

The proof-theoretic strength of Ramsey's theorem for pairs and two colors

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Abstract

Ramsey's theorem for n -tuples and k -colors (RT_k^n) asserts that every k -coloring of $[\mathbb{N}]^n$ admits an infinite monochromatic subset. We study the proof-theoretic strength of Ramsey's theorem for pairs and two colors, namely, the set of its Π_1^0 consequences, and show that RT_2^2 is Π_3^0 conservative over $\text{I}\Sigma_1^0$. This strengthens the proof of Chong, Slaman and Yang that RT_2^2 does not imply $\text{I}\Sigma_2^0$, and shows that RT_2^2 is finitistically reducible, in the sense of Simpson's partial realization of Hilbert's Program. Moreover, we develop general tools to simplify the proofs of Π_3^0 -conservation theorems.

1 Introduction

Ramsey's theorem for n -tuples and k -colors (RT_k^n) asserts that every k -coloring of $[\mathbb{N}]^n$ admits an infinite monochromatic subset. In the study of reverse mathematics, determining the strength of Ramsey's theorem for pairs is a long-term open problem. Hirst[22] showed that RT_2^2 implies the Σ_2^0 -bounding principle ($\text{B}\Sigma_2^0$). On the other hand, Cholak, Jockusch and Slaman [6] showed that $\text{WKL}_0 + \text{RT}_2^2 + \text{I}\Sigma_2^0$ is a Π_1^1 -conservative extension of $\text{I}\Sigma_2^0$, where WKL_0 stands for weak König's lemma and $\text{I}\Sigma_n^0$ is the Σ_n^0 -induction scheme. Thus, the first-order strength of Ramsey's theorem for pairs and two colors is in between $\text{B}\Sigma_2^0$ and $\text{I}\Sigma_2^0$. After this work, the project of deciding the first-order strength of RT_2^2 has been strongly carried out using forcing constructions or priority arguments on nonstandard models of $\text{B}\Sigma_2^0$, mainly by Chong, Slaman and Yang [10, 9]. They proved in particular that $\text{WKL}_0 + \text{RT}_2^2$ does not imply $\text{I}\Sigma_2^0$ [8].

In this paper, we use a hybrid of forcing construction, indicator arguments, and proof-theoretic technique to show that $\text{RT}_2^2 + \text{WKL}_0$ is a Π_3^0 -conservative extension of $\text{I}\Sigma_1^0$. This answers Question 13.2 of Cholak, Jockusch and Slaman [6], and strengthens the above recent result by Chong,

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Slaman and Yang since the Π_3^0 -consequences of IS_2^0 are different from those of IS_1^0 . Our conservation result gives actually a weak answer to the original question, namely, the proof-theoretic strength of RT_2^2 is the same as BS_2^0 , since BS_2^0 is a Π_3^0 -conservative extension of IS_1^0 . By the proof-theoretic strength of a theory T we mean the set of Π_1^0 sentences which are provable in T , or the proof-theoretic ordinal of T which is decided by the Π_2^0 -consequences of T . Deciding the proof-theoretic strength of RT_2^2 is an important problem from a philosophical point of view. In the sense of Simpson’s partial realization [40] of Hilbert’s Program, one would conclude that RT_2^2 is *finitistically reducible*.

1.1 Reverse mathematics

Reverse mathematics is a vast foundational program that seeks to determine which *set existence axioms* are needed to prove theorems from “ordinary” mathematics. It uses the framework of subsystems of second-order arithmetic. Indeed, Friedman realized that a large majority of theorems admitted a natural formulation in the language of second-order arithmetic. The base theory RCA_0 , standing for Recursive Comprehension Axiom, contains the basic axioms for first-order arithmetic (axioms of discrete ordered semi-ring) together with the Δ_1^0 -comprehension scheme and the Σ_1^0 -induction scheme. RCA_0 can be thought of as capturing *computable mathematics*.

Since then, thousands of theorems have been studied within the framework of reverse mathematics. A surprising phenomenon emerged from the early years of reverse mathematics: Most theorems studied require very weak axioms. Moreover, many of them happen to be *equivalent* to one of five main sets of axioms, that are referred to as the *Big Five*, namely, RCA_0 , weak König’s lemma (WKL_0), the arithmetic comprehension axiom (ACA_0), arithmetical transfinite recursion (ATR_0), and Π_1^1 -comprehension axiom ($\Pi_1^1\text{-CA}_0$). See Simpson [41] for an extensive study of the Big Five and mathematics within them. In this paper, we shall consider exclusively theorems which are provable in ACA_0 . See Hirschfeldt [20] for a gentle introduction to the reverse mathematics below ACA_0 .

1.2 Ramsey’s theorem and its consequences

Ramsey theory is a branch of mathematics studying the conditions under which some structure appears among a sufficiently large collection of objects. In the past two decades, Ramsey theory emerged as one of the most important topics in reverse mathematics. This theory provides a large class of theorems escaping the Big Five phenomenon, and whose strength is notoriously hard to gauge. Perhaps the most famous such theorem is *Ramsey’s theorem*.

Definition 1.1 (Ramsey’s theorem). A subset H of \mathbb{N} is *homogeneous* for a coloring $f : [\mathbb{N}]^n \rightarrow k$ (or *f-homogeneous*) if all the n -tuples over H are given the same color by f . RT_k^n is the statement “Every coloring $f : [\mathbb{N}]^n \rightarrow k$ has an infinite f -homogeneous set”.

Jockusch [23] conducted a computational analysis of Ramsey’s theorem, later formalized by Simpson [41] within the framework of reverse mathematics. Whenever $n \geq 3$, Ramsey’s theorem for n -tuples happens to be equivalent to ACA_0 . The status of Ramsey’s theorem for pairs was open for decades, until Seetapun [39] proved that RT_2^2 is strictly weaker than ACA_0 over RCA_0 . Cholak, Jockusch and Slaman [7] extensively studied Ramsey’s theorem for pairs.

In order to better understand the logical strength of RT_2^2 , Bovykin and Weiermann [4] decomposed Ramsey’s theorem for pairs into the Erdős-Moser theorem and the ascending descending sequence principle. The Erdős-Moser is a statement from graph theory.

Definition 1.2 (Erdős-Moser theorem). A tournament T is an irreflexive binary relation such that for all $x, y \in \mathbb{N}$ with $x \neq y$, exactly one of $T(x, y)$ or $T(y, x)$ holds. A tournament T is *transitive* if the corresponding relation T is transitive in the usual sense. EM is the statement “Every infinite tournament T has an infinite transitive subtournament.”

Definition 1.3 (Ascending descending sequence). Given a linear order (*i.e.*, a transitive tournament) $<_L$ on \mathbb{N} , an *ascending* (*descending*) sequence is a set S such that for every $x <_{\mathbb{N}} y \in S$, $x <_L y$ ($x >_L y$). ADS is the statement “Every infinite linear order admits an infinite ascending or descending sequence”.

The Erdős-Moser theorem provides together with the ascending descending principle an alternative decomposition of Ramsey’s theorem for pairs. Indeed, every coloring $f : [\mathbb{N}]^2 \rightarrow 2$ can be seen as a tournament R such that $R(x, y)$ holds if $x < y$ and $f(x, y) = 1$, or $x > y$ and $f(y, x) = 0$. Then, EM is saying “Every coloring $f : [\mathbb{N}]^n \rightarrow k$ has an infinite transitive subcoloring” and ADS is saying “Every transitive coloring $f : [\mathbb{N}]^n \rightarrow k$ has an infinite f -homogeneous set”. (In what follows, we always consider EM and ADS as these forms.) We therefore obtain the following equivalence.

Theorem 1.1 (Hirschfeldt/Shore [21], Bovykin/Weiermann [4]). $\text{RCA}_0 \vdash \text{RT}_2^2 \leftrightarrow \text{ADS} + \text{EM}$.

The ascending descending sequence has been introduced by Hirschfeldt and Shore [21]. They proved that ADS is strictly weaker than Ramsey’s theorem for pairs. On the other hand, Lerman, Solomon and Towsner [30] proved that the Erdős-Moser theorem is strictly weaker than RT_2^2 . For technical purposes, we shall consider a statement equivalent to the ascending descending principle. Pseudo Ramsey’s theorem for pairs has been introduced by Murakami, Yamazaki and the second author [33] to study a factorization theorem from automata theory.

Definition 1.4 (Pseudo Ramsey’s theorem for pairs). A set H is *pseudo-homogeneous* for a coloring $f : [\mathbb{N}]^2 \rightarrow k$ if there is a color $c < k$ such that every pair $\{x, y\} \in [H]^2$ are the endpoints of a finite sequence $x_0 < x_1 < \dots < x_n$ such that $f(x_i, x_{i+1}) = c$ for each $i < n$. psRT_k^2 is the statement “Every coloring $f : [\mathbb{N}]^n \rightarrow k$ has an infinite f -pseudo-homogeneous set”.

In particular, if $f : [\mathbb{N}]^2 \rightarrow 2$ is a transitive coloring, then any set H pseudo-homogeneous for f is already homogeneous for f . Thus, $\text{RCA}_0 + \text{psRT}_2^2$ implies ADS (see [33]). The first author [38] and Steila (see [42]) independently proved the reverse implication, namely, $\text{RCA}_0 + \text{ADS}$ implies psRT_2^2 .

1.3 Proof strength and conservation results

In the study of reverse mathematics, deciding the first-order or proof-theoretic strength of axioms and mathematical principles is one of the main topic. Especially, the conservation result for weak König’s lemma always plays the central role as a large part of mathematics can be proven within WKL_0 (see Simpson [41]). The following theorems show that one can use weak König’s lemma almost freely to seek for first-order consequences.

Theorem 1.2 (Harrington, see [41]). WKL_0 is a Π_1^1 -conservative extension of IS_1^0 .

Theorem 1.3 (Friedman[14], see [41]). WKL_0 is a Π_2^0 -conservative extension of PRA.

The Σ_2^0 -bounding principle ($\text{B}\Sigma_2^0$) informally asserts that a finite union of finite sets is finite. Many mathematical reasonings make an essential use of $\text{B}\Sigma_2^0$ and in particular RT_2^2 implies $\text{B}\Sigma_2^0$. The strength of the Σ_2^0 -bounding principle is therefore important for the study of combinatorial principles.

Theorem 1.4 (Hájek[16]). $\text{WKL}_0 + \text{BS}_2^0$ is a Π_1^1 -conservative extension of BS_2^0 .

Thankfully, BS_2^0 can be freely used for a restricted class of formulas. Let $\tilde{\Pi}_3^0$ be a class of formulas of the form $\forall X \varphi(X)$ where φ is a Π_3^0 -formula. The following is a parameterized version of the Parsons/Paris/Friedman conservation theorem.

Theorem 1.5 (see, e.g., [5] or [25]). BS_2^0 is a $\tilde{\Pi}_3^0$ -conservative extension of IS_1^0 .

Note that the above four conservation theorems are frequently used in this paper, and so we shall not mention them explicitly.

There are many advanced studies to investigate the first-order strength of Ramsey's theorem and related combinatorial principles. One of the most important methods for these studies consists in adapting computability-theoretic techniques for combinatorial principles. By this method, Chong, Slaman and Yang [10] showed that two weaker combinatorial principles, namely, the ascending descending sequence (ADS) and the chain antichain principle (CAC), introduced by Shore/Hirschfeldt[21], are Π_1^1 -conservative over WKL_0 . In [9], they showed that $\text{WKL}_0 + \text{SRT}_2^2$ does not imply IS_2^0 , and they improved the result and proved that $\text{WKL}_0 + \text{RT}_2^2$ does not imply IS_2^0 in [8]. More recently, Chong, Kreuzer and Yang [unpublished] showed that SRT_2^2 is Π_3^0 -conservative over $\text{WKL}_0 + \text{WF}(\omega^\omega)$, where $\text{WF}(\omega^\omega)$ asserts the well-foundedness of ω^ω .

Besides the computability-theoretic techniques, many other significant approaches can be found in the literature. Kohlenbach/Kreuzer [27] and Kreuzer [29] characterized the Π_2^0 -parts of RT_2^2 and CAC with several different settings by proof-theoretic approaches. Bovykin/Weiermann [4] and the second author [47] showed that indicators defined by Paris's density notion can approach the proof-theoretic strength of various versions of Ramsey's theorem, and by a similar method, the second author [48] also showed that $\text{RT}_k^n + \text{WKL}_0^*$ is fairly weak and is a Π_2^0 -conservative extension of RCA_0^* , where RCA_0^* is RCA_0 with only Σ_0^0 -induction and the exponentiation. In addition, the study of the Π_2^0 -parts of Ramsey-type theorems has some application to computer science [42].

1.4 Second-order structures of arithmetic and their cuts

A structure for the language of second-order arithmetic \mathcal{L}_2 is a pair (M, S) where $M = (M, +_M, \cdot_M, 0_M, 1_M, <_M)$ is a structure for the language of first-order (Peano) arithmetic \mathcal{L}_{PA} , and S is a subset of the power set of M .

Definition 1.5 (Cut). Given a structure M of the first-order arithmetic, a substructure $I \subseteq M$ is said to be a *cut* of M (abbreviated $(I \subseteq_e M)$) if $I = \{a \in M \mid \exists b \in I(a <_M b)\}$.

Here, the standard first-order structure ω can be considered as the smallest cut of any first-order structure. Given a structure (M, S) , a cut $I \subseteq_e M$ induces the second-order structure $(I, S \upharpoonright I)$, where $S \upharpoonright I := \{X \cap I \mid X \in S\}$. We sometimes consider S as a family of unary predicates on M and identify (M, S) as an $\mathcal{L}_{\text{PA}} \cup S$ -structure. Accordingly, $(I, S \upharpoonright I)$ can be considered as an $\mathcal{L}_{\text{PA}} \cup S$ -substructure of (M, S) . Note that $S \upharpoonright I$ may then be a multiset on I , but this is harmless without second-order equality. In this sense, one can easily check that $(I, S \upharpoonright I)$ is a Σ_0^0 -elementary substructure of (M, S) if I is closed under $+_M$ and \cdot_M .

We write $\text{Cod}(M)$ for the collection of all M -finite subsets, *i.e.*, subsets of M canonically coded by elements in M (as the usual binary expansion). Given some cut $I \subseteq_e M$, we write $\text{Cod}(M/I)$ for $\text{Cod}(M) \upharpoonright I$. If $I \subsetneq M$, then $\text{Cod}(M/I) = S \upharpoonright I$ for any $S \subseteq \mathcal{P}(M)$ such that $(M, S) \models \text{RCA}_0$, thus $(I, S \upharpoonright I)$ only depends on M and I .

A cut $I \subseteq_e M$ is said to be *semi-regular* if $I \cap X$ is bounded for any $X \in \text{Cod}(M)$ such that $|X| \in I$, where $|X|$ denotes the cardinality of X in M . A semi-regular cut is one of the central notions in the study of cuts, especially with the connection to second-order structures, since it characterizes the models of WKL_0 . We will use the following theorem throughout this paper without mentioning it explicitly.

Theorem 1.6 (see, e.g., [28]). *Let I be a cut of a first-order structure M . Then, I is semi-regular if and only if $(I, \text{Cod}(M/I)) \models \text{WKL}_0$.*

Bounding principles are also characterized by cuts with some elementarity condition. In this paper, we will use the following characterization.

