pi_L|_{L_1})$, or simply denoted as (L_1, π_L), is a Poisson Lie group.

Let G be a connected complex semi-simple Lie group and let the notation be as in §1.4. Fix, furthermore, a symmetric non-degenerate invariant bilinear form $\langle\, , \rangle$ on $\mathfrak{g}$, and denote also by $\langle\, , \rangle$ the induced bilinear form on $\mathfrak{h}^*$. Define $\Lambda \in \wedge^2\mathfrak{g}$ by

$$\Lambda = \sum_{\alpha \in \Delta_+} \frac{\langle \alpha, \alpha \rangle}{2} e_{-\alpha} \wedge e_\alpha \in \wedge^2 \mathfrak{g},$$

where for each $\alpha \in \Delta_+$, $\{h_\alpha, e_\alpha, e_{-\alpha}\}$ is an $\mathfrak{sl}(2,\mathbb{C})$-triple for α. Note that for any $\alpha \in \Delta_+$, the element $e_{-\alpha} \wedge e_\alpha \in \wedge^2\mathfrak{g}$ stays the same if the $\mathfrak{sl}(2,\mathbb{C})$-triple $\{h_\alpha, e_\alpha, e_{-\alpha}\}$ is changed to $\{h_\alpha, \lambda e_\alpha, \frac{1}{\lambda} e_{-\alpha}\}$ for $\lambda \in \mathbb{C}^\times$. Consequently, the element $\Lambda \in \wedge^2\mathfrak{g}$ depends on $\langle\, , \rangle$ but *not* on the choices of the $\mathfrak{sl}(2,\mathbb{C})$-triples for the positive roots. Let π_{st} be the bivector field on G given by

$$\pi_{\mathrm{st}}(g) = l_g(\Lambda) - r_g(\Lambda), \qquad g \in G,$$

where for $g \in G$, l_g and r_g respectively denote the left and right translations on G by g. Then (G, π_{st}) is a Poisson Lie group, called a *standard complex semi-simple Poisson Lie group* [8, §4.4]. Moreover, the Poisson structure π_{st} is invariant under the action of T by left translation, and the T-orbits of symplectic leaves, also called T-leaves, of π_{st} are precisely the so-called double

Bruhat cells $(BuB) \cap (B_-vB_-)$, where $u, v \in W$ (see [14, 16]). In particular, every BuB, where $u \in W$, is a Poisson submanifold of $(G, \pi_{\rm st})$, and every parabolic subgroup P of G containing B, being a union of (B, B)-double cosets in G, is a Poisson Lie subgroup of $(G, \pi_{\rm st})$. Similar statements hold if B is replaced by B_-.

We state another important property of $(G, \pi_{\rm st})$: let α be a simple root and consider the group homomorphism $\theta_\alpha : SL(2,\mathbb{C}) \to G$ defined in §1.4 corresponding to any choice of an $sl(2,\mathbb{C})$-triple $\{h_\alpha, e_\alpha, e_{-\alpha}\}$ for α. Equip $SL(2,\mathbb{C})$ with the multiplicative Poisson structure

$$\pi_{SL(2,\mathbb{C})}(g) = l_g(\Lambda_0) - r_g(\Lambda_0), \qquad g \in SL(2,\mathbb{C}), \tag{5}$$

where $\Lambda_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \wedge^2 \mathfrak{sl}(2,\mathbb{C})$. Then [17]

$$\theta_\alpha : \left(SL(2,\mathbb{C}),\, \frac{\langle \alpha, \alpha\rangle}{2}\pi_{SL(2,\mathbb{C})}\right) \longrightarrow (G,\, \pi_{\rm st}) \tag{6}$$

is a Poisson map. It follows that $\theta_\alpha(SL(2,\mathbb{C}))$ is a Poisson Lie subgroup of $(G, \pi_{\rm st})$. Moreover, let $g = u_{-\alpha}(z)$ and $g' = u_\alpha(z)\dot{s}_\alpha$, where $z \in \mathbb{C}$. Then

$$\pi_{\rm st}(g) = \frac{\langle \alpha, \alpha\rangle}{2} l_g(zh_\alpha \wedge e_{-\alpha}), \tag{7}$$

$$\pi_{\rm st}(g') = \frac{\langle \alpha, \alpha\rangle}{2} l_{g'}(zh_\alpha \wedge e_{-\alpha} - 2e_\alpha \wedge e_{-\alpha}) = \frac{\langle \alpha, \alpha\rangle}{2} r_{g'}(ze_\alpha \wedge h_\alpha + 2e_\alpha \wedge e_{-\alpha}). \tag{8}$$

2.2. **The definition of the Poisson structure π_n on $Z_{\mathbf{u}}$.** Recall that given a Poisson Lie group (L, π_L) and a Poisson manifold (Y, π_Y), a left Lie group action $\sigma : L \times Y \to Y$ of L on Y is said to be a Poisson action if σ is a Poisson map from the product Poisson manifold $(L \times Y,\, \pi_L \times \pi_Y)$ to (Y, π_Y). Right Poisson actions of Poisson Lie groups are similarly defined.

Let (Q, π_Q) be a Poisson Lie group, let (X, π_X) be a Poisson manifold with a right Poisson action by (Q, π_Q), and let (Y, π_Y) a Poisson submanifold with a left Poisson action by (Q, π_Q). Define the right action of Q on $X \times Y$ by

$$(x,\, y) \cdot q = (xq,\, q^{-1}y), \qquad x \in X,\, y \in Y,\, q \in Q,$$

and assume that the quotient space of $X \times Y$ by Q, denoted by $X \times_Q Y$, is a smooth manifold. Then (see [19, §7] and [22]) the direct product Poisson structure $\pi_X \times \pi_Y$ on $X \times Y$ projects to a well-defined Poisson structure on $X \times_Q Y$.

Example 2.1. Let (Q, π_Q) be a closed Poisson Lie subgroup of a Poisson Lie group (L, π_L), and let (Y, π_Y) be a Poisson manifold with a left Poisson action by (Q, π_Q). Consider the quotient manifold $Z = L \times_Q Y$, where Q acts on L by right translation. Then Z has the Poisson structure π_Z that is the projection to Z of the direct product Poisson structure $\pi_L \times \pi_Y$ on $L \times Y$. Denoting the image in Z of $(l, y) \in L \times Y$ by $[l, y]$, it follows from the multiplicativity of π_L that the left action of L on Z given by

$$l \cdot [l_1,\, y] = [ll_1,\, y], \qquad l, l_1 \in L,\, y \in Y, \tag{9}$$

is a Poisson action of the Poisson Lie group (L, π_L) on the Poisson manifold $(Z,\, \pi_Z)$. Moreover, since $\pi_L(e) = 0$, where e is the identity element of L, the inclusion $Y \hookrightarrow L \times Y, y \mapsto (e, y)$, $y \in Y$, is a Poisson embedding of (Y, π_Y) into $(L \times Y,\, \pi_L \times \pi_Y)$. Consequently,

$$Y \hookrightarrow Z,\;\; y \longmapsto [e,\, y], \qquad y \in Y,$$

is a Poisson embedding of (Y, π_Y) into the Poisson manifold (Z, π_Z). ◇

Consider now the standard semi-simple Poisson Lie group $(G, \pi_{\rm st})$ in §2.1. Let $\mathbf{u} = (s_1, \ldots, s_n)$ be any sequence of simple reflections in the Weyl group W. Then for each $1 \leq i \leq n$, the parabolic subgroup $P_{s_i} = B \cup Bs_iB$ is a Poisson Lie subgroup of $(G, \pi_{\rm st})$. By taking $(L, \pi_L) = (P_{s_i}, \pi_{\rm st})$ and $Q = B$ in Example 2.1 and repeat the construction therein, one sees that the direct product

Poisson structure $\pi_{\rm st}^n$, regarded as a Poisson structure on the product manifold $P_{s_1} \times \cdots \times P_{s_n}$, projects to a well-defined Poisson structure, denoted by π_n, on the Bott-Samelson variety $Z_{\mathbf{u}}$. It also follows from Example 2.1 that the left action of P_{s_1} on $Z_{\mathbf{u}}$ given by

$$p \cdot [p_1, p_2, \ldots, p_n] = [pp_1,\, p_2, \ldots,\, p_n], \qquad p \in P_{s_1},\, p_j \in P_{s_j},\ 1 \leq j \leq n, \tag{10}$$

is a Poisson action of the Poisson group $(P_{s_1}, \pi_{\rm st})$ on the Poisson manifold $(Z_{\mathbf{u}}, \pi_n)$. In particular, since $\pi_{\rm st}(t) = 0$ for $t \in T$, the action of T on $Z_{\mathbf{u}}$ via (10) is by Poisson isomorphisms of π_n.

2.3. **$\mathbb{P}^1$-extensions.** To prepare for the calculation of the Poisson structure π_n in coordinates, we first look at a special case of Example 2.1: let $(Y, \pi_{\scriptscriptstyle Y})$ be a Poisson manifold with a left Poisson action σ by the Poisson Lie subgroup $(B, \pi_{\rm st})$ of $(G, \pi_{\rm st})$, and let α be a simple root. One then has the quotient manifold $Z = P_{s_\alpha} \times_B Y$, which fibers over $P_{s_\alpha}/B \cong \mathbb{P}^1$ with fibers diffeomorphic to Y. Let $\pi_{\scriptscriptstyle Z}$ denote the projection to Z of the product Poisson structure $\pi_{\rm st} \times \pi_{\scriptscriptstyle Y}$ on $P_{s_\alpha} \times Y$. Choose any non-zero $e_\alpha \in \mathfrak{g}_\alpha$, giving rise to the $\mathfrak{sl}(2, \mathbb{C})$-triple $\{h_\alpha, e_\alpha, e_{-\alpha}\}$ for α, and let the notation be as in §1.4. Consider the two open subsets

$$Z_- = \{[u_{-\alpha}(z),\, y] : \, z \in \mathbb{C},\, y \in Y\} \quad \text{and} \quad Z_+ = \{[u_\alpha(z)\dot{s}_\alpha,\ y] : z \in \mathbb{C}, y \in Y\}$$

of Z with parametrization

$$\psi_- : \ \mathbb{C} \times Y \longrightarrow Z_-, \ \ \psi_-(z, y) = [u_{-\alpha}(z),\ y],$$
$$\psi_+ : \ \mathbb{C} \times Y \longrightarrow Z_+, \ \ \psi_+(z, y) = [u_\alpha(z)\dot{s}_\alpha,\ y].$$

We will compute $\psi_-^{-1}(\pi_{\scriptscriptstyle Z})$ and $\psi_+^{-1}(\pi_{\scriptscriptstyle Z})$ as bi-vector fields on $\mathbb{C} \times Y$. For $x \in \mathfrak{b}$, let η_x be the vector field on Y given by $\eta_x(y) = \frac{d}{dt}|_{t=0} \exp(tx)y$ for $y \in Y$. In the statement of the following Lemma 2.2, we use the obvious way of viewing vector fields on $\mathbb{C}$ and on Y as that on $\mathbb{C} \times Y$.

Lemma 2.2. *With the notation as above, one has*

$$\psi_-^{-1}(\pi_{\scriptscriptstyle Z})(z, y) = -\frac{\langle \alpha, \alpha \rangle}{2} z \frac{d}{dz} \wedge \eta_{h_\alpha}(y) + \pi_{\scriptscriptstyle Y}(y), \tag{11}$$

$$\psi_+^{-1}(\pi_{\scriptscriptstyle Z})(z, y) = \frac{\langle \alpha, \alpha \rangle}{2} \frac{d}{dz} \wedge (z\eta_{h_\alpha}(y) - 2\eta_{e_\alpha}(y)) + \pi_{\scriptscriptstyle Y}(y). \tag{12}$$

Proof. For $g \in P_{s_\alpha}$ and $y \in Y$, let

$$\lambda_g : \ Z \longrightarrow Z : \ [p,\ y'] \longmapsto [gp,\ y'], \quad p \in P_{s_\alpha},\, y' \in Y,$$
$$\rho_y : \ P_{s_\alpha} \longrightarrow Z : \ p \longmapsto [p,\ y], \quad p \in P_{s_\alpha}.$$

Fix $z \in \mathbb{C}$ and $y \in Y$, and let $g = u_{-\alpha}(z) \in P_{s_\alpha}$ and $q = [u_{-\alpha}(z),\, y] = \lambda_g([e,\, y]) \in Z$. By Example 2.1, $\pi_{\scriptscriptstyle Z}(q) = \lambda_g(\pi_{\scriptscriptstyle Z}([e, y])) + \rho_y(\pi_{\rm st}(g))$. Using (7), one has

$$\pi_{\scriptscriptstyle Z}(q) = \lambda_g(\pi_{\scriptscriptstyle Z}([e, y])) + \frac{\langle \alpha, \alpha \rangle}{2}(\rho_y l_g)(zh_\alpha \wedge e_{-\alpha}) = \lambda_g(\pi_{\scriptscriptstyle Z}([e, y])) + \frac{\langle \alpha, \alpha \rangle}{2}(\lambda_g \rho_y)(zh_\alpha \wedge e_{-\alpha}),$$

and thus

$$(\psi_-^{-1}(\pi_{\scriptscriptstyle Z}))(z, y) = \psi_-^{-1}(\pi_{\scriptscriptstyle Z}(q)) = (\psi_-^{-1} \circ \lambda_g)(\pi_{\scriptscriptstyle Z}([e, y])) + \frac{\langle \alpha, \alpha \rangle}{2}(\psi_-^{-1}\lambda_g \rho_y)(zh_\alpha \wedge e_{-\alpha}).$$

Since the inclusion $(Y, \pi_{\scriptscriptstyle Y}) \hookrightarrow (Z, \pi_{\scriptscriptstyle Z}) : y' \mapsto [e, y']$ is Poisson, $(\psi_-^{-1} \circ \lambda_g)(\pi_{\scriptscriptstyle Z}([e, y])) = \pi_{\scriptscriptstyle Y}(y)$. Direct calculations give

$$(\psi_-^{-1}\lambda_g \rho_y)(h_\alpha) = \eta_{h_\alpha}(y) \quad \text{and} \quad (\psi_-^{-1}\lambda_g \rho_y)(e_{-\alpha}) = \frac{d}{dz}.$$

One thus has (11). Similarly, for $z \in \mathbb{C}$ and $y \in Y$, letting $g' = u_\alpha(z)\dot{s}_\alpha$ and using (8), one has

$$\psi_+^{-1}(\pi_{\scriptscriptstyle Z})(z, y) = \pi_{\scriptscriptstyle Y}(y) + \frac{\langle \alpha, \alpha \rangle}{2}(\psi_+^{-1}\lambda_{g'}\rho_y)((zh_\alpha - 2e_\alpha) \wedge e_{-\alpha}).$$

Since $(\psi_+^{-1}\lambda_{g'}\rho_y)(h_\alpha) = \eta_{h_\alpha}$, $(\psi_+^{-1}\lambda_{g'}\rho_y)(e_\alpha) = \eta_{e_\alpha}$, and $(\psi_+^{-1}\lambda_{g'}\rho_y)(e_{-\alpha}) = -\frac{d}{dz}$, one has (12).

Q.E.D.

3. The Poisson structure π_n in affine coordinate charts, I

Throughout §3, we fix a sequence $\mathbf{u} = (s_1, \ldots, s_n)$ of simple reflections in W, and let $Z_{\mathbf{u}}$ be the Bott-Samelson variety associated to $\mathbf{u}$. Recall that Γ denotes the set of all simple roots. For $1 \leq j \leq n$, let $\alpha_j \in \Gamma$ be such that $s_j = s_{\alpha_j}$. To define local coordinates on $Z_{\mathbf{u}}$, we also fix a root vector e_α for each $\alpha \in \Gamma$ and let $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ be the unique element such that $[e_\alpha, e_{-\alpha}] = h_\alpha$. One then (see §1.4) has the one-parameter subgroups $u_{\pm\alpha} : \mathbb{C} \to G$ for each $\alpha \in \Gamma$ and the representative $\dot{s}_\alpha \in N_G(T)$ for the simple reflection $s_\alpha \in W$.

3.1. Affine coordinate charts on $Z_{\mathbf{u}}$. Let

$$\Upsilon_{\mathbf{u}} = \{e, s_1\} \times \{e, s_2\} \times \cdots \{e, s_n\},$$

where e denotes the identity element of W. Elements in $\Upsilon_{\mathbf{u}}$ will be called *subexpressions* of $\mathbf{u}$. When $\gamma = \mathbf{u}$, we say that γ is the *full subexpression* of $\mathbf{u}$. For $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}}$, let $\gamma^0 = e$ and $\gamma^i = \gamma_1\gamma_2 \cdots \gamma_i \in W$ for $1 \leq i \leq n$.

As a subgroup of P_{s_1}, the maximal torus T of G acts on $Z_{\mathbf{u}}$ via (10), with the fixed point set $(Z_{\mathbf{u}})^T = \{[\dot{\gamma}_1, \dot{\gamma}_2, \ldots, \dot{\gamma}_n] : (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}}\}$, where $\dot{e} = e$. For each $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}}$, let $\mathcal{O}^\gamma \subset Z_{\mathbf{u}}$ be the image of the embedding $\Phi_\gamma : \mathbb{C}^n \to Z_{\mathbf{u}}$ given by

$$\Phi_\gamma(z_1, \ldots, z_n) = [u_{-\gamma_1(\alpha_1)}(z_1)\dot{\gamma}_1,\, u_{-\gamma_2(\alpha_2)}(z_2)\dot{\gamma}_2,\, \ldots,\, u_{-\gamma_n(\alpha_n)}(z_n)\dot{\gamma}_n]. \tag{13}$$

The parametrization Φ_γ of $\mathcal{O}^\gamma$ by $\mathbb{C}^n$ depends on the choice of the root vectors $\{e_\alpha : \alpha \in \Gamma\}$ for the simple roots, but different choices of such root vectors only result in re-scalings of the coordinate functions. In particular, the affine chart $\mathcal{O}^\gamma$ is canonically defined. It is also easy to see that each $\mathcal{O}^\gamma$ is T-invariant with

$$t \cdot \Phi_\gamma(z_1, z_2, \ldots, z_n) = \Phi_\gamma\left(t^{-\gamma^1(\alpha_1)}z_1,\, t^{-\gamma^2(\alpha_2)}z_2,\, \ldots,\, t^{-\gamma^n(\alpha_n)}z_n\right), \tag{14}$$

where $t \in T$ and $(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$. Note also that $\bigcup_{\gamma \in \Upsilon_{\mathbf{w}}} \mathcal{O}^\gamma = Z_{\mathbf{u}}$, i.e., $Z_{\mathbf{u}}$ is covered by the 2^n T-invariant affine charts $\{\mathcal{O}^\gamma : \gamma \in \Upsilon_{\mathbf{u}}\}$.

