THE CONICAL KÄHLER-RICCI FLOW WITH WEAK INITIAL DATA ON FANO MANIFOLD

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Abstract. In this paper, we prove the long-time existence and uniqueness of the conical Kähler-Ricci flow with weak initial data which admits L^p density for some $p > 1$ on Fano manifold. Furthermore, we study the convergence behavior of this kind of flow.

1. Introduction

Conical Kähler-Einstein metric plays an important role in solving the Yau-Tian-Donaldson's conjecture (see [6, 7, 8, 44]). There has been renewed interest in conical Kähler-Einstein metric recently, see references [1, 3, 4, 18, 24, 26, 30, 42]etc. On the other hand, the conical Kähler-Ricci flow was introduced to attack the existence problem of conical Kähler-Einstein metric. The long-time existence and limit behaviour of the conical Kähler-Ricci flow has been widely studied. In Riemann surface case, Mazzeo-Rubinstein-Sesum [35] and H. Yin [47] [48] did it with different function spaces. In higher dimension case, Chen-Wang [11] studied the strong conical Kähler-Ricci flow and obtained the short-time existence, Y.Q. Wang [46] and the authors [34] got the long-time existence of the conical Kähler-Ricci flow respectively. In [34], the authors also considered the convergence of this flow on Fano manifold with positive twisted first Chern class, they proved that, for any cone angle $0 < 2\pi\beta < 2\pi$, the conical Kähler-Ricci flow converges to a conical Kähler-Einstein metric if there exists one. Chen-Wang [12] obtained the convergence result of this flow when the twisted first Chern class is negative or zero. Later, L.M. Shen [38][39] studied the unnormalized conical Kähler-Ricci flow, and G. Edwards [17] obtained the uniform bound of the scalar curvature when the twisted first Chern class is negative.

In [34], the authors studied the conical Kähler-Ricci flow which starts with a model metric

$$\omega_\beta = \omega_0 + \sqrt{-1}k\partial\bar{\partial}|s|_h^{2\beta} \tag{1.1}$$

on Fano manifold, where $\omega_0 \in c_1(M)$ is a smooth Kähler metric, s is the defining section of a smooth divisor $D \in |-\lambda K_M|$ and h is a smooth Hermitian metric on $-\lambda K_M$ with curvature $\lambda\omega_0$. In [11, 12, 46], Chen-Wang studied the existence of the conical Kähler-Ricci flow from initial (α, β) metric or weak (α, β) metric with other assumptions.

In this paper, we mainly study the long-time existence, uniqueness and convergence of the conical Kähler-Ricci flow with some weak initial data which admits L^p

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density with $p>1$ on Fano manifold. We still consider the conical Kähler-Ricci flow by using smooth approximation of twisted Kähler-Ricci flows as that in [34].

Let M be a Fano manifold with complex dimension n, $\omega_0\in c_1(M)$ be a smooth Kähler metric. For any $p\in(0,\infty]$, we define the class

$$\mathcal{E}_p(M,\omega_0)=\big\{\varphi\in\mathcal{E}(M,\omega_0)|\frac{(\omega_0+\sqrt{-1}\partial\bar{\partial}\varphi)^n}{\omega_0^n}\in L^p(M,\omega_0^n)\big\}, \tag{1.2}$$

where the class

$$\mathcal{E}(M,\omega_0)=\big\{\varphi\in PSH(M,\omega_0)|\int_M(\omega_0+\sqrt{-1}\partial\bar{\partial}\varphi)^n=\int_M\omega_0^n\big\} \tag{1.3}$$

defined in [21] is the largest subclass of $PSH(M,\omega_0)$ on which the operator $(\omega_0+\sqrt{-1}\partial\bar{\partial}\cdot)^n$ is well defined and the comparison principle is valid. When $p>1$, by S. Kolodziej's L^p estimate [29] and S. Dinew's uniqueness theorem [14] (see also Theorem B in [21]), we know that the functions in $\mathcal{E}_p(M,\omega_0)$ are Hölder continuous with respect to ω_0 on M.