Theorem 1.7 (Folklore, essentially due to Paris/Kirby[35], see also [45]). *Let $n \geq 1$. Let (M, S) be a model of $\text{I}\Sigma_{n-1}^0$, and let I be a cut of M such that $(I, S \upharpoonright I)$ be a Σ_n^0 -elementary substructure of (M, S) . Then, $(I, S \upharpoonright I) \models \text{B}\Sigma_{n+1}^0$.*

1.5 Hilbert's program and finitistic reductionism

During the early 20th century, mathematics went through a serious foundational crisis, with the discovery of various paradoxes and inconsistencies. Some great mathematicians such as Kronecker, Poincaré and Brouwer challenged the validity of infinitistic reasoning. Hilbert [18] proposed a three-step program to answer those criticisms. First, he suggested to identify the finitistic part of mathematics, then to axiomatize infinite reasoning, and eventually to give a finitistically correct consistency proof of this axiomatic system. However, his program was nip in the bud by Gödel's incompleteness theorems [15].

In 1986, Simpson [40] proposed a formal interpretation of Hilbert's program by taking primitive recursive arithmetic (PRA) as the base system for capturing finitistic reasoning. This choice was convincingly justified by Tait [43]. Simpson took second-order arithmetic (Z_2) as the big system capturing infinitistic reasoning, based on the work of Hilbert and Bernays [19]. In this setting, finitistic reductionism can be interpreted as proving that Z_2 is Π_1^0 -conservative over PRA. By Gödel incompleteness theorem, this cannot be the case. However, Simpson proposed to consider Π_1^0 -conservation of subsystems of second-order arithmetic over PRA as a partial realization of Hilbert's program. He illustrated his words with weak König's lemma (WKL) which was proven by Friedman to be Π_2^0 -conservative over PRA (Theorem 1.3). In this paper, we contribute to Hilbert's program by showing that $\text{WKL}_0 + \text{RT}_2^2$ is Π_2^0 -conservative over PRA, and therefore that RT_2^2 is finitistically reducible. Moreover, we develop general tools to simplify the proofs of Π_3^0 -conservation theorems, and thereby contribute to the simplification of the realization of Hilbert's program.

1.6 Notation

In order to avoid confusion between the theory and the meta-theory, we shall use ω to denote the set of (standard) natural numbers, and \mathbb{N} to denote the sets of natural numbers inside the system. Accordingly, we shall write ω for the ordinal ω in the system. We write $[a, b]_{\mathbb{N}}, (a, b)_{\mathbb{N}}, (a, \infty)_{\mathbb{N}}, \dots$ for intervals of natural numbers, e.g., $(a, b]_{\mathbb{N}} = \{x \in \mathbb{N} \mid a < x \leq b\}$. Given a set X and some $n \in \mathbb{N}$, $[X]^n$ is the collection of all sets of size n . $[X]^{<\mathbb{N}}$ is the union $\bigcup_n [X]^n$.

We use $\Pi_n, \Sigma_n, \Delta_n$ to denote first-order formulas without set parameters, whereas $\Pi_n^0, \Sigma_n^0, \Delta_n^0$ are second-order formulas, *i.e.*, with set parameters. A $\tilde{\Pi}_n^0$ -formula is a second-order formula of the form $(\forall X)\varphi(X)$ where φ is a Π_n^0 -formula.

Given two sets A, B , $A \oplus B = \{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$, $A \subseteq_{\text{fin}} B$ means that A is a finite subset of B , and $A \subseteq^* B$ means that the set A is included, up to finite changes, in B . We write $A < B$ for the formula $(\forall x \in A)(\forall y \in B)x < y$. Whenever $A = \{x\}$, we shall simply write $x < B$ for $A < B$. A set X can be seen as an infinite join $X = \bigoplus_i X_i$, where $x \in X_i$ iff $\langle i, x \rangle \in X$. We then write $X[i]$ for X_i . Given a set X or a string σ and some integer $m \in \omega$, we write $X \upharpoonright m$ for the initial segment of X (resp. σ) of length m .

1.7 Structure of this paper

The main target of this paper is the following conservation theorem.

Theorem. $\text{WKL}_0 + \text{RT}_2^2$ is a $\tilde{\Pi}_3^0$ -conservative extension of IS_1^0 .

We will prove this in the following way.

In Sections 2 and 3, we will explain that $\tilde{\Pi}_3^0$ -consequences of Ramsey's theorem and its variations are characterized by some largeness notions of finite sets. We will introduce largeness notion for Γ , where Γ is any of RT_2^2 , psRT_2^2 (which is equivalent to ADS), and EM. Roughly speaking, giving a bound for largeness for Γ within IS_1^0 provides $\tilde{\Pi}_3^0$ -conservation for $\text{WKL}_0 + \Gamma$ over IS_1^0 (Theorem 3.1).

According to the decomposition of RT_2^2 into ADS and EM and the amalgamation theorem (Theorem 3.5), the conservation for RT_2^2 can be decomposed into the conservation for ADS and the conservation for EM. In Section 4, we give a bound for the largeness notion for psRT_2^2 (Lemma 4.4) by using the Ketonen/Solovay theorem. It actually provides the conservation result for ADS (Corollary 4.5).

It is rather complicated to give a bound for the largeness notion for EM. For this, we will introduce a new combinatorial principle called the *grouping principle*. We mainly focus on the grouping principle for pairs and two colors GP_2^2 . Section 5 is devoted to the reverse mathematical study of GP_2^2 , especially from the view point of computability theory. In Section 6, we will prove a conservation theorem for GP_2^2 (Theorem 6.5). For this, we will modify the construction of a low solution for the stable version of GP_2^2 (Theorem 5.2) presented in the previous section.

In Section 7, we give a bound for the largeness notion for EM (Lemma 7.2) by using a finite version of the grouping principle, which is a consequence of GP_2^2 . It provides the conservation result for EM (Theorem 7.3). Then, combining this with the conservation result in Section 4 by the amalgamation theorem, we obtain the main theorem.

The main theorem can be formalized within WKL_0 , and that leads to the consistency equivalence of IS_1^0 and RT_2^2 . This is argued in Section 8.

2 Largeness

A family of finite sets of natural numbers $\mathbb{L} \subseteq [\mathbb{N}]^{<\mathbb{N}}$ is said to be a *largeness notion* if any infinite set has a finite subset in \mathbb{L} and \mathbb{L} is closed under supersets. A finite set $X \in [\mathbb{N}]^{<\mathbb{N}}$ is said to be \mathbb{L} -large if $X \in \mathbb{L}$. A (possibly largeness) notion $\mathbb{L} \subseteq [\mathbb{N}]^{<\mathbb{N}}$ is said to be *regular* if for any \mathbb{L} -large set F , any finite set $G \subseteq_{\text{fin}} \mathbb{N}$ for which there exists an order-preserving injection $h : F \rightarrow G$ such that $(\forall x \in F)h(x) \leq x$, then $G \in \mathbb{L}$. A Δ_0 -definable notion $\mathbb{L} \subseteq [\mathbb{N}]^{<\mathbb{N}}$ is said to be a (regular)

largeness notion provably in IS_1^0 if $\text{IS}_1^0 \vdash \text{“L is a (regular) largeness notion”}$. In this paper, we shall mainly consider regular largeness notions provably in IS_1^0 .

Example 2.1. The family $\text{L}_\omega = \{X \subseteq_{\text{fin}} \mathbb{N} : |X| > \min X\}$ is a regular largeness notion provably in IS_1^0 .

The notion of largeness enjoys a property similar to the pigeonhole principle, as states the following lemma.

Lemma 2.1 ($\text{WKL}_0 + \text{B}\Sigma_2^0$). *For any largeness notion L, for any infinite set X and for any $k, N_0 \in \mathbb{N}$, there exists $N_1 \in \mathbb{N}$ such that for any partition $X \cap [N_0, N_1]_{\mathbb{N}} = X_0 \sqcup \dots \sqcup X_{k-1}$, there exists an L-large set F such that $F \subseteq X_i$ for some $i < k$.*

Since $\text{WKL}_0 + \text{B}\Sigma_2^0$ is a $\tilde{\Pi}_3^0$ -conservative extension of IS_1^0 , this lemma for a Δ_1 -definable largeness notion provably in IS_1^0 is provable in IS_1^0 .

Proof. By $\text{B}\Sigma_2^0$, for any partition $X \cap [N_0, \infty)_{\mathbb{N}} = X_0 \sqcup \dots \sqcup X_{k-1}$, one of the X_i 's is infinite and thus it contains an L-large subset. Thus, a bound for such an L-large set can be obtained by the usual compactness argument which is available within WKL_0 . \square

Note that the use of $\text{B}\Sigma_2^0$ in the previous lemma is essential. If $\text{B}\Sigma_2^0$ fails, there exists a partition $X = X_0 \sqcup \dots \sqcup X_{k-1}$ such that each of the X_i 's is finite. Then, $\text{L} = \{F \mid \forall i < k (F \not\subseteq X_i)\}$ is a largeness notion failing the lemma. The use of WKL_0 is not essential since the argument can be formalized within $\text{RWKL} + \text{B}\Sigma_2^0$, where RWKL denotes the Ramsey-type weak König's lemma introduced by Flood [13].

2.1 α -largeness

From now on, we fix a primitive recursive notation for ordinals below ϵ_0 . In this paper, we actually use ordinals of the form $\alpha = \sum_{i < k} n_i \cdot \omega^i < \omega^\omega$ where $n_i \in \mathbb{N}$ and $n_0 \geq \dots \geq n_{k-1}$. (We write 1 for ω^0 , and $n \cdot k$ for $\sum_{i < k} n_i \cdot \omega^i$.) For a given $\alpha < \omega^\omega$ and $m \in \mathbb{N}$, define $0[m] = 0$, $\alpha[m] = \beta$ if $\alpha = \beta + 1$ and $\alpha[m] = \beta + n \cdot \omega^{i-1} \cdot m$ if $\alpha = \beta + n \cdot \omega^i$ for some $n \geq 1$.

Definition 2.2 (IS_1^0). Let $\alpha < \omega^\omega$. A set $X = \{x_0 < \dots < x_{\ell-1}\} \subseteq_{\text{fin}} \mathbb{N}$ is said to be α -large if $\alpha[x_0] \dots [x_{\ell-1}] = 0$. In other words, any finite set is 0-large, and X is said to be α -large if

- $X \setminus \{\min X\}$ is β -large if $\alpha = \beta + 1$,
- $X \setminus \{\min X\}$ is $(\beta + n \cdot \omega^{i-1} \cdot \min X)$ -large if $\alpha = \beta + n \cdot \omega^i$.

We let $\text{L}_\alpha = \{X \subseteq_{\text{fin}} \mathbb{N} \mid X \text{ is } \alpha\text{-large}\}$.

In particular, a set X is m -large iff $|X| \geq m$ and ω -large iff $|X| > \min X$. See [17] for the general definition of α -largeness. One can easily see that if $X \subseteq Y$ for some α -large set X and some finite set Y , then Y is α -large.

We say that X is α -small if it is not α -large. The following basic combinatorics have been proven in [17, Theorem II.3.21] in their full generality.

Lemma 2.2 (IS_1^0). *Fix any $k, n \in \mathbb{N}$.*

- (i) *A set X is $n \cdot k$ -large if and only if it is a union of some n -large finite sets $X_0 < \dots < X_{k-1}$.*
- (ii) *A set X is $n \cdot k$ -small if it is a union of n -small finite sets $X_0 < \dots < X_{k-1}$.*

In particular, $\{k\} \cup X_0 \cup \dots \cup X_{k-1}$ is $^{n+1}$ -large if each of X_i is n -large and $k < X_0 < \dots < X_{k-1}$. Similarly, if $\{k\} \cup X_0 \cup \dots \cup X_{k-1}$ is $^{n+1}$ -large and $k < X_0 < \dots < X_{k-1}$, then one of X_i 's is n -large.

The following theorem corresponds to the well-known fact that the proof-theoretic ordinal of IS_1^0 is ω^ω .

Theorem 2.3. *For any $n \in \omega$, IS_1^0 proves that L_{ω^n} is a regular largeness notion.*

Proof. One can easily check the regularity within IS_1^0 . We will see that $\text{IS}_1^0 \vdash$ “any infinite set contains an n -large subset” by (external) induction. The case $n = 0$ is trivial. We show the case $n = k + 1$. Within IS_1^0 , let an infinite set X be given. Then, by the induction hypothesis and Σ_1^0 -induction, one can find $\min X$ -many k -large sets $F_i \subseteq X$ such that $\min X < F_0 < \dots < F_{\min X - 1}$. By Lemma 2.2 and the discussion below it, $\{\min X\} \cup F_0 \cup \dots \cup F_{\min X - 1}$ is $^{k+1}$ -large. \square

2.2 Largeness for Ramsey-like statements

Many Ramsey-type theorems studied in reverse mathematics are statements of the form “For every coloring $f : [\mathbb{N}]^n \rightarrow k$, there is an infinite set H satisfying some structural properties”. The most notable example is Ramsey’s theorem, which asserts for every coloring $f : [\mathbb{N}]^n \rightarrow k$ the existence of an infinite f -homogeneous set. These statements can be seen as *mathematical problems*, whose *instances* are coloring, and whose *solutions* are the sets satisfying the desired structural properties.

Definition 2.3. A *Ramsey-like- Π_2^1 -formula* is a Π_2^1 -formula of the form

$$(\forall f : [\mathbb{N}]^n \rightarrow k)(\exists Y)(Y \text{ is infinite} \wedge \Psi(f, Y))$$

where $n, k \in \omega$ and $\Psi(f, Y)$ is of the form $(\forall G \subseteq_{\text{fin}} Y)\Psi_0(f \upharpoonright [[0, \max G]_{\mathbb{N}}]^n, G)$ such that Ψ_0 is a Δ_0^0 -formula.

In particular, RT_k^n is a Ramsey-like- Π_2^1 -statement where $\Psi(f, Y)$ is the formula “ Y is homogeneous for f ”. Similarly, psRT_k^2 and EM are Ramsey-like- Π_2^1 statements. On the other hand, SRT_2^2 is not a Ramsey-like- Π_2^1 -statement. However, SRT_2^2 is equivalent to the Ramsey-like- Π_2^1 -formula saying “for any 2-coloring f on $[\mathbb{N}]^2$, there exists an infinite set Y such that Y is homogeneous for f or there exists $a < \min Y$ witnessing the non-stability of f , that is, such that for any $x, y \in Y$ there exist $b, c \in [x, y]_{\mathbb{N}}$ such that $f(a, b) \neq f(a, c)$ ”. Although the definition of a Ramsey-like- Π_2^1 -formula seems very restrictive, we can show that it entails a much larger class of Π_2^1 -statements.

Definition 2.4. A *restricted- Π_2^1 -formula* is a Π_2^1 -formula of the form $\forall X \exists Y \Theta(X, Y)$ where Θ is a Σ_3^0 -formula.

Proposition 2.4. *For any restricted- Π_2^1 -formula $\forall X \exists Y \Theta(X, Y)$, there exists a Ramsey-like- Π_2^1 -formula $\forall X \exists Z (Z \text{ is infinite} \wedge \Psi(X, Z))$ such that*

$$\text{WKL}_0 \vdash \forall X (\exists Y \Theta(X, Y) \leftrightarrow \exists Z (Z \text{ is infinite} \wedge \Psi(X, Z))).$$

(Here, X is considered as a function $X : [\mathbb{N}]^1 \rightarrow 2$ in the definition of Ramsey-like- Π_2^1 -formula.)

Proof. We work within WKL_0 . Let $\forall X \exists Y \Theta(X, Y)$ be a restricted- Π_2^1 -formula. Without loss of generality, one can write $\Theta(X, Y) \equiv \forall n \exists m \theta(X \upharpoonright m, Y \upharpoonright m, n, m)$ where θ is Σ_3^0 since existential

number quantifier can be replaced with an existential set quantifier. Define a formula $\Psi(X, Z)$ as follows:

$$\Psi(X, Z) \equiv \forall F \subseteq_{\text{fin}} Z (F \neq \emptyset \rightarrow (\exists \sigma \in 2^{\max F})(\forall n < |F|)(\exists m < \max F)\theta(X \upharpoonright m, \sigma \upharpoonright m, n, m)).$$

We now show that $\exists Y \Theta(X, Y) \leftrightarrow \exists Z (Z \text{ is infinite} \wedge \Psi(X, Z))$.

