ON THE T-LEAVES AND THE RANKS OF A POISSON STRUCTURE ON TWISTED CONJUGACY CLASSES

JIANG-HUA LU

Abstract. Let G be a connected complex semisimple Lie group with a fixed maximal torus T and a Borel subgroup $B \supset T$. For an arbitrary automorphism θ of G, we introduce a holomorphic Poisson structure π_θ on G which is invariant under the θ-twisted conjugation by T and has the property that every θ-twisted conjugacy class of G is a Poisson subvariety with respect to π_θ. We describe the T-orbits of symplectic leaves, called T-leaves, of π_θ and compute the dimensions of the symplectic leaves (i.e, the ranks) of π_θ. We give the lowest rank of π_θ in any given θ-twisted conjugacy class, and we relate the lowest possible rank locus of π_θ in G with spherical θ-twisted conjugacy classes of G. In particular, we show that π_θ vanishes somewhere on G if and only if θ induces an involution on the Dynkin diagram of G, and that in such a case a θ-twisted conjugacy class C contains a vanishing point of π_θ if and only if C is spherical.

1. Introduction and statement of results

1.1. **Introduction.** Let G be a connected complex semi-simple Lie group and fix a Borel subgroup B of G and a maximal torus T contained in B. It is well-known that the choice of the pair (B, T) gives rise to a so-called *standard* multiplicative holomorphic Poisson structure $\pi_{\rm st}$ on G that vanishes on T, and that the Poisson Lie group $(G, \pi_{\rm st})$ is the semi-classical limit of the quantum group $\mathbb{C}_q[G]$, the Hopf algebra dual of the quantized enveloping algebra $U_q\mathfrak{g}$ of G (see [10, 16]). The Poisson group $(G, \pi_{\rm st})$ is reviewed in §2.1.

Let $\theta \in {\rm Aut}(G)$. Define the θ-twisted conjugation of G on itself by

$$g_1 \cdot_\theta g = g_1 g \theta(g_1)^{-1}, \quad g, g_1 \in G, \tag{1.1}$$

and call its orbits the θ-twisted conjugacy classes in G. Using a general construction from [21] and [23, Theorem 2.3], we define a holomorphic Poisson structure π_θ on G with the properties that every θ-twisted conjugacy class C in G is a Poisson submanifold with respect to π_θ and that (C, π_θ) is a Poisson homogeneous space [14] of the Poisson Lie group $(G, \pi_{\rm st})$ with respect to the θ-twisted conjugation. The precise definition of π_θ is given in §2.2.

The Poisson structure π_θ is invariant under the θ-twisted conjugation by elements in T. If $\Sigma \subset G$ is a symplectic leaf of π_θ in G, the set $T \cdot_\theta \Sigma = \cup_{h \in T} h \cdot_\theta \Sigma$ is called the T-orbit of Σ in G, or a T-leaf of π_θ. When $\theta = {\rm Id}_G$, the identity automorphism of G, the Poisson structure π_θ, in particular its T-leaves and some T-equivariant Poisson resolutions of certain Poisson subvarieties of (G, π_θ), are studied in [18].

In this paper, for an arbitrary $\theta \in \mathrm{Aut}(G)$, we describe the T-leaves of π_θ and compute the dimensions of the symplectic leaves, also called the ranks, of π_θ. We describe the lowest rank of π_θ in any given θ-twisted conjugacy class, and we relate the lowest possible rank locus of π_θ with spherical θ-twisted conjugacy classes in G. For the case of $\theta = \mathrm{Id}_G$, our result on the T-leaves of π_θ strengthens that given in [18]. Work in this paper makes use of the results in [9, 22] on intersections of Bruhat cells and θ-twisted conjugacy classes and on the element m_C in the Weyl group of G naturally associated to a θ-twisted conjugacy class C in G. This paper can be regarded as first applications of results in [8, 9, 22] to Poisson geometry (see [22, Remark 3.10]). It can also be regarded as a sequel to [18].

1.2. **Statement of results.** Recall that we fix, from the very beginning, a maximal torus T of G and a Borel subgroup B containing T. To simplify the statements on the Poisson structure π_θ, we assume in this section that $\theta(B) = B$ and $\theta(T) = T$. The case of an arbitrary $\theta \in \mathrm{Aut}(G)$ is treated in §6.

Let $W = N_G(T)/T$ be the Weyl group, where $N_G(T)$ is the stabilizer of T in G. Let $l : W \to \mathbb{N}$ be the length function on W and let $\leq$ be the Bruhat order on W. Let B_- be the Borel subgroup of G such that $B \cap B_- = T$. Then one has the Bruhat decompositions

$$G = \bigsqcup_{w \in W} BwB = \bigsqcup_{w \in W} BwB_-.$$

where BwB_- has co-dimension $l(w)$ in G for $w \in W$. For a θ-twisted conjugacy class C in G, let $W_C^- = \{w \in W : C \cap (BwB_-) \neq \emptyset\}$. The sets W_C^- were first studied in [9]. It particular, it is shown in [9, §2.4] that

$$W_C^- = \{w \in W : w \leq m_C\},$$

where m_C is the unique element in W such that $C \cap (Bm_CB)$ is dense in C. The elements m_C play an important role in the study of spherical conjugacy classes and their applications to the theory of representations of quantum groups (see works of N. Cantarini, G. Carnovale, and M. Costantini in [5, 6, 7, 11]). For G simple and $\theta = \mathrm{Id}_G$, the list of all m_C's, as C runs over all conjugacy classes in G, is given in [9, §3] using results from [5]. For G of classical type and for every conjugacy class C in G, the element m_C is explicitly computed in [8]. For G simple and θ an outer automorphism, the list of the m_C's, as C runs over all θ-twisted conjugacy classes in G, is given in [22].

Our main results on the T-leaves of π_θ are summarized in the following Theorem 1.1, where for $w \in W$, $1 + w\theta$ is regarded as a linear operator on the Lie algebra $\mathfrak{h}$ of T, with 1 denoting the identity operator on $\mathfrak{h}$.

Theorem 1.1. *The intersections $C \cap (BwB_-)$, where C is a θ-twisted conjugacy class in G and $w \in W$ is such that $w \leq m_C$, are precisely all the T-leaves of π_θ in G, and the symplectic leaves in $C \cap (BwB_-)$ have dimension equal to*

$$\dim C - l(w) - \dim \ker(1 + w\theta).$$

A slightly stronger version of Theorem 1.1, where T can be replaced by certain subtori of T, is proved in Theorem 3.7. Symplectic leaves in $C \cap (BwB_-)$ are described in Corollary 3.6 as connected components of certain submanifolds of $C \cap (BwB_-)$.

Recall that a θ-twisted conjugacy class in G is said to be *spherical* if it admits an open orbit under the θ-twisted conjugation by B. Our second main result relates the lowest possible rank of π_θ in G and the collection $\mathcal{S}_\theta$ of all spherical θ-twisted conjugacy classes in G. More precisely, for $g \in G$, let $\mathrm{rank}(\pi_\theta(g))$ be the rank of π_θ at g, i.e., the dimension of the symplectic leaf of π_θ passing through g, and let $\mathrm{rank}(1 - \theta^2)$ be the rank of the linear operator $1 - \theta^2$ on $\mathfrak{h}$.

Theorem 1.2. *One has $\mathrm{rank}(\pi_\theta(g)) \geq \mathrm{rank}(1 - \theta^2)$ for every $g \in G$, and*

$$\{g \in G : \mathrm{rank}(\pi_\theta(g)) = \mathrm{rank}(1 - \theta^2)\} = \bigsqcup_{C \in \mathcal{S}_\theta} C \cap (Bm_C B_-).$$

In particular, the set $\{g \in G : \mathrm{rank}(\pi_\theta(g)) = \mathrm{rank}(1 - \theta^2)\}$ is nonempty if and only if spherical θ-twisted conjugacy classes in G exist.

Denote by Γ both the Dynkin diagram of G and the set of simple roots determined by (T, B). Since the induced action of θ on $\mathfrak{h}^*$ permutes the simple roots, it induces an automorphism on Γ which is also denoted by θ. A consequence of Theorem 1.2 is the following conclusion on the zero locus $Z(\pi_\theta) = \{g \in G : \pi_\theta(g) = 0\}$ of π_θ.

Corollary 1.3. *1) $Z(\pi_\theta) \neq \emptyset$ if and only $\theta^2 = 1 \in \mathrm{Aut}(\Gamma)$.*

2) When $\theta^2 = 1 \in \mathrm{Aut}(\Gamma)$, one has

$$Z(\pi_\theta) = \bigsqcup_{C \in \mathcal{S}_\theta} C \cap (Bm_C B_-),$$

and $C \cap (Bm_C B_-)$ is a single T-orbit for every $C \in \mathcal{S}_\theta$.

Examples related to $SL(n+1, \mathbb{C})$ and the triality automorphism for D_4 are given in §3 and §5. In section §7 we study in some detail the Poisson structure $\pi_{\tilde{\theta}}$ on G^n, where $n \geq 2$, and $\tilde{\theta} \in \mathrm{Aut}(G^n)$ is given by

$$\tilde{\theta}(g_1, g_2, \ldots, g_n) = (g_2, \ldots, g_n, g_1), \qquad g_j \in G, 1 \leq j \leq n.$$

In particular, for $n = 2$, we show that $(G \times G, \pi_{\tilde{\theta}})$ is isomorphic to the Drinfeld double $(G \times G, \Pi_{\rm st})$ (see §2.1) of the Poisson Lie group $(G, \pi_{\rm st})$. Consequently, for the diagonal action of T on $G \times G$ by left (or right) translations, the T-leaves of $\Pi_{\rm st}$ in $G \times G$ are precisely the *double Bruhat cells associated to conjugacy classes*, i.e., the submanifolds

$$G_C^{u,v} = \{(k_1, k_2) \in BuB \times B_-vB_- : \, k_1k_2^{-1} \in C\} \subset G \times G,$$

where $u, v \in W$ and C is any conjugacy class C in G (see also Remark 7.8 for a more direct proof of this fact). Note that when $C = \{e\}$, $G_C^{u,v}$ is isomorphic to the double Bruhat cell $G^{u,v} = BuB \cap B_-vB_-$, which have been studied intensively [2, 19] in connection with total positivity and cluster algebras. Further studies of double Bruhat cells associated to conjugacy classes will be carried out elsewhere.

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2. Definition of the Poisson structure π_θ on G

2.1. **The Poisson Lie group** $(G, \pi_{\rm st})$ **and its dual** (G^*, π_{G^*})**.** As in §1.1, let G be a connected complex semi-simple Lie group with Lie algebra $\mathfrak{g}$ and fix a pair (B, T), where B is a Borel subgroup of G and $T \subset B$ a maximal torus of G. Let B_- be the Borel subgroup of G such that $B \cap B_- = T$, and let N and N_- be respectively the uniradicals of B and B_-. The Lie algebras of T, B, B_-, N and N_- will be denoted by $\mathfrak{h}, \mathfrak{b}, \mathfrak{b}_-, \mathfrak{n}$ and $\mathfrak{n}_-$ respectively.