3.2. The Poisson structure π_n in coordinates, I. For each $\gamma \in \Upsilon_{\mathbf{u}}$, we now give our first formula for the Poisson structure π_n on $Z_{\mathbf{u}}$ in the coordinates $(z_1, z_2, \ldots, z_n)$ on $\mathcal{O}^\gamma$ given in (13). A more detailed formula, expressing each Poisson bracket $\{z_i, z_k\}$, where $1 \leq i < k \leq n$, as a polynomial with coefficients explicitly given in terms of the structure constants of the Lie algebra $\mathfrak{g}$, will be given in §4.

For $1 \leq i \leq n-1$, let σ_i be the holomorphic vector field on the Bott-Samelson variety $Z_{(s_{i+1}, \ldots, s_n)}$ given by

$$\sigma_i(p) = \frac{d}{dt}|_{t=0}((\exp(te_{\alpha_i})) \cdot p), \qquad p \in Z_{(s_{i+1}, \ldots, s_n)}, \tag{15}$$

where $\cdot$ denotes the left action of $B \subset P_{s_i}$ on $Z_{(s_{i+1}, \ldots, s_n)}$ by left translation (see (10)).

Lemma 3.1. *Let $\gamma \in \Upsilon_{\mathbf{u}}$. In the coordinates $(z_1, \ldots z_n)$ on the affine chart $\mathcal{O}^\gamma$ given in (13), the Poisson structure π_n on $Z_{\mathbf{u}}$ is given by,*

$$\{z_i, z_k\} = \begin{cases} \langle \gamma^i(\alpha_i),\, \gamma^k(\alpha_k)\rangle z_i z_k, & \text{if } \gamma_i = e \\ -\langle \gamma^i(\alpha_i),\, \gamma^k(\alpha_k)\rangle z_i z_k - \langle \alpha_i, \alpha_i\rangle \sigma_i(z_k) & \text{if } \gamma_i = s_i \end{cases}, \quad 1 \leq i < k \leq n, \tag{16}$$

where $\sigma_i(z_k)$ denotes the action of the vector field σ_i on z_k as a local function on $Z_{(s_{i+1}, \ldots, s_n)}$.

Proof. Identify $\mathcal{O}^\gamma \cong \mathbb{C} \times \mathcal{O}^{\gamma'}$, where $\gamma' = (\gamma_2, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}'}$ and $\mathbf{u}' = (s_2, \ldots, s_n)$. Equip $\mathcal{O}^{\gamma'}$ with the Poisson structure π_{n-1} on $Z_{(s_2, \ldots, s_n)}$. One has, by Lemma 2.2,

$$\pi_n = \begin{cases} -\frac{\langle \alpha_1, \alpha_1\rangle}{2} z_1 \frac{d}{dz_1} \wedge \eta_1 + \pi_{n-1}, & \text{if } \gamma_1 = e, \\ \frac{\langle \alpha_1, \alpha_1\rangle}{2} \frac{d}{dz_1} \wedge (z_1\eta_1 - 2\sigma_1) + \pi_{n-1}, & \text{if } \gamma_1 = s_1, \end{cases} \tag{17}$$

where η_1 is the holomorphic vector field on $Z_{(s_2,\ldots,s_n)}$ given by

$$\eta_1(q) = \frac{d}{dt}|_{t=1}(\check{\alpha}_1(t) \cdot q), \qquad q \in Z_{(s_2,\ldots,s_n)}.$$

By (14), the vector field η_1 is given in the coordinates $(z_2, \ldots, z_n)$ on $\mathcal{O}^{\gamma'}$ by

$$\eta_1 = \sum_{k=2}^{n} (-\gamma_2 \cdots \gamma_k(\alpha_k))(h_{\alpha_1}) z_k \frac{\partial}{\partial z_k} = -\sum_{k=2}^{n} \frac{2\langle \gamma^1(\alpha_1), \gamma^k(\alpha_k)\rangle}{\langle \alpha_1, \alpha_1\rangle} z_k \frac{\partial}{\partial z_k}.$$

Lemma 3.1) now follows by repeatedly using (17).

Q.E.D.

Example 3.2. Consider $G = SL(3,\mathbb{C})$ with the standard choices of B and B_- consisting respectively of upper triangular and lower triangular matrices in $SL(3,\mathbb{C})$, and let the bilinear form $\langle\, , \rangle$ on $\mathfrak{sl}(3,\mathbb{C})$ be given by $\langle X, Y\rangle = \mathrm{tr}(XY)$ for $X, Y \in \mathfrak{sl}(3,\mathbb{C})$. Denote the two simple roots by α_1 and α_2 choose root vectors $e_{\alpha_1} = E_{12}$ and $e_{\alpha_2} = E_{23}$, where E_{ij} has 1 at the (i,j)-entry and 0 everywhere else. Let $\mathbf{u} = (s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_1})$. Using Lemma 3.1, one can compute directly the Poisson structure π_3 on $Z_{\mathbf{u}}$ in any of the eight affine coordinate charts with coordinates (z_1, z_2, z_3). For example, for $\gamma = \mathbf{u}$, one has

$$\{z_1, z_2\} = -z_1 z_2, \qquad \{z_1, z_3\} = z_1 z_3 - 2, \qquad \{z_2, z_3\} = -z_2 z_3, \tag{18}$$

and for $\gamma = (s_{\alpha_1}, e, e) \in \Upsilon_{\mathbf{u}}$, one has

$$\{z_1, z_2\} = z_1 z_2, \qquad \{z_1, z_3\} = -2z_1 z_3 + 2z_3^2, \qquad \{z_2, z_3\} = -z_2 z_3. \tag{19}$$

$\diamond$

3.3. **Some log-canonical charts for π_n.** Let $\gamma \in \Upsilon_{\mathbf{u}}$. We say that the affine coordinate chart $\mathcal{O}^\gamma$ of $Z_{\mathbf{u}}$ is *log-canonical* for the Poisson structure π_n, or that the Poisson structure π_n is *log-canonical* in the affine coordinate chart $\mathcal{O}^\gamma$, if the Poisson brackets between the coordinate functions $(z_1, z_2, \ldots, z_n)$ on $\mathcal{O}^\gamma$ have the form $\{z_i, z_k\} = \lambda_{ik} z_i z_k$ for some $\lambda_{ik} \in \mathbb{C}$ for each pair $1 \leq i < k \leq n$. By Lemma 3.1, π_n is log-canonical in $\mathcal{O}^\gamma$ if and only if

$$\{z_i, z_k\} = \epsilon_i \langle \gamma^i(\alpha_i), \gamma^k(\alpha_k)\rangle z_i z_k, \qquad 1 \leq i < k \leq n,$$

where $\epsilon_i = 1$ if $\gamma_i = e$ and $\epsilon_i = -1$ if $\gamma_i = s_i$. The following Lemma 3.3, which follows trivially from Lemma 3.1, says that π_n is log-canonical in the affine chart $\mathcal{O}^{(e,e,\ldots,e)}$.

Lemma 3.3. *In the coordinates $(z_1, z_2, \ldots, z_n)$ on $\mathcal{O}^{(e,e,\ldots,e)}$, one has*

$$\{z_i,\, z_k\} = \langle \alpha_i,\, \alpha_k\rangle z_i z_k, \qquad \forall\, 1 \leq i < k \leq n.$$

To exhibit other log-canonical affine coordinate charts for π_n, we make the following observation on the functions $\sigma_i(z_k)$, $1 \leq i < k \leq n$, in Lemma 3.1.

Lemma 3.4. *Let $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}}$, and let $1 \leq i \leq n$. If $\gamma_i = s_i$ and if $k > i$ is such that $s_j \neq s_i$ for all $i+1 \leq j \leq k$, then $\sigma_i(z_k) = 0$.*

Proof. For $i+1 \leq j \leq n$, let $z_j \in \mathbb{C}$ and $p_j = u_{-\gamma_j(\alpha_j)}(z_j)\dot{\gamma}_k$. For $t \in \mathbb{C}$, consider

$$[u_{\alpha_i}(t) p_{i+1},\ p_{i+2},\ \ldots,\ p_n] \in Z_{(s_{i+1},\ldots,s_n)}.$$

For each $i+1 \leq j \leq k$, since $p_{i+1}p_{i+2}\cdots p_j$ lies in the Levi subgroup of the parabolic subgroup of G determined by the set of simple roots in $\{\alpha_{i+1}, \ldots, \alpha_j\}$ which does not contain α_i, one has

$$(p_{i+1}p_{i+2}\cdots p_j)^{-1} u_{\alpha_i}(t) p_{i+1}p_{i+2}\cdots p_j \in N,$$

where N is the unipotent subgroup of G with Lie algebra $\mathfrak{n} = \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$. It thus follows from the definition of the vector field σ_i that $\sigma_i(z_k) = 0$, where z_k is now regarded as a local function on $Z_{(s_{i+1},\ldots,s_n)}$.

Q.E.D.

The next Lemma 3.5, which follows directly from Lemma 3.1 and Lemma 3.4, exhibits a log-canonical affine chart for π_n associated to each $s \in \{s_1, \ldots, s_n\}$.

Lemma 3.5. *Let $s \in \{s_1, s_2, \ldots, s_n\}$ and let $i_0 = \max\{i : 1 \leq i \leq n,\, s_i = s\}$. Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ be such that $\gamma_{i_0} = s$ and $\gamma_i = e$ for all $i \neq i_0$. Then in the coordinates $(z_1, z_2, \ldots, z_n)$ on $\mathcal{O}^\gamma$ and for all $1 \leq i < k \leq n$, one has*

$$\{z_i, z_k\} = \begin{cases} \langle \alpha_i, \alpha_k \rangle z_i z_k, & 1 \leq i < k < i_0 \;\text{ or }\; i_0 < i < k \leq n, \\ \langle \alpha_i, s(\alpha_k) \rangle z_i z_k, & 1 \leq i \leq i_0 \leq k \leq n,\; i \neq k. \end{cases} \tag{20}$$

The following Corollary 3.6 also follows directly from Lemma 3.1 and Lemma 3.4.

Corollary 3.6. *If $\mathbf{u} = (s_1, s_2, \ldots, s_n)$ is such that $s_i \neq s_j$ for all $i \neq j$, then the Poisson structure π_n on $Z_{\mathbf{u}}$ is log-canonical in every one of the 2^n affine coordinate charts $\{\mathcal{O}^\gamma : \gamma \in \Upsilon_{\mathbf{u}}\}$.*

4. The Poisson structure π_n in affine coordinates charts, II

Throughout §4, fix a sequence $\mathbf{u} = (s_1, \ldots, s_n)$ of simple reflections, and let $Z_{\mathbf{u}}$ be the corresponding Bott-Samelson variety. To better understand the Poisson structure π_n in the coordinates $(z_1, z_2, \ldots, z_n)$ on the affine chart $\mathcal{O}^\gamma$ defined in §3.1, where $\gamma \in \Upsilon_{\mathbf{u}}$, one needs to compute more explicitly the vector field σ_i in Lemma 3.1 on the Bott-Samelson variety $Z_{(s_{i+1}, \ldots, s_n)}$ for $1 \leq i \leq n-1$. For $x \in \mathfrak{b}$, define the vector field σ_x on $Z_{\mathbf{u}}$ by

$$\sigma_x(p) = \frac{d}{dt}|_{t=0}((\exp tx) \cdot p), \qquad p \in Z_{\mathbf{u}}, \tag{21}$$

where $\cdot$ denotes the left action of $B \subset P_{s_1}$ on $Z_{\mathbf{u}}$ given in (10). Using some facts on root strings of the root system of $\mathfrak{g}$ reviewed in §4.1, for any $\beta \in \Delta_+$ and $e_\beta \in \mathfrak{g}_\beta$, we give in §4.2 an explicit formula for σ_{e_β} in the coordinates $(z_1, z_2, \ldots, z_n)$ one each affine chart $\mathcal{O}^\gamma$ of $Z_{\mathbf{u}}$. The formula for σ_{e_β}, given in Theorem 4.10, is expressed explicitly in terms of the root strings and the structure constants of $\mathfrak{g}$. As a consequence (see Theorem 4.14), the Poisson structure π_n can also be expressed in each affine coordinate chart $\mathcal{O}^\gamma$ in terms of root strings and the structure constants of $\mathfrak{g}$. We believe that our formula for the vector fields σ_{e_β} is of interest irrespective of the Poisson structure π_n.

4.1. **Some lemmas on root strings.** In §4.1, let

$$\{h_\alpha\}_{\alpha \in \Gamma} \cup \{e_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}\}_{\alpha \in \Delta_+} \tag{22}$$

be any basis of $\mathfrak{g}$ such that $[e_\alpha, e_{-\alpha}] = h_\alpha$ for each $\alpha \in \Delta_+$. One then has the Lie group homomorphism $\theta_\alpha : SL(2, \mathbb{C}) \to G$ for each $\alpha \in \Delta_+$. Let the notation be as in §1.4. For α, $\beta \in \Delta$ such that $\alpha + \beta \in \Delta$, let $N_{\alpha,\beta} \neq 0$ be such that $[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$.

Lemma 4.1. *For $\alpha \in \Delta_+$, one has*

$$u_\alpha(t) u_\alpha(z) \dot{s}_\alpha = u_\alpha(t+z) \dot{s}_\alpha, \qquad t, z \in \mathbb{C}, \tag{23}$$

$$u_\alpha(t) u_{-\alpha}(z) = u_{-\alpha}\left(\frac{z}{1+tz}\right) u_\alpha(t(1+tz)) \alpha^\vee(1+tz), \qquad t, z \in \mathbb{C}, 1+tz \neq 0, \tag{24}$$

$$u_{-\alpha}(t) = u_\alpha\left(\frac{1}{t}\right) \dot{s}_\alpha u_\alpha(t) \alpha^\vee(t), \qquad t \in \mathbb{C}^\times. \tag{25}$$

For $\alpha, \beta \in \Gamma$ and $\alpha \neq \beta$, one has .

$$u_\beta(t) \beta^\vee(t) u_{-\alpha}(z) = u_{-\alpha}\left(t^{\frac{-2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle}} z\right) u_\beta(t) \beta^\vee(t), \qquad t \in \mathbb{C}^\times,\, z \in \mathbb{C}. \tag{26}$$

Proof. Identities (24) and (25) follow from computations in $\mathrm{SL}(2, \mathbb{C})$, and (26) follows from the fact that the two root subgroups corresponding to $-\alpha$ and β commute.

Q.E.D.

Let α and β be two linearly independent roots, $\alpha \in \Delta_+$, and let $\{\beta + j\alpha : -p \leq j \leq q\}$, where p and q are non-negative integers, be the α-string through β. Then the subspace

$$L = \sum_{j=-p}^{q} \mathfrak{g}_{\beta+j\alpha}$$

of $\mathfrak{g}$ becomes an $SL(2,\mathbb{C})$-module via the group homomorphism $\theta_\alpha : SL(2,\mathbb{C}) \to G$ and the adjoint representation of G on $\mathfrak{g}$. On the other hand, let L^{p+q} be the vector space of homogeneous polynomials in (x, y) of degree $p + q$ with the (left) action of $\mathrm{SL}(2,\mathbb{C})$ by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f\right)(x,y) = f\left((x,y)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = f\left(ax + cy,\, bx + dy\right), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}).$$

Let $\{u_0, \ldots, u_{p+q}\}$ be the basis of L^{p+q} given by

$$u_i = \varepsilon_0\varepsilon_1\cdots\varepsilon_{i-1}\binom{p+q}{i}x^i y^{p+q-i}, \qquad 0 \leq i \leq p+q, \tag{27}$$

where for $0 \leq j \leq p + q - 1$, $\varepsilon_j \in \mathbb{C}$ is defined by

$$\varepsilon_j = \frac{j+1}{N_{\alpha,\beta-(p-j)\alpha}}, \tag{28}$$

and it is understood that $\varepsilon_0\varepsilon_1\cdots\varepsilon_{i-1} = 1$ when $i = 0$ in (27).

Lemma 4.2. *With the notation as above, the linear map*

$$\chi:\ L \longrightarrow L^{p+q}:\quad \chi(e_{\beta+j\alpha}) = u_{p+j}, \quad -p \leq j \leq q, \tag{29}$$

is an $\mathrm{SL}(2,\mathbb{C})$*-equivariant isomorphism.*

Proof. The two irreducible representations of $SL(2,\mathbb{C})$ on L and on L^{p+q}, being of the same dimension, must be isomorphic, and by Schur's lemma, there is a unique $SL(2,\mathbb{C})$-equivariant isomorphism $\chi : L \to L^{p+q}$ such that $\chi(e_{\beta-p\alpha}) = u_0$. Straightforward calculations show that χ must be given as in (29). See also [4, Lemma 6.2.2].

Q.E.D.

The following Lemma 4.3 is the key to the proof of Theorem 4.10 in §4.2.