To show the left to right implication, take $Y \subseteq \mathbb{N}$ such that $\forall n \exists m \theta(X \upharpoonright m, Y \upharpoonright m, n, m)$. Define an infinite increasing sequence $\langle z_i \mid i \in \mathbb{N} \rangle$ as $z_0 = \min\{m + 1 \mid \theta(X \upharpoonright m, Y \upharpoonright m, 0, m)\}$, and $z_{i+1} = \min\{m > z_i \mid \theta(X \upharpoonright m, Y \upharpoonright m, i + 1, m)\}$. Let $Z = \{z_i \mid i \in \mathbb{N}\}$, then we have $Z \text{ is infinite} \wedge \Psi(X, Z)$ (given a non-empty set $F \subseteq_{\text{fin}} Z$, set $\sigma = Y \upharpoonright \max F$).

To show the right to left implication, take $Z \subseteq \mathbb{N}$ such that $Z \text{ is infinite} \wedge \Psi(X, Z)$. Define a tree T as

$$T = \{\sigma \in 2^{<\mathbb{N}} \mid \forall F \subseteq Z \cap [0, |\sigma|]_{\mathbb{N}} (F \neq \emptyset \rightarrow (\forall n < |F|)(\exists m < \max F)\theta(X \upharpoonright m, \sigma \upharpoonright m, k, n, m))\}.$$

Then, since $\Psi(X, Z)$ holds, the tree T is infinite. By WKL_0 , take $Y \in [T]$. One can check that $\forall n \exists m \theta(X \upharpoonright m, Y \upharpoonright m, n, m)$ since Z is infinite. \square

Definition 2.5 ($\text{I}\Sigma_1^0$). Fix an ordinal $\alpha < \omega$ and a Ramsey-like- Π_2^1 -statement $(\forall f : [\mathbb{N}]^n \rightarrow k)(\exists Y)(Y \text{ is infinite} \wedge \Psi(f, Y))$. A set $Z \subseteq_{\text{fin}} \mathbb{N}$ is said to be α -large(Γ) if for any $f : [[0, \max Z]_{\mathbb{N}}]^n \rightarrow k$, there exists an α -large set $Y \subseteq Z$ such that $\Psi(f, Y)$ holds.

By the definition of a Ramsey-like- Π_2^1 -formula, if $Z' \supseteq Z$ and Z is α -large(Γ), then Z' is α -large(Γ).

Example 2.6. A set Z is α -large(RT_k^n) if for every coloring $f : [[0, \max Z]_{\mathbb{N}}]^n \rightarrow k$, there is an α -large f -homogeneous set $Y \subseteq Z$.

Note that α -largeness(Γ) for $\Gamma \in \{\text{RT}_k^n, \text{psRT}_k^2, \text{EM}\}$ are all Δ_0 -definable notions, and $\text{I}\Sigma_1^0$ proves that they are all regular. However, it is not obvious within $\text{I}\Sigma_1^0$ that they are all largeness notions. Actually, showing within $\text{I}\Sigma_1^0$ that α -largeness(RT_2^2) is a largeness notion is the key to know the Π_3^0 -part of RT_2^2 . Our approach is to measure the size of α -large(Γ) sets by comparing them with α -large sets. The following classical theorem is fundamental for this purpose. (It is not hard to check that the proof works within $\text{I}\Sigma_1^0$.)

Theorem 2.5 (Ketonen/Solovay[26, Section 6]). *Let $k \in \omega$. The following is provable within $\text{I}\Sigma_1^0$. If a finite set X is $k+4$ -large and $\min X > 3$, then X is k -large(RT_k^2).*

3 Density and $\tilde{\Pi}_3^0$ -conservation

The goal of this section is to prove the following theorem.

Theorem 3.1 (Conservation through largeness). *Let Γ be a Ramsey-like- Π_2^1 -statement. If k -large(Γ)ness is a largeness notion provably in $\text{I}\Sigma_1^0$ for any $k \in \omega$, then, $\text{WKL}_0 + \Gamma$ is a $\tilde{\Pi}_3^0$ -conservative extension of $\text{I}\Sigma_1^0$.*

For this, we will introduce an iterated version of a largeness notion which is called ‘‘density’’. This notion is introduced by Paris in [34].

Definition 3.1 ($\text{I}\Sigma_1^0$, Density notion). Fix a Ramsey-like II_2^1 -statement

$$\Gamma = (\forall f : [\mathbb{N}]^n \rightarrow k)(\exists Y)(Y \text{ is infinite} \wedge \Psi(f, Y)).$$

We define the notion of m -density(Γ) of a finite set $Z \subseteq \mathbb{N}$ inductively as follows. First, a set Z is 0 -dense(Γ) if it is \aleph_1 -large and $\min Z > 1$. Assuming the notion of m -density(Γ) is defined, a set Z is $(m+1)$ -dense(Γ) if

- for any $f : [[0, \max Z]_{\mathbb{N}}]^n \rightarrow \max Z$, there is an m -dense(Γ) set $Y \subseteq Z$ such that $\Psi(f, Y)$ holds, and,
- for any partition $Z_0 \sqcup \dots \sqcup Z_{k-1} = Z$ such that $k \leq Z_0 < \dots < Z_{k-1}$, one of Z_i 's is m -dense(Γ).

Note that there exists a Δ_0 -formula $\theta(m, Z)$ saying that “ Z is m -dense(Γ).” (Here, we always assume $\min Z > 1$ to avoid technical annoyances.)

In case Γ is psRT_k^2 or RT_k^n for some $n, k \geq 2$, the second condition is implied from the first condition as follows: for a given partition $Z_0 \sqcup \dots \sqcup Z_{k-1} = Z$, set $f(x, y) = 1$ if $x, y \in Z_i$ for some $i < k$ and $f(x, y) = 0$ otherwise, then, f is a transitive coloring and any \aleph_1 -large homogeneous set $H \subseteq Z$ is included in some Z_i 's. (For more precise explanations, see [34] or [4].) On the other hand, the density notion for EM without the second condition does not work well (see [4]).

Definition 3.2. Let Γ be a Ramsey-like- II_2^1 -statement. Then, the statement $m\text{-}\widetilde{\text{PH}}(\Gamma)$ asserts that for any $X_0 \subseteq \mathbb{N}$, if X_0 is infinite then there exists an m -dense(Γ) set X such that $X \subseteq_{\text{fin}} X_0$.

The density notion for Γ provides a cut to be a model of $\text{WKL}_0 + \Gamma$.

Lemma 3.2. Let Γ be a Ramsey-like- II_2^1 -statement. Given a countable nonstandard model M of $\text{I}\Sigma_1$ and an M -finite set $Z \subseteq M$ which is a -dense(Γ) for some $a \in M \setminus \omega$, then there exists an initial segment I of M such that $(I, \text{Cod}(M/I)) \models \text{WKL}_0 + \Gamma$ and $I \cap Z$ is infinite in I .

Proof. Let Γ be a Ramsey-like- II_2^1 -statement of the form

$$(\forall f : [\mathbb{N}]^n \rightarrow k)(\exists Y)(Y \text{ is infinite} \wedge \Psi(f, Y))$$

where $n, k \in \omega$ and Ψ is of the form in Definition 2.3. Let $M \models \text{I}\Sigma_1$ be a countable nonstandard model, and $Z \subseteq M$ be M -finite set which is a -dense(Γ) for some $a \in M \setminus \omega$. Let $\{E_i\}_{i \in \omega}$ be an enumeration of all M -finite sets such that any M -finite set appears infinitely many times, and $\{f_i\}_{i \in \omega}$ be an enumeration of all M -finite functions from $[[0, \max Z]_{\mathbb{N}}]^n$ to $\max Z$ such that $\Phi(f_i)$ holds.

In the following, we will construct an ω -length sequence of M -finite sets $Z = Z_0 \supseteq Z_1 \supseteq \dots$ so that for each $i \in \mathbb{N}$, Z_i is $(a-i)$ -dense(Γ), $\min Z_i < \min Z_{i+3}$, $\Psi(f_i \upharpoonright [[0, \max Z_{3i}]_{\mathbb{N}}]^n, Z_{3i+1})$, and $(\min Z_{3i+2}, \max Z_{3i+2})_{\mathbb{N}} \cap E_i = \emptyset$ if $|E_i| < \min Z_{3i+1}$.

At the stage $s = 3i$, let Z_{3i} and f_i be given. Then, one can find $Z_{3i+1} \subseteq Z_{3i}$ which is $(a-3i-1)$ -dense(Γ) such that $\Psi(f_i \upharpoonright [[0, \max Z_{3i}]_{\mathbb{N}}]^n, Z_{3i+1})$ by the definition of density(Γ).

At the stage $s = 3i+1$, let Z_{3i+1} and E_i be given. If $\min Z_{3i+1} \leq |E_i|$, let $Z_{3i+2} = Z_{3i+1}$. If $\min Z_{3i+1} > |E_i|$, let $E_i = \{e_0, \dots, e_{l-1}\}$ where $e_0 < e_1 < \dots < e_{l-1}$, and put $W^0 = Z_{3i+1} \cap [0, e_0)_{\mathbb{N}}$, $W^j = Z_{3i+1} \cap [e_{j-1}, e_j)_{\mathbb{N}}$ for $1 \leq j < l$, and $W^l = Z_{3i+1} \cap [e_{l-1}, \infty)_{\mathbb{N}}$. Then, $Z_{3i+1} = W^0 \sqcup \dots \sqcup W^l$, thus one of W_j 's is $(a-3i-2)$ -dense(Γ). Put Z_{3i+2} to be such W_j .

At the stage $s = 3i+2$, Put $Z_{3i+3} = Z_{3i+2} \setminus \{\min Z_{3i+2}\}$.

Now, let $I = \sup\{\min Z_i \mid i \in \omega\} \subseteq_e M$. By the construction of the steps $s = 3i + 1$, I is a semi-regular cut, thus $(I, \text{Cod}(M/I)) \models \text{WKL}_0$. By the construction of the steps $s = 3i + 2$, $Z_i \cap I$ is infinite in I for any $i \in \omega$. To check that $(I, \text{Cod}(M/I)) \models \Gamma$, let $f : [I]^n \rightarrow k \in \text{Cod}(M/I)$. Then, there exists f_i such that $f = f_i \cap I$. By the construction, $M \models \Psi(f_i \upharpoonright [[0, \max Z_{3i}]_{\mathbb{N}}]^n, Z_{3i+1})$ holds, thus we have $(I, \text{Cod}(M/I)) \models \Psi(f, Z_{3i+1} \cap I) \wedge Z_{3i+1} \cap I$ is infinite. \square

Theorem 3.3. *Let Γ be a Ramsey-like- Π_2^1 -statement. Then, $\text{WKL}_0 + \Gamma$ is a $\tilde{\Pi}_3^0$ -conservative extension of $\text{RCA}_0 + \{m\text{-}\widetilde{\text{PH}}(\Gamma) \mid m \in \omega\}$.*

By Theorem 3.3 and Proposition 2.4, one can characterize the $\tilde{\Pi}_3^0$ -part of any restricted- Π_2^1 -statement. This is actually a (generalization of a) special case of the well-known indicator arguments (see, e.g., [34, 25]). In general, one can replace the second condition and the initial condition for 0-density in Definition 3.1 with suitable indicator conditions for a base system T , then the partial conservation for $T + \Gamma$ over $T + \{m\text{-}\widetilde{\text{PH}}(\Gamma) \mid m \in \omega\}$ holds, and one can even consider the $\tilde{\Pi}_4^0$ -part in some cases. See [47, 48].

Proof of Theorem 3.3. By the usual compactness argument, one can easily check that $\text{WKL}_0 + \Gamma$ implies $m\text{-}\widetilde{\text{PH}}(\Gamma)$ for any $m \in \omega$. Thus, $\text{WKL}_0 + \Gamma$ is an extension of $\text{RCA}_0 + \{m\text{-}\widetilde{\text{PH}}(\Gamma) \mid m \in \omega\}$.

To see that it is a $\tilde{\Pi}_3^0$ -conservative extension, let $\varphi_0 \equiv \forall X \forall x \exists y \forall z \varphi(X[z], x, y, z)$ be a $\tilde{\Pi}_3^0$ -sentence which is not provable in $\text{RCA}_0 + \{m\text{-}\widetilde{\text{PH}}(\Gamma) \mid m \in \omega\}$, where φ is Σ_0^0 . Take a countable nonstandard model $(M, S) \models \text{RCA}_0 + \{m\text{-}\widetilde{\text{PH}}(\Gamma) \mid m \in \omega\} + \neg\varphi_0$. Then, there exist $A \in S$ and $a \in M$ such that $(M, S) \models \forall y \exists z \neg\varphi(A[z], a, y, z)$. In (M, S) , define a sequence $\langle x_i \mid i \in M \rangle$ so that $x_0 = a$ and $x_{i+1} = \min\{x > x_i \mid \forall y < x_i \exists z < x \neg\varphi(A[z], a, y, z)\}$. By recursive comprehension in (M, S) , put $X = \{x_i \mid i \in M\} \in S$. Then, X is infinite in (M, S) . By $m\text{-}\widetilde{\text{PH}}(\Gamma)$ for $m \in \omega$, there exist m -dense(Γ) finite subsets of X for any $m \in \omega$. Thus, by overspill for Σ_1^0 -statement, there exists an m -dense(Γ) finite subset Z of X for some $m \in M \setminus \omega$. Now, by Lemma 3.2, there exists $I \subseteq_e M$ such that $(I, \text{Cod}(M/I)) \models \text{WKL}_0 + \Gamma$ and $Z \cap I$ is infinite in I . Note that $\text{Cod}(M/I) = \{W \cap I \mid W \text{ is } M\text{-finite}\} = \{W \cap I \mid W \in S\}$. Since $Z \subseteq X$, $a \leq \min Z \in I$ and for any $w, w' \in Z \cap I$ such that $w < w'$, $(I, \text{Cod}(M/I)) \models \forall y < w \exists z < w' \neg\varphi(A \cap I[z], a, y, z)$. Since $Z \cap I$ is unbounded in I , we have $(I, \text{Cod}(M/I)) \models \forall y \exists z \neg\varphi(A \cap I[z], a, y, z)$, which means $(I, \text{Cod}(M/I)) \models \neg\varphi_0$. Thus, $\text{WKL}_0 + \Gamma$ does not prove φ_0 . \square

The following generalized version of the Parsons theorem plays a key role in this section.

Theorem 3.4 (Generalized Parsons theorem). *Let $\psi(F)$ be a Δ_0 -formula. Assume that*

$$\text{IS}_1^0 \vdash \forall X \subseteq \mathbb{N} (X \text{ is infinite} \rightarrow \exists F \subseteq_{\text{fin}} X \psi(F)).$$

Then, there exists $n \in \omega$ such that

$$\text{IS}_1 \vdash \forall Z \subseteq_{\text{fin}} (0, \infty)_{\mathbb{N}} (Z \text{ is } \textit{\textit{n}}\text{-large} \rightarrow \exists F \subseteq Z \psi(F)).$$

Note that one can apply this theorem for any $\tilde{\Pi}_3^0$ -conservative extension of IS_1^0 , e.g., $\text{WKL}_0 + \text{BS}_2^0$.