Let $\langle \, , \, \rangle_{\mathfrak{g}}$ be the Killing form of $\mathfrak{g}$, and equip the direct product Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ with the symmetric bilinear form

$$\langle (x_1, y_1), \, (x_2, y_2) \rangle_{\mathfrak{g} \oplus \mathfrak{g}} = \langle x_1, x_2 \rangle_{\mathfrak{g}} - \langle y_1, y_2 \rangle_{\mathfrak{g}}, \quad x_1, x_2, y_1, y_2 \in \mathfrak{g}. \tag{2.1}$$

Let $\mathfrak{g}_{\rm diag} = \{(x, x) : x \in \mathfrak{g}\}$ and let

$$\mathfrak{g}_{\rm st}^* = \{(x_+ + y, \, -y + x_-) : \, x_+ \in \mathfrak{n}, x_- \in \mathfrak{n}_-, y \in \mathfrak{h}\} \subset \mathfrak{g} \oplus \mathfrak{g}.$$

Then $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_{\rm diag} + \mathfrak{g}_{\rm st}^*$ is a direct sum of vector spaces and both $\mathfrak{g}_{\rm diag}$ and $\mathfrak{g}_{\rm st}^*$ are *Lagrangian subalgebras* of the quadratic Lie algebra $(\mathfrak{g} \oplus \mathfrak{g}, \langle \, , \, \rangle_{\mathfrak{g} \oplus \mathfrak{g}})$ in the sense that they are Lie subalgebras of $\mathfrak{g} \oplus \mathfrak{g}$ and are maximal isotropic with respect to $\langle \, , \, \rangle_{\mathfrak{g} \oplus \mathfrak{g}}$. The decomposition $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_{\rm diag} + \mathfrak{g}_{\rm st}^*$ is called the *standard Lagrangian splitting* of $\mathfrak{g} \oplus \mathfrak{g}$. The connected complex subgroups of $G \times G$ with Lie algebras $\mathfrak{g}_{\rm diag}$ and $\mathfrak{g}_{\rm st}^*$ are respectively $G_{\rm diag} = \{(g,g) : g \in G\}$ and

$$G^* = \{(nh, mh^{-1}) : n \in N, m \in N_-, h \in T\}. \tag{2.2}$$

By the theory of Poisson Lie groups ([10, 16], [18, Appendix], [23, Proposition 2.2]), the Lagrangian splitting $\mathfrak{g}\oplus\mathfrak{g} = \mathfrak{g}_{\rm diag} + \mathfrak{g}_{\rm st}^*$ induces multiplicative holomorphic Poisson structures $\pi_{\rm st}$ on G and π_{G^*} on G^*, making $(G, \pi_{\rm st})$ and (G^*, π_{G^*}) a dual pair of complex Poisson Lie groups. The r-matrix associated to the splitting $\mathfrak{g}\oplus\mathfrak{g} = \mathfrak{g}_{\rm diag} + \mathfrak{g}_{\rm st}^*$ is the element $R \in \wedge^2(\mathfrak{g}\oplus\mathfrak{g})$ given by

$$R = \frac{1}{2}\sum_{j=1}^{n}(\xi_j \wedge x_j) \in \wedge^2(\mathfrak{g}\oplus\mathfrak{g}), \tag{2.3}$$

where $\{x_j\}_{j=1}^n$ is any basis of $\mathfrak{g}_{\rm diag}$ and $\{\xi_j\}_{j=1}^n$ the basis of $\mathfrak{g}_{\rm st}^*$ such that $\langle x_j, \xi_k\rangle_{\mathfrak{g}\oplus\mathfrak{g}} = \delta_{jk}$ for $1 \leq j, k \leq n$.

The product group $G\times G$ carries the multiplicative Poisson structure

$$\Pi_{\rm st} = R^r - R^l, \tag{2.4}$$

where R^l and R^r denote respectively the left and right invariant bivector fields on $G \times G$ with value R at the identity element. The Poisson Lie group $(G\times G, \Pi_{\rm st})$ is called a *Drinfeld double* [20] of the Poisson Lie group $(G, \pi_{\rm st})$. It follows from the definition that $\Pi_{\rm st}$ is invariant under the following action of $T \times T$ on $G\times G$:

$$(h_1, h_2)\cdot(g_1, g_2) = (h_1 g_1 h_2^{-1},\ h_1 g_2 h_2^{-1}), \quad h_1, h_2 \in T,\ g_1, g_2 \in G.$$

Moreover, the embeddings $(G, \pi_{\rm st}) \cong (G_{\rm diag}, \pi_{\rm st}) \hookrightarrow (G\times G, \Pi_{\rm st})$ and $(G^*, -\pi_{G^*}) \hookrightarrow (G\times G, \Pi_{\rm st})$ are Poisson [23, Proposition 2.2]. We will return to the Poisson structure $\Pi_{\rm st}$ on $G\times G$ in §7.

2.2. **The definition of π_θ.** Let $\theta \in {\rm Aut}(G)$, and let $G_\theta = \{(g, \theta^{-1}(g)) : g \in G\}$. Identify $(G\times G)/G_\theta$ with G by the isomorphism

$$\eta : \ (G\times G)/G_\theta \longrightarrow G : \quad (g_1, g_2)G_\theta \longmapsto g_1\theta(g_2)^{-1}, \quad g_1, g_2 \in G.$$

The natural left action of $G\times G$ on $(G\times G)/G_\theta$ becomes that of $G\times G$ on G by

$$(g_1, g_2)\cdot_\theta g = g_1 g \theta(g_2)^{-1}, \quad g_1, g_2, g \in G. \tag{2.5}$$

Note the action in (2.5) restricted to $G_{\rm diag} \subset G\times G$ is the θ-twisted conjugation of G on itself given in (1.1), and, by abuse of notation, we are denoting both actions by $\cdot_\theta$. Let κ be the Lie algebra anti-homomorphism from $\mathfrak{g}\oplus\mathfrak{g}$ to the Lie algebra of holomorphic vector fields on G induced by the action in (2.5), i.e.,

$$\kappa(x, y) = x^r - \theta(y)^l, \quad x, y \in \mathfrak{g},\ g \in G, \tag{2.6}$$

where for $x \in \mathfrak{g}$, x^r and x^l respectively denote the right and left invariant vector field on G with value x at the identity element of G. Define the bivector field π_θ on G by

$$\pi_\theta = (\kappa \wedge \kappa)(R),$$

where $R \in \wedge^2(\mathfrak{g} \oplus \mathfrak{g})$ is given in (2.3). The following Proposition 2.1 is a special case of [23, Proposition 2.2, Theorem 2.3].

Proposition 2.1. *1) π_θ is a Poisson bi-vector field on G, and every θ-twisted conjugacy class C in G is a Poisson submanifold of (G, π_θ);*

2) The twisted conjugation action

$$(G, \pi_{\rm st}) \times (G, \pi_\theta) \longrightarrow (G, \pi_\theta), \quad (g_1, g) \longmapsto g_1 \cdot_\theta g = g_1 g \theta(g_1)^{-1}, \quad g_1, g \in G,$$

is a Poisson action of the Poisson Lie group $(G, \pi_{\rm st})$ on the Poisson manifold (G, π_θ);

3) The symplectic leaves of π_θ in G are the connected components of the intersections of θ-twisted conjugacy classes and G^*-orbits in G, where G^* acts on G as a subgroup of $G \times G$ via (2.5).

Remark 2.2. The θ-twisted conjugacy class through the identity element of G is isomorphic to G/G^θ, where $G^\theta = \{g \in G : \theta(g) = g\}$. The Poisson manifold $(G/G^\theta, \pi_\theta)$ is an example of a *De Concini* Poisson homogeneous space of $(G, \pi_{\rm st})$ considered in [15, Section 6.4]. The standard Lagrangian splitting $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_{\rm diag} + \mathfrak{g}_{\rm st}^*$ induces a Poisson structure Π_0 on the variety $\mathcal{L}$ of Lagrangian subalgebras of $(\mathfrak{g} \oplus \mathfrak{g}, \langle \, , \, \rangle_{\mathfrak{g} \oplus \mathfrak{g}})$, which is considered in [17]. When G is of adjoint type, $G \cong (G \times G)/G_\theta$ can be identified with the $(G \times G)$-orbit in $\mathcal{L}$ through the point $\mathfrak{g}_\theta = \{(x, \theta^{-1}(x)) : x \in \mathfrak{g}\} \in \mathcal{L}$, and the induced Poisson structure on G coincides with π_θ.

Remark 2.3. For $g \in G$, let $\mathrm{Ad}_g : G \to G, \mathrm{Ad}_g(g_1) = g g_1 g^{-1}$ for $g_1 \in G$. If $\theta, \theta' \in \mathrm{Aut}(G)$ are such that $\theta = \mathrm{Ad}_g \theta'$ for some $g \in G$, it is easy to see from the definitions that the right translation on G by g intertwines the θ-twisted conjugation and the θ'-twisted conjugation and is a Poisson isomorphism from (G, π_θ) to $(G, \pi_{\theta'})$. For an arbitrary $\theta \in \mathrm{Aut}(G)$, since all Borel subgroups of G are conjugate to each other, there exists $g_1 \in G$ such that $\theta(B) = g_1 B g_1^{-1}$. Since T and $g_1^{-1}\theta(T) g_1$ are two maximal tori of G both contained in B, there exists $b \in B$ such that $b^{-1} g_1^{-1} \theta(T) g_1 b = T$. Let $g = g_1 b$, and let $\theta' = \mathrm{Ad}_{g^{-1}} \theta$. Then θ' stabilizes both B and T. To study the geometry of the Poisson structure π_θ and in particular the T-leaves of π_θ, we can thus assume that $\theta(B) = B$ and $\theta(T) = T$. This remark will be used in §6.

Assuming that $\theta(B) = B$ and $\theta(T) = T$, we now give an explicit formula for π_θ.

Let $\{y_i\}_{i=1}^k$ be a basis of $\mathfrak{h}$ such that $2\langle y_i, y_j \rangle_{\mathfrak{g}} = \delta_{ij}$ for $1 \leq i, j \leq k = \dim \mathfrak{h}$. Let $\Delta_+ \subset \mathfrak{h}^*$ be the set of positive roots determined by B, and for each $\alpha \in \Delta_+$, let E_α

and $E_{-\alpha}$ be root vectors for α and $-\alpha$ respectively such that $\langle E_\alpha, E_{-\alpha}\rangle_{\mathfrak{g}} = 1$. The following Lemma 2.4 in the case when $\theta = \mathrm{Id}_G$ is given in [18, §2.2].