Let D be a divisor on M. By saying a closed positive $(1,1)$-current $\omega\in 2\pi c_1(M)$ with locally bounded potential is a conical Kähler metric with angle $2\pi\beta$ ($0<\beta\leq 1$) along D, we mean that ω is a smooth Kähler metric on $M\setminus D$, and near each point $p\in D$, there exists local holomorphic coordinate $(z^1,\cdots,z^n)$ in a neighborhood U of p such that locally $D=\{z^n=0\}$, and ω is asymptotically equivalent to the model conical metric

$$\sqrt{-1}|z^n|^{2\beta-2}dz^n\wedge d\overline{z}^n+\sqrt{-1}\sum_{j=1}^{n-1}dz^j\wedge d\overline{z}^j \qquad on \quad U. \tag{1.4}$$

Assume that $D\in|-\lambda K_M|$ ($\lambda\in\mathbb{Q}$), $\mu=1-(1-\beta)\lambda$, $\hat{\omega}\in c_1(M)$ is a Kähler current which admits L^p density with respect to ω_0^n for some $p>1$ and $\int_M\hat{\omega}^n=\int_M\omega_0^n$. We study the long-time existence, uniqueness and convergence of the following conical Kähler-Ricci flow with weak initial data $\hat{\omega}$

$$\begin{cases}\frac{\partial\omega(t)}{\partial t}=-Ric(\omega(t))+\mu\omega(t)+(1-\beta)[D].\\ \\ \omega(t)|_{t=0}=\hat{\omega}\end{cases} \tag{1.5}$$

From now on, we denote the Kähler current $\hat{\omega}=\omega_0+\sqrt{-1}\partial\bar{\partial}\varphi_0:=\omega_{\varphi_0}$ with $\varphi_0\in\mathcal{E}_p(M,\omega_0)$ for some $p>1$.

Definition 1.1. *We call $\omega(t)$ a long-time solution to the conical Kähler-Ricci flow* (1.5) *if it satisfies the following conditions.*

- *For any $[\delta,T]$ ($\delta,T>0$), there exist constant C such that*
$$C^{-1}\omega_\beta\leq\omega(t)\leq C\omega_\beta \qquad on \quad [\delta,T]\times(M\setminus D);$$
- *On $(0,\infty)\times(M\setminus D)$, $\omega(t)$ satisfies the smooth Kähler-Ricci flow;*
- *On $(0,\infty)\times M$, $\omega(t)$ satisfies equation (1.5) in the sense of currents;*
- *There exists metric potential $\varphi(t)\in C^0\big([0,\infty)\times M\big)\cap C^\infty\big((0,\infty)\times(M\setminus D)\big)$ such that $\omega(t)=\omega_0+\sqrt{-1}\partial\bar{\partial}\varphi(t)$ and $\lim\limits_{t\to 0^+}\|\varphi(t)-\varphi_0\|_{L^\infty(M)}=0$;*
- *On $[\delta,T]$, there exist constant $\alpha\in(0,1)$ and C^* such that the above metric petential $\varphi(t)$ is C^α on M with respect to ω_0 and $\|\frac{\partial\varphi(t)}{\partial t}\|_{L^\infty(M\setminus D)}\leqslant C^*$.*

In Definition 1.1, by saying $\omega(t)$ satisfies equation (1.5) in the sense of currents on $(0,\infty)\times M := M_\infty$, we mean that for any smooth $(n-1,n-1)$-form $\eta(t)$ with compact support in $(0,\infty)\times M$, we have

$$\int_{M_\infty}\frac{\partial\omega(t)}{\partial t}\wedge\eta(t,x)dt=\int_{M_\infty}(-Ric(\omega(t))+\mu\omega(t)+(1-\beta)[D])\wedge\eta(t,x)dt,$$

where the integral on the left side can be written as

$$\int_{M_\infty}\frac{\partial\omega(t)}{\partial t}\wedge\eta(t,x)dt=-\int_{M_\infty}\omega(t)\wedge\frac{\partial\eta(t,x)}{\partial t}dt$$

it the sense of currents.