Lemma 4.3. *Let* $\alpha \in \Delta_+$ *and* $\beta \in \Delta$ *be linearly independent, and let* $\{\beta + j\alpha : -p \leq j \leq q\}$ *be the* α*-string through* β*. Then for any* $t \in \mathbb{C}$*, one has*

$$\mathrm{Ad}_{(u_\alpha(t)\dot{s}_\alpha)^{-1}}(e_\beta) = \sum_{j=0}^{q}(-1)^p\frac{\varepsilon_0\varepsilon_1\cdots\varepsilon_{p-1}}{\varepsilon_0\varepsilon_1\cdots\varepsilon_{q-j-1}}\binom{p+j}{j}t^j e_{s_\alpha(\beta)-j\alpha}, \tag{30}$$

$$\mathrm{Ad}_{(u_{-\alpha}(t))^{-1}}(e_\beta) = \sum_{j=0}^{p}(-1)^j\varepsilon_{p-j}\varepsilon_{p-j+1}\cdots\varepsilon_{p-1}\binom{q+j}{j}t^j e_{\beta-j\alpha}. \tag{31}$$

Proof. By Lemma 4.2, one has

$$\begin{aligned}\chi\left(\mathrm{Ad}_{(u_\alpha(t)\dot{s}_\alpha)^{-1}}(e_\beta)\right) &= \begin{pmatrix} 0 & 1 \\ -1 & t\end{pmatrix}\cdot u_p \\ &= \varepsilon_0\varepsilon_1\cdots\varepsilon_{p-1}\binom{p+q}{p}(-y)^p(x+ty)^q \\ &= \varepsilon_0\varepsilon_1\cdots\varepsilon_{p-1}\binom{p+q}{p}(-y)^p\left(\sum_{j=0}^{q}\binom{q}{j}t^jy^jx^{q-j}\right) \\ &= \sum_{j=0}^{q}(-1)^p\frac{\varepsilon_0\varepsilon_1\cdots\varepsilon_{p-1}}{\varepsilon_0\varepsilon_1\cdots\varepsilon_{q-j-1}}\binom{p+j}{j}t^ju_{q-j}.\end{aligned}$$

It follows that

$$\mathrm{Ad}_{(u_\alpha(t)\dot{s}_\alpha)^{-1}}(e_\beta) = \sum_{j=0}^{q}(-1)^p\frac{\varepsilon_0\varepsilon_1\cdots\varepsilon_{p-1}}{\varepsilon_0\varepsilon_1\cdots\varepsilon_{q-j-1}}\binom{p+j}{j}t^je_{\beta+(q-p-j)\alpha}.$$

As (see for example, [15, Proposition 25.1]) $\frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}=p-q$, one has, for any $j\in\mathbb{Z}$,

$$s_\alpha(\beta)-j\alpha=\beta-\frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha-j\alpha=\beta+(q-p-j)\alpha,$$

from which (30) follows. One proves (31) similarly (see also Lemma 6.2.1 in [4]).

Q.E.D.

To unify the two formulas in (30) and (31), for $\alpha\in\Delta_+$, $\kappa\in\{s_\alpha,e\}$, and $t\in\mathbb{C}$, let

(32) $$p_{\kappa,\alpha}(t)=u_{-\kappa(\alpha)}(t)\dot{\kappa}\in P_{s_\alpha},$$

and for $\beta\in\Delta$, $\beta\neq\pm\alpha$, as in Lemma 4.3, let

(33) $$c_{\alpha,\beta}^{\kappa,j}=(-1)^p\frac{\varepsilon_0\varepsilon_1\cdots\varepsilon_{p-1}}{\varepsilon_0\varepsilon_1\cdots\varepsilon_{q-j-1}}\binom{p+j}{j}, \qquad j=0,\ldots,q \text{ and } \kappa=s_\alpha,$$

(34) $$c_{\alpha,\beta}^{\kappa,j}=(-1)^j\varepsilon_{p-j}\varepsilon_{p-j+1}\cdots\varepsilon_{p-1}\binom{q+j}{j}, \qquad j=0,\ldots,p \text{ and } \kappa=e.$$

Lemma 4.3 can now be reformulated as follows.

Lemma 4.4. *Let $\alpha\in\Delta_+$ and $\beta\in\Delta$ be linearly independent. Then for any $\kappa\in\{s_\alpha,e\}$ and $t\in\mathbb{C}$,*

(35) $$\mathrm{Ad}_{(p_{\kappa,\alpha}(t))^{-1}}(e_\beta)=\sum_{\substack{j\geq 0,\\ \kappa(\beta)-j\alpha\in\Delta}}c_{\alpha,\beta}^{\kappa,j}\,t^j\,e_{\kappa(\beta)-j\alpha}.$$

Proof. Let $j\in\mathbb{Z}$ and $j\geq 0$. When $\kappa=e$, $\kappa(\beta)-j\alpha\in\Delta$ if and only if $\beta-j\alpha\in\Delta$, which is the same as $0\leq j\leq p$. When $\kappa=s_\alpha$, $\kappa(\beta)-j\alpha\in\Delta$ if and only if $s_\alpha(\beta+j\alpha)\in\Delta$, which is the same as $\beta+j\alpha\in\Delta$, which, in turn, is the same as $0\leq j\leq q$.

Q.E.D.

Remark 4.5. Recall that a basis $\{h_\alpha\}_{\alpha\in\Gamma}\cup\{e_\alpha\in\mathfrak{g}_\alpha\}_{\alpha\in\Delta}$ of $\mathfrak{g}$ is said to be a Chevalley basis if $[e_\alpha,e_{-\alpha}]=h_\alpha$ for all $\alpha\in\Delta$, and if for all $\alpha,\beta\in\Delta$ such that $\alpha+\beta\in\Delta$, one has $N_{\alpha,\beta}=-N_{-\alpha,-\beta}$. If $\{h_\alpha\}_{\alpha\in\Gamma}\cup\{e_\alpha\in\mathfrak{g}_\alpha\}_{\alpha\in\Delta}$ is a Chevalley basis of $\mathfrak{g}$, by [4, Theorem 4.1.2] and [15, Theorem 25.2], $N_{\alpha,\beta}=\pm(p+1)$ for any roots α and β such that $\alpha+\beta\in\Delta$, where p is the largest non-negative integer such that $\beta-p\alpha\in\Delta$. Thus, for α and β as in Lemma 4.3 and for every $0\leq j\leq p+q-1$, one has $\varepsilon_j=\pm1$, and consequently all the coefficients $c_{\alpha,\beta}^{\kappa,j}$'s appearing in (35) are integers. ◇

4.2. **The vector field σ_{e_β} in coordinates.** Fix again $\mathbf{u} = (s_1, \ldots, s_n) = (s_{\alpha_1}, \ldots, s_{\alpha_n})$ be a sequence of simple reflections, and let $Z_{\mathbf{u}}$ be the corresponding Bott-Samelson variety. Let $\{e_\alpha \in \mathfrak{g}_\alpha : \alpha \in \Gamma\}$ be a set of root vectors for the simple roots, and extend it to a basis $\{h_\alpha\}_{\alpha\in\Gamma} \cup \{e_\alpha \in \mathfrak{g}_\alpha\}_{\alpha\in\Delta}$ of $\mathfrak{g}$ such that $[e_\alpha, e_{-\alpha}] = h_\alpha$ for all $\alpha \in \Delta$. Recall from (21) that for any $x \in \mathfrak{b}$, σ_x is the vector field on $Z_{\mathbf{u}}$ generating the action of B on $Z_{\mathbf{u}}$ in the direction of x. For $\beta \in \Delta_+$, we then have the vector field σ_{e_β} on $Z_{\mathbf{u}}$ given by

$$\sigma_{e_\beta}(p) = \frac{d}{dt}|_{t=0}((\exp te_\beta) \cdot p), \qquad p \in Z_{\mathbf{u}}. \tag{36}$$

On the other hand, the choice $\{e_\alpha : \alpha \in \Gamma\}$ gives rise to coordinates $(z_1, \ldots, z_n)$ on the affine chart $\mathcal{O}^\gamma$ for each $\gamma \in \Upsilon_{\mathbf{u}}$. In this section, for every $\gamma \in \Upsilon_{\mathbf{u}}$, we use the results in §4.1 to compute the vector fields σ_{e_β}, $\beta \in \Delta_+$, in the coordinates $(z_1, \ldots, z_n)$ on $\mathcal{O}^\gamma$ in terms of root strings and structure constants of $\mathfrak{g}$ in the basis $\{h_\alpha\}_{\alpha\in\Gamma} \cup \{e_\alpha \in \mathfrak{g}_\alpha\}_{\alpha\in\Delta}$ of $\mathfrak{g}$.

For $x \in \mathfrak{b}$ and $1 \leq k \leq n$, consider also the vector field $\sigma_x^{(k)}$ on the Bott-Samelson variety $Z_{(s_k,\ldots,s_n)}$ defined by

$$\sigma_x^{(k)}(p) = \frac{d}{dt}|_{t=0}((\exp tx) \cdot p), \qquad p \in Z_{(s_k,\ldots,s_n)}, \tag{37}$$

where again $\cdot$ denotes the left action of B on $Z_{(s_k,\ldots,s_n)}$ (see (10)). Note that $\sigma_x = \sigma_x^{(1)}$ for $x \in \mathfrak{b}$.

Fix $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}}$ and let $(z_1, \ldots, z_n)$ be the coordinates on $\mathcal{O}^\gamma \subset Z_{\mathbf{u}}$. For $1 \leq k \leq n$, we also regard $(z_k, \ldots, z_n)$ as coordinates on the affine chart $\mathcal{O}^{(\gamma_k,\ldots,\gamma_n)}$ of $Z_{(s_k,\ldots,s_n)}$, so for $x \in \mathfrak{b}$ and $k \leq j \leq n$, $\sigma_x^{(k)}(z_j)$ is the action of $\sigma_x^{(k)}$ on z_j as a function on $\mathcal{O}^{(\gamma_k,\ldots,\gamma_n)} \subset Z_{(s_k,\ldots,s_n)}$.

The following Lemma 4.6 gives a recursive formula for σ_{e_β}, regarded as a vector field on $\mathcal{O}^\gamma$.

Lemma 4.6. *Let $\beta \in \Delta_+$ and $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}}$.*

1) $\beta = \alpha_1$ and $\gamma_1 = s_1$. In this case, $\sigma_{e_\beta}(z_1) = 1$ and $\sigma_{e_\beta}(z_k) = 0$ for all $k \geq 2$;

2) $\beta = \alpha_1$ and $\gamma_1 = e$. In this case, $\sigma_{e_\beta}(z_1) = -z_1^2$ and for $k \geq 2$,

$$\sigma_{e_\beta}(z_k) = \sigma_{e_\beta}^{(2)}(z_k) + z_1\sigma_{h_{\alpha_1}}^{(2)}(z_k);$$

3) $\beta \neq \alpha_1$. In this case, $\sigma_{e_\beta}(z_1) = 0$ and for $k \geq 2$,

$$\sigma_{e_\beta}(z_k) = \sum_{\substack{j\geq 0,\\ \gamma_1(\beta)-j\alpha_1\in\Delta_+}} c_{\alpha_1,\beta}^{\gamma_1,j}\, z_1^j\, \sigma_{e_{\gamma_1(\beta)-j\alpha_1}}^{(2)}(z_k).$$

Proof. Cases 1) and 2) follow from (23) and (24) respectively. Case 3) follows from Lemma 4.4 and the fact that, as $\beta \in \Delta_+$ and $\beta \neq \alpha_1$, all the roots in the α_1-string through $\gamma_1(\beta)$ are positive.

Q.E.D.

To combine the cases in Lemma 4.6, we note that when $\beta = \alpha_1$,

$$\{j_1 \geq 0 : \gamma_1(\beta) - j_1\alpha_1 \in \Delta_+\} = \begin{cases} \emptyset, & \text{if } \gamma_1 = s_1, \\ \{0\}, & \text{if } \gamma_1 = e. \end{cases}$$

For $\alpha \in \Gamma$, also set

$$c_{\alpha,\alpha}^{e,0} = 1. \tag{38}$$

We can now reformulate Lemma 4.6 as follows.

Lemma 4.7. *Let $\beta \in \Delta_+$ and $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}}$. Then*

$$\sigma_{e_\beta}(z_1) = \begin{cases} 1, & \text{if } \beta = \alpha_1 \text{ and } \gamma_1 = s_1, \\ -z_1^2, & \text{if } \beta = \alpha_1 \text{ and } \gamma_1 = e, \\ 0, & \text{if } \beta \neq \alpha_1, \end{cases} \tag{39}$$

and for $2 \le k \le n$,

(40)
$$\sigma_{e_\beta}(z_k) = \sum_{\substack{j_1 \ge 0, \\ \gamma_1(\beta) - j_1\alpha_1 \in \Delta_+}} c^{\gamma_1, j_1}_{\alpha_1, \beta}\, z_1^{j_1}\, \sigma^{(2)}_{e_{\beta_{\gamma_1(\beta)-j_1\alpha_1}}}(z_k) + \begin{cases} z_1 \sigma^{(2)}_{h_{\alpha_1}}(z_k), & \text{if } \beta = \alpha_1 \text{ and } \gamma_1 = e, \\ 0, & \text{otherwise.} \end{cases}$$

To obtain a closed formula for the vector field σ_{e_β} on $Z_{\mathbf{u}}$, we introduce more notation. Let $\mathbb{N}$ denote the set of non-negative integers.

Notation 4.8. For $\beta \in \Delta_+$ and $(j_1, \ldots, j_n) \in \mathbb{N}^n$, let $\beta_{(j_1)} = \gamma_1(\beta) - j_1\alpha_1 \in \mathfrak{h}^*$, and for $2 \le k \le n$, let

$$\begin{aligned} \beta_{(j_1,\ldots,j_k)} &= \gamma_k(\beta_{(j_1,\ldots,j_{k-1})}) - j_k\alpha_k \\ &= \gamma_k\gamma_{k-1}\cdots\gamma_2\gamma_1(\beta) - j_1\gamma_k\gamma_{k-1}\cdots\gamma_2(\alpha_1) - \ldots - j_{k-1}\gamma_k(\alpha_{k-1}) - j_k\alpha_k \in \mathfrak{h}^*, \\ J_k &= \left\{(j_1, \ldots, j_{k-1}) \in \mathbb{N}^{k-1} : \beta_{(j_1,\ldots,j_l)} \in \Delta_+, \forall\, 1 \le l \le k-1, \text{ and } \beta_{(j_1,\ldots,j_{k-1})} = \alpha_k\right\}. \end{aligned}$$

For $2 \le k \le n$ and for $(j_1, \ldots, j_{k-1}) \in J_k$, let

$$c^{\gamma}_{j_1,\ldots,j_{k-1}} = c^{\gamma_1,j_1}_{\alpha_1,\beta} \cdots c^{\gamma_{k-1},j_{k-1}}_{\alpha_{k-1},\beta_{(j_1,\ldots,j_{k-2})}} \ne 0. \tag{41}$$

Here is it understood that $\beta_{(j_1,\ldots,j_{k-2})} = \beta$ if $k = 2$. Also note that for $k \ge 2$ and $1 \le i \le k-1$, $c^{\gamma_i,j_i}_{\alpha_i,\beta_{(j_1,\ldots,j_{i-1})}}$ is defined in (33) and (34) when $\beta_{(j_1,\ldots,j_{i-1})} \ne \alpha_i$, and if $\beta_{(j_1,\ldots,j_{i-1})} = \alpha_i$, then $\gamma_i(\beta_{(j_1,\ldots,j_{i-1})}) - j_i\alpha_i \in \Delta_+$ only if $\gamma_i = e$ and $j_i = 0$, and in this case $c^{\gamma_i,j_i}_{\alpha_i,\beta_{(j_1,\ldots,j_{i-1})}} = 1$ as defined in (38).

For each $1 \le k \le n$, introduce two functions $\phi^\gamma_\beta(z_1, \ldots, z_{k-1})$ and $\psi^\gamma_\beta(z_1, \ldots, z_{k-1})$ as follows: for $k = 1$, let

$$\phi^\gamma_\beta(z_1, \ldots, z_{k-1}) = \begin{cases} 1 & \text{if } \beta = \alpha_1, \\ 0 & \text{if } \beta \ne \alpha_1, \end{cases} \quad \text{and} \quad \psi^\gamma_\beta(z_1, \ldots, z_{k-1}) = 0, \tag{42}$$

and for $2 \le k \le n$, let

$$\phi^\gamma_\beta(z_1, \ldots, z_{k-1}) = \sum_{(j_1,\ldots,j_{k-1}) \in J_k} c^\gamma_{j_1,\ldots,j_{k-1}}\, z_1^{j_1} z_2^{j_2} \cdots z_{k-1}^{j_{k-1}}, \tag{43}$$

$$\psi^\gamma_\beta(z_1, \ldots, z_{k-1}) = -\sum_{1 \le i \le k-1,\, \gamma_i = e} \frac{2\langle \gamma^i(\alpha_i), \gamma^k(\alpha_k)\rangle}{\langle \gamma^i(\alpha_i), \gamma^i(\alpha_i)\rangle} z_i \phi^\gamma_\beta(z_1, \ldots z_{i-1}), \tag{44}$$

where recall that $\gamma^i = \gamma_1\gamma_2\cdots\gamma_i$ for $1 \le i \le n$, and the function $\phi^\gamma_\beta(z_1, \ldots, z_{k-1})$ (resp. $\psi^\gamma_\beta(z_1, \ldots, z_{k-1})$) is defined to be 0 if the index set for the summation on the right hand side of (43) (resp. (44)) is empty.

Remark 4.9. Since a root string can have length at most 4, it follows from (43) and (44) that the powers of any coordinate z_i in the polynomials $\phi^\gamma_\beta(z_1, \ldots, z_{k-1})$ and $\psi^\gamma_\beta(z_1, \ldots, z_{k-1})$ can be at most 3 (and 1 when $\mathfrak{g}$ is simply-laced). ◇

The following Theorem 4.10 gives a purely combinatorial formula for the vector field σ_{e_β}.