Lemma 2.4. *One has*

$$\pi_\theta = \sum_{i=1}^{k} \theta(y_i)^l \wedge y_i^r - \sum_{\alpha\in\Delta_+} E_\alpha^r \wedge \theta(E_{-\alpha})^l + \frac{1}{2}\sum_{\alpha\in\Delta_+} \left(E_\alpha^r \wedge E_{-\alpha}^r + E_\alpha^l \wedge E_{-\alpha}^l\right). \tag{2.7}$$

Proof. Using the bases

$$\{x_i\} = \{(y_1, y_1),\ (y_2, y_2), \ldots, (y_k, y_k),\ (E_\alpha,\ E_\alpha),\ (E_{-\alpha}, E_{-\alpha}) : \alpha \in \Delta_+\} \tag{2.8}$$

$$\{\xi_i\} = \{(y_1, -y_1),\ (y_2, -y_2), \ldots, (y_k, -y_k),\ (0, -E_{-\alpha)}),\ (E_\alpha, 0) : \alpha \in \Delta_+\}, \tag{2.9}$$

for $\mathfrak{g}_{\mathrm{diag}}$ and $\mathfrak{g}_{\mathrm{st}}^*$ and by (2.3), one has

$$\begin{aligned} R = &\sum_{i=1}^{k}(y_i, 0) \wedge (0, y_i) + \sum_{\alpha\in\Delta_+} (E_\alpha, 0) \wedge (0, E_{-\alpha}) \\ &+ \frac{1}{2}\sum_{\alpha\in\Delta_+} ((E_\alpha, 0) \wedge (E_{-\alpha}, 0) + (0, E_\alpha) \wedge (0, E_{-\alpha})). \end{aligned} \tag{2.10}$$

It follows from the definition of π_θ that

$$\pi_\theta = \sum_{i=1}^{k} \theta(y_i)^l \wedge y_i^r - \sum_{\alpha\in\Delta_+} E_\alpha^r \wedge \theta(E_{-\alpha})^l + \frac{1}{2}\sum_{\alpha\in\Delta_+} \left(E_\alpha^r \wedge E_{-\alpha}^r + \theta(E_\alpha)^l \wedge \theta(E_{-\alpha})^l\right).$$

Using the facts that θ preserves $\langle\, ,\, \rangle_{\mathfrak{g}}$ and that the induced action of θ on $\mathfrak{h}^*$ permutes the positive roots, one proves (2.7).

Q.E.D.

3. T-leaves of π_θ

3.1. **The intersections $C \cap (BwB_-)$.** Throughout §3-§5, we assume that $\theta(B) = B$ and $\theta(T) = T$. Then $\theta(B_-) = B_-$. Recall the subgroup $G^* \subset B \times B_-$ given in (2.2) and let G^* act on G via (2.5). Since $\theta(B) = B$ and $\theta(B_-) = B_-$, every G^*-orbit in G lies in BwB_- for a unique $w \in W$. By Proposition 2.1, every symplectic leaf of π_θ in G lies in the intersection $C \cap (BwB_-)$ for a unique θ-twisted conjugacy class C and a unique $w \in W$. One thus needs to first study the intersections $C \cap (BwB_-)$ and in particular to know when such an intersection is nonempty. We now recall some results from [9].

Let C be an arbitrary θ-twisted conjugacy class in G. Since C is irreducible, there is a unique element $m_C \in W$ such that $C \cap (Bm_CB)$ is dense in C. Let

$$W_C = \{w \in W : C \cap (BwB) \neq \emptyset\} \quad \text{and} \quad W_C^- = \{w \in W : C \cap (BwB_-) \neq \emptyset\}.$$

It is clear that $W_C \subset W_C^-$. By [5, §1], m_C is the unique maximal element of W_C with respect to the Bruhat order $\leq$ on W. Since $\theta(T) = T$, one has $\theta(N_G(T)) = N_G(T)$ and thus an induced map

$$\theta: \quad W \longrightarrow W, \quad \theta(gT) = \theta(g)T, \qquad g \in N_G(T). \tag{3.1}$$

Note that $\theta(w)(h) = \theta w(\theta^{-1}(h))$ for $h \in T$. For $w \in W$, define the θ-twisted conjugacy class of w in W to be $\{vw\theta(v^{-1}) : v \in W\}$. It is shown in [9, §2.4] that for every θ-twisted conjugacy class C in G, m_C is the unique maximal length element in its θ-twisted conjugacy class in W. Moreover [9, Corollary 3.3 and Theorem 3.8], $m_C^2 = 1$ and $\theta(m_C) = m_C$. Thus

$$(m_C\theta)^2 = \theta^2 \in \mathrm{End}(\mathfrak{h}). \tag{3.2}$$

Proposition 3.1. [9, §2.4] *One has $W_C^- = \{w \in W : \ w \leq m_C\}$ for every θ-twisted conjugacy class C in G.*

Example 3.2. Let $G = SL(n+1, \mathbb{C})$ and $\theta = \mathrm{Id}_G$. Let B and B_- be the subgroups of G consisting, respectively, of all upper-triangular and lower triangular matrices. Identify the Weyl group of G with the symmetric group S_{n+1} on $n+1$ letters. For a conjugacy class C in $SL(n+1, \mathbb{C})$, let $r(C) = \min\{\mathrm{rank}(g - cI) : c \in \mathbb{C}\}$ for any $g \in C$, and let $l(C) = \min(r(C), [(n+1)/2])$, where $[(n+1)/2] = m$ if $n = 2m$ or $n = 2m-1$. Let $m_0 = 1$, and for an integer $1 \leq l \leq [(n+1)/2]$, let $m_l \in S_{n+1}$ be the involution with the cycle decomposition

$$m_l = (1, n+1)(2, n) \cdots (l, \ n+2-l).$$

By [9, Corollary 4.3], $m_C = m_{l(C)}$ for any conjugacy class C in $SL(n+1, \mathbb{C})$. This example will be continued in Example 5.7.

Lemma 3.3. *For any θ-twisted conjugacy class C in G and any $w \in W$ such that $w \leq m_C$, the intersection $C \cap (BwB_-)$ is a smooth and connected submanifold of G and has dimension equal to $\dim C - l(w)$.*

Proof. Since $G_{\mathrm{diag}} \cap (B \times B_-)$ is connected, it follows from [25, Corollary 1.5] that the intersection $C \cap (BwB_-)$ is transversal and irreducible, so $C \cap (BwB_-)$ is smooth and connected, and

$$\dim(C \cap (BwB_-)) = \dim C + \dim(BwB_-) - \dim G = \dim C - l(w).$$

Q.E.D.

3.2. **Symplectic leaves in $C \cap (BwB_-)$.** In this subsection, we fix a θ-twisted conjugacy class C in G and $w \in W$ such that $w \leq m_C$. We will consider the symplectic leaves of π_θ contained in $C \cap (BwB_-)$, which, by Proposition 2.1, are the connected components of intersections of C with G^*-orbits in BwB_-. Since $\pi_{\rm st}(h) = 0$ for every $h \in T$, it follows from 2) of Proposition 2.1 that the Poisson structure π_θ on G is invariant under the θ-twisted conjugation by elements in T. Thus for any $h \in T$, the map $h\cdot_\theta : C \cap (BwB_-) \to C \cap (BwB_-)$ maps a symplectic leaf of π_θ to another. If Σ is a symplectic leaf of π_θ, the union

$$T \cdot_\theta \Sigma = \bigcup_{h \in T} h \cdot_\theta \Sigma$$

will be called a T-orbit of symplectic leaves of π_θ, or, a T*-leaf* of π_θ in short.

Fix a representative $\bar{w} \in N_G(T)$, and let $N^w = N \cap \bar{w} N \bar{w}^{-1}$. Then one has the unique factorization $BwB_- = N^w T \bar{w} N_-$ and the $(T \times T)$-action on BwB_- given by

$$(h_1, h_2) \cdot_\theta (nh\bar{w}m) = \mathrm{Ad}_{h_1}(n) h_1 (w\theta)(h_2^{-1}) h \bar{w} \mathrm{Ad}_{\theta(h_2)}(m), \tag{3.3}$$

where $n \in N^w$, $h \in T$, $m \in N_-$, and for $h \in T$, $(w\theta)(h) = \bar{w}(\theta(h))\bar{w}^{-1} \in T$. Let

$$T_{w\theta} = \{h(w\theta)(h) : \ h \in T\} \quad \text{and} \quad \mathfrak{h}_{w\theta} = \{x + (w\theta)(x) : \ x \in \mathfrak{h}\}.$$

It follows from (3.3) that every G^*-orbit in BwB_- is of the form

$$G^* \cdot_\theta (h_0 \bar{w}) = N^w (T_{w\theta} h_0) \bar{w} N_- \subset BwB_-$$

for some $h_0 \in T$, and $\dim(G^* \cdot_\theta (h_0\bar{w})) = \dim G - l(w) - \dim \ker(1 + w\theta)$, where $1 + w\theta : \mathfrak{h} \to \mathfrak{h}$, with 1 denoting the identity operator on $\mathfrak{h}$. Define

$$\tau_{\bar{w}} : \ BwB_- \longrightarrow T, \ \ \tau_{\bar{w}}(nh\bar{w}m) = h, \quad n \in N^w, h \in T, m \in N_-.$$

It follows from (3.3) that

$$\tau_{\bar{w}}((h_1, h_2) \cdot_\theta g) = h_1 (w\theta)(h_2^{-1}) \tau_{\bar{w}}(g), \quad h_1, h_2 \in T, \ g \in BwB_-. \tag{3.4}$$

For $h \in T$, set

$$S_C^{\bar{w}}(h) = C \cap (G^* \cdot_\theta (h\bar{w})) = \{g \in C \cap (BwB_-) : \tau_{\bar{w}}(g) \in H_{w\theta} h\}. \tag{3.5}$$

Note that $S_C^{\bar{w}}(h_1) = S_C^{\bar{w}}(h_2)$ if and only if $H_{w\theta} h_1 = H_{w\theta} h_2$. By (3.4), one has

$$h_1 \cdot_\theta S_C^{\bar{w}}(h) = S_C^{\bar{w}}(h_1 (w\theta)(h_1^{-1}) h) = S_C^{\bar{w}}(h_1^2 h), \quad \forall \ h_1, h \in T. \tag{3.6}$$

By Proposition 2.1, symplectic leaves of π_θ in $C \cap (BwB_-)$ are precisely the connected components of the $S_C^{\bar{w}}(h)$'s, where $h \in T$. See [18, Remark 2.4] for an example where $S_C^{\bar{w}}(h)$ is not connected.

Lemma 3.4. *1) For any $h \in T$, $S_C^{\bar{w}}(h) \neq \emptyset$ and*

$$\dim S_C^{\bar{w}}(h) = \dim C - l(w) - \dim \ker(1 + w\theta). \tag{3.7}$$

2) Let T' be a subtorus of T such that $T'T_{w\theta} = T$. Then for any $h_1, h_2 \in T$ there exist $h' \in T'$ such that $h' \cdot_\theta S_C^{\bar{w}}(h_1) = S_C^{\bar{w}}(h_2)$;

3) For $h_0, h \in T$, $h \cdot_\theta S_C^{\bar{w}}(h_0) = S_C^{\bar{w}}(h_0)$ if and only if $h^2 \in T_{w\theta}$;

4) For $h_0 \in T$, $x \in \mathfrak{h}$ and $g \in S_C^{\bar{w}}(h_0)$, $\kappa(x, x)(g) \in T_g S_C^{\bar{w}}(h_0)$ if and only if $x \in \mathfrak{h}_{w\theta}$, where $\kappa(x, x)$ is given in (2.6) and $T_g S_C^{\bar{w}}(h_0)$ is the tangent space of $S_C^{\bar{w}}(h_0)$ at g.