This theorem quickly implies the Parsons theorem (see, e.g., [5]), namely, any Π_2 -statement provably in IS_1 is bounded by a primitive recursive function, as follows. If a Π_2 -statement $\forall x \exists y \theta(x, y)$ is provable within IS_1^0 , then put $\psi(F) \equiv (\forall x < \min F)(\exists y < \max F)\theta(x, y)$. Then, IS_1^0 proves that any infinite set contains a finite set F such that $\psi(F)$ holds. By this theorem, there exists $n \in \omega$ such that $(Z \text{ is } \textit{\textit{n}}\text{-large} \rightarrow \exists F \subseteq Z \psi(F))$. One can easily find a primitive recursive function h such that $[a, h(a)]_{\mathbb{N}}$ is $\textit{\textit{n}}\text{-large}$ for any $a \in \omega$. Thus, we have $\text{IS}_1 \vdash \forall x \exists y < h(x)\theta(x, y)$.

The following proof is a generalization of the proof of Theorem 1.3, which is the Parsons theorem extended into WKL_0 .

Proof of Theorem 3.4. Assume that $\text{IS}_1 \not\vdash \forall Z \subseteq_{\text{fin}} (0, \infty)_{\mathbb{N}} (Z \text{ is } ^n\text{-large} \rightarrow \exists F \subseteq Z \psi(F))$ for any $n \in \omega$, where $\Psi(F)$ is Σ_1 . Then, there exists a countable model $M \models \text{IS}_1 + \{\exists Z \subseteq_{\text{fin}} (0, \infty)_{\mathbb{N}} (Z \text{ is } ^n\text{-large} \wedge \forall F \subseteq Z \neg \psi(F)) \mid n \in \omega\}$ such that $M \not\cong \omega$. By overspill, there exist $a \in M \setminus \omega$ and an M -finite set $Z \subseteq M$ with $\min Z > 0$ such that Z is a -large and $\forall F \subseteq Z \neg \psi(F)$. Let $\{E_i\}_{i \in \omega}$ be an enumeration of all M -finite sets such that any M -finite set appears infinitely many times. Now, we will construct an ω -length sequence of M -finite sets $Z = Z_0 \supseteq Z_1 \supseteq \dots$ so that Z_i is $^{a-i}$ -large, $\min Z_i < \min Z_{i+1}$ and $(\min Z_{i+1}, \max Z_{i+1}) \cap E_i = \emptyset$ if $|E_i| < \min Z_i$. Let Z_i and E_i be given such that Z_i is $^{a-i}$ -large. If $\min Z_i \leq |E_i|$, let $Z_{i+1} = Z_i \setminus \{\min Z_i\}$, then Z_{i+1} is $^{a-i-1}$ -large. If $\min Z_i > |E_i|$, let $E_i = \{e_0, \dots, e_{l-1}\}$ where $e_0 < e_1 < \dots < e_{l-1}$, and let $W = Z_i \setminus \{\min Z_i\}$, $W^0 = W \cap [0, e_0]_{\mathbb{N}}$, $W^j = W \cap [e_{j-1}, e_j]_{\mathbb{N}}$ for $1 \leq j < l$, and $W^l = W \cap [e_{l-1}, \infty)_{\mathbb{N}}$. Then, W is $(^{a-i-1} \cdot \min Z_i)$ -large and $W = W^0 \sqcup \dots \sqcup W^l$. Since $l+1 \leq \min Z_i$, there exists $j \leq l$ such that W^j is $^{a-i-1}$ -large by Lemma 2.2. Then, set $Z_{i+1} = W^j$.

Now, let $I = \sup\{\min Z_i \mid i \in \omega\} \subseteq_e M$. By the construction, I is a semi-regular cut, thus $(I, \text{Cod}(M/I)) \models \text{IS}_1^0$. On the other hand, $Z \cap I \in \text{Cod}(M/I)$ and $(I, \text{Cod}(M/I)) \models Z \cap I$ is infinite $\wedge \forall F \subseteq Z \cap I \neg \psi(F)$ since $\neg \psi(F)$ is Π_1 . This contradicts to $\text{IS}_1^0 \vdash \forall X \subseteq \mathbb{N} (X \text{ is infinite} \rightarrow \exists F \subseteq_{\text{fin}} X \psi(F))$. \square

We are now ready to prove the main conservation theorem of the section.

Proof of Theorem 3.1. Let Γ be a Ramsey-like- Π_2^1 -statement, and assume that for any $k \in \omega$,

$$\text{IS}_1^0 \vdash \forall X \subseteq \mathbb{N} (X \text{ is infinite} \rightarrow \exists F \subseteq_{\text{fin}} X (F \text{ is } ^k\text{-large}(\Gamma))).$$

Then, by Theorem 3.4, for each $k \in \omega$ there exists n_k such that

$$\text{IS}_1^0 \vdash \forall Z \subseteq_{\text{fin}} (0, \infty)_{\mathbb{N}} (Z \text{ is } ^{n_k}\text{-large} \rightarrow Z \text{ is } ^k\text{-large}(\Gamma)).$$

Now, put $h : \omega \rightarrow \omega$ as $h(0) = 1$ and $h(m+1) = \max\{n_{h(m)}, h(m) + 1\}$. We will check

$$\text{IS}_1^0 \vdash \forall Z \subseteq_{\text{fin}} (0, \infty)_{\mathbb{N}} (Z \text{ is } ^{h(m)}\text{-large} \rightarrow Z \text{ is } m\text{-dense}(\Gamma)).$$

by induction. The case $m = 0$ follows from the definition. The case $m = m' + 1$, $^{h(m')}$ -large sets are m' -dense(Γ) by the induction hypothesis. Then, the first condition of the $m' + 1$ -density follows from $h(m' + 1) \geq n_{h(m')}$, and the second condition follows from $h(m' + 1) \geq h(m') + 1$ and Lemma 2.2. Thus, by Theorem 2.3, IS_1^0 proves that any infinite set contains an m -dense(Γ) set for any $m \in \omega$. Hence, by Theorem 3.3, $\text{WKL}_0 + \Gamma$ is a $\tilde{\Pi}_3^0$ -conservative extension of IS_1^0 . \square

When two conservation results are obtained, one can often amalgamate those results. For example, if two Π_2^1 -theories T_1 and T_2 are Π_1^1 -conservative over a base Π_2^1 -theory T_0 , then $T_1 + T_2$ is also Π_1^1 -conservative over T_0 (see [46]). Similar amalgamation property holds for $\tilde{\Pi}_3^0$ -conservation as follows.

Theorem 3.5 (Amalgamation). *Fix $n \geq 1$. Let T be a theory extending IS_1^0 which consists of sentences of the form $\forall X \exists Y \theta(X, Y)$ where θ is Π_{n+2}^0 , and let Γ_1 and Γ_2 be sentences of the same form as T . If $T + \Gamma_i$ is a $\tilde{\Pi}_{n+2}^0$ -conservative extension of T for $i = 1, 2$, then, $T + \Gamma_1 + \Gamma_2$ is a $\tilde{\Pi}_{n+2}^0$ -conservative extension of T .*

Proof. The proof is essentially the same as the case for the amalgamation of two Π_2^1 -theories which are Π_1^1 -conservative over a base theory in [46]. Here, we consider the \mathcal{L}_2 -structure (M, S) as a two-sorted structure, namely, M and S are disjoint sets and \in is a relation on $M \times S$. In this understanding, $\tilde{\Pi}_{n+2}^0$ -conservation implies that any model of T has a Σ_{n+2}^0 -elementary extension which is a model of $T + \Gamma_i$ for $i = 1, 2$. (This is because if $(M, S) \models T$, then $\text{Th}_{\mathcal{L}_2 \cup M \cup S}(M, S) \cap \Sigma_{n+2}^0 + T + \Gamma_i$ is consistent.) Now, assume $T \not\models \forall X \psi(X)$ where ψ is Π_{n+2}^0 , and take a model $(M_0, S_0) \models T + \exists X \neg \psi(X)$. Then, one can construct a Σ_{n+2}^0 -elementary chain of models $(M_0, S_0) \subseteq (M_1, S_1) \subseteq \dots$ such that $(M_{2j+i}, S_{2j+i}) \models T + \Gamma_i$ for $i = 1, 2$ and $j \in \omega$. By the usual elementary chain argument, $(\bar{M}, \bar{S}) = (\bigcup_{k \in \omega} M_k, \bigcup_{k \in \omega} S_k)$ is a Σ_{n+2}^0 -elementary extension of (M_0, S_0) , and therefore $(\bar{M}, \bar{S}) \models T + \Gamma_1 + \Gamma_2 + \exists X \neg \psi(X)$. Hence $T + \Gamma_1 + \Gamma_2 \not\models \forall X \psi(X)$. \square

Note that $\text{I}\Sigma_1^0$, WKL_0 and any Ramsey-like statement is of the form $\forall X \exists Y \theta(X, Y)$ where θ is Π_3^0 . Therefore, one can always use the amalgamation theorem for $\tilde{\Pi}_3^0$ -conservation. In particular, to prove the $\tilde{\Pi}_3^0$ -conservation theorem for RT_2^2 , we only need to prove $\tilde{\Pi}_3^0$ -conservation theorems for ADS and EM.

4 Conservation theorem for ADS

In this section, we will show that $\text{WKL}_0 + \text{ADS}$ is a $\tilde{\Pi}_3^0$ -conservative extension of $\text{I}\Sigma_1^0$. Actually, this is just a weakening of the following theorem by Chong, Slaman and Yang, where CAC is the chain antichain principle, since ADS is a consequence of CAC over RCA_0 [21].

Theorem 4.1 (Chong, Slaman, Yang [10]). *$\text{WKL}_0 + \text{CAC}$ is a Π_1^1 -conservative extension of $\text{B}\Sigma_2^0$.*

Here, we will give an alternative proof by calculating the size of k -large(psRT_2^2) sets. To simplify the proof below, we will use a slightly modified α -largeness notion.

Definition 4.1 ($\text{I}\Sigma_1^0$). Any set is said to be 0-large*. Given some $\alpha < {}^\omega$, $X \subseteq_{\text{fin}} \mathbb{N}$ is said to be α -large* if

- $X \setminus \{\min X\}$ is β -large* if $\alpha = \beta + 1$,
- X is $(\beta + {}^{n-1} \cdot \min X)$ -large* if $\alpha = \beta + {}^n$.

Trivially, if $X \subseteq_{\text{fin}} \mathbb{N}$ is α -large, then X is α -large*.

Lemma 4.2. *For any $k \in \omega$, the following is provable within $\text{I}\Sigma_1^0$. For any $\alpha < {}^k$ and for any $X \subseteq_{\text{fin}} \mathbb{N}$, X is α -large if X is $\alpha + 1$ -large*.*

Proof. By Π_1^0 -transfinite induction up to k , which is available within RCA_0 . \square

Lemma 4.3. *The following is provable within $\text{I}\Sigma_1^0$. For any $k, n \in \mathbb{N}$, if X is a disjoint union of X_0, \dots, X_{k-1} such that $X_i < X_{i+1}$ for any $i < k - 1$ and each X_i is n -large*, then X is ${}^n \cdot k$ -large*. Thus, if $k \leq \min X_0$, X is ${}^{n+1}$ -large*.*

Proof. Similar to Lemma 2.2. \square

Lemma 4.4. *For any $k \in \omega$, the following is provable within $\text{I}\Sigma_1^0$. If a finite set $X \subseteq \mathbb{N}$ is ${}^{2k+6}$ -large and $\min X > 3$, then X is k -large(psRT_2^2).*

Proof. Given a coloring $f : [[0, \max X]_{\mathbb{N}}]^2 \rightarrow 2$, define the coloring $\bar{f} : [X]^2 \rightarrow 2k + 2$ as $\bar{f}(x, y) = 2j + i$ if $f(x, y) = i$ and $j = \min\{j' < k \mid \neg(\exists H \subseteq [x, y]_{\mathbb{N}} \cap X \ x \in H, H \text{ is } j'+1\text{-large}^* \text{ and } H \cup \{y\} \text{ is pseudo-homogeneous for } f \text{ with the color } i)\} \cup \{k\}$.

By Theorem 2.5, take $Y \subseteq X$ such that Y is l -large and \bar{f} -homogeneous. Let $Y = \{y_0 < y_1 < \dots < y_l\}$, and $\bar{f}([Y]^2) = 2j + i$. Then, $l \geq y_0$. By the definition of \bar{f} , for $s = 0, \dots, l - 1$, one can take $H_s \subseteq [y_s, y_{s+1}]_{\mathbb{N}}$ such that $y_s \in H_s$, H_s is j -large* and $[H_s \cup \{y_{s+1}\}]^2$ is pseudo-homogeneous for f with the color i . By Lemma 4.3, $H = \bigcup_{s=0}^{l-1} H_s$ is $j+1$ -large*, and $[H \cup \{y_k\}]^2$ is pseudo-homogeneous for f with the color i . This H assures that $\bar{f}(y_0, y_k) \neq 2j + i$ or $j = k$. Thus, we have $j = k$. Hence H is pseudo-homogeneous for f and $k+1$ -large*, thus it is k -large. \square

Corollary 4.5. $\text{WKL}_0 + \text{psRT}_2^2$, or equivalently, $\text{WKL}_0 + \text{ADS}$ is a $\tilde{\Pi}_3^0$ -conservative extension of IS_1^0 .

Proof. By Theorems 2.3, 3.1 and Lemma 4.4. \square

Note that we could have proven Corollary 4.5 by working with ADS directly. However, the unnatural formulation of ADS as a Ramsey-like- Π_2^1 -statement introduces additional technicalities in the proof. Indeed, the standard formulation of ADS involves linear orders, whereas a Ramsey-like statement is about arbitrary coloring functions. In this framework, a solution to ADS is either an infinite homogeneous set, or a set whose minimal element witnesses the non-transitivity of the coloring.

5 Grouping principle

In this section, we introduce the grouping principle, which is a consequence of Ramsey's theorem. The grouping principle will be used in the conservation proof of the Erdős-Moser theorem, although it is currently unknown how the two statements relate in reverse mathematics. The grouping principle seems interesting to study in its own right, and we conduct a study of its relations with other Ramsey-type principles already studied in reverse mathematics.

Definition 5.1 (RCA₀, grouping principle). Given a largeness notion L and a coloring $f : [X]^n \rightarrow k$, an L -grouping for f is an infinite family of L -large finite sets $\{F_0 < F_1 < \dots\} \subseteq L$ such that

$$\forall i_1 < \dots < i_n \exists c < k \forall x_1 \in F_{i_1}, \dots, \forall x_n \in F_{i_n} f(x_1, \dots, x_n) = c$$

Now $\text{GP}_k^n(L)$ (grouping principle for L) asserts that for any coloring $f : [\mathbb{N}]^n \rightarrow k$, there exists an infinite L -grouping for f . We write GP_k^n for the statement saying that for any largeness notion L , $\text{GP}_k^n(L)$ holds, GP^n for $\forall k \text{GP}_k^n$, and GP for $\forall n \text{GP}^n$.

Note that being a largeness notion is a Π_1^1 -statement. Therefore, an instance of GP_2^2 is a pair $\langle L, f \rangle$ where L is a collection of finite sets, and $f : [\mathbb{N}]^2 \rightarrow 2$ is a coloring. A solution to an instance $\langle L, f \rangle$ is either an L -grouping for f , or an infinite set witnessing that L is not a largeness notion, that is, an infinite set with no finite subset in L .

In order to simplify the analysis of Ramsey's theorem for pairs, Cholak, Jockusch and Slaman [7] split the proof of RT_2^2 into *cohesiveness* and a stable restriction of RT_2^2 . A coloring $f : [\mathbb{N}]^2 \rightarrow k$ is *stable* if for every x , $\lim_y f(x, y)$ exists. SRT_k^2 is the restriction of RT_2^2 to stable colorings.

Definition 5.2 (Cohesiveness). An infinite set C is \vec{R} -cohesive for a sequence of sets R_0, R_1, \dots if for each $i \in \mathbb{N}$, $C \subseteq^* R_i$ or $C \subseteq^* \overline{R_i}$. COH is the statement "Every uniform sequence of sets \vec{R} has an \vec{R} -cohesive set."