Theorem 4.10. *Let $\beta \in \Delta_+$ and let $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}}$. The vector field σ_{e_β} acts on the coordinate functions $(z_1, \ldots, z_n)$ on the affine chart $\mathcal{O}^\gamma$ as follows: for $1 \le k \le n$,*

$$\sigma_{e_\beta}(z_k) = \begin{cases} \phi^\gamma_\beta(z_1, \ldots, z_{k-1}) + \psi^\gamma_\beta(z_1, \ldots, z_{k-1})z_k, & \text{if } \gamma_k = s_k, \\ -\phi^\gamma_\beta(z_1, \ldots, z_{k-1})z_k^2 + \psi^\gamma_\beta(z_1, \ldots, z_{k-1})z_k, & \text{if } \gamma_k = e. \end{cases} \tag{45}$$

Proof. When $k = 1$, Theorem 4.10 holds by (42) and by Lemma 4.7. Let $k \ge 2$. Let

$$J'_k = \left\{(j_1, \ldots, j_{k-1}) \in \mathbb{N}^{k-1} : \beta_{(j_1,\ldots,j_l)} \in \Delta_+, \forall\, 1 \le l \le k-1\right\},$$

and define $c^{\gamma}_{j_1,\ldots,j_{k-1}} \in \mathbb{C}^\times$ for $(j_1,\ldots,j_{k-1}) \in J'_k$ as in (41). Then by Lemma 4.7,

$$\sigma_{e_\beta}(z_k) = \sum_{j_1 \in J'_2} c^{\gamma_1, j_1}_{\alpha_1,\beta} z_1^{j_1} \sigma^{(2)}_{e_{\beta(j_1)}}(z_k) + \begin{cases} z_1 \sigma^{(2)}_{h_{\alpha_1}}(z_k), & \text{if } \beta = \alpha_1 \text{ and } \gamma_1 = e, \\ 0, & \text{otherwise.} \end{cases} \tag{46}$$

By repeatedly using (46), one has

$$\begin{aligned}\sigma_{e_\beta}(z_k) = & \sum_{(j_1,\ldots,j_{k-1}) \in J'_k} c^{\gamma}_{j_1,\ldots,j_{k-1}} z_1^{j_1} \cdots z_{k-1}^{j_{k-1}} \sigma^{(k)}_{e_{\beta(j_1,\ldots,j_{k-1})}}(z_k) \\ & + \sum_{1 \le i \le k-1,\, \gamma_i = e} \phi^{\gamma}_{\beta}(z_1,\ldots,z_{i-1}) z_i \sigma^{(i+1)}_{h_{\alpha_i}}(z_k).\end{aligned}$$

Let $z'_k = 1$ if $\gamma_k = s_k$ and $z'_k = -z_k^2$ if $\gamma_k = e$. By Lemma 4.6, for $(j_1,\ldots,j_{k-1}) \in J'_k$, one has $\sigma^{(k)}_{e_{\beta(j_1,\ldots,j_{k-1})}}(z_k) = 0$ unless $\beta_{(j_1,\ldots,j_{k-1})} = \alpha_k$, in which case $\sigma^{(k)}_{e_{\beta(j_1,\ldots,j_{k-1})}}(z_k) = z'_k$. Thus

$$\begin{aligned}\sigma_{e_\beta}(z_k) &= \sum_{(j_1,\ldots,j_{k-1}) \in J_k} c^{\gamma}_{j_1,\ldots,j_{k-1}} z_1^{j_1} \cdots z_{k-1}^{j_{k-1}} z'_k + \sum_{1 \le i \le k-1,\, \gamma_i = e} \phi^{\gamma}_{\beta}(z_1,\ldots,z_{i-1}) z_i \sigma^{(i+1)}_{h_{\alpha_i}}(z_k) \\ &= \phi^{\gamma}_{\beta}(z_1,\ldots,z_{k-1}) z'_k + \sum_{1 \le i \le k-1,\, \gamma_i = e} \phi^{\gamma}_{\beta}(z_1,\ldots,z_{i-1}) z_i \sigma^{(i+1)}_{h_{\alpha_i}}(z_k)\end{aligned}$$

On the other hand, for each $1 \le i \le k-1$ with $\gamma_i = e$,

$$\sigma^{(i+1)}_{h_{\alpha_i}}(z_k) = -\frac{2\langle \alpha_i, \gamma_{i+1} \cdots \gamma_k(\alpha_k)\rangle}{\langle \alpha_i, \alpha_i \rangle} z_k = -\frac{2\langle \gamma^i(\alpha_i), \gamma^k(\alpha_k)\rangle}{\langle \gamma^i(\alpha_i), \gamma^i(\alpha_i)\rangle} z_k.$$

It follows that

$$\sigma_{e_\beta}(z_k) = \phi^{\gamma}_{\beta}(z_1,\ldots,z_{k-1}) z'_k + \psi^{\gamma}_{\beta}(z_1,\ldots,z_{k-1}) z_k.$$

Q.E.D.

Remark 4.11. In the context of Theorem 4.10, for a given $\gamma = (\gamma_1,\ldots,\gamma_n) \in \Upsilon_{\mathbf{u}}$ and $1 \le k \le n$, let $\gamma' = (\gamma_1,\ldots,\gamma_{k-1},\gamma_k s_k,\gamma'_{k+1},\ldots,\gamma'_n) \in \Upsilon_{\mathbf{u}}$, where $\gamma'_j \in \{e, s_j\}$ are arbitrary for $k+1 \le j \le n$, and let $(z'_1,\ldots,z'_n)$ be the coordinates on $\mathcal{O}^{\gamma'}$. Then $z_j = z'_j$ for $1 \le j \le k-1$, and $z'_k = 1/z_k$. By (43) and (44),

$$\phi^{\gamma}_{\beta}(z_1,\ldots,z_{k-1}) = \phi^{\gamma'}_{\beta}(z_1,\ldots,z_{k-1}) \quad \text{and} \quad \psi^{\gamma}_{\beta}(z_1,\ldots,z_{k-1}) = -\psi^{\gamma'}_{\beta}(z_1,\ldots,z_{k-1}).$$

One can thus derive one case of the formula (45) from the other case using the change of coordinates $z'_k = 1/z_k$. ◇

Example 4.12. Let β be a simple root and let $\gamma = (e,e,\ldots,e) \in \Upsilon_{\mathbf{u}}$. Then in the affine chart $\mathcal{O}^{(e,e,\ldots,e)}$ with coordinates $(z_1,\ldots,z_n)$ given in (13), the vector field σ_{e_β} is given by

$$\sigma_{e_\beta}(z_k) = -\frac{2\langle \beta, \alpha_k\rangle}{\langle \beta, \beta\rangle} \left(\sum_{1 \le i \le k-1,\, \alpha_i = \beta} z_i \right) z_k + \begin{cases} 0, & \text{if } \alpha_k \ne \beta, \\ -z_k^2, & \text{if } \alpha_k = \beta, \end{cases} \qquad 1 \le k \le n. \tag{47}$$

Indeed, let $1 \le k \le n$. By Theorem 4.10, one has,

$$\sigma_{e_\beta}(z_k) = -\phi^{\gamma}_{\beta}(z_1,\ldots,z_{k-1}) z_k^2 + \psi^{\gamma}_{\beta}(z_1,\ldots,z_{k-1}) z_k.$$

As β is a simple root, one sees from the definition of ϕ^{γ}_{β} that $\phi^{\gamma}_{\beta}(z_1,\ldots,z_{k-1}) = 1$ if $\alpha_k = \beta$ and $\phi^{\gamma}_{\beta}(z_1,\ldots,z_{k-1}) = 0$ if $\alpha_k \ne \beta$. It follows from the definition of ψ^{γ}_{β} that

$$\psi^{\gamma}_{\beta}(z_1,\ldots,z_{k-1}) = -\frac{2\langle \beta, \alpha_k\rangle}{\langle \beta, \beta\rangle} \left(\sum_{1 \le i \le k-1,\, \alpha_i = \beta} z_i \right).$$

This proves (47). Applying Lemma 3.1 and (47), one sees that in the affine chart $\mathcal{O}^{(s_1,e,\ldots,e)}$, the Poisson structure Π is given by

$$\{z_i, z_k\} = \langle\alpha_i, \alpha_k\rangle z_i z_k, \quad \text{if } 2 \leq i < k \leq n,$$

$$\{z_1, z_k\} = \begin{cases} -\langle\alpha_1, \alpha_k\rangle \left(z_1 - 2\sum_{2\leq i\leq j-1,\, \alpha_i=\alpha_1} z_i\right) z_k, & \text{if } 2 \leq k \leq n \text{ and } \alpha_k \neq \alpha_1, \\ -\langle\alpha_1, \alpha_1\rangle \left(z_1 - 2\sum_{2\leq i\leq j-1,\, \alpha_i=\alpha_1} z_i - z_k\right) z_k, & \text{if } 2 \leq k \leq n \text{ and } \alpha_k = \alpha_1. \end{cases}$$

On the other hand, by Lemma 3.3, in the coordinates $(\xi_1, \ldots, \xi_n)$ on $\mathcal{O}^{(e,e,\ldots,e)}$ given by

$$(\xi_1, \xi_2, \ldots, \xi_n) \longmapsto [u_{-\alpha_1}(\xi_1),\, u_{-\alpha_2}(\xi_2),\, \ldots,\, u_{-\alpha_n}(\xi_n)],$$

the Poisson structure π_n is given by $\{\xi_i, \xi_k\} = \langle\alpha_i, \alpha_k\rangle\xi_i\xi_k$ for all $1 \leq i < k \leq n$. It is easy to see that on the intersection $\mathcal{O}^{(e,e,\ldots,e)} \cap \mathcal{O}^{(s_1,e,\ldots,e)}$, the changes between the coordinates $(\xi_1, \xi_2, \ldots, \xi_n)$ on $\mathcal{O}^{(e,e,\ldots,e)}$ and the coordinates $(z_1, z_2, \ldots, z_n)$ on $\mathcal{O}^{(s_1,e,\ldots,e)}$ are given by $z_1 = 1/\xi_1$, and for $2 \leq k \leq n$,

$$z_k = \left(\sum_{\substack{\alpha_i=\alpha_1 \\ 1\leq i\leq k-1}} \xi_i\right)^{\frac{-2\langle\alpha_1,\alpha_2\rangle}{\langle\alpha_1,\alpha_1\rangle}} \text{ if } \alpha_k \neq \alpha_1, \text{ and } z_k = \frac{\xi_k}{\left(\sum_{\substack{\alpha_i=\alpha_1 \\ 1\leq i\leq k-1}} \xi_i\right)\left(\sum_{\substack{\alpha_i=\alpha_1 \\ 1\leq i\leq k}} \xi_i\right)} \text{ if } \alpha_k = \alpha_1.$$

It is remarkable (see [7] for some details of the calculations) that these changes of coordinates indeed change the quadratic Poisson structure expressed in the coordinates $(z_1, \ldots, z_n)$ to the log-canonical one in the coordinates $(\xi_1, \ldots, \xi_n)$. ◇

4.3. **The Poisson structure π_n in coordinates, II.** Let again $\{e_\alpha \in \mathfrak{g}_\alpha : \alpha \in \Gamma\}$ be a set of root vectors for the simple roots, which gives rise to the coordinates $(z_1, \ldots, z_n)$ on each affine chart $\mathcal{O}^\gamma$ via (13). Recall from Lemma 3.1 that the Poisson structure π_n can be expressed in the coordinates $(z_1, \ldots, z_n)$ on $\mathcal{O}^\gamma$ in terms of the vector fields σ_i, $1 \leq i \leq n-1$ on the Bott-Samelson variety $Z^{(s_{i+1},\ldots,s_n)}$, given in (15). We now apply Theorem 4.10 to the vector fields σ_i.

To this end, extend the set $\{e_\alpha \in \mathfrak{g}_\alpha : \alpha \in \Gamma\}$ to a basis $\{h_\alpha\}_{\alpha\in\Gamma} \cup \{e_\alpha \in \mathfrak{g}_\alpha\}_{\alpha\in\Delta}$ of $\mathfrak{g}$ such that $[e_\alpha, e_{-\alpha}] = h_\alpha$ for all $\alpha \in \Delta$. Fix $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}}$. For $1 \leq i < k \leq n$, define two polynomials in the variables $(z_{i+1}, \ldots, z_{k-1})$ by

$$\phi^\gamma_{i,k}(z_{i+1}, \ldots, z_{k-1}) \stackrel{\text{def}}{=} \phi^{(\gamma_{i+1},\ldots,\gamma_n)}_{\alpha_i}(z_{i+1}, \ldots, z_{k-1}), \tag{48}$$

$$\psi^\gamma_{i,k}(z_{i+1}, \ldots, z_{k-1}) \stackrel{\text{def}}{=} \psi^{(\gamma_{i+1},\ldots,\gamma_n)}_{\alpha_i}(z_{i+1}, \ldots, z_{k-1}) \tag{49}$$

by taking $\beta = \alpha_i$ and replacing $\mathbf{u}$ by $(s_{i+1}, \ldots, s_n)$ and γ by $(\gamma_{i+1}, \ldots, \gamma_n)$ in (43) and (44). Here recall that when $k = i+1$, it is understood that $\mathbb{C}[z_{i+1}, \ldots, z_{k-1}] = \mathbb{C}$. Let $1 \leq i \leq n-1$. By Theorem 4.10, the vector field σ_i is given in the coordinates $(z_{i+1}, \ldots, z_n)$ on the affine chart $\mathcal{O}^{(\gamma_{i+1},\ldots,\gamma_n)}$ of $Z_{(s_{i+1},\ldots,s_n)}$ by

$$\sigma_i(z_k) = \begin{cases} \phi^\gamma_{i,k}(z_{i+1}, \ldots, z_{k-1}) + \psi^\gamma_{i,k}(z_{i+1}, \ldots, z_{k-1})z_k, & \text{if } \gamma_k = s_k, \\ -\phi^\gamma_{i,k}(z_{i+1}, \ldots, z_{k-1})z_k^2 + \psi^\gamma_{i,k}(z_{i+1}, \ldots, z_{k-1})z_k, & \text{if } \gamma_k = e, \end{cases} \quad i < k \leq n. \tag{50}$$

Lemma 4.13. *The polynomials $\phi^\gamma_{i,k}(z_{i+1}, \ldots, z_{k-1})$ and $\psi^\gamma_{i,k}(z_{i+1}, \ldots, z_{k-1})$, where $\gamma \in \Upsilon_{\mathbf{u}}$ and $1 \leq i < k \leq n$, are independent of the extension of $\{e_\alpha : \alpha \in \Gamma\}$ to the basis $\{h_\alpha\}_{\alpha\in\Gamma} \cup \{e_\alpha \in \mathfrak{g}_\alpha\}_{\alpha\in\Delta}$ of $\mathfrak{g}$.*

Proof. The coordinates $(z_1, \ldots, z_n)$ on $\mathcal{O}^\gamma$ and the definition of the vector fields σ_i, $1 \leq i \leq n-1$, on $Z^{(s_{i+1},\ldots,s_n)}$ depend only on the choice of $\{e_\alpha : \alpha \in \Gamma\}$ and not on its extension to the basis $\{h_\alpha\}_{\alpha\in\Gamma} \cup \{e_\alpha \in \mathfrak{g}_\alpha\}_{\alpha\in\Delta}$ of $\mathfrak{g}$.

Q.E.D.

The following Theorem 4.14, which expresses more explicitly the formula for the Poisson structure π_n on $Z_{\mathbf{u}}$ in the affine coordinates given in Lemma 3.1, is a combination of Lemma 3.1 and Theorem 4.10.

Theorem 4.14. *Let $\{e_\alpha : \alpha \in \Gamma\}$ be any choice of a set of root vectors for the simple roots and let $\gamma \in \Upsilon_{\mathbf{u}}$. Then in the coordinates $(z_1, \ldots, z_n)$ on the affine chart $\mathcal{O}^\gamma$ of $Z_{\mathbf{u}}$ determined by $\{e_\alpha : \alpha \in \Gamma\}$, the Poisson structure π_n is given by*

$$\{z_i, z_k\} = \begin{cases} \langle \gamma^i(\alpha_i),\, \gamma^k(\alpha_k)\rangle z_i z_k, & \text{if } \gamma_i = e \\ -\langle \gamma^i(\alpha_i),\, \gamma^k(\alpha_k)\rangle z_i z_k - \langle \alpha_i, \alpha_i\rangle \sigma_i(z_k) & \text{if } \gamma_i = s_i \end{cases}, \quad 1 \leq i < k \leq n, \tag{51}$$

where for $1 \leq i < k \leq n$, $\sigma_i(z_k) \in \mathbb{C}[z_{i+1}, \ldots, z_k]$ is given in (50). In particular, when $\gamma = \mathbf{u}$ is the full subexpression, $\sigma_i(z_k) \in \mathbb{C}[z_{i+1}, \ldots, z_{k-1}]$ for all $1 \leq i < k \leq n$.

5. The polynomial Poisson algebras $(\mathbb{C}[z_1, \ldots, z_n], \pi_\gamma)$

Throughout §5, fix a Bott-Samelson variety $Z_{\mathbf{u}}$, where with $\mathbf{u} = (s_1, \ldots, s_n) = (s_{\alpha_1}, \ldots, s_{\alpha_n})$, with $\alpha_i \in \Gamma$ for $1 \leq i \leq n$,

Definition 5.1. Given a set $\{e_\alpha : \alpha \in \Gamma\}$ of root vectors for the simple roots, for each $\gamma \in \Upsilon_{\mathbf{u}}$, let π_γ denote the Poisson structure on the polynomial algebra $\mathbb{C}[z_1, \ldots, z_n]$ given by (51) in Theorem 4.14.

The coordinates $(z_1, \ldots, z_n)$ on the affine charts $\mathcal{O}^\gamma$ of $Z_{\mathbf{u}}$ depend on the choice of the set $\{e_\alpha : \alpha \in \Gamma\}$ of root vectors for the simple roots. A different choice of such a set gives rise to re-scalings of the coordinates and thus may result in a different Poisson bracket on the polynomial algebra of the coordinate functions. We show in §5.1 that this is not the case. In §5.2, we show that each $(\mathbb{C}[z_1, \ldots, z_n], \pi_\gamma)$, where $\gamma \in \Upsilon_{\mathbf{u}}$, is a Poisson-Ore extension of $\mathbb{C}$ compatible with the T-action given in (14). When $\gamma = \mathbf{u}$ is the full expression of $\mathbf{u}$, we show in §5.3 that the Poisson-Ore extension is nilpotent and symmetric in the sense of [12, Definition 4].