We study the conical Kähler-Ricci flow (1.5) by using the following twisted Kähler-Ricci flow with weak initial data ω_{φ_0}.

$$\begin{cases}\frac{\partial\omega_\varepsilon(t)}{\partial t}=-Ric(\omega_\varepsilon(t))+\mu\omega_\varepsilon(t)+\theta_\varepsilon,\\ \\ \omega_\varepsilon(t)|_{t=0}=\omega_{\varphi_0},\end{cases}\tag{1.6}$$

where $\theta_\varepsilon=(1-\beta)(\lambda\omega_0+\sqrt{-1}\partial\overline{\partial}\log(\varepsilon^2+|s|_h^2))$ is a smooth closed positive (1,1)-form, s is the definition section of D and h is a smooth Hermitian metric on $-\lambda K_M$ with curvature $\lambda\omega_0$. The smooth case of the twisted Kähler-Ricci flow was studied in [13, 17, 19, 20, 32, 33, 34, 38, 46], etc.

There are some important results on the Kähler-Ricci flow (as well as its twisted versions with smooth twisted form) from weak initial data, such as Chen-Ding [5], Chen-Tian [9], Chen-Tian-Zhang [10], Guedj-Zeriahi [23], Nezza-Lu [36], Song-Tian [41], Székelyhidi-Tosatti [43]. Here, we first obtain the long-time existence, uniqueness and regularity of the flow (1.6) by following Song-Tian's arguments in [41]. Then we study the long-time existence of the conical Kähler-Ricci flow (1.5) by approximating method. In this process, in addition to getting the locally uniform regularity of the twisted Kähler-Ricci flow (1.6), the most important step is to prove that $\varphi(t)$ converges to φ_0 in L^∞-norm as $t\to 0^+$ (i.e the 4th property in Definition 1.1), where $\varphi(t)$ is a metric potential of $\omega(t)$ with respect to the metric ω_0. Here we need a new idear because of Song-Tian's method in [41] is invalid. At the same time, we prove the uniqueness of the conical Kähler-Ricci flow by Jeffres' trick [25] and an improvement of the arguments in [46]. In fact, we obtain the following theorem.

Theorem 1.2. *Let M be a Fano manifold with complex dimension n, $\omega_0\in c_1(M)$ be a smooth Kähler metric on M, divisor $D\in|-\lambda K_M|$ ($\lambda\in\mathbb{Q}$) and $\hat{\omega}\in c_1(M)$ be a Kähler current which admits L^p density with respect to ω_0^n for some $p>1$. For any $\beta\in(0,1)$, there exists a unique solution $\omega(t,\cdot)$ to the conical Kähler-Ricci flow* (1.5) *with weak initial data $\hat{\omega}$.*

Then we consider the convergence of the conical Kähler-Ricci flow (1.5). When $\lambda>0$ and there is no nontrivial holomorphic field which is tangent to D along D, Tian-Zhu [45] proved a Moser-Trudinger type inequality for conical Kähle-Einstein manifold and gave a new proof of Donaldson's openness theorem [16]. Using the Moser-Trudinger type inequality in [45] and following the arguments in [34], we obtain the following convergence result of the conical Kähler-Ricci flow (1.5).

Theorem 1.3. *Assume that $\lambda > 0$ and there is no nontrivial holomorphic field on M tangent to D, if there exists a conical Kähler-Einstein metric with cone angle $2\pi\beta$ ($0 < \beta < 1$) along D, then the conical Kähler-Ricci flow* (1.5) *must converge to this conical Kähler-Einstein metric in C^{∞}_{loc} topology outside divisor D and globally in the sense of currents on M.*

Remark 1.4. *In this paper, we only study the convergence with positive twisted first Chern class, i.e. $\mu = 1 - (1-\beta)\lambda > 0$. When $\mu \leq 0$, one can also get the convergence of the conical Kähler-Ricci flow by following Chen-Wang's argument in* [12].