Cohesiveness is a statement from standard computability theory. Cholak, Jockusch and Slaman [7] claimed with an erroneous proof that it is a strict consequence of RT_2^2 over RCA_0 . Mileti [31] fixed the proof. Hirschfeldt and Shore [21] proved that COH is a consequence of ADS . Since then, many statements in reverse mathematics have been split into their cohesive and their stable part [21]. Accordingly, we will consider the stable version SGP_2^2 which stands for GP_2^2 for stable colorings. One can prove that $\text{RCA}_0 \vdash \text{COH} + \text{SGP}_2^2 \rightarrow \text{GP}_2^2$ by the same argument as $\text{RCA}_0 \vdash \text{COH} + \text{SRT}_2^2 \rightarrow \text{RT}_2^2$ [7]. Stable Ramsey's theorem for pairs admits a nice computability-theoretic characterization in terms of infinite subsets of a Δ_2^0 set. We can give a similar characterization for the stable grouping principle for pairs.

Definition 5.3 (RCA_0). Given a largeness notion L , an L -grouping for a set A is an infinite family of L -large finite sets $\{F_0 < F_1 < \dots\} \subseteq \text{L}$ such that $(\forall i)[F_i \subseteq A \vee F_i \subseteq \overline{A}]$

The argument can be carried-out within $\text{B}\Sigma_2^0$. Unlike SRT_2^2 , it is currently unknown whether SGP_2^2 implies $\text{B}\Sigma_2^0$ over RCA_0 .

Lemma 5.1. $\text{RCA}_0 + \text{B}\Sigma_2^0 \vdash \text{SGP}_2^2(\text{L}) \leftrightarrow$ “Every Δ_2^0 set has an infinite L -grouping”.

We will now show the existence of an ω -model of SGP_2^2 containing only low sets. Recall that an instance is a pair $\langle \text{L}, f \rangle$, and a solution is either a witness that L is not a largeness notion, or an L -grouping for f . We need therefore to show that given any computable collection of finite sets L and any Δ_2^0 set A , there is either an infinite low set Y witnessing that L is not a largeness notion, or a low L -grouping for A . In what follows, we denote by LOW the collection of all low sets.

Theorem 5.2. For every computable set L which is a largeness notion on (ω, LOW) , every Δ_2^0 set has an infinite low L -grouping.

Proof. Fix a Δ_2^0 set A . We will construct an infinite low L -grouping for A by an effective forcing notion whose conditions are tuples $c = (F_0, \dots, F_k, X_0, \dots, X_m)$ such that

- (i) $F_i \in \text{L}$ and $F_i \subseteq_{\text{fin}} A$ or $F_i \subseteq_{\text{fin}} \overline{A}$ for each $i \leq k$,
- (ii) $F_i < F_{i+1}$ for each $i < k$,
- (iii) $X_0 \sqcup \dots \sqcup X_m$ is a low partition of ω .

A condition $d = (F_0, \dots, F_\ell, Y_0, \dots, Y_n)$ extends a condition $c = (F_0, \dots, F_k, X_0, \dots, X_m)$ (written $d \leq c$) if $\ell \geq k$, for every $i \in (k, \ell]_{\mathbb{N}}$, $F_i \subseteq X_j$ for some $j \leq m$ and Y_0, \dots, Y_n refines X_0, \dots, X_m , that is, for each $i \leq n$, there is some $j \leq m$ such that $Y_i \subseteq X_j$. An *index* of the condition c is a tuple (F_0, \dots, F_k, e) where $\Phi_e^{\emptyset'}$ decides the jump of the partition X_0, \dots, X_m . We first claim that the finite sequence of sets can be extended.

Claim. For every condition $c = (F_0, \dots, F_k, X_0, \dots, X_m)$, there is an extension $d = (F_0, \dots, F_\ell, X_0, \dots, X_m)$ of c such that $\ell > k$. Moreover, an index of d can be found \emptyset' -uniformly in an index of c .

Proof of the claim. We first show that there is a set $F > F_k$ such that $F \in \text{L} \cap X_i$ for some $i \leq m$ and $F \subseteq A$ or $F \subseteq \overline{A}$. Let $i \leq m$ be such that X_i is infinite. We claim that there is some finite set $F \in \text{L}$ such that $F \subseteq X_i \cap A \setminus [0, \max(F_k)]_{\mathbb{N}}$ or $F \subseteq X_i \cap \overline{A} \setminus [0, \max(F_k)]_{\mathbb{N}}$. Suppose for the sake of contradiction that there is no such set. Then the Π_1^{0, X_i} class of all sets Z such that for every $F \in \text{L}$, $F \not\subseteq X_i \cap Z \setminus [0, \max(F_k)]_{\mathbb{N}}$ and $F \not\subseteq X_i \cap \overline{Z} \setminus [0, \max(F_k)]_{\mathbb{N}}$ is non-empty. By the low basis

theorem, there is a Z such that $Z \oplus X_i$ is low over X_i , hence low. The set Z or its complement contradicts the fact that \mathbf{L} is a largeness notion on (ω, \mathbf{LOW}) .

Knowing that such a set F exists, we can find it \emptyset' -uniformly in c and a Δ_2^0 index of the set A . The condition $(F_0, \dots, F_k, F, X_0, \dots, X_m)$ is a valid extension of c . Note that such a set F does not need to be part of an infinite $X_i \cap A$ or $X_i \cap \bar{A}$. The choice of an infinite part has simply been used to claim the existence of any such set. \square

We say that an \mathbf{L} -grouping for $A \langle E_0 < E_1 < \dots \rangle$ satisfies a condition $c = (F_0, \dots, F_k, X_0, \dots, X_m)$ if $E_0 = F_0, \dots, E_k = F_k$ and for every $i > k$, there is some $j \leq m$ such that $E_i \subseteq X_j$. A condition c forces formula $\varphi(G)$ if $\varphi(\vec{E})$ holds for every \mathbf{L} -grouping \vec{E} satisfying c .

Claim. *For every condition c and every index $e \in \omega$, there is an extension d forcing either $\Phi_e^G(e) \downarrow$ or $\Phi_e^G(e) \uparrow$. Moreover, an index of d can be found \emptyset' -uniformly in an index of c and e .*

Proof of the claim. Fix a condition $c = (F_0, \dots, F_k, X_0, \dots, X_m)$. We have two cases.

In the first case, for every 2-partition $Z_0 \cup Z_1 = \omega$, there is a sequence of finite sets F_{k+1}, \dots, F_ℓ such that $F_k < F_{k+1} < \dots < F_\ell$, $\Phi_e^{F_0, \dots, F_\ell}(e) \downarrow$, and for every $i \in (k, \ell]$, $F_i \in \mathbf{L}$ and there is some $j \leq m$ $F_i \subseteq Z_0 \cap X_j$ or $F_i \subseteq Z_1 \cap X_j$. In particular, taking $Z_0 = A$ and $Z_1 = \bar{A}$, there is a sequence of finite sets F_{k+1}, \dots, F_ℓ such that $d = (F_0, \dots, F_\ell, X_0, \dots, X_m)$ is a valid extension of c and $\Phi_e^{F_0, \dots, F_\ell}(e) \downarrow$. Such an extension can be found \emptyset' -uniformly in an index of c , e and a Δ_2^0 index of A .

In the second case, the $\Pi_1^{0, \vec{X}}$ class of all the 2-partitions $Z_0 \cup Z_1 = \omega$ such that $\Phi_e^{F_0, \dots, F_\ell}(e) \uparrow$ for every sequence of finite sets F_{k+1}, \dots, F_ℓ such that $F_k < F_{k+1} < \dots < F_\ell$, and for every $i \in (k, \ell]_{\mathbb{N}}$, $F_i \in \mathbf{L}$ and there is some $j \leq m$ $F_i \subseteq Z_0 \cap X_j$ or $F_i \subseteq Z_1 \cap X_j$ is non-empty. By the low basis theorem [24] relativized to \vec{X} , there is a such a 2-partition $Z_0 \cup Z_1 = \omega$ such that $Z_0 \oplus Z_1 \oplus \vec{X}$ is low. Moreover, a lowness index for $Z_0 \oplus Z_1 \oplus \vec{X}$ can be found uniformly in a lowness index for \vec{X} . The condition $d = (F_0, \dots, F_k, X_0 \cap Z_0, X_0 \cap Z_1, \dots, X_m \cap Z_0, X_m \cap Z_1)$ is an extension of c forcing $\Phi_e^G(e) \uparrow$.

Moreover, we can \vec{X}' -decide (hence \emptyset' -decide) whether the $\Pi_1^{0, \vec{X}}$ class is empty, thus we can find the extension d \emptyset' -effectively in an index of c and e . \square

Thanks to the claims, define an infinite, uniformly \emptyset' -computable decreasing sequence of conditions $c_0 = (\varepsilon, \omega) \geq c_1 \geq c_2 \geq \dots$, where $c_s = (F_0, \dots, F_{k_s}, X_0^s, \dots, X_{m_s}^s)$ such that for each $s \in \omega$

- (a) $k_s \geq s$
- (b) c_{s+1} forces $\Phi_s^G(s) \uparrow$ or $\Phi_s^G(s) \downarrow$

This sequence yields a \mathbf{L} -grouping for $A \langle F_0, F_1, \dots \rangle$ which is infinite by (a) and whose jump is Δ_2^0 by (b). This finishes the proof of Theorem 5.2. \square

Corollary 5.3. $\text{SGP}_2^2 + \text{SADS} + \text{WKL}_0$ has an ω -model with only low sets.

Proof. By Lemma 5.1 and the low basis theorem [24] in a relativized form. As explained, for every collection \mathbf{L} and every stable coloring $f : [\omega]^2 \rightarrow 2$, one need either to add a low set witnessing that \mathbf{L} is not a notion of largeness, or to add an infinite low \mathbf{L} -grouping for f . \square

Corollary 5.4. $\text{RCA}_0 + \text{SGP}_2^2 + \text{SADS} + \text{WKL}_0$ implies neither SRT_2^2 , nor SEM .

Proof. Downey, Hirschfeldt, Lempp and Solomon [12] built a computable instance of SRT_2^2 with no low solution. Corollary 5.3 enables us to conclude that $\text{RCA}_0 + \text{SGP}_2^2 + \text{SADS} + \text{WKL}_0$ does not imply SRT_2^2 . Since $\text{RCA}_0 + \text{SADS} + \text{SEM}$ implies SRT_2^2 (see [30]) then $\text{RCA}_0 + \text{SGP}_2^2 + \text{SADS} + \text{WKL}_0$ does not imply SEM. \square

Among the computability-theoretic properties used to separate Ramsey-type theorems in reverse mathematics, the framework of preservation of hyperimmunity has been especially fruitful.

Definition 5.4 (Hyperimmunity). The *principal function* of a set $B = \{x_0 < x_1 < \dots\}$ is the function p_B defined by $p_B(i) = x_i$ for each $i \in \mathbb{N}$. A set X is *hyperimmune* if its principal function is not dominated by any computable function.

Wang [44] recently used the notion of preservation of the arithmetic hierarchy to separate various theorems in reverse mathematics. The first author showed [36] that a former separation of the Erdős-Moser theorem from stable Ramsey's theorem for pairs due to Lerman, Solomon and Towsner [30] could be reformulated in a similar framework, yielding the notion of preservation of hyperimmunity.

Definition 5.5 (Preservation of hyperimmunity). A Π_2^1 -statement P *admits preservation of hyperimmunity* if for each set Z , each Z -hyperimmune sets A_0, A_1, \dots , and each P -instance $X \leq_T Z$, there is a solution Y to X such that the A 's are $Y \oplus Z$ -hyperimmune.

In particular, if a Π_2^1 -statement P admits preservation of hyperimmunity but another statement Q does not, then P does not imply Q over RCA_0 . We now show that the grouping principle enjoys preservation of hyperimmunity and deduce several separations from it.

Theorem 5.5. SGP_2^2 *admits preservation of hyperimmunity.*

Proof. Let C be a set and B_0, B_1, \dots be a sequence of C -hyperimmune sets. Let S be the collection of all sets X such that the B 's are $X \oplus C$ -hyperimmune. By Lemma 5.1, it suffices to show that for every $\Delta_2^{0,C}$ set A and every C -computable largeness notion L on (ω, S) there is an infinite L -grouping $\vec{F} = \langle F_0 < F_1 < \dots \rangle$ for A such that the B 's are $\vec{F} \oplus C$ -hyperimmune. Therefore, every instance $\langle L, A \rangle$ will have a solution $Y \in S$, which will be either a witness that L is not a largeness notion, or an L -grouping for A .

Fix A and L . We will construct an infinite L -grouping for A by a forcing argument whose conditions are tuples (F_0, \dots, F_k, X) where

- (i) $F_i \in L$ and $F_i \subseteq_{\text{fin}} A$ or $F_i \subseteq_{\text{fin}} \overline{A}$ for each $i \leq k$.
- (ii) $F_i < F_{i+1}$ for each $i < k$
- (iii) X is an infinite set such that the B 's are $X \oplus C$ -hyperimmune.

A condition $d = (F_0, \dots, F_\ell, Y)$ *extends* a condition $c = (F_0, \dots, F_k, X)$ (written $d \leq c$) if $\ell \geq k$ and for every $i \in (k, \ell]_{\mathbb{N}}$, $F_i \subseteq X$. The proof of the following claim is exactly the same as in Theorem 5.2, using the hyperimmune-free basis theorem instead of the low basis theorem.

Claim. *For every condition $c = (F_0, \dots, F_k, X)$, there is an extension $d = (F_0, \dots, F_\ell, Y)$ of c such that $\ell > k$.*

The following claim shows that every sufficiently generic filter yields a sequence \vec{F} such that the B 's are $\vec{F} \oplus C$ -hyperimmune. The notion of satisfaction and of forcing a formula $\varphi(G)$ are defined as in Theorem 5.2.

Claim. For every condition c and every pair of indices $e, i \in \omega$, there is an extension d forcing $\Phi_e^{G \oplus C}$ not to dominate p_{B_i} .

Proof of the claim. Fix a condition $c = (F_0, \dots, F_k, X)$. Let f be the function which on input x , searches for a finite set of integers U such that for every 2-partition $Z_0 \cup Z_1 = X$, there is some finite sequence of sets F_{k+1}, \dots, F_ℓ such that $F_k < \dots < F_\ell$, $\Phi_e^{(F_0, \dots, F_\ell) \oplus C}(x) \downarrow \in U$ and for every $i \in (k, \ell]_{\mathbb{N}}$, $F_i \in \mathbb{L} \cap Z_0$ or $F_i \in \mathbb{L} \cap Z_1$. If such a set U is found, $f(x) = 1 + \max(U)$, otherwise $f(x) \uparrow$. The function f is partial $X \oplus C$ -computable. We have two cases.

- Case 1: f is total. By $X \oplus C$ -hyperimmunity of B_i , there is some x such that $f(x) \leq p_{B_i}(x)$. Let U be the finite set witnessing $f(x) \downarrow$. By taking $Z_0 = X \cap A$ and $Z_1 = X \cap \bar{A}$, there is a finite sequence of sets F_{k+1}, \dots, F_ℓ such that $\Phi_e^{(F_0, \dots, F_\ell) \oplus C}(x) \downarrow \in U$ and $d = (F_0, \dots, F_\ell, X)$ is a valid extension of c . The condition d forces $\Phi_e^{G \oplus C}(x) < f(x)$.
- Case 2: there is some x such that $f(x) \uparrow$. By compactness, the $\Pi_1^{0, X \oplus C}$ class \mathcal{C} of sets $Z_0 \oplus Z_1$ such that $Z_0 \cup Z_1 = X$ and $\Phi_e^{(F_0, \dots, F_\ell) \oplus C}(x) \uparrow$ for every finite sequence of sets F_{k+1}, \dots, F_ℓ such that $F_k < \dots < F_\ell$ and for every $i \in (k, \ell]_{\mathbb{N}}$, $F_i \in \mathbb{L} \cap Z_0$ or $F_i \in \mathbb{L} \cap Z_1$ is not empty. By the hyperimmune-free basis theorem [24], there exists some partition $Z_0 \oplus Z_1 \in \mathcal{C}$ such that the B 's are $Z_0 \oplus Z_1 \oplus C$ -hyperimmune. The set Z_j is infinite for some $j < 2$ and the condition $d = (F_0, \dots, F_k, Z_j)$ is an extension of c forcing $\Phi_e^{G \oplus C}(x) \uparrow$.