5.1. Re-scaling of coordinates.

Let $\{e_\alpha : \alpha \in \Gamma\}$ and $\{e'_\alpha : \alpha \in \Gamma\}$ be two sets of choices of root vectors for the simple roots, and let $\{e_{-\alpha} \in \mathfrak{g}_{-\alpha} : \alpha \in \Gamma\}$ and $\{e'_{-\alpha} \in \mathfrak{g}_{-\alpha} : \alpha \in \Gamma\}$ be the corresponding root vectors such that $[e_\alpha, e_{-\alpha}] = [e'_\alpha, e'_{-\alpha}] = h_\alpha$ for each $\alpha \in \Gamma$.

For $\alpha \in \Gamma$, let $u_{\pm\alpha}, u'_{\pm\alpha} : \mathbb{C} \to G$ be the one-parameter subgroups of G respectively determined by the $\mathfrak{sl}(2)$-triples $\{e_\alpha, e_{-\alpha}, h_\alpha\}$ and $\{e'_\alpha, e'_{-\alpha}, h_\alpha\}$ (see §1.4), and let

$$\dot{s}_\alpha = u_\alpha(-1) u_{-\alpha}(1)\, u_\alpha(-1) \in N_G(T) \quad \text{and} \quad \dot{s}'_\alpha = u'_\alpha(-1) u'_{-\alpha}(1)\, u'_\alpha(-1) \in N_G(T).$$

For $z \in \mathbb{C}$, and $\kappa \in \{e, s_\alpha\}$, let

$$p_{\kappa,\alpha}(z) = u_{-\kappa(\alpha)}(z)\dot{\kappa} \in P_{s_\alpha} \quad \text{and} \quad p'_{\kappa,\alpha}(z) = u'_{-\kappa(\alpha)}(z)\dot{\kappa}' \in P_{s_\alpha},$$

where recall that $\dot{e} = \dot{e}' = e \in G$. For each $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}}$, one then has two sets of coordinates $(z_1, \ldots, z_n)$ and $(z'_1, \ldots, z'_n)$ on $\mathcal{O}^\gamma$, respectively by

$$\mathbb{C}^n \ni (z_1, \ldots, z_n) \longmapsto [p_{\gamma_1,\alpha_1}(z_1),\, \ldots,\, p_{\gamma_n,\alpha_n}(z_n)], \tag{52}$$

$$\mathbb{C}^n \ni (z'_1, \ldots, z'_n) \longmapsto [p'_{\gamma_1,\alpha_1}(z'_1),\, \ldots,\, p'_{\gamma_n,\alpha_n}(z'_n)]. \tag{53}$$

The main result of §5.1 is the following Proposition 5.2.

Proposition 5.2. *Let $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}}$ and let the two sets of coordinates $(z_1, \ldots, z_n)$ and $(z'_1, \ldots, z'_n)$ on $\mathcal{O}^\gamma$ be given as in* (52) *and* (53)*. For $1 \leq i < k \leq n$, let $\{z_i, z_k\} = f_{i,k}(z_1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n]$. Then*

$$\{z'_i,\, z'_k\} = f_{i,k}(z'_1,\, \ldots,\, z'_n), \quad 1 \leq i < k \leq n.$$

Remark 5.3. It is easy to see that the two sets of coordinates are related by re-scalings, i.e., there exist $\delta_1, \ldots, \delta_n \in \mathbb{C}^\times$ such that $z_i' = \delta_i z_i$ for each $1 \leq i \leq n$. One thus has

$$\{z_i',\, z_k'\} = \delta_i\delta_k\{z_i,\, z_k\} = \delta_i\delta_k f_{i,k}(z_1,\, \ldots,\, z_n) = \delta_i\delta_k f_{i,k}(\delta_1^{-1}z_1',\, \ldots,\, \delta_n^{-1}z_n'), \qquad 1 \leq i < k \leq n.$$

Proposition 5.2 states that the polynomials f_{ik} for $1 \leq i < k \leq n$ satisfy

$$\delta_i\delta_k f_{i,k}(\delta_1^{-1}z_1',\, \ldots,\, \delta_n^{-1}z_n') = f_{i,k}(z_1',\, \ldots,\, z_n').$$

$\diamond$

We first prove two lemmas which show that the re-scalings between the two sets of coordinates are not arbitrary.

Lemma 5.4. *For $\alpha \in \Gamma$, $\kappa \in \{e, s_\alpha\}$, and $z \in \mathbb{C}$, one has*

$$p'_{\kappa,\alpha}(z) = \begin{cases} p_{\kappa,\alpha}(\lambda_\alpha z)\alpha^\vee(1/\lambda_\alpha), & \kappa = s_\alpha, \\ p_{\kappa,\alpha}(z/\lambda_\alpha), & \kappa = e. \end{cases} \tag{54}$$

Proof. Let $\theta_\alpha, \theta'_\alpha : SL(2,\mathbb{C}) \to G$ be the Lie group homomorphisms respectively determined by the $\mathfrak{sl}(2)$-triples $\{e_\alpha, e_{-\alpha}, h_\alpha\}$ and $\{e'_\alpha, e'_{-\alpha}, h_\alpha\}$ (see §1.4). Then

$$\theta'_\alpha = \mathrm{Ad}_{\alpha^\vee(\sqrt{\lambda_\alpha})} \circ \theta_\alpha,$$

where $\mathrm{Ad}_{\alpha^\vee(\sqrt{\lambda_\alpha})} : G \to G$ denotes conjugation by $\alpha^\vee(\sqrt{\lambda_\alpha}) \in T$. It follows that

$$\dot{s}'_\alpha = \mathrm{Ad}_{\alpha^\vee(\sqrt{\lambda_\alpha})}(\dot{s}_\alpha) = \dot{s}_\alpha\, \alpha^\vee(1/\lambda_\alpha), \tag{55}$$

and thus

$$p'_{\kappa,\alpha}(z) = \mathrm{Ad}_{\alpha^\vee(\sqrt{\lambda_\alpha})}(p_{\kappa,\alpha}(z)) = \begin{cases} p_{\kappa,\alpha}(\lambda_\alpha z)\alpha^\vee(1/\lambda_\alpha), & \kappa = s_\alpha, \\ p_{\kappa,\alpha}(z/\lambda_\alpha), & \kappa = e. \end{cases}$$

Q.E.D.

For $\alpha \in \Gamma$, let $e'_\alpha = \lambda_\alpha e_\alpha$, and choose either one of the two square roots of λ_α in $\mathbb{C}^\times$ and denote it by $\sqrt{\lambda_\alpha}$. Note that $e'_{-\alpha} = \lambda_\alpha^{-1} e_{-\alpha}$ for each $\alpha \in \Gamma$. Choose any $t \in T$ such that

$$t^\alpha = \lambda_\alpha, \qquad \forall\, \alpha \in \Gamma. \tag{56}$$

Such an element indeed exists, as it can be taken to be any of the preimages in $T \subset G$ of the unique such element in the maximal torus $T/Z(G)$ of $G_{\mathrm{ad}} \stackrel{\mathrm{def}}{=} G/Z(G)$, where $Z(G)$ is the center of G. Recall from (10) that $\cdot$ denotes the left action of B on $Z_{\mathbf{u}}$.

Lemma 5.5. *For any $t \in T$ satisfying (56) and for any $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Upsilon_{\mathbf{u}}$, one has*

$$t \cdot [p_{\gamma_1,\alpha_1}(z_1),\, \ldots,\, p_{\gamma_n,\alpha_n}(z_n)] = [p'_{\gamma_1,\alpha_1}(z_1),\, \ldots,\, p'_{\gamma_n,\alpha_n}(z_n)], \qquad (z_1, \ldots, z_n) \in \mathbb{C}^n. \tag{57}$$

Proof. We prove Lemma 5.5 by induction on n. When $n = 1$, $t^{-\gamma_1(\alpha_1)} = t^{\alpha_1} = \lambda_{\alpha_1}$ if $\gamma_1 = s_1$ and $t^{-\gamma_1(\alpha_1)} = t^{-\alpha_1} = 1/\lambda_{\alpha_1}$ if $\gamma_1 = e$, so by Lemma 5.4,

$$t \cdot [p_{\gamma_1,\alpha_1}(z_1)] = [p_{\gamma_1,\alpha_1}(t^{-\gamma_1(\alpha_1)}z)] = [p'_{\gamma_1,\alpha_1}(z)].$$

Let $n \geq 2$ and assume that Lemma 5.5 holds for $n-1$. Then

$$t\cdot[p_{\gamma_1,\alpha_1}(z_1),\, \ldots,\, p_{\gamma_n,\alpha_n}(z_n)] = [p_{\gamma_1,\alpha_1}(t^{-\gamma_1(\alpha_1)}z_1),\, \gamma_1(t)p_{\gamma_2,\alpha_2}(z_2),\, p_{\gamma_3,\alpha_3}(z_3),\, \ldots,\, p_{\gamma_n,\alpha_n}(z_n)].$$

If $\gamma_1 = e$, then $p_{\gamma_1,\alpha_1}(t^{-\gamma_1(\alpha_1)}z_1) = p_{\gamma_1,\alpha_1}(z_1/\lambda_{\alpha_1}) = p'_{\gamma_1,\alpha_1}(z_1)$, so (57) holds by the induction assumption. Assume that $\gamma_1 = s_1$. Then by Lemma 5.4,

$$t\cdot[p_{\gamma_1,\alpha_1}(z_1),\, \ldots,\, p_{\gamma_n,\alpha_n}(z_n)] = [p'_{\gamma_1,\alpha_1}(z_1),\, \alpha_1^\vee(\lambda_{\alpha_1})s_1(t)p_{\gamma_2,\alpha_2}(z_2),\, p_{\gamma_3,\alpha_3}(z_3),\, \ldots,\, p_{\gamma_n,\alpha_n}(z_n)].$$

Consider now the element $\alpha_1^\vee(\lambda_{\alpha_1})s_1(t) \in T$. For every $\alpha \in \Gamma$, one has

$$(\alpha_1^\vee(\lambda_{\alpha_1})s_1(t))^\alpha = \lambda_{\alpha_1}^{\frac{2\langle\alpha,\alpha_1\rangle}{\langle\alpha_1,\alpha_1\rangle}} t^{s_1(\alpha)} = t^{\frac{2\langle\alpha,\alpha_1\rangle}{\langle\alpha_1,\alpha_1\rangle}\alpha_1 + s_1(\alpha)} = t^\alpha = \lambda_\alpha.$$

By the induction assumption, one then has

$$\alpha_1^\vee(\lambda_{\alpha_1})s_1(t)[p_{\gamma_2,\alpha_2}(z_2),\ p_{\gamma_3,\alpha_3}(z_3),\ \ldots,\ p_{\gamma_n,\alpha_n}(z_n)] = [p'_{\gamma_2,\alpha_2}(z_2),\ \ldots,\ p'_{\gamma_n,\alpha_n}(z_n)] \in Z_{(s_1,\ldots,s_n)},$$

and hence (57) holds.

Q.E.D.

Proof of Proposition 5.2: Let t be any element in T satisfying (56). By setting

$$[p_{\gamma_1,\alpha_1}(z_1),\ \ldots,\ p_{\gamma_n,\alpha_n}(z_n)] = [p'_{\gamma_1,\alpha_1}(z'_1),\ \ldots,\ p'_{\gamma_n,\alpha_n}(z'_n)] \in \mathcal{O}^\gamma,$$

and by Lemma 5.5, one has

$$[p_{\gamma_1,\alpha_1}(z'_1),\ \ldots,\ p_{\gamma_n,\alpha_n}(z'_n)] = t^{-1}\cdot[p_{\gamma_1,\alpha_1}(z_1),\ \ldots,\ p_{\gamma_n,\alpha_n}(z_n)].$$

It follows from (14) that

$$z'_i = (t^{-1})^* z_i = t^{\gamma^i(\alpha_i)} z_i, \qquad 1 \le i \le n,$$

where $(t^{-1})^* : \mathrm{Reg}(\mathcal{O}^\gamma) \to \mathrm{Reg}(\mathcal{O}^\gamma)$ is given by $((t^{-1})^* f)(q) = f(t^{-1}\cdot q)$ for $f \in \mathrm{Reg}(\mathcal{O}^\gamma)$ and $q \in \mathcal{O}^\gamma$, and $\mathrm{Reg}(\mathcal{O}^\gamma)$ is the algebra of regular functions on $\mathcal{O}^\gamma$. As the action of T on $(Z_{\mathbf{u}}, \pi_n)$ is by Poisson isomorphisms (see §2.2), one has, for any $1 \le i, k \le n$,

$$\{z'_i,\ z'_k\} = \{(t^{-1})^* z_i,\ (t^{-1})^* z_k\} = (t^{-1})^*\{z_i,\ z_k\} = ((t^{-1})^* f_{i,k})(z_1, \ldots, z_n) = f_{i,k}(z'_1, \ldots, z'_n).$$

This finishes the proof of Proposition 5.2.

5.2. **The Poisson algebra** $(\mathbb{C}[z_1, \ldots, z_n], \pi_\gamma)$ **as a** T**-Poisson-Ore extension of** $\mathbb{C}$**.** Recall [11, 18, 21] that a Poisson polynomial algebra $A = (\mathbb{C}[z_1, \ldots, z_n], \{\ ,\ \})$ is said to be a *Poisson-Ore extension* of $\mathbb{C}$ if the Poisson bracket $\{\ ,\ \}$ satisfies

$$\{z_i, \mathbb{C}[z_{i+1}, \ldots, z_n]\} \subset z_i\mathbb{C}[z_{i+1}, \ldots, z_n] + \mathbb{C}[z_{i+1}, \ldots, z_n], \qquad 1 \le i \le n-1.$$

In such a case, define

$$\{z_i, f\} = z_i a_i(f) + b_i(f), \qquad 1 \le i \le n-1, \quad f \in \mathbb{C}[z_{i+1}, \ldots, z_n]. \tag{58}$$

Then [21] for each $1 \le i \le n-1$, a_i is a Poisson derivation, and b_i an a_i-Poisson derivation, of the Poisson subalgebra $\mathbb{C}[z_{i+1}, \ldots, z_n]$ of the Poisson algebra A, i.e.,

$$a_i\{f,\ g\} = \{a_i(f),\ g\} + \{f,\ a_i(g)\}, \tag{59}$$

$$b_i\{f,\ g\} = \{b_i(f),\ g\} + \{f,\ b_i(g)\} + a_i(f)b_i(g) - b_i(f)a_i(g) \tag{60}$$

for $f, g \in \mathbb{C}[z_{i+1}, \ldots, z_n]$. In this case, the Poisson algebra A is also denoted as

$$A = \mathbb{C}[z_n]\,[z_{n-1};\ a_{n-1},\ b_{n-1}] \cdots [z_2;\ a_2,\ b_2]\,[z_1;\ a_1,\ b_1]. \tag{61}$$

A Poisson-Ore extension of $\mathbb{C}$ as in (61) is said to be *nilpotent* [12, Definition 4] if b_i is a locally nilpotent derivation of $\mathbb{C}[z_{i+1}, \ldots, z_n]$ for each $1 \le i \le n-1$.

The following Definition 5.6 follows [12, Definition 4] but emphasizes on the torus actions.