Proof. We prove 2) first. Let $h_1, h_2 \in T$. Since $T'T_{w\theta} = T$, there exists $h_3 \in T', h_4 \in T_{w\theta}$ such that $h_2 = h_3 h_4 h_1$. Let $h' \in T'$ be such that $h_3 = (h')^2$. Then $h_2 = (h')^2 h_4 h_1 \in T_{w\theta}(h')^2 h_1$, so by (3.6), $h' \cdot_\theta S_C^{\bar{w}}(h_1) = S_C^{\bar{w}}(h_2)$. This proves 2). Since $C \cap (BwB_-) \neq \emptyset$, $S_C^{\bar{w}}(h_1) \neq \emptyset$ for some $h_1 \in T$. Applying 2) to $T' = T$, one sees that $S_C^{\bar{w}}(h) \neq \emptyset$ for all $h \in T$. Since the intersection $C \cap (G^* \cdot_\theta (h_0 \bar{w}))$ is transversal, one has (3.7). 3) follows directly from (3.6). To prove 4), assume that $x \in \mathfrak{h}$ and $g \in S_C^{\bar{w}}(h_0)$ are such that $\kappa(x, x)(g) \in T_g(S_C^{\bar{w}}(h_0))$. Then there exists $y^* = (y_+ + y, \, -y + y_-) \in \mathfrak{g}_{\rm st}^*$, where $y_+ \in \mathfrak{n}, y_- \in \mathfrak{n}_-$ and $y \in \mathfrak{h}$, such that $y^* - (x, x) \in \mathrm{Ad}_{(g,e)}\{(z, \theta^{-1}(z)) : z \in \mathfrak{g}\}$, or $y_+ + y - x = \mathrm{Ad}_g \theta(y_- - y - x)$. Write $g = n h h_0 \bar{w} n_-$ for some $n \in N^w, h \in \mathfrak{h}_{w\theta}$, and $n_- \in N_-$. One has

$$\mathrm{Ad}_{nhh_0}^{-1}(y_+ + y - x) = \mathrm{Ad}_{\bar{w}} \mathrm{Ad}_{n_-} \theta(y_- - y - x) \in \mathfrak{b} \cap \mathrm{Ad}_{\bar{w}} \mathfrak{b}_- = \mathfrak{h} + \mathfrak{n} \cap \mathrm{Ad}_{\bar{w}} \mathfrak{n}_-,$$

whose $\mathfrak{h}$-component gives $y - x = -w\theta(y + x)$, and thus

$$x = \frac{1}{2}(x - w\theta(x)) + \frac{1}{2}(x + w\theta(x)) = \frac{1}{2}(x + y + w\theta(x + y)) \in \mathfrak{h}_{w\theta}.$$

Q.E.D.

Proposition 3.5. *Let T' be a subtorus of T such that $T'T_{w\theta} = T$. Then $C \cap (BwB_-)$ is one single T'-orbit of symplectic leaves of π_θ under the θ-twisted conjugation.*

Proof. Consider the action map

$$\sigma: \; T' \times \Sigma \longrightarrow C \cap (BwB_-), \;\; (h, g) \longmapsto hg\theta(h^{-1}), \quad h \in T', \, g \in \Sigma.$$

Let $\mathfrak{h}' \subset \mathfrak{h}$ be the Lie algebra of T'. For any $g \in \Sigma$, the kernel of the differential $\sigma_*(e, g)$ of σ at (e, g) is isomorphic to $\{x \in \mathfrak{h}' : \kappa(x, x)(g) \in T_g \Sigma\}$, which, by Lemma 3.4, is $\mathfrak{h}' \cap \mathfrak{h}_{w\theta}$. Let $\mathfrak{h}''$ be any complement of $\mathfrak{h}' \cap \mathfrak{h}_{w\theta}$ in $\mathfrak{h}'$. It follows from $T'T_{w\theta} = T$ that $\mathfrak{h}' + \mathfrak{h}_{w\theta} = \mathfrak{h}$ and thus $\mathfrak{h} = \mathfrak{h}'' + \mathfrak{h}_{w\theta}$ is a direct sum. Thus $\sigma_*(e, g)$ maps $\mathfrak{h}'' \times T_g \Sigma$ injectively to $T_g(C \cap (BwB_-))$. Since $\dim(\mathfrak{h}'' \times T_g \Sigma) = \dim T_g(C \cap (BwB_-))$, σ is a submersion at (e, g). Since T is abelian, σ is a submersion everywhere. Thus $T' \cdot_\theta \Sigma$ is an open subset of $C \cap (BwB_-)$. If Σ' is another symplectic leaf of π_θ in $C \cap (BwB_-)$, $T' \cdot_\theta \Sigma'$ is also an open subset of $C \cap (BwB_-)$, and either $T' \cdot_\theta \Sigma = T' \cdot_\theta \Sigma'$ or

$(T' \cdot_\theta \Sigma) \cap (T' \cdot_\theta \Sigma') = \emptyset$. Since $C \cap (BwB_-)$ is connected, one must have $T' \cdot_\theta \Sigma = T' \cdot_\theta \Sigma'$. This shows that $T' \cdot_\theta \Sigma = C \cap (BwB_-)$.

Q.E.D.

Corollary 3.6. *With the notation as in Proposition 3.5, let $h \in T$ and let Σ be any connected component of $S_C^{\bar{w}}(h)$. Then Σ is a symplectic leaf of π_θ in $C \cap (BwB_-)$, and every symplectic leaf of π_θ in $C \cap (BwB_-)$ is of the form $h' \cdot_\theta \Sigma$ for some $h' \in T'$.*

3.3. **The T-leaves of π_θ.** Summarizing the results in §3.1 and §3.2, we have the following Theorem 3.7.

Theorem 3.7. *Let T' be a subtorus of T such that $T'T_{w\theta} = T$ for every $w \in W$. Then the T'-orbits of symplectic leaves of π_θ in G are precisely all the intersections $C \cap (BwB_-)$, where C is a θ-twisted conjugacy class in G and $w \in W$ is such that $w \leq m_C$. Every such an intersection $C \cap (BwB_-)$ is non-empty and the symplectic leaves in $C \cap (BwB_-)$ have dimension equal to $\dim C - l(w) - \dim \ker(1 + w\theta)$.*

Theorem 1.1 is now a special case of Theorem 3.7 by taking $T' = T$.

In §7, we apply Theorem 3.7 to an example where T' is a proper subtorus of T.

4. The lowest rank of π_θ in a θ-twisted conjugacy class

4.1. **Some linear algebra facts.** Recall that for an n-dimensional real or complex vector space V, $S \in \mathrm{End}(V)$ is called a reflection if $\dim \ker(1 - S) = n - 1$ and $\dim \ker(1 + S) = 1$, where 1 also denotes the identity map on V.

Lemma 4.1. *Let V be an n-dimensional real vector space and let $S \in \mathrm{End}(V)$ be a reflection. Then for any $A \in \mathrm{End}(V)$,*

$$-1 + \dim \ker(1 + A) \leq \dim \ker(1 + AS) \leq 1 + \dim \ker(1 + A); \tag{4.1}$$

$$-1 + \mathrm{rank}(1 + A) \leq \mathrm{rank}(1 + AS) \leq 1 + \mathrm{rank}(1 + A). \tag{4.2}$$

Moreover, $\mathrm{rank}(1 + A) = 1 + \mathrm{rank}(1 + AS)$ *if and only if* $\ker(1 + S) \subset \mathrm{im}(1 + A)$ *and* $\ker(1 + S) \cap \mathrm{im}(1 + AS) = 0$.

Proof. Let $x_0 \in \ker(1 + S)$, $x_0 \neq 0$. Then $\mathrm{im}(1 - S) = \mathbb{R}x_0$. It follows from $1 + AS = (1 + A) - A(1 - S)$ that $\mathrm{im}(1 + AS) \subset \mathrm{im}(1 + A) + \mathbb{R}x_0$. Exchanging A and AS, one has

$$\mathrm{im}(1 + A) + \mathbb{R}x_0 = \mathrm{im}(1 + AS) + \mathbb{R}x_0, \tag{4.3}$$

from which both inequalities in (4.2) follow. Using $\dim \ker(A) = n - \mathrm{rank}(A)$ for $A \in \mathrm{End}(V)$, one proves (4.1). The last statement also follows from (4.3).

Q.E.D.

Lemma 4.2. *Let V be a finite dimensional real vector space with an inner product $\langle\,,\,\rangle$, and let $O(V,\langle\,,\,\rangle)$ be the group of all linear operators on V preserving $\langle\,,\,\rangle$. Then for all $A, S \in O(V,\langle\,,\,\rangle)$ and S a reflection, one has*

$$\dim\ker(1+AS) = \dim\ker(1+A) \pm 1 \quad \textit{and} \quad \operatorname{rank}(1+AS) = \operatorname{rank}(1+A) \pm 1.$$

Proof. If $\lambda \in \mathbb{C}$ is an eigenvalue for A, so is $\bar{\lambda}$, and $|\lambda|^2 = 1$. It follows that $\det(A) = (-1)^{\dim\ker(1+A)}$. Since $\det(AS) = -\det(A)$, $\dim\ker(1+AS) \neq \dim\ker(1+A)$. It follows from Lemma 4.1 that $\dim\ker(1+AS) = \dim\ker(1+A) \pm 1$, and that

$$\operatorname{rank}(1+AS) = \dim V - \dim\ker(1+AS) = \operatorname{rank}(1+A) \pm 1.$$

Q.E.D.

Lemma 4.3. *For any finite dimensional real vector space V and $A \in \operatorname{End}(V)$, one has* $\ker(1+A) \subset \operatorname{im}(1-A)$ *and* $\operatorname{rank}(1-A) - \dim\ker(1+A) = \operatorname{rank}(1-A^2)$.

Proof. Since $2x = (x+Ax)+(x-Ax)$ for every $x \in V$, one has $\ker(1+A) \subset \operatorname{im}(1-A)$. Since the map $(1+A)|_{\operatorname{im}(1-A)} : \operatorname{im}(1-A) \to \operatorname{im}(1-A^2)$ is surjective and has $\ker(1+A)$ as its kernel, one has $\operatorname{rank}(1-A) - \dim\ker(1+A) = \operatorname{rank}(1-A^2)$.

Q.E.D.