□

Let \mathcal{F} be a sufficiently generic filter for this notion of forcing. The filter \mathcal{F} yields a sequence $\vec{F} = \langle F_0, F_1, \dots \rangle$ which is infinite by the first claim, and such that the B 's are $\vec{F} \oplus C$ -hyperimmune by the second claim. This finishes the proof of Theorem 5.5. □

Corollary 5.6. $\text{RCA}_0 + \text{GP}_2^2 + \text{COH} + \text{EM} + \text{WKL}_0$ does not imply ADS.

Proof. By the hyperimmune-free basis theorem [24], WKL_0 admits preservation of hyperimmunity. The first author proved in [36] that COH and EM admit preservation of hyperimmunity, but that ADS does not. Last, GP_2^2 admits preservation of hyperimmunity since $\text{COH} + \text{SGP}_2^2$ implies GP_2^2 over RCA_0 . □

Definition 5.6 (Diagonally non-computable function). A function f is *diagonally non-computable* (d.n.c.) relative to X if for every e , $f(e) \neq \Phi_e^X(e)$. 2-DNC is the statement “For every set X , there is a function d.n.c. relative to the jump of X ”.

Beware, the notation 2-DNC may cause some confusion with DNC_2 , the restriction to $\{0, 1\}$ -valued d.n.c. functions which is equivalent to WKL_0 over RCA_0 . The following proof is an adaptation of the proof that the Erdős-Moser implies 2-DNC over RCA_0 [37].

Theorem 5.7. $\text{RCA}_0 \vdash \text{GP}_2^2(\mathbb{L}_\omega) \rightarrow 2\text{-DNC}$

Proof. Fix a set X . Let $g(\cdot, \cdot)$ be a total X -computable function such that $\Phi_e^{X'}(e) = \lim_s g(e, s)$ if the limit exists, and $\Phi_e^{X'}(e) \uparrow$ if the limit does not exist. Also fix for each $e \in \mathbb{N}$ an enumeration $D_{e,0}, D_{e,1}, \dots$ of all finite sets of cardinal 3^{e+1} . We define the function $f : [\mathbb{N}]^2 \rightarrow 2$ by primitive recursion. Let f_0 be the function nowhere defined. At stage $s + 1$, do the following. Start with $f_{s+1} = f_s$. Then, for each $e < s$, take the first pair $\{x, y\} \in [(D_{e,g(e,s)} \cap [0, s]_{\mathbb{N}}) \setminus \bigcup_{k < e} D_{k,g(k,s)}]^2$ if it exists, and set $f_{s+1}(x, s) = 0$ and $f_{s+1}(y, s) = 1$. Finally, set $f(z, s) = 0$ for any $z < s$ such

that $f_{s+1}(s, z)$ remains undefined. This finishes the construction of f_{s+1} . Note that f_s is defined on $[[0, s]_{\mathbb{N}}]^2$. Thus, $f = \bigcup_s f_s$ must exist and is total on $[\mathbb{N}]^2$.

By $\text{GP}_2^2(\mathbb{L}_\omega)$, let $\vec{F} = \langle F_0, F_1, \dots \rangle$ be an infinite \mathbb{L}_ω -grouping for f . Let $h(e)$ be such that $D_{e, h(e)} \subseteq F$ for some $F \in \vec{F}$. Such an F exists since $D_{e,0}, D_{e,1}, \dots$ enumerates all finite sets of cardinality 3^{e+1} , and \vec{F} contains sets of arbitrary size. We claim that $h(e) \neq \Phi_e^{X'}(e)$ for all e , which would prove 2-DNC. Suppose otherwise, i.e., suppose that $\Phi_e^{X'}(e) = h(e)$ for some e . Let $F \in \vec{F}$ be such that $D_{e, h(e)} \subseteq F$. Then there is a stage s_0 such that $h(e) = g(e, s)$ for all $s \geq s_0$ or equivalently $D_{e, g(e, s)} = D_{e, h(e)} \subseteq F$ for all $s \geq s_0$. We claim that for any s be bigger than both $\max(F)$ and s_0 , there are some $x, y \in D_{e, h(e)} \subseteq F$ such that $f(x, s) \neq f(y, s)$, which contradicts the fact that \vec{F} is an \mathbb{L}_ω -grouping for f .

To see this, let s be such a stage. At that stage s of the construction of f , a pair $\{x, y\} \in [(D_{e, g(e, s)} \cap [0, s]_{\mathbb{N}}) \setminus \bigcup_{k < e} D_{k, g(k, s)}]^2$ is selected by a cardinality argument since $|D_{e, s} \cap [0, s]_{\mathbb{N}}| = |D_{e, s}| = 3^{e+1} > \sum_{k < e} 3^{k+1} = |\bigcup_{k < e} D_{k, g(k, s)}|$. Since $D_{e, g(e, s)} = D_{e, h(e)} \subseteq F$, this pair is contained in F . At this stage, we set $f(x, s) \neq f(y, s)$, therefore, \vec{F} is not an \mathbb{L}_ω -grouping for f , contradiction. \square

In particular, SRT_2^2 does not imply $\text{GP}_2^2(\mathbb{L}_\omega)$ over RCA_0 since Chong, Slaman and Yang [9] built a (non-standard) model of $\text{SRT}_2^2 + \text{B}\Sigma_2^0$ containing only low sets, whereas provably in RCA_0 , there is no Δ_2^0 d.n.c. function relative to \emptyset' .

Definition 5.7 (Rainbow Ramsey theorem). Fix $n, k \in \mathbb{N}$. A coloring function $f : [\mathbb{N}]^n \rightarrow \mathbb{N}$ is k -bounded if for every $y \in \mathbb{N}$, $\text{card } f^{-1}(y) \leq k$. A set R is a *rainbow* for f (or an f -rainbow) if f is injective over $[R]^n$. RRT_k^n is the statement ‘‘Every k -bounded function $f : [\mathbb{N}]^n \rightarrow \mathbb{N}$ has an infinite f -rainbow’’.

Miller [32] proved that the statement 2-DNC is equivalent to the rainbow Ramsey theorem for pairs (RRT_2^2) over RCA_0 .

Corollary 5.8. $\text{RCA}_0 \vdash \text{GP}_2^2 \rightarrow \text{RRT}_2^2$.

Seetapun and Slaman [39] defined a *Cardinality scheme* for a set of formulas Γ as follows. For every $\varphi(x, y) \in \Gamma$, $\text{C}\Gamma$ contains the universal closure of the formula ‘‘If $\varphi(x, y)$ defines an injective function, then its range is unbounded’’. Conidis and Slaman [11] proved that the rainbow Ramsey theorem for pairs implies the Σ_2 cardinality scheme ($\text{C}\Sigma_2$).

Corollary 5.9. $\text{RCA}_0 \vdash \text{GP}_2^2 \rightarrow \text{C}\Sigma_2$.

In particular, this shows that GP_2^2 is not Π_1^1 -conservative over $\text{RCA}_0 + \text{I}\Sigma_1^0$ since a Skolem hull argument shows that $\text{C}\Sigma_2$ is not provable in $\text{I}\Sigma_1$ (see Seetapun and Slaman [39]).

6 Conservation theorem for GP_2^2

In this section, we will prove a conservation result for the grouping principle. To calculate the size of α -large(EM) sets in Section 7, we will use a finite version of the grouping principle within $\text{I}\Sigma_1^0$. Instead of proving the finite grouping principle within $\text{I}\Sigma_1^0$ directly, we would like to show a conservation theorem for the infinite grouping principle over $\text{I}\Sigma_1^0$.

In Section 5, we have seen that SGP_2^2 has an ω -model with only low sets. It is well-known that a low solution construction is often able to be converted into a forcing construction of a solution satisfying Σ_1^0 -induction within a countable nonstandard model, which leads to a Π_1^1 -conservation

over IS_1^0 . Unfortunately, our construction of a low solution for SGP_2^2 in Section 5 requires BS_2^0 and thus it is not formalizable within IS_1^0 . To overcome this situation, we show a general conservation theorem characterized by using recursively saturated models.

Theorem 6.1. *Fix $n \geq 1$. Let Γ be a formula of the form $\forall X \exists Y \theta(X, Y)$ where θ is Π_{n+1}^0 . Then, $\text{RCA}_0 + \text{BS}_{n+1}^0 + \Gamma$ is a $\tilde{\Pi}_{n+2}^0$ -conservative extension of IS_n^0 if the following condition holds:*

(†) *for any countable recursively saturated model $(M, S) \models \text{BS}_{n+1}^0$ such that S is finite and for any $X \in S$, there exists $Y \subseteq M$ such that $(M, S \cup \{Y\}) \models \text{IS}_n^0 + \theta(X, Y)$.*

To show this theorem, we use the following property of recursively saturated models and resplendent models, which are introduced by Barwise and Schlipf. See [2] for the historical information of recursively saturated models and resplendent models.

Theorem 6.2 (see Sections 1.8 and 1.9 of [28]). *Let $\mathcal{L} \supseteq \mathcal{L}_{\text{PA}}$ be a finite language, and let \mathcal{M} be a countable \mathcal{L} -structure. Then the following are equivalent.*

1. \mathcal{M} is recursively saturated.
2. \mathcal{M} is resplendent, i.e., for any recursive set of sentences T of a finite language $\mathcal{L}' \supseteq \mathcal{L}$ such that $\text{Th}(\mathcal{M}) \cup T$ is consistent, there exists an expansion \mathcal{M}' of \mathcal{M} such that $\mathcal{M}' \models T$.
3. \mathcal{M} is chronically resplendent, i.e., \mathcal{M} is resplendent with the extra condition that the expansion \mathcal{M}' is resplendent as an \mathcal{L}' -structure.

Proof of Theorem 6.1. Let $\Gamma \equiv \forall X \exists Y \theta(X, Y)$ where θ is Π_{n+1}^0 satisfy the condition (†), and let $\varphi_0 \equiv \forall X \forall x \varphi(X, x)$ be a $\tilde{\Pi}_{n+2}^0$ -sentence where φ is Σ_{n+1}^0 . We will show that $\text{IS}_n^0 \not\models \varphi_0$ implies that $\text{RCA}_0 + \text{BS}_{n+1}^0 + \Gamma \not\models \varphi_0$. Assume that $\text{IS}_n^0 \not\models \varphi_0$, and take a countable recursively saturated model $(M, S) \models \text{IS}_n^0$ such that $(M, S) \models \neg \varphi_0$. Then, there exists $a \in M$ and $A \in S$ such that $(M, \{A\}) \models \neg \varphi(A, a)$. We will construct an (ω -length) sequence of cuts $M = I_0 \supseteq_e I_1 \supseteq_e \dots$ and a sequence of sets $A_i \subseteq I_i$ such that

- $(I_i, \{A_0, \dots, A_i\} \upharpoonright I_i) \models \text{IS}_n^0$ and $(I_i, \{A_0, \dots, A_i\} \upharpoonright I_i)$ is recursively saturated,
- if $i < j$, then $(I_j, \{A_0, \dots, A_i\} \upharpoonright I_j)$ is a Σ_n^0 -elementary substructure of $(I_i, \{A_0, \dots, A_i\} \upharpoonright I_i)$, and,
- $(I_{i+1}, \{A_0, \dots, A_i, Z_i, A_{i+1}\} \upharpoonright I_{i+1}) \models \theta(Z_i, A_{i+1})$, where Z_i is a k -th Δ_1^0 -definable set in $(I_j, \{A_0, \dots, A_j\} \upharpoonright I_j)$ if $i = (j, k)$.

Set $I_0 = M$ and $A_0 = A \oplus \{a\}$. Now, given $(I_i, \{A_0, \dots, A_i\} \upharpoonright I_i)$, we will first find a cut $I_{i+1} \subsetneq_e I_i$ such that $(I_{i+1}, \{A_0, \dots, A_i\} \upharpoonright I_{i+1})$ is Σ_n^0 -elementary substructure of $(I_i, \{A_0, \dots, A_i\} \upharpoonright I_i)$ and recursively saturated. Let J_0 be the set of all Σ_n^0 -definable elements in $(I_i, \{A_0, \dots, A_i\} \upharpoonright I_i)$. Since $(I_i, \{A_0, \dots, A_i\} \upharpoonright I_i)$ is recursively saturated, J_0 is not cofinal in I_i , thus $J = \sup J_0$ forms a proper cut of I_i and it is a Σ_n^0 -elementary substructure. Therefore, a recursive $\mathcal{L}_{\text{PA}} \cup \{A_0, \dots, A_i, J\}$ -theory T saying that $(J, \{A_0, \dots, A_i\} \upharpoonright J)$ is a Σ_n^0 -elementary proper cut of $(I_i, \{A_0, \dots, A_i\} \upharpoonright I_i)$ is consistent with $\text{Th}((I_i, \{A_0, \dots, A_i\} \upharpoonright I_i))$. (One can state Σ_n^0 -elementarity by using the truth predicate.) Thus, by the chronic resplendency of Theorem 6.2, there exists $J' \subseteq I_i$ such that $(I_i, \{A_0, \dots, A_i, J'\} \upharpoonright I_i)$ satisfies T and is recursively saturated. Let I_{i+1} be such J' , then $(I_{i+1}, \{A_0, \dots, A_i\} \upharpoonright I_{i+1})$ is Σ_n^0 -elementary substructure of $(I_i, \{A_0, \dots, A_i\} \upharpoonright I_i)$ and recursively saturated.

By Theorem 1.7, $(I_{i+1}, \{A_0, \dots, A_i\} \upharpoonright I_{i+1}) \models \text{BS}_{n+1}^0$, and thus $(I_{i+1}, \{A_0, \dots, A_i, Z_i\} \upharpoonright I_{i+1}) \models \text{BS}_{n+1}^0$. ($Z_i \cap I_{i+1}$ is Δ_1^0 -definable in $(I_{i+1}, \{A_0, \dots, A_i\} \upharpoonright I_{i+1})$ by Σ_n^0 -elementarity.) Thus, by the

condition (†), there exists $B \subseteq I_{i+1}$ such that $(I_{i+1}, \{A_0, \dots, A_i, Z_i, B\} \upharpoonright I_{i+1}) \models \theta(Z_i, B)$. By using chronic respendency as above, one can re-choose $B \subseteq I_{i+1}$ so that $(I_{i+1}, \{A_0, \dots, A_i, Z_i, B\} \upharpoonright I_{i+1})$ is recursively saturated. Then, put A_{i+1} as such B .

Claim. $\bar{I} = \bigcap_{i \in \omega} I_i$ is a cut of M , and $(\bar{I}, \{A_0, \dots, A_i\} \upharpoonright \bar{I})$ is a Σ_n^0 -elementary substructure of $(I_i, \{A_0, \dots, A_i\} \upharpoonright I_i)$.