Definition 5.6. Let $A = (\mathbb{C}[z_1, \ldots, z_n], \{\ ,\ \})$ be a polynomial Poisson algebra and $\mathbb{T}$ a complex algebraic torus with Lie algebra $\mathfrak{t}$ acting on A rationally [11] by Poisson algebra automorphisms. A is said to be a $\mathbb{T}$-*Poisson-Ore extension of* $\mathbb{C}$ (with respect to the given T-action) if each z_i, $1 \le i \le n$, is a weight vector for the T-action with weight $\lambda_i \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^\times)$, and if

$$A = \mathbb{C}[z_n]\,[z_{n-1};\ a_{n-1},\ b_{n-1}] \cdots [z_2;\ a_2,\ b_2]\,[z_1;\ a_1,\ b_1]$$

is a Poisson-Ore extension of $\mathbb{C}$ such that there exist $h_1, \ldots, h_{n-1} \in \mathfrak{t}$ satisfying $\lambda_i(h_i) \neq 0$ and $a_i = h_i|_{\mathbb{C}[z_{i+1},\ldots,z_n]}$ for each $1 \le i \le n-1$. Such a $\mathbb{T}$-Poisson-Ore extension of $\mathbb{C}$ is said to be *symmetric* if $b_i(z_k) \in \mathbb{C}[z_{i+1}, \ldots, z_{k-1}]$ for $1 \leq i < k \leq n$, and if, in addition to $h_1, \ldots, h_{n-1} \in \mathfrak{t}$ as above, there exists $h_n \in \mathfrak{t}$ such that $\lambda_n(h_n) \neq 0$ and

$$\lambda_i(h_k) = \lambda_k(h_i), \qquad 1 \leq i,\, k \leq n. \tag{62}$$

Remark 5.7. For a $\mathbb{T}$-Poisson-Ore extension of $\mathbb{C}$ as in Definition 5.6, one has

$$\{z_i,\, z_k\} = a_i(z_k) + b_i(z_k) = \lambda_k(h_i) z_i z_k + b_i(z_k) \in \lambda_k(h_i) z_i z_k + \mathbb{C}[z_{i+1}, \ldots, z_n], \qquad 1 \leq i < k \leq n,$$

a property referred to as *semi-quadratic* in [12, Definition 4]. ⋄

Remark 5.8. Let A be a $\mathbb{T}$-Poisson-Ore extension of $\mathbb{C}$ as in Definition 5.6. Then

$$[h|_{\mathbb{C}[z_{i+1}, \ldots, z_n]},\, b_i] = \lambda_i(h) b_i, \qquad 1 \leq i \leq n-1, \;\; h \in \mathfrak{t}, \tag{63}$$

where the left hand side is the commutator bracket between the two derivations $h|_{\mathbb{C}[z_{i+1}, \ldots, z_n]}$ and b_i of $\mathbb{C}[z_{i+1}, \ldots, z_n]$. In fact, (63) is equivalent to $[h|_{\mathbb{C}[z_{i+1}, \ldots, z_n]},\, b_i](z_k) = \lambda_i(h) b_i(z_k)$ for all $1 \leq i < k \leq n$ and $h \in \mathfrak{t}$, which, by the fact the z_j is a $\mathbb{T}$-weight vector with weight λ_j for each $1 \leq j \leq n$, is in turn equivalent to $h(\{z_i, z_k\}) = \{h(z_i), z_k\} + \{z_i, h(z_k)\}$ for all $h \in \mathfrak{t}$ and $1 \leq i < k \leq n$, which is equivalent to $\mathbb{T}$ acting on A by Poisson automorphisms. In particular,

$$[a_i,\, b_i] = \lambda_i(h_i) b_i, \qquad 1 \leq i \leq n-1.$$

Let $1 \leq i \leq n-1$ and consider the 2-dimensional Lie bialgebra $\mathfrak{b}_2 = \mathbb{C}x + \mathbb{C}y$ with Lie bracket $[x, y] = -\lambda_i(h_i) y$ and Lie co-bracket $\delta : \mathfrak{b}_2 \to \wedge^2 \mathfrak{b}_2$ given by $\delta(x) = 0$ and $\delta(y) = x \wedge y$. Consider the Poisson subalgebra $A_{i+1} = \mathbb{C}[z_{i+1}, \ldots, z_n]$ of A and let $\mathrm{Der}_{\mathbb{C}}(A_{i+1})$ be the Lie algebra of derivations (for the commutative algebra structure) of A_{i+1}. Define the Lie algebra anti-homomorphism $\sigma : \mathfrak{b}_2 \to \mathrm{Der}_{\mathbb{C}}(A_{i+1})$ by $\sigma(x) = a_i$ and $\sigma(y) = -b_i$. Then (59) and (60) are equivalent to σ being a *left Poisson action of the Lie bialgebra* $(\mathfrak{b}_2, \delta)$ on the Poisson algebra A_{i+1} (see [19, §2]). Let x^* and y^* be the dual basis of $\mathfrak{b}_2^*$ which is a Lie bialgebra with Lie bracket $[x^*, y^*] = y^*$ and Lie co-bracket $x^* \mapsto 0$ and $y^* \mapsto -\lambda_i(h_i) x^* \wedge y^*$. Let $\rho : \mathfrak{b}_2^* \to \mathrm{Der}_{\mathbb{C}}\mathbb{C}[z_i]$ be the Lie algebra homomorphism given by $\rho(x^*) = -z_i \partial/\partial z_i$ and $\rho(y^*) = \partial/\partial z_i$. Then ρ is a *right Poisson action of the Lie bialgebra* $\mathfrak{b}_2^*$ on $\mathbb{C}[z_i]$ with the trivial Poisson bracket. The Poisson-Ore extension $A_i := \mathbb{C}[z_i, z_{i+1}, \ldots, z_n]$ of A_{i+1} with the Poisson bracket given in (58) can now be interpreted as the *mixed product Poisson structure* on $A_i = \mathbb{C}[z_i] \otimes A_{i+1}$ defined by the pair (ρ, σ) of Poisson actions of Lie bialgebras introduced in [19]. ⋄

Remark 5.9. A symmetric T-Poisson-Ore extension of $\mathbb{C}$ is automatically nilpotent. Indeed, let $1 \leq i \leq n-1$ and let the notation be as in Definition 5.6. To show that b_i is locally nilpotent as a derivation of $\mathbb{C}[z_{i+1}, \ldots, z_n]$, observe first that for integers $m, N \geq 1$ and $f_1, f_2, \ldots, f_m \in \mathbb{C}[z_{i+1}, \ldots, z_n]$, $b_i^N(f_1 f_2 \cdots f_m)$ is a linear combination of terms of the form $b_i^{N_1}(f_1) b_i^{N_2}(f_1) \cdots b_i^{N_m}(f_m)$ with $N_1 + N_2 + \cdots + N_m = N$. Thus b_i is locally nilpotent if for each $i < k \leq n$, $b_i^{N_k}(z_k) = 0$ for some integer $N_k \geq 1$. As $b_i(z_{i+1}) \in \mathbb{C}$, one has $b_i^2(z_{i+1}) = 0$. Assume that there exist $N_j \geq 1$ such that $b_i^{N_j}(z_j) = 0$ for $i+1 \leq j \leq k-1$. As $b_i(z_k) \in \mathbb{C}[z_{i+1}, \ldots, z_{k-1}]$, the above observation shows that there is an integer $N_k \geq 1$ such that $b_i^{N_k}(b_k) = 0$. Induction on k now shows that b_i is locally nilpotent. Observe also that if A is a symmetric T-Poisson-Ore extension of $\mathbb{C}$, then for $1 \leq i < k \leq n$,

$$\{z_i,\, z_k\} = \lambda_k(h_i) z_i z_k + b_i(z_k) \in \lambda_k(h_i) z_i z_k + \mathbb{C}[z_{i+1}, \ldots, z_{k-1}] \subset \mathbb{C}[z_i, \ldots, z_k]. \tag{64}$$

Consequently, $\mathbb{C}[z_i, \ldots, z_k]$ is a Poisson subalgebra of A for all $1 \leq i < k \leq n$. ⋄

Lemma 5.10. *Suppose that $A = (\mathbb{C}[z_1, \ldots, z_n], \{\, ,\, \})$ is a symmetric $\mathbb{T}$-Poisson-Ore extension of $\mathbb{C}$. Then, with respect to the same $\mathbb{T}$-action, A is a $\mathbb{T}$-Poisson-Ore extension of $\mathbb{C}$ in the reversed order of the variables. More precisely, with the notation as in Definition 5.6, for each $2 \leq k \leq n$, $\mathbb{C}[z_1, \ldots, z_{k-1}]$ is a Poisson subalgebra of A, and*

$$\{f,\, z_k\} = a_k'(f) z_k + b_k'(f), \qquad f \in \mathbb{C}[z_1, \ldots, z_{k-1}], \tag{65}$$

where $a'_k = h_k|_{\mathbb{C}[z_1,\ldots,z_{k-1}]}$ *as a derivation of* $\mathbb{C}[z_1,\ldots,z_{k-1}]$ *and* b'_k *is the unique derivation of* $\mathbb{C}[z_1,\ldots,z_{k-1}]$ *such that* $b'_k(z_i) = b_i(z_k) \in \mathbb{C}[z_{i+1},\ldots,z_{k-1}]$ *for* $1 \leq i \leq k-1$. *Moreover, for any* $h \in \mathfrak{t}$, $[h|_{\mathbb{C}[z_1,\ldots,z_{k-1}]}, b'_k] = \lambda_k(h)b'_k$ *as derivations of* $\mathbb{C}[z_1,\ldots,z_{k-1}]$.

Proof. It follows from (64) that $\mathbb{C}[z_1,\ldots,z_{k-1}]$ is a Poisson subalgebra of A for every $2 \leq k \leq n$. The assumption that $\lambda_i(h_k) = \lambda_k(h_i)$ for all $1 \leq i, k \leq n$ and the definition of the b'_k's imply that (65) holds for $f = z_i$ for each $i < k$, so it holds for all $f \in \mathbb{C}[z_1,\ldots,z_{k-1}]$. Let $h \in \mathfrak{t}$ and $2 \leq k \leq n$. Then for each $1 \leq i \leq k-1$, using (63), one has $h(b_i(z_k)) - b_i(h(z_k)) = \lambda_i(h)b_i(z_k)$, from which one has $h(b_i(z_k)) - \lambda_i(h)b_i(z_k) = b_i(h(z_k)) = \lambda_k(h)b_i(z_k)$, and it follows that

$$h(b'_k(z_i)) - b'_k(h(z_i)) = h(b_i(z_k)) - \lambda_i(h)b_i(z_k) = \lambda_k(h)b_i(z_k) = \lambda_k(h)b'_k(z_i).$$

This proves that $[h|_{\mathbb{C}[z_1,\ldots,z_{k-1}]}, b'_k] = \lambda_k(h)b'_k$ as derivations of $\mathbb{C}[z_1,\ldots,z_{k-1}]$.

Q.E.D.

Notation 5.11. In the context of Lemma 5.10, we also write

$$A = \mathbb{C}[z_1]\,[z_2;\ a'_2,\ b'_2] \cdots [z_{n-1};\ a'_{n-1},\ b'_{n-1}]\,[z_n;\ a'_n,\ b'_n]. \tag{66}$$

Returning to the Bott-Samelson variety $(Z_{\mathbf{u}}, \pi_n)$, where $\mathbf{u} = (s_1,\ldots,s_n) = (s_{\alpha_1},\ldots,s_{\alpha_n})$, choose again any set $\{e_\alpha : \alpha \in \Gamma\}$ of root vectors for the simple roots, so that one has coordinates $(z_1,\ldots,z_n)$ on $\mathcal{O}^\gamma$ for each $\gamma \in \Upsilon_{\mathbf{u}}$. Fix $\gamma \in \mathcal{O}^\gamma$ and consider the Poisson polynomial algebra $(\mathbb{C}[z_1,\ldots,z_n], \pi_\gamma)$. Recall again that the maximal torus T acts on $\mathcal{O}^\gamma$ by (14), which gives rise to a rational action of T on $(\mathbb{C}[z_1,\ldots,z_n], \pi_\gamma)$ by Poisson automorphisms. More precisely,

$$t \cdot z_i = t^{-\gamma^i(\alpha_i)} z_i, \qquad 1 \leq i \leq n. \tag{67}$$

For $h \in \mathfrak{h} = \mathrm{Lie}(T)$, denote by ∂_h the Poisson derivation of $(\mathbb{C}[z_1,\ldots,z_n], \pi_\gamma)$ generating the T-action in the direction of h, i.e,

$$\partial_h(z_i) = -\gamma^i(\alpha_i)(h)z_i, \qquad 1 \leq i \leq n,\ h \in \mathfrak{h}. \tag{68}$$

Note that both the T-action and the derivations ∂_x on $\mathbb{C}[z_1,\ldots,z_n]$ depend on γ, but for notational simplicity we do not include the dependence on γ in the notation. For $1 \leq i \leq n-1$, recall also the vector field σ_i on the Bott-Samelson variety $Z_{(s_{i+1},\ldots,s_n)}$ defined in (15), and recall that the induced derivation on $\mathbb{C}[z_{i+1},\ldots,z_n]$, identified with the algebra of regular functions on $\mathcal{O}^{(s_{i+1},\ldots,s_n)} \subset Z_{(s_{i+1},\ldots,s_n)}$ is also denoted by σ_i.

Proposition 5.12. *For each* $\gamma \in \Upsilon_{\mathbf{u}}$, $(\mathbb{C}[z_1,\ldots,z_n], \pi_\gamma)$ *is a* T*-Poisson-Ore extension of* $\mathbb{C}$ *with respect to the* T*-action on* $\mathbb{C}[z_1,\ldots,z_n]$ *given in* (67). *More explicitly,*

$$(\mathbb{C}[z_1,\ldots,z_n], \pi_\gamma) = \mathbb{C}[z_n]\,[z_{n-1};\ a_{n-1},\ b_{n-1}] \cdots [z_2;\ a_2,\ b_2]\,[z_1;\ a_1,\ b_1], \tag{69}$$

where for $1 \leq i \leq n-1$,

$$a_i = -\frac{\langle \alpha_i, \alpha_i \rangle}{2} \partial_{\gamma^{i-1}(h_{\alpha_i})}|_{\mathbb{C}[z_{i+1},\ldots,z_n]}, \qquad b_i = \begin{cases} 0, & \text{if } \gamma_i = e, \\ -\langle \alpha_i, \alpha_i \rangle \sigma_i, & \text{if } \gamma_i = s_i. \end{cases} \tag{70}$$

When $\gamma = \mathbf{u}$, *the extension is symmetric. More explicitly, for* $\gamma = \mathbf{u}$, *one also has*

$$A = \mathbb{C}[z_1]\,[z_2;\ a'_2,\ b'_2] \cdots [z_{n-1};\ a'_{n-1},\ b'_{n-1}]\,[z_n;\ a'_n,\ b'_n], \tag{71}$$

where for $2 \leq k \leq n$, $a'_k = -\frac{\langle \alpha_k, \alpha_k \rangle}{2} \partial_{\gamma^{k-1}(h_{\alpha_k})}|_{\mathbb{C}[z_1,\ldots,z_{k-1}]}$, *and* b'_k *is the unique derivation of* $\mathbb{C}[z_1,\ldots,z_{k-1}]$ *such that* $b'_k(z_i) = -\langle \alpha_i, \alpha_i \rangle \sigma_i(z_k)$ *for* $1 \leq i \leq k-1$.

Proof. Let $\gamma = (\gamma_1,\ldots,\gamma_n) \in \Upsilon_{\mathbf{u}}$ and let $\lambda_i = -\gamma^i(\alpha_i)$ for $1 \leq i \leq n$. By (67), z_i is a weight vector for the T-action on $\mathbb{C}[z_1,\ldots,z_n]$ with weight λ_i. For $1 \leq i \leq n$, define $h_i \in \mathfrak{h} = \mathrm{Lie}(T)$ by

$$h_i = -\frac{\langle \alpha_i, \alpha_i \rangle}{2} \gamma^{i-1}(h_{\alpha_i}) = \begin{cases} -\frac{\langle \alpha_i, \alpha_i \rangle}{2} \gamma^i(h_{\alpha_i}), & \text{if } \gamma_i = e, \\ \frac{\langle \alpha_i, \alpha_i \rangle}{2} \gamma^i(h_{\alpha_i}), & \text{if } \gamma_i = s_i. \end{cases} \tag{72}$$

Then for $1 \leq i < k \leq n$,

$$\partial_{h_i}(z_k) = \lambda_k(h_i)z_k = -\gamma^k(\alpha_k)(h_i)z_k = \langle \gamma^{i-1}(\alpha_i),\, \gamma^k(\alpha_k)\rangle z_k.$$

It now follows from Theorem 4.14 that (69) holds with the a_i's and b_i's given by (70). Moreover, for each $1 \leq i \leq n$, $\lambda_i(h_i) \neq 0$, as

$$\lambda_i(h_i) = \langle \gamma^{i-1}(\alpha_i),\, \gamma^i(\alpha_i)\rangle = \langle \alpha_i,\, \gamma_i(\alpha_i)\rangle = \begin{cases} \langle \alpha_i,\, \alpha_i\rangle, & \gamma_i = e, \\ -\langle \alpha_i,\, \alpha_i\rangle, & \gamma_i = s_i. \end{cases}$$

Thus $(\mathbb{C}[z_1, \ldots, z_n], \pi_\gamma)$ is a T-Poisson-Ore extension of $\mathbb{C}$ with respect to the T-action in (67).

Assume now that $\gamma = \mathbf{u}$ is the full subexpression of $\mathbf{u}$. In this case,

$$h_i = -\frac{\langle \alpha_i,\, \alpha_i\rangle}{2}\gamma^{i-1}(h_{\alpha_i}) = -\frac{\langle \alpha_i,\, \alpha_i\rangle}{2} s_1 s_2 \cdots s_{i-1}(h_{\alpha_i}) \in \mathfrak{h}, \qquad 1 \leq i \leq n.$$

With $\lambda_i = -\gamma^i(\alpha_i) = s_1 s_2 \cdots s_{i-1}(\alpha_i)$, one has, for $1 \leq i, k \leq n$,

$$\lambda_i(h_k) = -\langle \gamma^i(\alpha_i),\, \gamma^k(\alpha_k)\rangle = -\langle s_1 s_2 \cdots s_{i-1}(\alpha_i),\, s_1 s_2 \cdots s_{k-1}(\alpha_k)\rangle = \lambda_k(h_i).$$

By Theorem 4.14, one also has $b_i(z_k) \in \mathbb{C}[z_{i+1}, \ldots, z_{k-1}]$ for $1 \leq i < k \leq n$. This shows that $(\mathbb{C}[z_1, \ldots, z_n], \pi_\gamma)$, for $\gamma = \mathbf{u}$, is a *symmetric* T-Poisson-Ore extension of $\mathbb{C}$ with respect to the T-action given in (67). By Lemma 5.10, (71) holds.

Q.E.D.