The paper is organized as follows. In section 2, we prove the long-time existence and uniqueness of the twisted Kähler-Ricci flow (1.6) by adapting Song-Tian's methods in [41]. In section 3, we obtain the existence of a long-time solution to the conical Kähler-Ricci flow (1.5) by limiting the twisted Kähler-Ricci flows, and prove that $\varphi(t)$ converges to φ_0 in L^{∞}-norm as $t \to 0^+$, where $\varphi(t)$ is a metric potential of $\omega(t)$ with respect to the metric ω_0. We also prove the uniqueness of the conical Kähler-Ricci flow with weak initial data ω_{φ_0}. In section 4, by using the uniform Perelman's estimates along the twisted Kähler-Ricci flows obtained in [34], we prove the convergence theorem under the assumptions in Theorem 1.3.

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2. The long-time existence of the twisted Kähler-Ricci flow with weak initial data

In this section, we prove the long-time existence and uniqueness of the twisted Kähler-Ricci flow (1.6) by following Song-Tian's arguments in [41]. For further consideration in the next section, we shall pay attention to the estimates which are independent of ε. In the following arguments, for the sake of brevity, we only consider the flow (1.6) in the case of $\lambda = 1$ (i.e. $\mu = \beta$), where $\beta \in (0, 1)$. Our arguments are also valid for any λ, only if the coefficient β before $\omega_{\varepsilon}(t)$ in the case of $\lambda = 1$ is replaced by $\mu = 1 - (1-\beta)\lambda$. We denote

$$F = \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_0)^n}{\omega_0^n} \in L^p(M, \omega_0^n).$$

Recall that $C^{\infty}(M)$ is dense in $L^p(M, \omega_0^n)$. Therefore there exists a sequence of positive functions $F_j \in C^{\infty}(M)$ such that $\int_M F_j \omega_0^n = \int_M \omega_0^n$ and

$$\lim_{j\to\infty} \|F_j - F\|_{L^p(M)} = 0.$$

By considering the complex Monge-Ampère equation

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_{0,j})^n = F_j\omega_0^n \tag{2.1}$$

and using the stability theorem in [28] (see also [15] or [22]), we have

$$\lim_{j\to\infty} \|\varphi_{0,j} - \varphi_0\|_{L^{\infty}(M)} = 0, \tag{2.2}$$

where $\varphi_{0,j} \in PSH(M, \omega_0) \cap C^\infty(M)$ satisfy $\sup_M(\varphi_0 - \varphi_{0,j}) = \sup_M(\varphi_{0,j} - \varphi_0)$.

Let $\omega_{\varphi_{0,j}} = \omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi_{0,j}$. We prove the long-time existence of the twisted Kähler-Ricci flow (1.6) by using a sequence of smooth twisted Kähler-Ricci flows

$$\begin{cases} \frac{\partial \omega_{\varepsilon,j}(t)}{\partial t} = -Ric(\omega_{\varepsilon,j}(t)) + \beta\omega_{\varepsilon,j}(t) + \theta_\varepsilon. \\ \\ \omega_{\varepsilon,j}(t)|_{t=0} = \omega_{\varphi_{0,j}} \end{cases} \tag{2.3}$$

Since the twisted Kähler-Ricci flow preserves the Kähler class, we can write the flow (2.3) as the parabolic Monge-Ampére equation on potentials,

$$\begin{cases} \frac{\partial \varphi_{\varepsilon,j}(t)}{\partial t} = \log \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_{\varepsilon,j}(t))^n}{\omega_0^n} + F_0 + \beta\varphi_{\varepsilon,j}(t) + \log(\varepsilon^2 + |s|_h^2)^{1-\beta}, \\ \\ \varphi_{\varepsilon,j}(0) = \varphi_{0,j} \end{cases} \tag{2.4}$$

where F_0 satisfies $-Ric(\omega_0) + \omega_0 = \sqrt{-1}\partial\overline{\partial}F_0$, $\frac{1}{V}\int_M e^{-F_0} dV_0 = 1$ and $dV_0 = \frac{\omega_0^n}{n!}$. By using the function