Proof of the claim. Clearly \bar{I} forms a cut, and thus it is always a Σ_0^0 -elementary substructure. We show Σ_n^0 -elementarity by induction on the complexity of formulas. Let $0 < k \leq n$ and assume that $(I_i, \{A_0, \dots, A_i\} \upharpoonright I_i) \models \exists x \psi(x, c)$ where $c \in \bar{I}$ and ψ is Π_{k-1}^0 . By IS_n^0 in $(I_i, \{A_0, \dots, A_i\} \upharpoonright I_i)$, take the least $d \in I_i$ such that $\psi(d, c)$ holds. If $d \notin \bar{I}$, then there exists $j > i$ such that $d \notin I_j$, thus $(I_j, \{A_0, \dots, A_i\} \upharpoonright I_j) \models \neg \exists x \psi(x, c)$. This contradicts the second condition of the construction of I_i 's, thus $d \in \bar{I}$. Hence, $(\bar{I}, \{A_0, \dots, A_i\} \upharpoonright \bar{I}) \models \psi(d, c)$ by the induction hypothesis. \square

Put $\bar{S} = \Delta_1^0\text{-Def}(\bar{I}, \{A_i \mid i \in \omega\} \upharpoonright \bar{I})$. Then, by the claim and Theorem 1.7, $(\bar{I}, \bar{S}) \models \text{RCA}_0 + \text{B}\Sigma_{n+1}^0$. By the third condition of the construction of A_i 's and the Σ_n^0 -elementarity, we have $(\bar{I}, \bar{S}) \models \Gamma + \neg \varphi(A, a)$. Thus, $\text{RCA}_0 + \text{B}\Sigma_{n+1}^0 + \Gamma \not\models \varphi_0$. \square

Now, we apply Theorem 6.1 to SGP_2^2 . For this, we reformulate the low solution construction for SGP_2^2 as follows.

Lemma 6.3. *Let (M, S_0) be a countable model of $\text{B}\Sigma_2^0$ such that S_0 is finite and $f : [\mathbb{N}]^2 \rightarrow 2$ and $\mathbb{L} \subseteq [\mathbb{N}]^{<\mathbb{N}}$ are Δ_1^0 -definable in (M, S_0) . Then, there exists $G \subseteq M$ such that*

$$(M, S_0 \cup \{G\}) \models \text{IS}_1^0 + \text{“}G \text{ is a witness that } \mathbb{L} \text{ is not a largeness notion”}, \text{ or,}$$

$$(M, S_0 \cup \{G\}) \models \text{IS}_1^0 + \text{“}G \text{ is an infinite } \mathbb{L}\text{-grouping for } A_f = \{x \in \mathbb{N} \mid \lim_{y \rightarrow \infty} f(x, y) = 1\} \text{”}.$$

Proof. Let (M, S) be a countable model of $\text{B}\Sigma_2^0$ such that S_0 is finite and $f : [\mathbb{N}]^2 \rightarrow 2$ and $\mathbb{L} \subseteq [\mathbb{N}]^{<\mathbb{N}}$ are Δ_1^0 -definable in (M, S_0) . By Hájek[16], we can always find an ω -extension $S \supseteq S_0$ such that $(M, S) \models \text{B}\Sigma_2^0 + \text{WKL}_0$ (see also Belanger[3]). Then, $f, \mathbb{L} \in S$. Thus, we will work on (M, S) instead of (M, S_0) . If \mathbb{L} is not a largeness notion in (M, S) , take a witness $G \in S$ of not being a largeness notion, then, we have done. Otherwise, we will construct an infinite \mathbb{L} -grouping for A_f .

The following construction is a “model-theoretic interpretation” of Theorem 5.2. To simplify the coding, we will only consider a minimal \mathbb{L} -sequence, namely, a sequence of the form $\langle F_i \in \mathbb{L} \mid i < k \rangle$, $F_i \subseteq X_0$, $F_i < F_j$ if $i < j$ and $F_i \setminus \{\max F_i\} \notin \mathbb{L}$, i.e., each F_i is minimal. A code for a minimal \mathbb{L} -sequence is a binary sequence $\sigma \in 2^{<M}$ (which is coded in M) such that $\{x < |\sigma| \mid \sigma(x) = 1\} = \bigcup_{i < k} F_i$ for some minimal \mathbb{L} -sequence $\langle F_i \mid i < k \rangle$. (By the minimality, one can effectively decode a binary sequence to obtain the \mathbb{L} -sequence.) Note that σ may code extra 0's after $\max F_{k-1}$. Thus, one can identify a code σ with a pair $(\langle F_i \mid i < k \rangle, d)$ where $d = |\sigma|$. With this identification, we let $\|\sigma\|_{\mathbb{L}} = k$. Given an (M) -finite sequence of sets $\vec{Y} = \langle Y_j \mid j < l \rangle \in S$ and an (M) -finite set F , F is said to be consistent with \vec{Y} if for any $j < l$, $(F \subseteq Y_j \vee F \subseteq \bar{Y}_j)$. A code σ for a minimal \mathbb{L} -sequence $\langle F_i \mid i < k \rangle$ is said to be consistent with \vec{Y} if for any $i < k$, F_i is consistent with \vec{Y} . Given $t \in M$, we let $A_{f,t} = \{x < t \mid x \in X_0, \lim_{y \rightarrow \infty, y \in X_0} f(x, y) = 1\}$. Note that $A_{f,t} \in S$ since $(M, S) \models \text{B}\Sigma_2^0$.

Now, we will construct $G \subseteq M$ by arithmetical forcing. Let $\dot{\sigma}$ be the set of all pairs of the form (σ, \vec{Y}) such that

- $\vec{Y} = \langle Y_j \mid j < l \rangle$ is an (M) -finite sequence of sets in S ,

- σ is a code for a minimal L-sequence which is consistent with \vec{Y} and $A_{f,|\sigma|}$,

and we let $(\sigma, \vec{Y}) \preceq (\tau, \vec{Z})$ if $\sigma \supseteq \tau$ and $\vec{Y} \supseteq \vec{Z}$. Take an (M, S) -generic filter \mathcal{G} of $(\ , \preceq)$ and put $G = \bigcup \{\sigma \mid \exists \vec{Y} \in S(\sigma, \vec{Y}) \in \mathcal{G}\}$. Then, this G is the desired. It is clear by construction that G is a minimal L-sequence which is consistent with A_f . To see that G preserves $\text{I}\Sigma_1^0$ and G is infinite in (M, S) , we need to check the following are dense in $(\ , \preceq)$:

$$\mathcal{D}_{\theta, b}^1 := \left\{ (\sigma, \vec{Y}) \mid \begin{array}{l} \forall \tau \supseteq \sigma (\tau \text{ is a minimal L-sequence consistent with } \vec{Y} \rightarrow \forall n \leq |\tau| \neg \theta(b, n, \tau \upharpoonright n)) \\ \vee \exists c \leq b (\forall d < c \forall \tau \supseteq \sigma (\tau \text{ is a minimal L-sequence consistent with } \vec{Y} \\ \rightarrow \forall n \leq |\tau| \neg \theta(d, n, \tau \upharpoonright n)) \wedge \exists n \leq |\sigma| \theta(c, n, \sigma \upharpoonright n)) \end{array} \right\},$$

where $b \in M$ and $\theta(x, n, \sigma) \in \Sigma_0^0$ with parameters from (M, S) ,

$$\mathcal{D}_e^2 := \left\{ (\sigma, \vec{Y}) \mid \|\sigma\|_{\mathbb{L}} \geq e \right\}, \text{ where } e \in M.$$

One can easily see that if $(\sigma, \vec{Y}) \in \mathcal{D}_{\theta, b}^1$, then (σ, \vec{Y}) forces “if $\exists n \theta(b, n, G \upharpoonright n)$, there exists least $c \leq b$ such that $\exists n \theta(c, n, G \upharpoonright n)$ ”, which guarantees Σ_1^0 -least number principle, and if $(\sigma, \vec{Y}) \in \mathcal{D}_e^2$, then (σ, \vec{Y}) forces “ $\|G\|_{\mathbb{L}} \geq e$ ”, which means G is an infinite minimal L-sequence.

To see that $\mathcal{D}_{\theta, b}^1$ is dense, let (σ, \vec{Y}) be given. Let $\Theta(x)$ be a Σ_1^0 -formula saying that “there exists t such that for any $\vec{Z} = \langle Z_j \subseteq [|\sigma|, t]_{\mathbb{N}} \mid j < 2^x \rangle$ there exists $\tau \supseteq \sigma$ with $|\tau| \leq t$ such that $(\tau$ is a minimal L-sequence consistent with $\vec{Y} \upharpoonright \vec{Z} \wedge \exists n \leq |\tau| \theta(x, n, \tau \upharpoonright n)$).” We consider the two cases.

Case I $\Theta(b)$ fails in (M, S) .

In this case, by WKL_0 in (M, S) , there exists $\vec{Z} = \langle Z_j \subseteq [|\sigma|, \infty)_{\mathbb{N}} \mid j < 2^b \rangle$ such that $\forall \tau \supseteq \sigma (\tau$ is a minimal L-sequence consistent with $\vec{Y} \upharpoonright \vec{Z} \rightarrow \forall n \leq |\tau| \neg \theta(b, n, \tau \upharpoonright n)$). Take such \vec{Z} . Then, $(\sigma, \vec{Y} \upharpoonright \vec{Z}) \in \mathcal{D}_{\theta, b}^1$.

Case II $\Theta(b)$ holds in (M, S) .

In this case, by $\text{I}\Sigma_1^0$ in (M, S) , there exists the least $c \leq b$ such that $\Theta(c)$ holds. Then, by WKL_0 , there exists $\vec{W}^d = \langle W_j^d \subseteq [|\sigma|, \infty)_{\mathbb{N}} \mid j < 2^d \rangle$ such that $\forall \tau \supseteq \sigma (\tau$ is a minimal L-sequence consistent with $\vec{Y} \upharpoonright \vec{W}^d \rightarrow \forall n \leq |\tau| \neg \theta(d, n, \tau \upharpoonright n)$) for any $d < c$. Now, take the witness $t \in M$ for $\Theta(c)$, and put $\vec{Z} = \langle W_j^d \cap [|\sigma|, t]_{\mathbb{N}} \mid j < 2^d, d < c \rangle \frown \langle A_{f, t} \cap [|\sigma|, t]_{\mathbb{N}} \rangle$. Then, by $\Theta(c)$, there exists $\tau \supseteq \sigma$ with $|\tau| \leq t$ such that τ is a minimal L-sequence consistent with $\vec{Y} \upharpoonright \vec{Z}$ and $\exists n \leq |\tau| \theta(c, n, \tau \upharpoonright n)$. Then $(\tau, \vec{Y} \upharpoonright \langle W_j^d \mid j < 2^d, d < c \rangle)$ is a condition in $\mathcal{D}_{\theta, b}^1$ and $(\tau, \vec{Y} \upharpoonright \langle W_j^d \mid j < 2^d, d < c \rangle) \in \mathcal{D}_{\theta, b}^1$.

To see that \mathcal{D}_e^2 is dense, let (σ, \vec{Y}) be given where σ is a code for $\langle F_i \mid i < k \rangle$ and $\vec{Y} = \langle Y_j \mid j < l \rangle$. By applying Lemma 2.1 e times in (M, S) , one can find $\langle t_0, \dots, t_e \rangle$ such that $t_0 = |\sigma|$ and for any $s < e$ and for any 2^{l+1} splitting $M = \bigsqcup_{p < 2^{l+1}} W_p$, there exists a finite subset of $[t_s, t_{s+1})_{\mathbb{N}}$ which is L-large and included in one of W_p . ($\text{I}\Sigma_1^0$ is enough for this iteration.) Thus, there exists $E_s \subseteq [t_s, t_{s+1})_{\mathbb{N}}$ which is L-large and consistent with \vec{Y} and A_{f, t_e} for any $s < e$. Let $\tau \supseteq \sigma$ be a code for a sequence $\langle F_i \mid i < k \rangle \frown \langle E_s \mid s < e \rangle$. Then, (τ, \vec{Y}) is a condition in \mathcal{D}_e^2 and $\|\tau\|_{\mathbb{L}} \geq e$, thus, $(\tau, \vec{Y}) \in \mathcal{D}_e^2$. \square

Theorem 6.4. $\text{RCA}_0 + \text{SGP}_2^2$ is a $\tilde{\Pi}_3^0$ -conservative extension of $\text{I}\Sigma_1^0$.

Proof. Straightforward from Lemma 5.1, Theorem 6.1 and Lemma 6.3. \square

Thus, by the amalgamation theorem, we have the following conservation result.

Theorem 6.5. $\text{WKL}_0 + \text{GP}_2^2 + \text{ADS}$ is a $\tilde{\Pi}_3^0$ -conservative extension of $\text{I}\Sigma_1^0$.

Proof. Apply Theorem 3.5 for the conservation results Corollary 4.5 and Theorem 6.4, then we can see that $\text{WKL}_0 + \text{ADS} + \text{SGP}_2^2$ is a $\tilde{\Pi}_3^0$ -conservative extension of $\text{IS}\Sigma_1^0$. By Hirschfeldt/Shore[21], ADS implies COH. Thus, $\text{WKL}_0 + \text{GP}_2^2 + \text{ADS}$ is a $\tilde{\Pi}_3^0$ -conservative extension of $\text{IS}\Sigma_1^0$. \square

7 Conservation theorem for RT_2^2

In this section, we will show that $\text{WKL}_0 + \text{RT}_2^2$ is a $\tilde{\Pi}_3^0$ -conservative extension of $\text{IS}\Sigma_1^0$ by showing that $\text{WKL}_0 + \text{EM}$ is a $\tilde{\Pi}_3^0$ -conservative extension of $\text{IS}\Sigma_1^0$. For this, we will bound the size of $^k\text{-large}(\text{EM})$ sets by $^n\text{-large}$ sets using the following finite grouping principle.

Definition 7.1 ($\text{IS}\Sigma_1^0$, finite grouping principle). Given largeness notions L_1, L_2 and a coloring $f : [X]^n \rightarrow k$, (L_1, L_2) -grouping for f is a finite family of finite sets $\langle F_i \subseteq X \mid i < l \rangle$ such that

- $\forall i < j < l \max F_i < \min F_j$,
- $\forall i < l F_i \in \text{L}_1$,
- for any $H \subseteq_{\text{fin}} \mathbb{N}$, if $H \cap F_i \neq \emptyset$ for any $i < l$, then $H \in \text{L}_2$, and,
- $\forall i_1 < \dots < i_n \exists c < k \forall x_1 \in F_{i_1}, \dots, \forall x_n \in F_{i_n} f(x_1, \dots, x_n) = c$.

Note that if L_2 is regular, then the third condition can be replaced with $\{\max F_i \mid i < l\} \in \text{L}_2$. Now $\text{FGP}_k^n(\text{L}_1, \text{L}_2)$ (finite grouping principle for (L_1, L_2)) asserts that for any infinite set $X_0 \subseteq \mathbb{N}$, there exists a finite set $X \subseteq X_0$ such that for any coloring $f : [X]^n \rightarrow k$, there exists a (L_1, L_2) -grouping for f .

Theorem 7.1. *Let L_1 and L_2 be Δ_0 -definable regular largeness notions provably in $\text{IS}\Sigma_1^0$. Then, $\text{IS}\Sigma_1^0$ proves $\text{FGP}_2^2(\text{L}_1, \text{L}_2)$.*

Proof. One can easily check that $\text{FGP}_2^2(\text{L}_1, \text{L}_2)$ is a $\tilde{\Pi}_3^0$ -statement which is provable from $\text{WKL}_0 + \text{GP}_2^2(\text{L}_1)$. Thus, $\text{IS}\Sigma_1^0$ proves $\text{FGP}_2^2(\text{L}_1, \text{L}_2)$ by Theorem 6.5. \square

Now we apply the generalized Parsons theorem to the finite grouping principle. Actually, its upper bound also bounds $^k\text{-large}(\text{EM})$ sets as follows.