Remark 5.13. We already know from Remark 5.8 that for $h \in \mathfrak{t}$ and $1 \leq i \leq n-1$, the two derivations $a_h := \partial_h|_{\mathbb{C}[z_{i+1}, \ldots, z_n]}$ and b_i on $\mathbb{C}[z_{i+1}, \ldots, z_n]$ in Proposition 5.12 satisfy $[a_h, b_i] = \lambda_i(h) b_i$. This can also be checked directly: it clearly holds when $\gamma_i = e$. Assume that $\gamma_i = s_i$. In the notation of (37) and by Lemma 2.2, one has $a_h = \sigma^{(i+1)}_{(\gamma^i)^{-1}(h)}$ and $b_i = -\langle \alpha_i, \alpha_i\rangle \sigma^{(i+1)}_{e_{\alpha_i}}$. Thus

$$[a_h,\, b_i] = -\langle \alpha_i, \alpha_i\rangle \left[\sigma^{(i+1)}_{(\gamma^i)^{-1}(h)},\, \sigma^{(i+1)}_{e_{\alpha_i}}\right] = \langle \alpha_i, \alpha_i\rangle \sigma^{(i+1)}_{[(\gamma^i)^{-1}(h), e_{\alpha_i}]} = \lambda_i(h) b_i.$$

$\diamond$

Remark 5.14. For an arbitrary $\gamma \in \Upsilon_{\mathbf{u}}$, the derivations b_i on $\mathbb{C}[z_{i+1}, \ldots, z_n]$ in Proposition 5.12 are not necessarily locally nilpotent: in Example 3.2 for $\gamma = (s_{\alpha_1}, e, e)$, the derivation b_1 on $\mathbb{C}[z_2, z_3]$ is given by $b_1(z_2) = 0$ and $b_1(z_3) = 2z_3^2$ which is not locally nilpotent. $\diamond$

5.3. **The Poisson structure π_n in $\mathcal{O}^{\mathbf{u}}$.** We now look in more detail at the Poisson polynomial algebra $(\mathbb{C}[z_1, \ldots, z_n], \pi_\gamma)$, where $\gamma = \mathbf{u}$ is the full subexpression of $\mathbf{u}$. In this case, the action of T on $\mathbb{C}[z_1, \ldots, z_n]$ is given by

$$t \cdot z_i = t^{s_1 s_2 \cdots s_{i-1}(\alpha_i)} z_i, \qquad t \in T,\ 1 \leq i \leq n, \tag{73}$$

and the Poisson structure π_γ on $\mathbb{C}[z_1, \ldots, z_n]$ is given by

$$\{z_i, z_k\} = c_{i,k} z_i z_k - \langle \alpha_i, \alpha_i\rangle \sigma_i(z_k) = c_{i,k} z_i z_k + b'_k(z_i), \qquad 1 \leq i < k \leq n, \tag{74}$$

where for $1 \leq i, k \leq n$,

$$c_{i,k} = -\langle \gamma^i(\alpha_i),\, \gamma^k(\alpha_k)\rangle = -\langle s_1 s_2 \cdots s_{i-1}(\alpha_i),\, s_1 s_2 \ldots s_{k-1}(\alpha_k)\rangle, \tag{75}$$

σ_i is the derivation on $\mathbb{C}[z_{i+1}, \ldots, z_{k-1}]$ corresponding to the vector field on the Bott-Samelson variety $Z^{(s_{i+1}, \ldots, s_n)}$ generating the B-action on $Z^{(s_{i+1}, \ldots, s_n)}$ in the direction of e_{α_i} (see (15)), and b'_k is the unique derivation on $\mathbb{C}[z_1, \ldots, z_{k-1}]$ such that $\mathfrak{b}'_k(z_i) = -\langle \alpha_i, \alpha_i\rangle \sigma_i(z_k)$.

In §5.3, we give the geometric meaning of the derivation b'_k on $\mathbb{C}[z_1, \ldots, z_{k-1}]$.

To this end, consider the quotient manifold

$$F'_{-n} = B_-\backslash G \times_{B_-} G \times \cdots \times_{B_-} G$$

of G^n by $(B_-)^n$, where $(B_-)^n$ acts on G^n from the left by

$$(b_1, b_2, \ldots, b_n) \cdot (g_1, g_2, \ldots, g_n) = (b_1 g_1 b_2^{-1},\, b_2 g_2 b_3^{-1}, \ldots, b_n g_n), \qquad b_j \in B_-,\ g_j \in G. \tag{76}$$

Let $\rho_- : G^n \to F'_{-n}$ be the natural projection. Similar to the case of the quotient manifold F_n in (1), the product Poisson structure $\pi_{\rm st}^n$ on G^n projects by ρ_- to a well-defined Poisson structure on F'_{-n}, which will be denoted by π'_{-n}. Let $P_{-s_i} = B_- \cup B_- s_i B_-$ for $1 \leq i \leq n$. As each P_{-s_i} is a Poisson submanifold of $(G, \pi_{\rm st})$, the closed submanifold

$$Z'_{-\mathbf{u}} = B_- \backslash P_{-s_1} \times_{B_-} P_{-s_2} \times \cdots \times_{B_-} P_{-s_n}$$

of F'_{-n} is a Poisson submanifold with respect to π'_{-n}. We will also call $Z'_{-\mathbf{u}}$ a Bott-Samelson variety. Note that for each $1 \leq i \leq n$, one has

$$u_{\alpha_i}(z)\dot{s}_i = \dot{s}_i u_{-\alpha_i}(-z), \qquad z \in \mathbb{C}.$$

Setting $\rho_-(g_1, g_2, \ldots, g_n) = [g_1, g_2, \ldots, g_n]_- \in F'_{-n}$ for $(g_1, g_2, \ldots, g_n) \in G^n$, it follows that one has the open affine chart

$$\mathcal{O}_-^{\prime, \mathbf{u}} := B_- \backslash (B_- s_1 B_-) \times_{B_-} (B_- s_2 B_-) \times \cdots \times_{B_-} (B_- s_n B_-)$$

of $Z'_{-\mathbf{u}}$, with the parametrization by $\mathbb{C}^n$ via

$$\mathbb{C}^n \ni (z_1, z_2, \ldots, z_n) \longmapsto [u_{\alpha_1}(z_1)\dot{s}_{\alpha_1},\ u_{\alpha_2}(z_2)\dot{s}_{\alpha_2},\ \ldots,\ u_{\alpha_n}(z_n)\dot{s}_{\alpha_n}]_- \in \mathcal{O}_-^{\prime, \mathbf{u}}. \tag{77}$$

The restriction of the Poisson structure π'_{-n} to $\mathcal{O}_-^{\prime, \mathbf{u}}$ will also be denoted by π'_{-n}.

Proposition 5.15. *The map $I : (\mathcal{O}^{\mathbf{u}}, \pi_n) \to (\mathcal{O}_-^{\prime, \mathbf{u}},\ \pi'_{-n})$ given by*

$$[u_{\alpha_1}(z_1)\dot{s}_{\alpha_1},\ u_{\alpha_2}(z_2)\dot{s}_{\alpha_2},\ \ldots,\ u_{\alpha_n}(z_n)\dot{s}_{\alpha_n}] \longmapsto [u_{\alpha_1}(z_1)\dot{s}_{\alpha_1},\ u_{\alpha_2}(z_2)\dot{s}_{\alpha_2},\ \ldots,\ u_{\alpha_n}(z_n)\dot{s}_{\alpha_n}]_-,$$

where $(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$, is a Poisson anti-isomorphism.

Proof. Let $\rho : G^n \to F_n$ be the natural projection, so that $\pi_n = \rho(\pi_{\rm st}^n)$. It is proved in [19, §8] that the pair

$$\rho: \ \ (G^n,\, \pi_{\rm st}^n) \longrightarrow (F_n,\, \pi_n) \qquad \text{and} \qquad \rho_-: \ \ (G^n,\, \pi_{\rm st}^n) \longrightarrow \left(F'_{-n},\ \pi'_{-n}\right)$$

of Poisson submersions is a weak Poisson pair (see §A the Appendix), i.e., the map

$$(\rho,\ \rho_-): \ \ (G^n,\, \pi_{\rm st}^n) \longrightarrow (F_n \times F'_{-n},\ \pi_n \times \pi'_{-n}), \ \ (g,\ g') \longmapsto (\rho(g),\ \rho_-(g')), \qquad g, g' \in G^n,$$

is a Poisson map. For $\alpha \in \Gamma$, let Σ_α be the symplectic leaf of $\pi_{\rm st}$ in G through the point $\dot{s}_\alpha \in G$. To describe the two-dimensional symplectic manifold $(\Sigma_\alpha, \pi_{\rm st}|_{\Sigma_\alpha})$, consider the surface

$$\Sigma = \{(p, q, t) \in \mathbb{C}^3 : \ t^2(1 - pq) = 1\}$$

in $\mathbb{C}^3$ and equip Σ with the Poisson structure π given by

$$\{p, q\} = 2(1 - pq), \qquad \{p, t\} = pt, \qquad \{q, t\} = -qt. \tag{78}$$

A calculation in $SL(2, \mathbb{C})$ shows that the embedding

$$J: \ \ \Sigma \longrightarrow SL(2, \mathbb{C}), \quad (p, q, t) \longmapsto \begin{pmatrix} pt & -t \\ t & -qt \end{pmatrix}, \qquad (p, q, t) \in \Sigma,$$

identifies (Σ, π) as the symplectic leaf through $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL(2, \mathbb{C})$ of the Poisson structure $\pi_{SL(2,\mathbb{C})}$ on $SL(2, \mathbb{C})$ in (5). Using the Poisson homomorphism θ_α in (6), one sees [16] that

$$\Sigma_\alpha = \{g_\alpha(p, q, t) : \ (p, q, t) \in \Sigma\},$$

and $\pi_{\rm st}|_{\Sigma_\alpha} = \frac{\langle \alpha, \alpha \rangle}{2}(\theta_\alpha \circ J)(\pi)$, where for $(p, q, t) \in \Sigma$,

$$g_\alpha(p, q, t) = \theta_\alpha \begin{pmatrix} pt & -t \\ t & -qt \end{pmatrix} = u_\alpha(p)\dot{s}_\alpha \check{\alpha}(t) u_\alpha(-q) = u_{-\alpha}(q)\check{\alpha}(t)\dot{s}_\alpha u_{-\alpha}(-p). \tag{79}$$

Consider now the product manifold $\Sigma_{\mathbf{u}} = \Sigma_{\alpha_1} \times \Sigma_{\alpha_2} \times \cdots \times \Sigma_{\alpha_n}$ and denote the restriction of the product Poisson structure $\pi_{\rm st}^n$ to $\Sigma_{\mathbf{u}}$ still by $\pi_{\rm st}^n$. It follows from (79) that

$$\rho(\Sigma_{\mathbf{u}}) = \mathcal{O}^{\mathbf{u}} \qquad \text{and} \qquad \rho_-(\Sigma_{\mathbf{u}}) = \mathcal{O}_-^{\prime, \mathbf{u}},$$

and, denoting again by ρ (resp. ρ_-) the induced map from $\Sigma_{\mathbf{u}}$ to $\mathcal{O}^{\mathbf{u}}$ (resp. to $\mathcal{O}_-^{\prime,\mathbf{u}}$),

$$\rho: \ (\Sigma_{\mathbf{u}},\, \pi_{\rm st}^n) \longrightarrow (\mathcal{O}^{\mathbf{u}},\, \pi_n) \quad \text{and} \quad \rho_-: \ (\Sigma_{\mathbf{u}},\, \pi_{\rm st}^n) \longrightarrow (\mathcal{O}_-^{\prime,\mathbf{u}},\, \pi'_{-n}) \tag{80}$$

are Poisson submersions and form a weak Poisson pair (in fact a symplectic dual pair). Moreover, the submanifold

$$L := \{(u_{\alpha_1}(z_1)\dot{s}_{\alpha_1},\ u_{\alpha_2}(z_2)\dot{s}_{\alpha_2},\ \ldots,\ u_{\alpha_n}(z_n)\dot{s}_{\alpha_n}):\ (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n\}$$

of $\Sigma_{\mathbf{u}}$ is Lagrangian with respect to $\pi_{\rm st}^n$, and it is clear that $\rho|_L : L \to \mathcal{O}^{\mathbf{u}}$ is a diffeomorphism. It now follows from Lemma A.1 in the Appendix that $I = \rho_- \circ (\rho|_L)^{-1} : (\mathcal{O}^{\mathbf{u}},\, \pi_n) \to (\mathcal{O}_-^{\prime,\mathbf{u}},\, \pi'_{-n})$ is a Poisson anti-isomorphism.

Q.E.D.

We now prove a fact similar to that in Lemma 2.2: let (X, π_X) be a Poisson manifold with a right Poisson action by the Poisson Lie group $(B_-, \pi_{\rm st})$, let α be a simple root, and consider the quotient manifold $Z = X \times_{B_-} P_{-s_\alpha}$ (see notation in §2.2) equipped with Poisson structure π_Z which is the projection to Z of the product Poisson structure $\pi_X \times \pi_{\rm st}$ on $X \times P_{-s_\alpha}$. Denote by $[x, p]$ the image of $(x, p) \in X \times P_{-s_\alpha}$ in Z. Fix any $\mathfrak{sl}(2, \mathbb{C})$-triple $\{e_\alpha, e_{-\alpha}, h_\alpha\}$ and consider

$$\phi: \ \ X \times \mathbb{C} \longrightarrow Z_0, \ \ (x, z) \longmapsto [x,\ u_\alpha(z)\dot{s}_\alpha], \qquad x \in X,\, z \in \mathbb{C}.$$

Then ϕ is an embedding, and we regard ϕ as a diffeomorphism from $X \times \mathbb{C}$ to $Z_0 = \phi(X \times \mathbb{C})$. For $\xi \in \mathfrak{b}$, let σ'_ξ be the vector field on X defined by

$$\sigma'_\xi(x) = \frac{d}{dt}|_{t=0}(x \exp(t\xi)), \qquad x \in X.$$

Using the second part of (8), the proof of the following Lemma 5.16 is similar to that of Lemma 2.2 and is omitted.

Lemma 5.16. *With the notation as above, one has*

$$\phi^{-1}(\pi_Z)(x, z) = \pi_X(x) + \frac{\langle \alpha, \alpha \rangle}{2} \frac{d}{dz} \wedge \left(z\sigma'_{h_\alpha}(x) + 2\sigma'_{e_{-\alpha}}(x) \right).$$

Returning now to the Bott-Samelson variety $Z'_{-\mathbf{u}}$ for $\mathbf{u} = (s_1, \ldots, s_n) = (s_{\alpha_1}, \ldots, s_{\alpha_n})$, let $2 \le k \le n$, and consider

$$Z'_{-(s_1,\ldots,s_{k-1})} = B_-\backslash P_{-s_1} \times_{B_-} P_{-s_2} \times \cdots \times_{B_-} P_{-s_{k-1}}.$$

Denote again by $[p_1, \ldots, p_{k-1}]_-$ the image of $(p_1, \ldots, p_{k-1}) \in P_{-s_1} \times \cdots \times P_{-s_{k-1}}$ in $Z'_{-(s_1,\ldots,s_{k-1})}$, and let B_- act on $Z'_{-(s_1,\ldots,s_{k-1})}$ from the right by

$$[p_1,\ \ldots,\ p_{k-2},\ p_{k-1}]_- \cdot b_- = [p_1,\ \ldots,\ p_{k-2},\ p_{k-1}b_-], \qquad b_- \in B_-,\ p_i \in P_{-s_i},\ 1 \le i \le k-1.$$

For $\xi \in \mathfrak{b}_-$, denote by $\sigma_\xi^{\prime,(k-1)}$ the vector field on $Z'_{-(s_1,\ldots,s_{k-1})}$ given by

$$\sigma_\xi^{\prime,(k-1)}([p_1, \ldots, p_{k-2}, p_{k-1}]) = \frac{d}{dt}|_{t=0}[p_1, \ldots, p_{k-2}, p_{k-1}\exp(t\xi)]_-, \tag{81}$$

where $p_i \in P_{-s_i}$ for $1 \le i \le k-1$, so $\sigma_\xi^{\prime,(k-1)}$ generates the action of B_- on $Z'_{-(s_1,\ldots,s_{k-1})}$ in the direction of ξ. Let

$$\sigma'_k = \sigma_{e_{-\alpha}}^{\prime,(k-1)}. \tag{82}$$

Consider the coordinates $(z_1, z_2, \ldots, z_n)$ on $\mathcal{O}_-^{\prime,\mathbf{u}}$ given in (77). Then $(z_1, \ldots, z_{k-1})$ can be considered as coordinates on the open submanifold

$$\begin{aligned} \mathcal{O}_-^{\prime,(s_1,\ldots,s_{k-1})} &= B_-\backslash(B_-s_1B_-) \times_{B_-} (B_-s_2B_-) \times \cdots \times_{B_-} (B_-s_{k-1}B_-) \\ &= \{[u_{\alpha_1}(z_1)\dot{s}_{\alpha_1},\ \ldots,\ u_{\alpha_{k-1}}(z_{k-1})\dot{s}_{\alpha_{k-1}}]_- :\ (z_1, \ldots, z_{k-1}) \in \mathbb{C}^{k-1}\} \end{aligned}$$

of $Z'_{-(s_1,\ldots,s_{k-1})}$, and σ'_k can be regarded as a derivation on $\mathbb{C}[z_1, \ldots, z_{k-1}]$.

Lemma 5.17. *In the coordinates* $(z_1, z_2, \ldots, z_n)$ *on* $\mathcal{O}_-^{\prime,\mathbf{u}}$ *given in* (77)*, the Poisson structure* π'_{-n} *is given by*

$$\{z_i,\, z_k\} = -c_{i,k} z_i z_k - \langle \alpha_k,\, \alpha_k\rangle \sigma'_k(z_i), \qquad 1 \le i < k \le n, \tag{83}$$

where for $1 \le i, k \le n$, $c_{i,k}$ *is given in* (75).

Proof. By repeatedly applying Lemma 5.16 to the Poisson manifold $(\mathcal{O}_-^{\prime,\mathbf{u}}, \pi'_{-n})$, one sees that π'_{-n} is given in the coordinates $(z_1, z_2, \ldots, z_n)$ on $\mathcal{O}_-^{\prime,\mathbf{u}}$ by (see notation in (81))

$$\{z_i,\, z_k\} = -\frac{\langle \alpha_k,\, \alpha_k\rangle}{2} z_k \sigma_{h_{\alpha_k}}^{\prime,(k-1)}(z_i) - \langle \alpha_k,\, \alpha_k\rangle \sigma'_k(z_i), \qquad 1 \le i < k \le n,$$

For $h \in \mathfrak{t}$, one checks directly from the definition of the vector field $\sigma_h^{\prime,(k-1)}$ that

$$\sigma_h^{\prime,(k-1)}(z_i) = (s_{k-1}s_{k-2}\cdots s_{i+1}(\alpha_i)(h))z_i, \qquad 1 \le i \le k-1. \tag{84}$$

(83) now follows from

$$\begin{aligned}\frac{\langle \alpha_k,\, \alpha_k\rangle}{2}\sigma_{h_{\alpha_k}}^{\prime,(k-1)}(z_i) &= \langle s_{k-1}s_{k-2}\cdots s_{i+1}(\alpha_i),\ \alpha_k\rangle z_i = -\langle s_1 s_2\cdots s_{i-1}(\alpha_i),\ s_1 s_2 \cdots s_{k-1}(\alpha_k)\rangle z_i \\ &= c_{i,k} z_i.\end{aligned}$$

Q.E.D.