We now apply the above linear algebra facts to the case of $V = \mathfrak{h}$ with the inner product $\langle\,,\,\rangle$ induced from the Killing form of $\mathfrak{g}$. We continue to assume that $\theta(B) = B$ and $\theta(T) = T$, so $\theta \in O(\mathfrak{h},\langle\,,\,\rangle)$. For $w \in W$, let

$$L_\theta(w) = l(w) + \dim\ker(1+w\theta) \quad \text{and} \quad L'_\theta(w) = l(w) + \operatorname{rank}(1-w\theta).$$

By Lemma 4.3, one has

$$L'_\theta(w) - L_\theta(w) = \operatorname{rank}(1-(w\theta)^2), \quad w \in W. \tag{4.4}$$

For a root $\alpha \in \mathfrak{h}^*$, let $s_\alpha \in W$ be the corresponding reflection on $\mathfrak{h}$.

Lemma 4.4. *If $w, w' \in W$ are such that $w \le w'$, then*

$$L_\theta(w) \le L_\theta(w') \quad \textit{and} \quad L'_\theta(w) \le L'_\theta(w'), \tag{4.5}$$

and both $L_\theta(w') - L_\theta(w)$ and $L'_\theta(w') - L'_\theta(w)$ are even integers.

Proof. By [3, Corollary 2.2.2], there exists a sequence $(w_1, \ldots, w_k)$ in W such that $w = w_1 < \cdots < w_k = w'$ and $l(w_{j+1}) = l(w_j) + 1$ for $1 \le j \le k-1$. We can thus assume that $l(w') = l(w) + 1$. Then $w' = ws_\alpha$ for a positive root α. By Lemma 4.2,

$$L_\theta(w) = l(w) + \dim\ker(1+w'\theta s_{\theta^{-1}(\alpha)}) = l(w') - 1 + \dim\ker(1+w'\theta) \pm 1,$$

so $L_\theta(w') - L_\theta(w) \in \{0, 2\}$. Similarly, one has $L'_\theta(w') - L'_\theta(w) \in \{0, 2\}$.

Q.E.D.

4.2. **The lowest rank of π_θ in a θ-twisted conjugacy class.** For $g \in G$, denote by $\mathrm{rank}(\pi_\theta(g))$ the dimension of the symplectic leaf of π_θ through g. By Theorem 1.1,

$$\mathrm{rank}(\pi_\theta(g)) = \dim C - L_\theta(w), \quad \forall\, g \in C \cap (BwB_-), \tag{4.6}$$

where C is any θ-twisted conjugacy class and $w \in W$ is such that $w \le m_C$. By Lemma 4.4, if $w, w' \in W$ are such that $w \le w'$, then

$$\mathrm{rank}(\pi_\theta(g)) \ge \mathrm{rank}(\pi_\theta(g')), \quad \text{if } \forall\, g \in C \cap (BwB_-),\ g' \in C \cap (Bw'B_-). \tag{4.7}$$

Proposition 4.5. *For any θ-twisted conjugacy class C,*

$$\min\{\mathrm{rank}(\pi_\theta(g)) : g \in C\} = \dim C - L_\theta(m_C),$$

and for any $g \in C$, $\mathrm{rank}(\pi_\theta(g)) = \dim C - L_\theta(m_C)$ if and only if $g \in C \cap (Bm_CB_-)$.

Proof. By (4.7), $\dim C - L_\theta(m_C) = \mathrm{rank}(\pi_\theta(g))$ for any $g \in C \cap (Bm_CB_-)$ is the minimal rank of π_θ in C. It remains to show that if $w \in W$ is such that $w < m_C$, then $L_\theta(w) \le L_\theta(m_C) - 2$.

Let $w \in W$ be such that $w < m_C$. By [3, Corollary 2.2.2], there exists a sequence $(w_1, \ldots, w_k)$ in W such that $w = w_1 < \cdots < w_k = m_C$ and $l(w_{j+1}) = l(w_j) + 1$ for $1 \le j \le k-1$. By (4.7), $\mathrm{rank}(\pi_\theta(g)) \ge \dim C - L_\theta(w_{k-1})$ for all $g \in C \cap (BwB_-)$. It thus suffices to show that $L_\theta(w) = L_\theta(m_C) - 2$ for any $w \in W$ such that $w < m_C$ and $l(w) = l(m_C) - 1$.

Assume thus that $w < m_C$ and $l(w) = l(m_C) - 1$. Then $w = m_C s_\beta$ for some positive root β such that $m_C(\beta) < 0$. By the proof of Lemma 4.4, $L_\theta(w) = L_\theta(m_C)$ or $L_\theta(w) = L_\theta(m_C) - 2$. Suppose that $L_\theta(w) = L_\theta(m_C)$. Then

$$\mathrm{rank}(1 + m_C\theta) = 1 + \mathrm{rank}(1 + w\theta) = 1 + \mathrm{rank}(1 + m_C\theta s_{\theta^{-1}(\beta)}).$$

Regarding both m_C and θ as linear operators on $\mathfrak{h}^*$, one sees from Lemma 4.1 that $\theta^{-1}(\beta) \in \mathrm{im}(1 + m_C\theta)$, and thus by (3.2),

$$\theta^{-1}(\beta) - m_C(\beta) = (1 - m_C\theta)\theta^{-1}(\beta) \in \mathrm{im}(1 - (m_C\theta)^2) = \mathrm{im}(1 - \theta^2).$$

Let $\lambda \in \mathfrak{h}^*$ be such that $\theta^{-1}(\beta) - m_C(\beta) = (1 - \theta^2)(\lambda)$. Let $\{\alpha_1, \ldots, \alpha_k\}$ be the set of all simple roots and write $\lambda = \sum_{j=1}^k a_j\alpha_j$ and $\theta^2(\lambda) = \sum_{j=1}^k b_j\alpha_j$. Then

$$\theta^{-1}(\beta) - m_C(\beta) = \sum_{j=1}^k (a_j - b_j)\alpha_j. \tag{4.8}$$

Since both $\theta^{-1}(\beta)$ and $-m_C(\beta)$ are positive roots, each $a_j - b_j$ is a non-negative integer and $\sum_{j=1}^k (a_j - b_j) > 0$. On the other hand, since θ^2 permutes the simple roots, $\sum_{j=1}^k a_j = \sum_{j=1}^k b_j$. We have thus arrived at a contradiction and we conclude that $L_\theta(w) = L_\theta(m_C) - 2$.

Q.E.D.

Remark 4.6. From Proposition 4.5 and the linear algebra fact that if $t \to A(t) \in \operatorname{End}(V)$ is a continuous map from an open neighborhood of $0 \in \mathbb{C}$ to $\operatorname{End}(V)$ then $\operatorname{rank}(A(t)) \geq \operatorname{rank}(A(0))$ for t closed to 0, one concludes that $C \cap (Bm_C B_-)$ is closed in C. To see this from another way, observe first that

$$Bm_C B_- \subset \bigsqcup_{w \in W, w \geq m_C} BwB$$

by [13, Corollary 1.2], so by the definition of m_C, $C \cap (Bm_C B_-) \subset Bm_C B$. Since

$$(Bm_C B_-) \cap (Bm_C B) = Bm_C,$$

one has $C \cap (Bm_C B_-) = C \cap (Bm_C)$. In particular, $C \cap (Bm_C B_-)$ is closed in C. Similarly, $C \cap (Bm_C B_-) = C \cap (m_C B_-)$.

5. Spherical conjugacy classes and the lowest rank of π_θ in G

5.1. **Spherical θ-twisted conjugacy classes.** Recall that a θ-twisted conjugacy class C in G is said to be spherical if it has an open orbit under the θ-twisted conjugation by some Borel subgroup of G. For $g \in G$, let $B \cdot_\theta g$ be the B-orbit through g under the θ-twisted conjugation. The following Lemma 5.1 on the dimensions of B-orbits in G is a generalization of [5, Theorem 5] and is proved in [12].

Lemma 5.1. *For any $w \in W$ and $g \in BwB$, one has*

$$\dim(B \cdot_\theta g) \geq l(w) + \operatorname{rank}(1 - w\theta).$$

In particular, $\dim C \geq l(m_C) + \operatorname{rank}(1 - m_C\theta)$ *for any θ-twisted conjugacy class C.*

Generalizing a result in [5] of N. Cantarini, G. Carnovale, and M. Costantini for $\theta = \mathrm{Id}_G$, the following characterization of spherical θ-twisted conjugacy classes is proved in [22] for the case of characteristic zero and in [7] for good odd characteristics.

Theorem 5.2. [5, 7, 22] *For $\theta \in \operatorname{Aut}(G)$ such that $\theta(B) = B$ and $\theta(T) = T$, a θ-twisted conjugacy class C in G is spherical if and only if*

$$\dim C = l(m_C) + \operatorname{rank}(1 - m_C\theta).$$

Corollary 5.3. *If C is a spherical θ-twisted conjugacy class in G, then*

$$\min\{\operatorname{rank}(\pi_\theta(g)) : g \in C\} = \operatorname{rank}(1 - \theta^2),$$

and for $g \in C$, $\operatorname{rank}(\pi_\theta(g)) = \operatorname{rank}(1 - \theta^2)$ *if and only if $g \in C \cap (Bm_C B_-)$.*

Proof. If C is spherical, then

$$\dim C - L_\theta(m_C) = L'_\theta(m_C) - L_\theta(m_C) = \mathrm{rank}(1-(m_C\theta)^2) = \mathrm{rank}(1-\theta^2),$$

and Corollary 5.3 now follows directly from Proposition 4.5.

Q.E.D.

Remark 5.4. Note that as θ is a linear operator on $\mathfrak{h}$ preserving the inner product on $\mathfrak{h}$ induced by the Killing form of $\mathfrak{g}$, one can decompose $\mathfrak{h}$ into the direct sum of $\ker(1-\theta^2)$ and 2-dimensional subspaces on which θ^2 acts as non-trivial rotations. Consequently, $\mathrm{rank}(1-\theta^2)$ is always an even integer.

Example 5.5. Assume that G is of type D_4 and $\theta \in \mathrm{Aut}(G)$ has order 3 such that the induced $\theta \in \mathrm{Aut}(\Gamma)$ fixes α_2 and maps α_1 to α_3, α_3 to α_4, and α_4 to α_1, where α_2 is the simple root not orthogonal to any of the other three. Then the fixed point set G^θ of θ is of type G_2 and the θ-twisted conjugacy class C through the identity element of e is spherical [12, §4.5]. In this case (see [12, §4.5] and [22, Example 3.9]), $m_C = w_0 s_{\alpha_2}$, where w_0 is the longest element in W, and one has $\dim C = 14 = l(w_0 s_{\alpha_2}) + \mathrm{rank}(1 - w_0 s_{\alpha_2}\theta)$. Since $\mathrm{rank}(1-\theta^2) = 2$, the lowest rank of π_θ in C is 2.