Lemma 7.2. *For any $k \in \omega$, there exists $n \in \omega$ such that*

$$\text{IS}\Sigma_1^0 \vdash \forall Z \subseteq_{\text{fin}} (3, \infty)_{\mathbb{N}} (Z \text{ is } ^n\text{-large} \rightarrow Z \text{ is } ^k\text{-large}(\text{EM})).$$

Proof. We will prove this by (external) induction. For the case $k = 1$, $n = 6$ is enough by Theorem 2.5. Assume now $k > 1$ and $^n\text{-largeness}$ implies $^{k-1}\text{-large}(\text{EM})$ ness in $\text{IS}\Sigma_1^0$. By Theorems 3.4 and 7.1, take $n \in \omega$ so that $\text{IS}\Sigma_1^0$ proves $\forall Z \subseteq_{\text{fin}} \mathbb{N} (Z \text{ is } ^n\text{-large} \rightarrow \text{any coloring } f : [Z]^2 \rightarrow 2 \text{ has an } (^n, ^6)\text{-grouping})$. Within $\text{IS}\Sigma_1^0$, given an $^n\text{-large}$ set $Z \subseteq (3, \infty)_{\mathbb{N}}$ and $f : [[0, \max Z]_{\mathbb{N}}]^2 \rightarrow 2$, we want to find $H \subseteq Z$ such that f is transitive on $[H]^2$ and H is $^k\text{-large}$. By the assumption, there exists an $(^n, ^6)\text{-grouping}$ $\langle F_i \mid i < l \rangle \subseteq Z$ for f . Since each F_i is $^n\text{-large}$, it is $^{k-1}\text{-large}(\text{EM})$, thus, there exists $H_i \subseteq F_i$ such that H_i is $^{k-1}\text{-large}$ and f is transitive on $[H_i]^2$. On the other hand, $\{\max F_i \mid i < l\}$ is $^6\text{-large}$, thus, there exists $\tilde{H} \subseteq \{\max F_i \mid i < l\}$ such that \tilde{H} is $^6\text{-large}$ and f is constant on $[\tilde{H}]^2$ by Theorem 2.5. Put $H = \bigcup \{H_i \mid i < l, \max F_i \in \tilde{H}\}$. Then, one can easily check that H is $^k\text{-large}$. We now show that f is transitive on $[H]^2$. Let $a, b, c \in H$ and $a < b < c$. If there exists $i < l$ such that $a, b, c \in H_i$, then f is transitive for a, b, c since f is transitive on $[H_i]^2$. If for some $i_0 < i_1 < l$, $a, b \in H_{i_0}$ and

$c \in H_{i_1}$, then, $f(a, c) = f(b, c)$, so f is transitive for a, b, c . The case $a \in H_{i_0}$ and $b, c \in H_{i_1}$ for some $i_0 < i_1 < l$ is similar. Finally, if for some $i_0 < i_1 < i_2 < l$, $a \in H_{i_0}$, $b \in H_{i_1}$ and $c \in H_{i_2}$, then $f(a, b) = f(\max F_{i_0}, \max F_{i_1}) = f(\max F_{i_0}, \max F_{i_2}) = f(a, c)$, thus f is transitive for a, b, c . \square

Theorem 7.3. $\text{WKL}_0 + \text{EM}$ is a $\tilde{\Pi}_3^0$ -conservative extension of IS_1^0 .

Proof. By Theorems 2.3, 3.1 and Lemma 7.2. \square

Now the main theorem follows from the amalgamation theorem.

Theorem 7.4. $\text{WKL}_0 + \text{RT}_2^2$ is a $\tilde{\Pi}_3^0$ -conservative extension of IS_1^0 .

Proof. By Theorem 3.5, Corollary 4.5 and Theorem 7.3. \square

Seetapun and Slaman [39] asked whether $\text{RCA}_0 + \text{RT}_2^2$ proves the consistency of IS_1^0 , and Cholak, Jockusch and Slaman [6] asked whether $\text{RCA}_0 + \text{RT}_2^2$ proves the totality of Ackermann function. We answer negatively through the main theorem.

Corollary 7.5. $\text{WKL}_0 + \text{RT}_2^2$ does not imply the consistency of IS_1^0 nor the totality of Ackermann function.

Note that one can avoid using the amalgamation theorem by directly combining the bounds for large(psRT_2^2)ness and large(EM)ness in order to obtain a bound for k -large(RT_2^2) sets and reprove the main conservation theorem.

Proposition 7.6. For any $k \in \omega$, there exists $n \in \omega$ such that

$$\text{IS}_1^0 \vdash \forall Z \subseteq_{\text{fin}} (3, \infty)_{\mathbb{N}} (Z \text{ is } {}^n\text{-large} \rightarrow Z \text{ is } {}^k\text{-large}(\text{RT}_2^2)).$$

Proof. Given $k \in \omega$, take $n \in \omega$ so that IS_1^0 proves $\forall Z \subseteq_{\text{fin}} (3, \infty)_{\mathbb{N}} (Z \text{ is } {}^n\text{-large} \rightarrow Z \text{ is } {}^{2k+6}\text{-large}(\text{EM}))$ by Lemma 7.2. Then, within IS_1^0 , given $Z \subseteq_{\text{fin}} (3, \infty)_{\mathbb{N}}$ which is n -large and $f : [[0, \max Z]_{\mathbb{N}}]^2 \rightarrow 2$, there exists $H_0 \subseteq Z$ such that H_0 is ${}^{2k+6}$ -large and f is transitive on $[H_0]^2$. One can easily find a transitive coloring $f' : [[0, \max H_0]_{\mathbb{N}}]^2 \rightarrow 2$ such that $f' \upharpoonright [H_0]^2 = f \upharpoonright [H_0]^2$. (Set $f'(x, y) = 0$ if $x \notin H_0$, $f'(x, y) = 1$ if $x \in H_0$ and $y \notin H_0$, and $f'(x, y) = f(x, y)$ if $x, y \in H_0$.) Then, by Lemma 4.4, there exists $H \subseteq H_0$ such that H is k -large and f is constant on $[H]^2$. Thus, Z is k -large(RT_2^2). \square

Then, Theorem 7.4 follows from Theorems 2.3, 3.1 and Proposition 7.6.

8 Formalizing the conservation proof

In this section, we will formalize Theorem 7.4 within PRA. Actually, most arguments we used are straightforwardly formalizable within WKL_0 . We however need to take care of the use of external induction and non-computable construction of models. We fix a standard provability predicate \vdash .

Lemma 8.1. The following are provable within WKL_0 .

- (1) $\forall \varphi \in \Pi_1^1 ((\text{WKL}_0 \vdash \varphi) \rightarrow (\text{IS}_1^0 \vdash \varphi))$ (Theorem 1.2).
- (2) $\forall n \in \mathbb{N} (\text{IS}_1^0 \vdash \text{any infinite set has an } {}^n\text{-large subset})$ (Theorem 2.3).
- (3) $\forall k \in \mathbb{N} (\text{IS}_1^0 \vdash X \text{ is } {}^{k+4}\text{-large} \wedge \min X > 3 \rightarrow X \text{ is } {}^k\text{-large}(\text{RT}_k^2))$ (Theorem 2.5).

- (4) *The generalized Parsons theorem (Theorem 3.4).*
- (5) *The amalgamation theorem (Theorem 3.5).*
- (6) $(\forall n \in \mathbb{N}(\mathbb{I}\Sigma_1^0 \vdash \text{any infinite set has an } n\text{-large}(\Gamma) \text{ subset})) \rightarrow (\forall \varphi \in \tilde{\Pi}_3^0((\text{WKL}_0 + \Gamma \vdash \varphi) \rightarrow (\mathbb{I}\Sigma_1^0 \vdash \varphi)))$ for $\Gamma \equiv \text{psRT}_2^2, \text{EM}$ (Theorem 3.1).
- (7) $\forall n \in \mathbb{N}(\mathbb{I}\Sigma_1^0 \vdash \text{any infinite set has an } n\text{-large}(\text{psRT}_2^2) \text{ subset})$ (Lemma 4.4).
- (8) $\forall \varphi \in \tilde{\Pi}_3^0((\text{WKL}_0 + \text{GP}_2^2 \vdash \varphi) \rightarrow (\mathbb{I}\Sigma_1^0 \vdash \varphi))$ (Theorem 6.5).
- (9) $\forall k \in \mathbb{N}(\mathbb{I}\Sigma_1^0 \vdash \text{FGP}_2^2(\mathbb{L}_{\omega^k}, \mathbb{L}_{\omega^6}))$ (Theorem 7.1).
- (10) $\forall n \in \mathbb{N}(\mathbb{I}\Sigma_1^0 \vdash \text{any infinite set has an } n\text{-large}(\text{EM}) \text{ subset})$ (Lemma 7.2).

Proof. We reason within WKL_0 . For (1), several formalized proofs are known. See, e.g., [1, 16]. For (2), the induction used here is on provability, thus it is a Σ_1^0 -induction. For (3), the original Ketonen and Solovay's proof is directly formalizable (see [26, Section 6]). For (4) and (5), we can directly formalize our model-theoretic proofs of Theorems 3.4 and 3.5 by using the completeness theorem which is available within WKL_0 . For (6), we can formalize the proofs of Lemma 3.2 and Theorem 3.3 by using the completeness theorem. To formalize the proof of Theorem 3.1, we use the induction on provability. For (7), formalize the proof of Lemma 4.4. Formalization is direct since we only deal with finite objects. For (8), formalize the argument in Section 6. To formalize the proof of Theorem 6.1, an ω -extension to be a model of $\text{B}\Sigma_2^0 + \text{WKL}_0$ is available within WKL_0 by formalizing the argument by Hájek[16] or Belanger[3]. The existence of a countable recursively saturated model is provable in WKL_0 (see [41, Section IX]) and the Theorem 6.2 can be formalized similarly. To formalize the proof of Lemma 6.3, one can take a generic by the Baire category theorem which is available within RCA_0 . (9) is straightforward from (8). For (10), formalize the proof of Lemma 7.2. The induction used here is again on provability. \square

Thus, we have the following formalized conservation theorem.

Theorem 8.2. *PRA proves that $\text{WKL}_0 + \text{RT}_2^2 \vdash \psi$ implies $\mathbb{I}\Sigma_1^0 \vdash \psi$ for any $\tilde{\Pi}_3^0$ -sentences.*

Now the consistency equivalence of $\mathbb{I}\Sigma_1^0$ and $\text{WKL}_0 + \text{RT}_2^2$ follows from this formalized conservation theorem.

Corollary 8.3. *Over PRA, $\text{Con}(\mathbb{I}\Sigma_1^0)$ is equivalent to $\text{Con}(\text{WKL}_0 + \text{RT}_2^2)$.*

9 Open questions

In their paper [8], Chong, Slaman and Yang asked whether RT_2^2 is a Π_1^1 -conservative extension of $\text{RCA}_0 + \text{B}\Sigma_2^0$. This question remains open, and a positive answer would strengthen our main conservation result since $\text{B}\Sigma_2^0$ is Π_3^0 -conservative over $\mathbb{I}\Sigma_1^0$ (Theorem 1.5).

Question 9.1 (Chong, Slaman, Yang). *Is RT_2^2 a Π_1^1 -conservative extension of $\text{RCA}_0 + \text{B}\Sigma_2^0$?*

In particular, they proved [10] that the chain anti-chain principle is Π_1^1 -conservative over $\text{RCA}_0 + \text{B}\Sigma_2^0$. Therefore, in order to answer Question 9.1, one needs only to prove that this is also the case for the Erdős-Moser theorem.

For the purposes of our conservation proof, we introduced the grouping principle, which seems to be interesting to study in its own right. The standard proofs that a principle implies $\text{B}\Sigma_2^0$ does

not seem to be directly applicable to the grouping principle. The relation between the grouping principle and the Σ_2^0 -bounding principle is currently unknown.

Question 9.2. Does GP_2^2 imply $\text{B}\Sigma_2^0$ over RCA_0 ?

A proof of Π_1^1 -conservation of the grouping principle over $\text{B}\Sigma_2^0$ would significantly simplify our conservation argument since the finite grouping principle can be easily derived from its infinite version.

Question 9.3. Is GP_2^2 a Π_1^1 -conservative extension of $\text{RCA}_0 + \text{B}\Sigma_2^0$?

The grouping principle has been used to establish a density bound for the Erdős-Moser theorem. The stable grouping principle does not imply the stable version of the Erdős-Moser theorem since the former admits low solutions whereas the latter does not. It is however unknown whether the full version of the two principles coincide.

Question 9.4. Does EM imply GP_2^2 over RCA_0 ?

Our conservation proof contains almost no information about the size of the proof, but it is interesting to know whether RT_2^2 gives shorter proofs for $\tilde{\Pi}_3^0$ -consequences of $\text{I}\Sigma_1^0$ or not.

Question 9.5. Does $\text{WKL}_0 + \text{RT}_2^2$ or $\text{WKL}_0 + \text{psRT}_2^2$ have a significant speed-up over $\text{I}\Sigma_1^0$?

Note that there is no significant speed-up between WKL_0 and $\text{I}\Sigma_1^0$ (see Avigad[1]). A killer example for this question is an existence of m -dense sets, *i.e.*, $m\text{-}\widetilde{\text{PH}}(\text{RT}_2^2)$ or $m\text{-}\widetilde{\text{PH}}(\text{psRT}_2^2)$. It is not hard to see that $m\text{-}\widetilde{\text{PH}}(\text{RT}_2^2)$ can be proved from $\text{WKL}_0 + \text{RT}_2^2$ by using RT_2^2 m -times, and the case of psRT_2^2 is similar. On the other hand, to prove $m\text{-}\widetilde{\text{PH}}(\text{psRT}_2^2)$ from $\text{I}\Sigma_1^0$, what we know is the following.

Proposition 9.1. *For any $m \in \omega$, the following is provable within $\text{I}\Sigma_1^0$. If a finite set $X \subseteq \mathbb{N}$ is 3^{m+1} -large and $\min X > 3$, then X is $m\text{-dense}(\text{psRT}_2^2)$.*

Proof. Easy induction by using Lemma 4.4. □

Thus, within $\text{I}\Sigma_1^0$, we can obtain an $m\text{-dense}(\text{psRT}_2^2)$ -set by using Σ_1^0 -induction 3^{m+1} -times. This might indicate that there is at most exponential speed-up for the case of psRT_2^2 . For the case of RT_2^2 , the situation is more difficult. An only upper bound for $m\text{-dense}(\text{RT}_2^2)$ sets obtained from our argument is the following.

Proposition 9.2. *There exists a primitive recursive function $h : \omega \rightarrow \omega$ such that for any $m \in \omega$, the following is provable within $\text{I}\Sigma_1^0$. If a finite set $X \subseteq \omega$ is $h(m)$ -large then X is $m\text{-dense}(\text{RT}_2^2)$.*

Proof. By formalizing the proof of Proposition 7.6 and applying Σ_1^0 -induction, we obtain

$$\forall m \in \mathbb{N} \exists k \in \mathbb{N} (\text{I}\Sigma_1^0 \vdash X \text{ is } k\text{-large} \rightarrow X \text{ is } m\text{-dense}(\text{RT}_2^2)).$$

Then, by the Parsons theorem, there exists a primitive recursive function $h : \omega \rightarrow \omega$ such that $\text{I}\Sigma_1^0$ proves

$$\forall m \in \mathbb{N} \exists k \leq h(m) (\text{I}\Sigma_1^0 \vdash_{h(m)} X \text{ is } k\text{-large} \rightarrow X \text{ is } m\text{-dense}(\text{RT}_2^2)),$$

where \vdash_x means that there exists a proof whose Gödel number is smaller than x . For $m \in \omega$, k and $h(m)$ in the above are standard, thus, $\text{I}\Sigma_1^0$ truly proves “ X is k -large $\rightarrow X$ is $m\text{-dense}(\text{RT}_2^2)$ ”. □

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