Corollary 5.18. *In the notation in Proposition 5.12 for the case of* $\gamma = \mathbf{u}$*, one has*

$$b'_k = \langle \alpha_k, \alpha_k\rangle \sigma'_k, \qquad 2 \le k \le n.$$

Proof. By Proposition 5.15 and Lemma 5.17, the Poisson structure π_n is given in the coordinates $(z_1, \ldots, z_n)$ on the affine chart $\mathcal{O}^{\mathbf{u}}$ by

$$\{z_i,\, z_k\} = c_{i,k} z_i z_k + \langle \alpha_k,\, \alpha_k\rangle \sigma'_k(z_i), \qquad 1 \le i < k \le n.$$

It follows from the definition of b'_k that $b'_k = \langle \alpha_k, \alpha_k\rangle \sigma'_k$ for $2 \le k \le n$.

Q.E.D.

Remark 5.19. We already know from Lemma 5.10 that for any $h \in \mathfrak{t}$ and $2 \le k \le n$, $[a'_h,\, b'_k] = \lambda_k(h) b'_k$, as derivations of $\mathbb{C}[z_1, \ldots, z_{k-1}]$, where $a'_h = \partial_h|_{\mathbb{C}[z_1,\ldots,z_{k-1}]}$ and $\lambda_k = s_1 s_2 \cdots s_{k-1}(\alpha_k)$. This fact can also be checked directly from Corollary 5.18. Indeed, that in the notation of (81), it follows from (84) that $a'_h = -\sigma^{\prime,(k-1)}_{s_{k-1}\cdots s_2 s_1(h)}$ and $b'_k = \langle \alpha_k, \alpha_k\rangle \sigma^{\prime,(k-1)}_{e_{-\alpha_k}}$, so

$$[a'_h,\, b'_k] = -\langle \alpha_k, \alpha_k\rangle \left[\sigma^{\prime,(k-1)}_{s_{k-1}\cdots s_2 s_1(h)},\ \sigma^{\prime,(k-1)}_{e_{-\alpha_k}}\right] = \lambda_k(h) b'_k.$$

$\diamond$

6. The polynomial rings $(\mathbb{Z}[z_1, \ldots, z_n], \pi_\gamma)$

6.1. π_γ **is defined over** $\mathbb{Z}$**.** Recall from §2 that once the Borel subgroup B and the maximal torus $T \subset B$ of G are fixed, the definition of the Poisson structure π_n on $Z_{\mathbf{u}}$ depends only on the choice of a symmetric non-degenerate invariant bilinear form $\langle\, ,\, \rangle$ on $\mathfrak{g}$ and not on the choices of root vectors e_α for $\alpha \in \Delta$. Although a choice of the set $\{e_\alpha : \alpha \in \Gamma\}$ of root vectors for the simple roots is needed to define the coordinates $(z_1, \ldots, z_n)$ on $\mathcal{O}^\gamma$ for $\gamma \in \Upsilon_{\mathbf{u}}$, we proved in Proposition 5.2 that the polynomials $f_{i,k} := \{z_i, z_k\} \in \mathbb{C}[z_1, \ldots, z_n]$ for $1 \le i, k \le n$ are independent on the choices of the root vectors for the simple roots. For each $\gamma \in \Upsilon_{\mathbf{u}}$, one thus has a well-defined Poisson polynomial algebra $(\mathbb{C}[z_1, \ldots, z_n], \pi_\gamma)$.

Theorem 6.1. *Suppose that the symmetric non-degenerate invariant bilinear form* $\langle\, ,\, \rangle$ *on* $\mathfrak{g}$ *is chosen such that* $\frac{1}{2}\langle \alpha,\, \alpha\rangle \in \mathbb{Z}$ *for each* $\alpha \in \Delta$*. Then for any* $\gamma \in \Upsilon_{\mathbf{u}}$*, the Poisson structure* π_γ *on* $\mathbb{C}[z_1, \ldots, z_n]$ *has the property that* $\{z_i, z_k\} \in \mathbb{Z}[z_i, \ldots, z_k] \subset \mathbb{Z}[z_1, \ldots, z_n]$ *for all* $1 \le i < k \le n$.

Proof. Choose any set $\{e_\alpha : \alpha \in \Gamma\}$ of root vectors for the simple roots and extend it to a Chevalley basis of $\mathfrak{g}$. Theorem 6.1 now follows from Remark 4.5 and the fact that for any $\alpha, \beta \in \Delta$,

$$\langle \alpha,\ \beta \rangle = \frac{2\langle \alpha, \beta \rangle}{\alpha, \alpha \rangle} \frac{\langle \alpha, \alpha \rangle}{2} \in \mathbb{Z}.$$

Q.E.D.

Note that a canonical choice of the bilinear form $\langle\, , \rangle$ on $\mathfrak{g}$ is such that $\langle \alpha, \alpha \rangle = 2$ for the short roots for each of the simple factors of $\mathfrak{g}$.

Remark 6.2. By Theorem 6.1, each $\gamma \in \Upsilon_{\mathbf{u}}$ gives rise to a Poisson algebra $(\mathbf{k}[z_1, \ldots, z_n], \pi_\gamma)$ over any field $\mathbf{k}$ of arbitrary characteristic. In particular, it follows from (51) in Theorem 4.14 that the Poisson structure π_γ on $\mathbf{k}[z_1, \ldots, z_n]$ is log-canonical for every $\gamma \in \Upsilon_{\mathbf{u}}$ if $\mathrm{char}(\mathbf{k}) = 2$. ⋄

6.2. **Examples.** Assume that $\mathfrak{g}$ is simple and let $\langle\, , \rangle$ be such that $\langle \alpha, \alpha \rangle = 2$ for the short roots of $\mathfrak{g}$. Based on Theorem 4.14, the first author has written a computer program in the GAP language [9] which allows one to compute the Poisson structure π_γ on $\mathbb{Z}[z_1, \ldots, z_n]$ for any $\mathbf{u} = (s_1, \ldots, s_n)$ and any $\gamma \in \Upsilon_{\mathbf{u}}$. We given some examples.

Example 6.3. Consider G_2 with the two simple roots α_1 and α_2 satisfying $\langle \alpha_2, \alpha_2 \rangle = 3\langle \alpha_1, \alpha_1 \rangle = 3$. Let $\mathbf{u} = (s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_1}, s_{\alpha_2})$ and note that $s_{\alpha_1} s_{\alpha_2} s_{\alpha_1} s_{\alpha_2} s_{\alpha_1} s_{\alpha_2}$ is the longest element in the Weyl group of G_2. For $\gamma = \mathbf{u}$, one has

$$\begin{aligned}
&\{z_1, z_2\} = -3z_1z_2, \quad \{z_1, z_3\} = -z_1z_3 - 2z_2, \quad \{z_1, z_4\} = -6z_3^2,\\
&\{z_1, z_5\} = z_1z_5 - 4z_3, \quad \{z_1, z_6\} = 3z_1z_6 - 6z_5, \quad \{z_2, z_3\} = -3z_2z_3\\
&\{z_2, z_4\} = -6z_3^3 - 3z_2z_4, \quad \{z_2, z_5\} = -6z_3^2, \quad \{z_2, z_6\} = 3z_2z_6 - 18z_3z_5 + 6z_4\\
&\{z_3, z_4\} = -3z_3z_4, \quad \{z_3, z_5\} = -z_3z_5 - 2z_4, \quad \{z_3, z_6\} = -6z_5^2\\
&\{z_4, z_5\} = -3z_4z_5, \quad \{z_4, z_6\} = -6z_5^3 - 3z_4z_6, \quad \{z_5, z_6\} = -3z_5z_6.
\end{aligned}$$

For the same $\mathbf{u}$ but $\gamma = (s_{\alpha_1}, s_{\alpha_2}, e, e, s_{\alpha_1}, s_{\alpha_2})$, one has

$$\begin{aligned}
&\{z_1, z_2\} = -3z_1z_2, \quad \{z_1, z_3\} = 2z_2z_3^2 + z_1z_3,\\
&\{z_1, z_4\} = -6z_2z_3z_4 + 6z_3z_4^2 - 3z_1z_4, \quad \{z_1, z_5\} = -4z_2z_3z_5 + 6z_3z_4z_5 - z_1z_5 - 2z_2 + 2z_4,\\
&\{z_1, z_6\} = 6z_3z_5^3z_6^2 + 6z_5^2z_6^2 + 6z_2z_3z_6 - 6z_3z_4z_6, \quad \{z_2, z_3\} = 3z_2z_3,\\
&\{z_2, z_4\} = -6z_2z_4 + 6z_4^2, \quad \{z_2, z_5\} = -3z_2z_5 + 6z_4z_5,\\
&\{z_2, z_6\} = 6z_5^3z_6^2 + 3z_2z_6 - 6z_4z_6, \quad \{z_3, z_4\} = -3z_3z_4, \quad \{z_3, z_5\} = -2z_3z_5,\\
&\{z_3, z_6\} = 3z_3z_6, \quad \{z_4, z_5\} = 3z_4z_5, \quad \{z_4, z_6\} = -3z_4z_6, \quad \{z_5, z_6\} = 3z_5z_6.
\end{aligned}$$

⋄

Example 6.4. Consider $G = SL(2)$ with the only simple root denoted by α and $s = s_\alpha$. Let $\mathbf{u} = (s, s, s, s, s)$. For $\gamma = \mathbf{u}$, one has

$$\begin{aligned}
&\{z_1, z_2\} = 2z_1z_2 - 2, \quad \{z_1, z_3\} = -2z_1z_3, \quad \{z_1, z_4\} = 2z_1z_4, \quad \{z_1, z_5\} = -2z_1z_5,\\
&\{z_2, z_3\} = 2z_2z_3 - 2, \quad \{z_2, z_4\} = -2z_2z_4, \quad \{z_2, z_5\} = 2z_2z_5, \quad \{z_3, z_4\} = 2z_3z_4 - 2,\\
&\{z_3, z_5\} = -2z_3z_5, \quad \{z_4, z_5\} = 2z_4z_5 - 2.
\end{aligned}$$

For $\gamma = (s, e, e, e, s)$, one has

$$\begin{aligned}
&\{z_1, z_2\} = -2z_1z_2 + 2z_2^2, \quad \{z_1, z_3\} = -2z_1z_3 + 4z_2z_3 + 2z_3^2,\\
&\{z_1, z_4\} = -2z_1z_4 + 4z_2z_4 + 4z_3z_4 + 2z_4^2, \quad \{z_1, z_5\} = 2z_1z_5 - 4z_2z_5 - 4z_3z_5 - 4z_4z_5 - 2,\\
&\{z_2, z_3\} = 2z_2z_3, \quad \{z_2, z_4\} = 2z_2z_4, \quad \{z_2, z_5\} = -2z_2z_5,\\
&\{z_3, z_4\} = 2z_3z_4, \quad \{z_3, z_5\} = -2z_3z_5, \quad \{z_4, z_5\} = -2z_4z_5.
\end{aligned}$$

In general, it is easy to see from Theorem 4.14 that for the sequence $\mathbf{u} = (s, s, \ldots, s)$ of length n, and $\gamma = \mathbf{u}$, the Poisson structure π_γ on $\mathbb{Z}[z_1, \ldots, z_n]$ is given by

$$\{z_i, z_{i+1}\} = 2z_iz_{i+1} - 2, \qquad 1 \leq i \leq n-1,$$
$$\{z_i, z_k\} = 2(-1)^{k-j+1}z_iz_k, \qquad 1 \leq i < k \leq n,\ k-i \geq 2.$$

The coefficient 2 in all the Poisson brackets results from that fact that $\langle \alpha, \alpha \rangle = 2$.

Appendix A. Weak Poisson pairs

In [19, §8], a *weak Poisson dual pair* is defined to be a pair of surjective Poisson submersions

$$\rho_Y :\ (X,\, \pi_X) \longrightarrow (Y,\, \pi_Y) \quad \text{and} \quad \rho_Z :\ (X,\, \pi_X) \longrightarrow (Z,\, \pi_Z) \tag{85}$$

between Poisson manifolds such that the map

$$(\rho_Y, \rho_Z):\ \ (X,\, \pi_X) \longrightarrow (Y \times Z,\ \pi_Y \times \pi_Z), \quad x \longmapsto (\rho_Y(x),\ \rho_Z(x)), \qquad x \in X,$$

is Poisson. When (X, π_X) is symplectic and when the tangent spaces to the fibers of ρ_Y and ρ_Z are the symplectic orthogonals of each other, the pair $(\rho_Y,\, \rho_Z)$ is a *symplectic dual pair*.

If (Y, π_Y) and (Z, π_Z) are two Poisson manifolds, the projections from the product Poisson manifold $(Y \times Z,\, \pi_Y \times \pi_Z)$ to the two factors clearly form a Poisson dual pair. Moreover, for a differentiable map $\phi : Y \to Z$, it is well-known [23] that $\phi : (Y, \pi_Y) \to (Z, \pi_Z)$ is anti-Poisson if and only if the graph of ϕ, i.e.,

$$\mathrm{Graph}(\phi) = \{(y,\, \phi(y) :\, y \in Y\} \subset Y \times Z,$$

is a coisotropic submanifold of $(Y \times Z,\ \pi_Y \times \pi_Z)$. The following Lemma A.1 is a (partial) generalization of this fact to the case of weak Poisson dual pairs.

Lemma A.1. *Let (ρ_Y, ρ_Z) be a weak Poisson dual pair as in* (85). *Suppose that X' is a coisotropic submanifold of (X, π_X) such that $\rho_Y|_{X'} : X' \to Y$ is a diffeomorphism. Then*

$$\phi = \rho_Z \circ (\rho_Y|_{X'})^{-1} :\ \ (Y, \pi_Y) \longrightarrow (Z, \pi_Z)$$

is an anti-Poisson map.

Proof. Fix $x \in X'$ and let $\rho_Y(x) = y$ and $z = \rho_Z(x) \in Z$. Let

$$\rho_{Y,x} :\ \ T_xX \longrightarrow T_yY \quad \text{and} \quad \rho_{Z,x} :\ \ T_xX \longrightarrow T_zZ$$

be respectively the differentials of ρ_Y and ρ_Z at x. Lemma A.1 now follows from the following Lemma A.2 by taking $(V, \pi) = (T_xX, \pi_X(x))$, $V_1 = \ker \rho_{Y,x}$, $V_2 = \ker \rho_{Z,x}$, and $U = T_xX'$.

Q.E.D.

In the following Lemma A.2, for a finite dimensional vector space V and a subspace $U_1 \subset V$, set $U_1^0 = \{\xi \in V^* : \xi|_{U_1} = 0\} \subset V^*$, and U_1 is said to be coisotropic with respect to $\pi \in \wedge^2 V$ if $\pi \in U_1 \wedge V$, where for any subspace U_2 of V,

$$U_1 \wedge U_2 = (\wedge^2 V) \cap (U_1 \otimes U_2 + U_2 \otimes U_1) \subset \wedge^2 V.$$

Lemma A.2. *Let V be a finite dimensional vector space, let $\pi \in \wedge^2 V$, and let V_1 and V_2 be two vector subspaces of V such that $\pi(V_1^0, V_2^0) = 0$. For $j = 1, 2$, let $\rho_j : V \to V/V_j$ be the projections so that $\rho_j(\pi) \in \wedge^2(V/V_j)$. Assume that U is a coisotropic subspace of V and that $\rho_1|_U : U \to V/V_1$ is an isomorphism. Let $\psi = \rho_2 \circ (\rho_1|_U)^{-1} : V/V_1 \to V/V_2$. Then $\psi(\rho_1(\pi)) = -\rho_2(\pi)$.*

Proof. For $\pi' = \sum_j v_j \wedge v_j' \in \wedge^2 V$ and $\xi \in V^*$, let $\xi\rfloor\pi' = \sum_j(\langle \xi, v_j\rangle v_j' - \langle \xi, v_j'\rangle v_j)$, where $\langle\, , \rangle$ denotes the pairing between V and V^*. Then the condition $\pi(V_1^0, V_2^0) = 0$ is equivalent to $\xi\rfloor\pi \in V_2$ for all $\xi \in V_1^0$. By assumption, $V = U + V_1$ is a direct sum. As U is coisotropic with respect to π, one can uniquely write $\pi = \pi_U + \pi_1$, where $\pi_U \in \wedge^2 U$ and $\pi_1 \in U \wedge V_1$. Let $\{u_1, \ldots, u_m\}$ be a basis of U and let $\xi_i \in V_1^0$, $1 \le i \le m$, be such that $\langle u_i, \xi_j\rangle = \delta_{i,j}$ for $1 \le i, j \le m$. Then

$$\pi_U = \frac{1}{2}\sum_{i=1}^m u_i \wedge (\xi_i\rfloor\pi_U) \quad \text{and} \quad \pi_1 = \sum_{i=1}^m u_i \wedge (\xi_i\rfloor\pi_1).$$

For $1 \le i \le m$, let $x_i = \xi_i\rfloor\pi = \xi_i\rfloor(\pi_U + \pi_1)$. Then

$$\begin{aligned}\pi &= \frac{1}{2}\sum_{i=1}^m u_i \wedge (\xi_i\rfloor\pi_U) + \sum_{i=1}^m u_i \wedge (\xi_i\rfloor(\pi_U + \pi_1)) - \sum_{i=1}^m u_i \wedge (\xi_i\rfloor\pi_U)\\ &= -\frac{1}{2}\sum_{i=1}^m u_i \wedge (\xi_i\rfloor\pi_U) + \sum_{i=1}^m u_i \wedge x_i = -\pi_U + \sum_{i=1}^m u_i \wedge x_i.\end{aligned}$$

As $x_i \in V_2$ for each $1 \le i \le m$, $\rho_2(\sum_{i=1}^m u_i \wedge x_i) = 0$, so $\psi(\rho_1(\pi)) = -\rho_2(\pi)$.

Q.E.D.

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