Remark 5.6. When $\theta^2 = 1 \in \mathrm{Aut}(\Gamma)$, or when G is simple, spherical θ-twisted conjugacy classes exist for any $\theta \in \mathrm{Aut}(G)$. Indeed, by Remark 2.3, we may assume that $\theta(B) = B$ and $\theta(T) = T$. Fix a root vector e_α for each $\alpha \in \Gamma$ and let $\lambda \in \mathbb{C}$ be such that $\theta(e_\alpha) = \lambda_\alpha e_{\theta(\alpha)}$. Choose any $h \in T$ such that $h^\alpha = \frac{1}{\lambda_\alpha}$ for each $\alpha \in \Gamma$, and let $\theta_1 = \theta \mathrm{Ad}_h = \mathrm{Ad}_{\theta(h)}\theta \in \mathrm{Aut}(G)$. Then $\theta_1(\alpha) = \theta(\alpha)$ and $\theta_1(e_\alpha) = e_{\theta(\alpha)}$ for every $\alpha \in \Gamma$, so $\theta_1 \in \mathrm{Aut}(G)$ has the same order as $\theta \in \mathrm{Aut}(\Gamma)$. Suppose that $\theta^2 = 1 \in \mathrm{Aut}(\Gamma)$. Then $\theta_1^2 = \mathrm{Id}_G$, and the θ_1-twisted conjugacy class through the identity element is trivial if $\theta_1 = \mathrm{Id}_G$ and a symmetric space, and hence spherical, if θ_1 has order 2, and thus the θ-twisted conjugacy class C through $\theta(h)$ is spherical. If G is simple, $\theta \in \mathrm{Aut}(\Gamma)$ has order 2 or 3, with the latter (triality case) possible only for G of type D_4, and we saw in Example 5.5 that spherical θ-twisted conjugacy classes exist in this case. When G is simple, spherical conjugacy classes in G (i.e. the case for $\theta = \mathrm{Id}_G$) have been classified in [5, 24].

5.2. **Proof of Theorem 1.2.** For any θ-twisted conjugacy class C in G and any $g \in C$, one has

$$\mathrm{rank}(\pi_\theta(g)) \geq \dim C - L_\theta(m_C) \geq L'_\theta(m_C) - L_\theta(m_C) = \mathrm{rank}(1-\theta^2).$$

We have already proved in Corollary 5.3 that $\mathrm{rank}(\pi_\theta(g)) = \mathrm{rank}(1-\theta^2)$ for all $g \in C \cap (Bm_C B_-)$ if C is spherical. Suppose that $g \in G$ is such that $\mathrm{rank}(\pi_\theta(g)) =$

$\text{rank}(1-\theta^2)$ and let C be the θ-twisted conjugacy class of g. By Proposition 4.5, $\text{rank}(\pi_\theta(g_1)) = \text{rank}(1-\theta^2)$ for all $g_1 \in C \cap (Bm_C B_-)$, and thus

$$\dim C = L_\theta(m_C) + \text{rank}(1-\theta^2) = L'_\theta(m_C).$$

By Theorem 5.2, C is spherical, and by Proposition 4.5, $g \in C \cap (Bm_C B_-)$.

Q.E.D.

5.3. **Proof of Corollary 1.3.** Corollary 1.3 follows directly from Theorem 1.1, Theorem 1.2 and the statement in Remark 5.6 that spherical θ-twisted conjugacy classes exist when $\theta^2 = 1 \in \text{Aut}(\Gamma)$.

Example 5.7. For $G = SL(n+1,\mathbb{C})$, a conjugacy class C in G is spherical if and only if either $C = C^{\text{ss}}_{l,\lambda,\lambda'}$, where $1 \leq l \leq (n+1)/2$ is an integer and $C^{\text{ss}}_{l,\lambda,\lambda'}$ is the semisimple conjugacy class with two distinguished eigenvalues λ and λ' of respective multiplicities l and $n+1-l$, or $C = C^{\text{uni}}_{l,\lambda}$, where $0 \leq l \leq (n+1)/2$ is an integer, and $C^{\text{uni}}_{l,\lambda}$ is the conjugacy class of λg with g an upper triangular Jordan matrix having 1 as the only eigenvalue and $n+1-l$ size 1 Jordan blocks and l size 2 Jordan blocks (see [4, 5]). Let again B and B_- be the subgroups of G consisting of all upper-triangular and lower triangular matrices respectively. For $C = C^{\text{ss}}_{l,\lambda,\lambda'}$ or $C = C^{\text{uni}}_{l,\lambda}$ as above, it follows from Example 3.2 that m_C has the cycle decomposition

$$m_C = m_l = (1, n+1)(2, n)\cdots(l,\ n+2-l),$$

and thus $\dim(C \cap Bm_C B_-) = \dim C - l(m_C) = \text{rank}(1-m_C) = l$, so by Corollary 1.3, $C \cap (Bm_C B_-) \cong (\mathbb{C}^\times)^l$. Explicit parameterizations of $C \cap (Bm_C B_-)$ by $(\mathbb{C}^\times)^l$ were given in [26], where the Poisson structure π_θ for $G = SL(n+1,\mathbb{C})$ and $\theta = \text{Id}_G$ was treated in more detail. For an integer k, let I_k be the $k \times k$ identity matrix and J_k the $k \times k$ anti-diagonal matrix with 1's on the anti-diagonal. Let $0 \leq l \leq (n+1)/2$ be an integer and let $\lambda, \lambda' \in \mathbb{C}$ be such that $\lambda^{n+1-l}(\lambda')^l = 1$. For $(x_1,\ldots,x_l) \in (\mathbb{C}^\times)^l$, let

$$g(\lambda,\lambda',x_1,\ldots,x_l) = \begin{pmatrix} (\lambda+\lambda')I_l & 0 & \lambda' X J_l \\ 0 & \lambda I_{n+1-2l} & 0 \\ -\lambda X' J_l & 0 & 0 \end{pmatrix} \in SL(n+1,\mathbb{C}),$$

where $X = \text{diag}(x_1, x_2, \ldots, x_l)$ and $X' = \text{diag}(x_l^{-1}, \ldots, x_2^{-1}, x_1^{-1})$. By [26, §5.3],

$$(C^{\text{ss}}_{l,\lambda,\lambda'}) \cap (Bm_C B_-) = \{g(\lambda,\lambda',x_1,\ldots,x_l) :\ (x_1,\ldots,x_l) \in (\mathbb{C}^\times)^l\},$$
$$(C^{\text{uni}}_{l,\lambda}) \cap (Bm_C B_-) = \{g(\lambda,\lambda,x_1,\ldots,x_l) :\ (x_1,\ldots,x_l) \in (\mathbb{C}^\times)^l\}.$$

We conclude that the zero-locus $Z(\pi_\theta)$ of π_θ in $SL(n+1,\mathbb{C})$ is given by

$$Z(\pi_\theta) = \bigcup_{l=0}^{[(n+1)/2]} \{g(\lambda,\lambda',x_1,\ldots,x_l) :\ \lambda^{n+1-l}(\lambda')^l = 1,\ (x_1,\ldots,x_l) \in (\mathbb{C}^\times)^l\}.$$

6. The Poisson structure π_θ for an arbitrary $\theta \in \mathrm{Aut}(G)$

Recall that we fixed, from the very beginning, the pair (B, T) of a Borel subgroup and a maximal torus $T \subset B$. Although the Poisson structure π_θ on G is defined in §2.2 for any $\theta \in \mathrm{Aut}(G)$ and Proposition 2.1 holds for π_θ, we made the assumption that θ stabilizes both B and T throughout §3-§5.

Assume now that $\theta \in \mathrm{Aut}(G)$ is arbitrary. By Remark 2.3, there exists $g_0 \in G$ such that $\theta(B) = g_0 B g_0^{-1}$ and $\theta(T) = g_0 T g_0^{-1}$. Choose any such g_0 and let $\theta' = \mathrm{Ad}_{g_0^{-1}}\theta$. Then θ' stabilizes both B and T, and

$$r_{g_0}: \quad (G, \pi_\theta) \longrightarrow (G, \pi_{\theta'}), \quad g \longmapsto g g_0, \qquad g \in G,$$

is a Poisson isomorphism. Applying results in §3-§5 to $\pi_{\theta'}$, we arrive at properties of π_θ via the Poisson isomorphism r_{g_0}. In particular, Theorem 1.1, Theorem 1.2, and Corollary 1.3 hold if BwB_- is replaced by $BwB_-g_0^{-1}$ for $w \in W$.

7. An example: the Poisson structure $\pi_{\tilde{\theta}}$ on G^n

7.1. **The Poisson structure $\pi_{\tilde{\theta}}$ on G^n.** In this section, we assume that G is non-trivial and consider the direct product group $\widetilde{G} = G^n$, where $n \geq 2$, and $\tilde{\theta} \in \mathrm{Aut}(G^n)$ is given by

$$\tilde{\theta}(g_1, g_2, \ldots, g_n) = (g_2, g_3, \ldots, g_n, g_1), \qquad g_j \in G,\ j = 1, 2, \ldots, n.$$

Let $\widetilde{B} = B^n$, $\widetilde{B}_- = B_-^n$, and $\widetilde{T} = T^n$. Then $\tilde{\theta}(\widetilde{B}) = \widetilde{B}$ and $\tilde{\theta}(\widetilde{T}) = \widetilde{T}$. Applying the construction in §2.2 to G^n using the pair $(\widetilde{B}, \widetilde{T})$ and $\tilde{\theta} \in \mathrm{Aut}(G^n)$, one obtains the holomorphic Poisson structure $\pi_{\tilde{\theta}}$ on G^n. We describe some properties of $\pi_{\tilde{\theta}}$.

The $\tilde{\theta}$-twisted conjugation of G^n on itself is given by

$$(g_1, g_2, \ldots, g_n) \cdot_{\tilde{\theta}} (k_1, k_2, \ldots, k_n) = (g_1 k_1 g_2^{-1},\ g_2 k_2 g_3^{-1},\ \ldots,\ g_n k_n g_1^{-1}). \tag{7.1}$$

Consider the multiplication map

$$\mu_n: \quad G^n \longrightarrow G, \quad (g_1, g_2, \ldots, g_n) \longmapsto g_1 g_2 \cdots g_n.$$

For a conjugacy class C in G, let $\widetilde{C} = \mu_n^{-1}(C) \subset G^n$.

Lemma 7.1. *$\tilde{\theta}$-twisted conjugacy classes in G^n are precisely the subsets $\widetilde{C}$ of G^n, where C is a conjugacy class in G.*

Proof. Let $\tilde{k} = (k_1, k_2, \ldots, k_n) \in G^n$, let $\mathcal{O}_{(k_1, k_2, \ldots, k_n)}$ be the $\tilde{\theta}$-twisted conjugacy class of $\tilde{k}$, and let $C_{k_1 k_2 \cdots k_n}$ be the conjugacy class of $k_1 k_2 \cdots k_n$ in G. It is clear from (7.1) that $\mathcal{O}_{(k_1, k_2, \ldots, k_n)} \subset \mu_n^{-1}(C_{k_1 k_2 \cdots k_n})$. Conversely, let $\tilde{h} = (h_1, h_2, \ldots, h_n) \in \mu_n^{-1}(C_{k_1 k_2 \cdots k_n})$ and let $g_1 \in G$ be such that $h_1 h_2 \cdots h_n = g_1 k_1 k_2 \cdots k_n g_1^{-1}$. For $2 \leq j \leq n$, let $g_j \in G$ be given inductively by $g_{j-1} k_{j-1} g_j^{-1} = h_{j-1}$. It follows from $h_1 h_2 \cdots h_n =$

$g_1k_1k_2\cdots k_ng_1^{-1}$ that $h_n = g_nk_ng_1^{-1}$, and thus $\tilde{g} \cdot_{\tilde{\theta}} \tilde{k} = \tilde{h}$, where $\tilde{g} = (g_1, g_2, \ldots, g_n)$. This shows that $\mu_n^{-1}(C_{k_1k_2\cdots k_n}) \subset \mathcal{O}_{(k_1,k_2,\ldots,k_n)}$. Hence $\mu_n^{-1}(C_{k_1k_2\cdots k_n}) = \mathcal{O}_{(k_1,k_2,\ldots,k_n)}$, and Lemma 7.1 follows.