$$\chi(\varepsilon^2 + |s|_h^2) = \frac{1}{\beta}\int_0^{|s|_h^2} \frac{(\varepsilon^2 + r)^\beta - \varepsilon^{2\beta}}{r} dr \tag{2.5}$$

which was given by F. Campana, H. Guenancia and M. Păun in [4], we can rewrite the flow (2.4) as

$$\begin{cases} \frac{\partial \phi_{\varepsilon,j}(t)}{\partial t} = \log \frac{(\omega_\varepsilon + \sqrt{-1}\partial\bar{\partial}\phi_{\varepsilon,j}(t))^n}{\omega_\varepsilon^n} + F_\varepsilon + \beta(\phi_{\varepsilon,j}(t) + k\chi(\varepsilon^2 + |s|_h^2)), \\ \\ \phi_{\varepsilon,j}(0) = \varphi_{0,j} - k\chi(\varepsilon^2 + |s|_h^2) := \phi_{\varepsilon,0,j} \end{cases} \tag{2.6}$$

where $\phi_{\varepsilon,j}(t) = \varphi_{\varepsilon,j}(t) - k\chi(\varepsilon^2 + |s|_h^2)$, $\omega_\varepsilon = \omega_0 + \sqrt{-1}k\partial\overline{\partial}\chi(\varepsilon^2 + |s|_h^2)$, $F_\varepsilon = F_0 + \log(\frac{\omega_\varepsilon^n}{\omega_0^n} \cdot (\varepsilon^2 + |s|_h^2)^{1-\beta})$. We know that $\chi(\varepsilon^2 + |s|_h^2)$ and F_ε are uniformly bounded (see (15) and (25) in [4]).

Proposition 2.1. *For any $T > 0$, there exists constant C depending only on $\|\varphi_0\|_{L^\infty(M)}$, β, n, ω_0 and T such that for any $t \in [0, T]$, $\varepsilon > 0$ and $j \in \mathbb{N}^+$,*

$$\|\phi_{\varepsilon,j}(t)\|_{L^\infty(M)} \leq C. \tag{2.7}$$

Furthermore, for any j, k, we have

$$\|\phi_{\varepsilon,j}(t) - \phi_{\varepsilon,k}(t)\|_{L^\infty([0,T]\times M)} \leq e^{\beta T}\|\varphi_{0,j} - \varphi_{0,k}\|_{L^\infty(M)}. \tag{2.8}$$

In particular, $\{\varphi_{\varepsilon,j}(t)\}$ satisfies

$$\lim_{j.k\to\infty} \|\varphi_{\varepsilon,j}(t) - \varphi_{\varepsilon,k}(t)\|_{L^\infty([0,T]\times M)} = 0. \tag{2.9}$$

Proof: From equation (2.6), we have

$$\begin{aligned} \frac{\partial e^{-\beta t}\phi_{\varepsilon,j}(t)}{\partial t} &= e^{-\beta t}\log \frac{(e^{-\beta t}\omega_\varepsilon + \sqrt{-1}\partial\bar{\partial}e^{-\beta t}\phi_{\varepsilon,j}(t))^n}{(e^{-\beta t}\omega_\varepsilon)^n} \\ &\quad + e^{-\beta t}(F_\varepsilon + k\beta\chi(\varepsilon^2 + |s|_h^2)) \\ &\leq e^{-\beta t}\log \frac{(e^{-\beta t}\omega_\varepsilon + \sqrt{-1}\partial\bar{\partial}e^{-\beta t}\phi_{\varepsilon,j}(t))^n}{(e^{-\beta t}\omega_\varepsilon)^n} + Ce^{-\beta t}, \end{aligned}$$

which is equivalent to

$$\frac{\partial}{\partial t}\big(e^{-\beta t}(\phi_{\varepsilon,j}(t)+\frac{C}{\beta})\big)\le e^{-\beta t}\log\frac{\big(e^{-\beta t}\omega_\varepsilon+\sqrt{-1}\partial\bar\partial e^{-\beta t}(\phi_{\varepsilon,j}(t)+\frac{C}{\beta})\big)^n}{(e^{-\beta t}\omega_\varepsilon)^n},$$

where constant C depends only on $\|\varphi_0\|_{L^\infty(M)}$, β, n and ω_0.