Q.E.D.

Identify the Weyl group $\widetilde{W}$ of G^n with W^n. Then for $\tilde{w} = (w_1, w_2, \ldots, w_n) \in W^n$,

$$\widetilde{B}\tilde{w}\widetilde{B}_- = (Bw_1B_-) \times (Bw_2B_-) \times \cdots \times (Bw_nB_-).$$

Recall that we assume that $n \geq 2$.

Lemma 7.2. *For any conjugacy class C in G and any $\tilde{w} = (w_1, w_2, \ldots, w_n) \in W^n$,*

$$\widetilde{C} \cap (\widetilde{B}\tilde{w}\widetilde{B}_-) \neq \emptyset.$$

Proof. Let C be a conjugacy class in G. By Proposition 3.1, it is enough to show that $\widetilde{C} \cap (\widetilde{B}\tilde{w}_0\widetilde{B}_-) \neq \emptyset$, where $\tilde{w}_0 = (w_0, w_0, \ldots, w_0) \in W^n$, and w_0 is the longest element in W. Since $\widetilde{B}\tilde{w}_0\widetilde{B}_- = (Bw_0)^n$, one needs to show that $C \cap \mu_n((Bw_0)^n) \neq \emptyset$.

If $n \geq 2$ is even, since $Bw_0Bw_0 = BB_- \ni e$, one has

$$\mu_n((Bw_0)^n) \supset Bw_0Bw_0 = BB_- \supset B,$$

and since $C \cap B \neq \emptyset$, one has $C \cap \mu_n((Bw_0)^n) \neq \emptyset$. Suppose that $n \geq 3$ is odd. Then $\mu_n((Bw_0)^n) \supset Bw_0Bw_0Bw_0 = Bw_0BB_-$. We claim that $Bw_0BB_- = G$ and thus $C \cap \mu_n((Bw_0)^n) \neq \emptyset$. To show that $Bw_0BB_- = G$, first note that Bw_0BB_- is the union of some (B, B_-)-double cosets. For any $w \in W$, since $w \leq w_0$, $BwB_- \cap Bw_0B \neq \emptyset$ by [13, Corollary 1.2], so $BwB_- \subset Bw_0BB_-$. Since $w \in W$ is arbitrary, one has $Bw_0BB_- = G$.

Q.E.D.

In the notation of §3.3, for every conjugacy class C in G one has $\widetilde{W}_{\widetilde{C}}^- = \widetilde{W}$ and $m_{\widetilde{C}} = (w_0, w_0, \ldots, w_0)$. Applying Theorem 1.1, we have the following description of $\widetilde{T}$-orbits of symplectic leaves of $\pi_{\tilde{\theta}}$ on G^n, where $\widetilde{T}$ acts on G^n by the $\tilde{\theta}$-twisted conjugation given in (7.1).

Proposition 7.3. *For any conjugacy class C and $\tilde{w} = (w_1, w_2, \ldots, w_n) \in W^n$,*

$$\widetilde{C} \cap (\widetilde{B}\tilde{w}\widetilde{B}_-) = \{(g_1, g_2, \ldots, g_n) \in (Bw_1B_-) \times (Bw_2B_-) \times \cdots \times (Bw_nB_-) : g_1g_2\cdots g_n \in C\}$$

is nonempty, and the $\widetilde{T}$-orbits of symplectic leaves of $\pi_{\tilde{\theta}}$ in G^n are precisely all such intersections; The dimension of the symplectic leaves of $\pi_{\tilde{\theta}}$ in $\widetilde{C} \cap (\widetilde{B}\tilde{w}\widetilde{B}_-)$ is

$$\dim C + (n-1)\dim G - (l(w_1) + l(w_2) + \cdots + l(w_n)) - \dim\ker(1 - (-1)^n w_1w_2\cdots w_n),$$

where $1-(-1)^n w_1 w_2 \cdots w_n$ is regarded as a linear operator on the Lie algebra $\mathfrak{h}$ of T.

Proof. We only need to prove the last statement on the dimension of the symplectic leaves in $\widetilde{C} \cap (\widetilde{B}\tilde{w}\widetilde{B}_-)$, which by Theorem 1.1, is equal to

$$\dim \widetilde{C} - (l(w_1) + l(w_2) + \cdots + l(w_n)) - \dim \ker(1 + (w_1, w_2, \ldots, w_n)\tilde{\theta}),$$

where $1+(w_1, w_2, \ldots, w_n)\tilde{\theta}$ is regarded as a linear operator on the Lie algebra of $\widetilde{T}$. It is easy to see that $\ker(1 + (w_1, w_2, \ldots, w_n)\tilde{\theta}) \cong \ker(1 - (-1)^n w_1 w_2 \cdots w_n)$, and since $\widetilde{C} \cong G^{n-1} \times C$, so $\dim \widetilde{C} = \dim C + (n-1)\dim G$, and the claim is proved.

Q.E.D.

7.2. **Spherical $\tilde{\theta}$-twisted conjugacy classes.** We now classify spherical $\tilde{\theta}$-twisted conjugacy classes in G^n. Recall that we assume G to be non-trivial.

Lemma 7.4. *1) If $n > 3$ or if $n = 3$ and the rank of G is greater than 1, there are no spherical $\tilde{\theta}$-twisted conjugacy classes in G^n;*

2) For $n = 3$ and G having rank 1, or for $n = 2$ and G arbitrary, $\widetilde{C}$ is spherical if and only if $C = \{z\}$, where z is in the center of G.

Proof. We already know that $\dim \widetilde{C} = \dim C + (n-1)\dim G$ for any conjugacy class C in G. On the other hand, the dimension of any spherical θ-twisted conjugacy class in G^n can not be larger than $\dim \widetilde{B} = n \dim B$. Since

$$\begin{aligned}(n-1)\dim G - n\dim B &= (n-1)(2\dim B - \dim T) - n\dim B \\ &= (n-3)\dim(B/T) + (\dim(B/T) - \dim T),\end{aligned}$$

and since $\dim(B/T)$ is the number of positive roots and $\dim T$ is the number of simple roots, $\dim \widetilde{C} > n \dim B$ for every conjugacy class C in G if either $n > 3$ or $n = 3$ and the rank of G is more than 1, and in such cases $\widetilde{C}$ can not be spherical for any C.

Let $n = 3$ and $G = SL(2, \mathbb{C})$ or $G = PSL(2, \mathbb{C})$. Then $\dim \widetilde{C} > 3 \dim B$ if C is not a trivial conjugacy class and thus $\widetilde{C}$ can not be spherical. Assume now that C is a trivial conjugacy class consisting of a single central element. Then $\dim \widetilde{C} = 2\dim G = 6$. Since $m_{\widetilde{C}} = (w_0, w_0, w_0)$ and since $l(m_{\widetilde{C}}) + \operatorname{rank}(1 - m_{\widetilde{C}}\tilde{\theta}) = 3 + 3 = 6$, it follows from from Theorem 5.2 that $\widetilde{C}$ is spherical.

It remains to consider the case of $n = 2$ and an arbitrary (semi-simple) G. In this case, for an arbitrary conjugacy class C in G, $\dim \widetilde{C} = \dim C + \dim G$ and

$$l(m_{\widetilde{C}}) + \operatorname{rank}(1 - m_{\widetilde{C}}\tilde{\theta}) = 2l(w_0) + \operatorname{rank}(1 - (w_0, w_0)\tilde{\theta}).$$

The linear operator $1 - (w_0, w_0)\tilde{\theta}$ on $\mathfrak{h} \oplus \mathfrak{h}$ is given by $(x, y) \to (x - w_0(y), y - w_0(x))$ and its kernel is given by $\{(w_0(y), y) : y \in \mathfrak{h}\}$. Thus $l(m_{\widetilde{C}}) + \operatorname{rank}(1 - m_{\widetilde{C}}\tilde{\theta}) =$

$2l(w_0) + \dim \mathfrak{h} = \dim G$. By Theorem 5.2, $\widetilde{C}$ is spherical if and only if $\dim C = 0$, i.e., if and only if C consists of a single central element.

Q.E.D.

Remark 7.5. Let $n \geq 2$ and let C be a conjugacy class in G. For $g \in C$, the stabilizer of $(g, e, \ldots, e) \in \widetilde{C} = \mu_n^{-1}(C)$ in G^n for the $\tilde{\theta}$-twisted conjugation is $(Z_g)_{\rm diag} = \{(g, g, \ldots, g) : g \in Z_g\}$, where $Z_g = \{k \in G : kg = gk\}$ is the centralizer of g in G. Thus $\widetilde{C} \cong G^n/(Z_g)_{\rm diag}$ for any $g \in C$. When $n = 2$ and C consists of a single central element g, $Z_g = G$, and it is well-known that $(G \times G)/G_{\rm diag}$ is symmetric and thus spherical. The conclusion in 2) of Lemma 7.4 can then be re-interpreted as saying that for a subgroup Z of G that is the centralizer of an element in G, the homogeneous space $(G \times G)/Z_{\rm diag}$ is spherical if and only if $Z = G$. We thank the referee for pointing this out.