For any $\delta>0$, we denote $\tilde\phi_{\varepsilon,j}(t)=e^{-\beta t}(\phi_{\varepsilon,j}(t)+\frac{C}{\beta})-\delta t$. Let (t_0,x_0) be the maximum point of $\tilde\phi_{\varepsilon,j}(t)$ on $[0,T]\times M$. If $t_0>0$, by maximum principle, we have

$$\begin{aligned}0 &\le \frac{\partial}{\partial t}\big(e^{-\beta t}(\phi_{\varepsilon,j}(t)+\frac{C}{\beta})\big)(t_0,x_0)-\delta\\ &\le e^{-\beta t}\log\frac{\big(e^{-\beta t}\omega_\varepsilon+\sqrt{-1}\partial\bar\partial\tilde\phi_{\varepsilon,j}(t)\big)^n}{(e^{-\beta t}\omega_\varepsilon)^n}(t_0,x_0)-\delta\\ &\le -\delta,\end{aligned}$$

which is impossible. Hence $t_0=0$, then

$$\phi_{\varepsilon,j}(t)\le e^{\beta t}\sup_M\phi_{\varepsilon,j}(0)+\delta Te^{\beta T}+\frac{C}{\beta}(e^{\beta T}-1).$$

Let $\delta\to0$, we obtain

$$\phi_{\varepsilon,j}(t)\le e^{\beta t}\sup_M\phi_{\varepsilon,j}(0)+\frac{C}{\beta}(e^{\beta T}-1). \tag{2.10}$$

By the same arguments, we can get the lower bound of $\phi_{\varepsilon,j}(t)$

$$\phi_{\varepsilon,j}(t)\ge e^{\beta t}\inf_M\phi_{\varepsilon,j}(0)-\frac{C}{\beta}(e^{\beta T}-1). \tag{2.11}$$

Combining (2.10) and (2.11), we have

$$\|\phi_{\varepsilon,j}(t)\|_{L^\infty(M)}\le e^{\beta T}\|\phi_{\varepsilon,j}(0)\|_{L^\infty(M)}+\frac{C}{\beta}(e^{\beta T}-1)\le C,$$

where constant C depends only on $\|\varphi_0\|_{L^\infty(M)}$, β, n, ω_0 and T.

Let $\psi_{\varepsilon,j,k}(t)=\phi_{\varepsilon,j}(t)-\phi_{\varepsilon,k}(t)$, then $\psi_{\varepsilon,j,k}$ satisfies the following equation

$$\begin{cases}\frac{\partial\psi_{\varepsilon,j,k}(t)}{\partial t}=\log\frac{\big(\omega_\varepsilon+\sqrt{-1}\partial\bar\partial\phi_{\varepsilon,k}(t)+\sqrt{-1}\partial\bar\partial\psi_{\varepsilon,j,k}(t)\big)^n}{(\omega_\varepsilon+\sqrt{-1}\partial\bar\partial\phi_{\varepsilon,k}(t))^n}+\beta\psi_{\varepsilon,j,k}(t).\\ \psi_{\varepsilon,j,k}(0)=\varphi_{0,j}-\varphi_{0,k}\end{cases} \tag{2.12}$$

By the same arguments as that in the first part, we have

$$\|\psi_{\varepsilon,j,k}(t)\|_{L^\infty([0,T]\times M)}\le e^{\beta T}\|\varphi_{0,j}-\varphi_{0,k}\|_{L^\infty(M)}.$$

Since $\{\varphi_{0,j}\}$ is a Cauchy in L^∞-norm, we conclude

$$\lim_{j.k\to\infty}\|\varphi_{\varepsilon,j}(t)-\varphi_{\varepsilon,k}(t)\|_{L^\infty([0,T]\times M)}=0.$$

□

We now prove the uniform equivalence of the volume forms along the complex Monge-Ampère flow (2.6).