7.3. **The Poisson structure $\pi_{\tilde{\theta}}$ on $G \times G$.** We look at the case of $n = 2$ in more detail. To this end, recall from §2.1 that the Drinfeld double of the Poisson Lie group $(G, \pi_{\rm st})$ is the Poisson Lie group $(G \times G, \Pi_{\rm st})$, where $G \times G$ has the product Lie group structure and $\Pi_{\rm st}$ is the multiplicative Poisson structure on $G \times G$ defined by $\Pi_{\rm st} = R^r - R^l$ in (2.4). Recall also from §2.1 that $\Pi_{\rm st}$ is invariant under the $T \times T$ given by

$$(h_1, h_2) \cdot (g_1, g_2) = (h_1 g_1 h_2^{-1},\, h_1 g_2 h_2^{-1}), \quad h_1, h_2 \in T,\, g_1, g_2 \in G. \tag{7.2}$$

Fix any representative $\bar{w}_0$ of the longest elements w_0 of W in $N_T(G)$, and let $r_{(\bar{w}_0, \bar{w}_0)} : G \times G \to G \times G$ be the right translation on $G \times G$ by $(\bar{w}_0, \bar{w}_0)$. Define

$$\tau : \ G \times G \longrightarrow G \times G, \ \ \tau(g_1, g_2) = (g_1, g_2^{-1}), \quad g_1, g_2 \in G. \tag{7.3}$$

Proposition 7.6. *As Poisson structures on $G \times G$, one has $\tau(\pi_{\tilde{\theta}}) = r_{(\bar{w}_0, \bar{w}_0)}(\Pi_{\rm st})$;*

Proof. We compare the formulas for $\tau(\pi_{\tilde{\theta}})$ and $r_{(\bar{w}_0, \bar{w}_0)}(\Pi_{\rm st})$. By the definition of $\Pi_{\rm st}$,

$$\begin{aligned}(r_{(\bar{w}_0, \bar{w}_0)}(\Pi_{\rm st}))(g_1, g_2) &= r_{(\bar{w}_0, \bar{w}_0)}(\Pi_{\rm st}(g_1 \bar{w}_0^{-1}, g_2 \bar{w}_0^{-1})) \\ &= l_{(g_1, g_2)} \left(R - {\rm Ad}_{(\bar{w}_0^{-1}, \bar{w}_0^{-1})} R\right) + \Pi_{\rm st}(g_1, g_2), \quad g_1, g_2 \in G.\end{aligned}$$

Thus $r_{(\bar{w}_0, \bar{w}_0)}(\Pi_{\rm st}) = R^r - \left({\rm Ad}_{(\bar{w}_0^{-1}, \bar{w}_0^{-1})} R\right)^l$. Let $\{y_i, 1 \leq i \leq k = \dim \mathfrak{h}\}$ and $\{E_\alpha, E_{-\alpha} : \alpha \in \Delta_+\}$ be as in §2.2. Let $R_{\mathfrak{h}} = \sum_{i=1}^k (y_i, 0) \wedge (0, y_i) \in \wedge^2(\mathfrak{g} \oplus \mathfrak{g})$

and

$$R_1 = \sum_{\alpha\in\Delta_+} (E_\alpha, 0) \wedge (0, E_{-\alpha}), \qquad R_2 = \sum_{\alpha\in\Delta_+} (0, E_\alpha) \wedge (E_{-\alpha}, 0),$$
$$R_3 = \frac{1}{2} \sum_{\alpha\in\Delta_+} \left((E_\alpha, 0) \wedge (E_{-\alpha}, 0) + (0, E_\alpha) \wedge (0, E_{-\alpha})\right).$$

Using the facts that $\mathrm{Ad}_{\bar{w}_0}$ preserves $\mathfrak{h}$ and $\langle\, ,\, \rangle_{\mathfrak{g}}$ and that $-\mathrm{Ad}_{\bar{w}_0}$ permutes the positive root spaces, one gets $R = R_{\mathfrak{h}} + R_1 + R_3$ and $\mathrm{Ad}_{(\bar{w}_0^{-1}, \bar{w}_0^{-1})} R = R_{\mathfrak{h}} - R_2 - R_3$. Thus

$$r_{(\bar{w}_0, \bar{w}_0)}(\Pi_{\mathrm{st}}) = R_{\mathfrak{h}}^r - R_{\mathfrak{h}}^l + (R_1 + R_3)^r + (R_2 + R_3)^l. \tag{7.4}$$

On the other hand, applying the explicit formula (2.7) to $\pi_{\tilde{\theta}}$, one has

$$\pi_{\tilde{\theta}} = \sum_{i=1}^{k} \tilde{\theta}(y_i, 0)^l \wedge (y_i, 0)^r + \tilde{\theta}(0, y_i)^l \wedge (0, y_i)^r$$
$$- \sum_{\alpha\in\Delta_+} \left((E_\alpha, 0)^r \wedge \tilde{\theta}(E_{-\alpha}, 0)^l + (0, E_\alpha)^r \wedge \tilde{\theta}(0, E_{-\alpha})^l\right) + R_3^r + R_3^l.$$

Using the definitions of $\tilde{\theta}$ and τ and the fact that $\tau(R_3^r + R_3^l) = R_3^r + R_3^l$, one has

$$\tau(\pi_{\tilde{\theta}}) = \sum_{i=1}^{k} ((y_i, 0)^r \wedge (0, y_i)^r - (y_i, 0)^l \wedge (0, y_i)^l)$$
$$+ \sum_{\alpha\in\Delta_+} \left((E_\alpha, 0)^r \wedge (0, E_{-\alpha})^r + (0, E_\alpha)^l \wedge (E_{-\alpha}, 0)^l\right) + R_3^r + R_3^l,$$
$$= R_{\mathfrak{h}}^r - R_{\mathfrak{h}}^l + R_1^r + R_2^l + R_3^r + R_3^l.$$

Comparing with (7.4), one sees that $\tau(\pi_{\tilde{\theta}}) = r_{(\bar{w}_0, \bar{w}_0)}(\Pi_{\mathrm{st}})$.

Q.E.D.

By Proposition 7.6, one has the $(T \times T)$-equivariant Poisson isomorphism

$$\phi := r_{(\bar{w}_0^{-1}, \bar{w}_0^{-1})}\tau : \quad (G \times G, \pi_{\tilde{\theta}}) \longrightarrow (G \times G, \Pi_{\mathrm{st}}), \quad (g_1, g_2) \longmapsto (g_1 \bar{w}_0^{-1},\, g_2^{-1} \bar{w}_0^{-1}).$$

where $T \times T$ acts on $(G \times G, \pi_{\tilde{\theta}})$ and on $(G \times G, \Pi_{\mathrm{st}})$ respectively by

$$(h_1, h_2) \cdot_{\tilde{\theta}} (g_1, g_2) = (h_1 g_1 h_2^{-1},\, h_2 g_2 h_1^{-1}),$$
$$(h_1, h_2) \bullet (g_1, g_2) = (h_1, w_0(h_2)) \cdot (g_1, g_2) = (h_1 g_1 w_0(h_2^{-1}),\, h_1 g_1 w_0(h_2^{-1})).$$

For $u, v \in W$ and a conjugacy class C in G, let

$$G_C^{u,v} = \{(k_1, k_2) \in BuB \times B_- v B_- :\, k_1 k_2^{-1} \in C\} \subset G \times G.$$

We will refer to $G_C^{u,v}$ the *double Bruhat cell associated to u, v and the conjugacy class C*. Note that when $C = \{e\}$, $G_C^{u,v}$ is isomorphic to the double Bruhat cell

$$G^{u,v} = BuB \cap B_- v B_-$$

which have been studied intensively [2, 19] in connection with total positivity and cluster algebras. Note that

$$\phi(\widetilde{C} \cap (\widetilde{B}\tilde{w}\widetilde{B}_-)) = G_C^{w_1 w_0, w_2^{-1} w_0}, \quad w_1, w_2 \in W. \tag{7.5}$$

Corollary 7.7. *Let $T \times T$ act on $G \times G$ by (7.2). The orbits of symplectic leaves of $\Pi_{\rm st}$ in $G \times G$ under both $T \times \{e\}$ and $\{e\} \times T$ are the subsets $G_C^{u,v}$, where $u, v \in W$ and C a conjugacy class in G. Every $G_C^{u,v}$ is a non-empty connected smooth submanifold of G of dimension equal to $\dim C + l(u) + l(v) + \dim T$, and the symplectic leaves of $\Pi_{\rm st}$ in $G_C^{u,v}$ have dimension equal to $\dim C + l(u) + l(v) + {\rm rank}(1 - uv^{-1})$.*

Proof. In the notation of §3.2, $\widetilde{T}_{\tilde{w}\tilde{\theta}} = \{(h_1 w_1(h_2), h_2 w_2(h_1)) : h_1, h_2 \in T\}$, and

$$(T \times \{e\})\widetilde{T}_{\tilde{w}\tilde{\theta}} = (\{e\} \times T)\widetilde{T}_{\tilde{w}\tilde{\theta}} = \widetilde{T}$$

for any $\tilde{w} = (w_1, w_2) \in W \times W$. It follows from Theorem 3.7 that under the action $\cdot_{\tilde{\theta}}$, the three tori $T \times \{e\}$, $\{e\} \times T$, and $T \times T$ have the same orbits of symplectic leaves of $\pi_{\tilde{\theta}}$, namely, the subsets

$$\widetilde{C} \cap (\widetilde{B}\tilde{w}\widetilde{B}_-) = \{(g_1, g_2) \in Bw_1B_- \times Bw_2B_- : g_1 g_2 \in C\},$$

where $\tilde{w} = (w_1, w_2) \in W \times W$ and C is a conjugacy class in G. Setting $u = w_1 w_0, v = w_2^{-1} w_0$ and using (7.5) and Proposition 7.3, Corollary 7.7 is proved.

Q.E.D.

Remark 7.8. Corollary 7.7 is proved by relating $\Pi_{\rm st}$ with the Poisson structure $\pi_{\tilde{\theta}}$ and applying Theorem 1.1 to $\pi_{\tilde{\theta}}$. A direct proof of Corollary 7.7 is outlined as follows: first, as the Drinfeld double of the Poisson Lie group $(G, \pi_{\rm st})$, the symplectic leaves of $\Pi_{\rm st}$ in $G \times G$ are the connected components of intersections of $(G_{\rm diag}, G_{\rm diag})$-double cosets and (G^*, G^*)-double cosets (this statement can be proved in a way similar to the proof of a statement on the symplectic leaves of the Poisson structure $\pi_+ = R^r + R^l$ on $G \times G$ that can be found in [1] and [18, Lemma 6.4]). For $u \in W$, define

$$h_u : \; BuB \longrightarrow T : \quad h_u(n\bar{u}hn') = h, \quad n, n' \in N, \, h \in T, \tag{7.6}$$

$$h'_u : \; B_-uB_- \longrightarrow T : \quad h'_u(n_-\bar{u}h'n'_-) = h', \quad n_-, n'_- \in N_-, \, h' \in T. \tag{7.7}$$

For $u, v \in W$, let $T_{u,v} = \{(h^u)^{-1}h^v : \; h \in T\}$, where for $h \in T$, $h^u = u^{-1}(h)$. Note that $\dim T_{u,v} = {\rm rank}(1 - uv^{-1})$. It is straightforward to prove that every (G^*, G^*)-double coset in $G \times G$ is of the form $G^*(\bar{u}h, \, \bar{v})G^*$ for unique $u, v \in W$ and $h \in T$, and

$$G^*(\bar{u}h, \, \bar{v})G^* = \{(g_1, g_2) \in BuB \times B_-vB_- : \;\; h_u(g_1)h'_v(g_2) \in hT_{u,v}\}.$$

Using arguments similar to those in the proof of Proposition 3.5, one can show that each $G_C^{u,v} = (G_{\rm diag}(C, e)G_{\rm diag}) \cap (BuB \times B_-vB_-)$ is a smooth connected submanifold of $G \times G$ with the given dimension and is a $T \times \{e\}$ and a $(\{e\} \times T)$-leaf for $\Pi_{\rm st}$ in $G \times G$.

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Jiang-Hua Lu, Department of Mathematics, Hong Kong University, Pokfulam Rd., Hong Kong

E-mail address: jhlu@maths.hku.hk