Lemma 2.2. *For any $T>0$, there exists constant C depending only on $\|\varphi_0\|_{L^\infty(M)}$, n, β, ω_0 and T such that for any $t\in(0,T]$, $\varepsilon>0$ and $j\in\mathbb{N}^+$,*

$$\frac{t^n}{C}\leq\frac{(\omega_\varepsilon+\sqrt{-1}\partial\bar{\partial}\phi_{\varepsilon,j}(t))^n}{\omega_\varepsilon^n}\leq e^{\frac{C}{t}}. \tag{2.13}$$

Proof: Let $\Delta_{\varepsilon,j}$ be the Laplacian operator associated to the Kähler form $\omega_{\varepsilon,j}(t)=\omega_\varepsilon+\sqrt{-1}\partial\bar{\partial}\phi_{\varepsilon,j}(t)$. Straightforward calculations show that

$$(\frac{\partial}{\partial t}-\Delta_{\varepsilon,j})\dot{\phi}_{\varepsilon,j}(t)=\beta\dot{\phi}_{\varepsilon,j}(t). \tag{2.14}$$

Let $H^+_{\varepsilon,j}(t)=t\dot{\phi}_{\varepsilon,j}(t)-A\phi_{\varepsilon,j}(t)$, where A is a sufficiently large number (for example $A=\beta T+2$). Then $H^+_{\varepsilon,j}(0)=-A\phi_{\varepsilon,j}(0)$ is uniformly bounded by a constant C depending only on $\|\varphi_0\|_{L^\infty(M)}$, β, n, ω_0 and T.

$$\begin{aligned}(\frac{\partial}{\partial t}-\Delta_{\varepsilon,j})H^+_{\varepsilon,j}(t) &= (1+\beta t-A)\dot{\phi}_{\varepsilon,j}(t)+A\Delta_{\varepsilon,j}\phi_{\varepsilon,j}(t)\\ &\leqslant (1+\beta t-A)\dot{\phi}_{\varepsilon,j}(t)+An.\end{aligned} \tag{2.15}$$

By the maximum principle, $H^+_{\varepsilon,j}(t)$ is uniformly bounded from above by a constant C depending only on $\|\varphi_0\|_{L^\infty(M)}$, n, β, ω_0 and T.

Let $H^-_{\varepsilon,j}(t)=\dot{\phi}_{\varepsilon,j}(t)+\phi_{\varepsilon,j}(t)-n\log t$. Then $H^-(t)$ tends to $+\infty$ as $t\to0^+$ and

$$(\frac{\partial}{\partial t}-\Delta_{\varepsilon,j})H^-_{\varepsilon,j}(t)=(\beta+1)\dot{\phi}_{\varepsilon,j}(t)+tr_{\omega_{\varepsilon,j}(t)}\omega_\varepsilon-n-\frac{n}{t}. \tag{2.16}$$

Assume that (t_0,x_0) is the minimum point of $H^-_{\varepsilon,j}(t)$ on $[0,T]\times M$. We conclude that $t_0>0$ and there exists constant C_1, C_2 and C_3 such that

$$\begin{aligned}(\frac{\partial}{\partial t}-\Delta_{\varepsilon,j})H^-_{\varepsilon,j}(t)|_{(t_0,x_0)} &\geq \big(C_1(\frac{\omega_\varepsilon^n}{\omega_{\varepsilon,j}^n(t)})^{\frac{1}{n}}+C_2\log\frac{\omega^n_{\varepsilon,j}(t)}{\omega^n_\varepsilon}-\frac{C_3}{t}\big)|_{(t_0,x_0)}\\ &\geq \big(\frac{C_1}{2}(\frac{\omega_\varepsilon^n}{\omega_{\varepsilon,j}^n(t)})^{\frac{1}{n}}-\frac{C_3}{t}\big)|_{(t_0,x_0)},\end{aligned} \tag{2.17}$$

where constant C_1 depends only on n, C_2 depends only on β and C_3 depends only on n, ω_0, $\|\varphi_0\|_{L^\infty(M)}$, β and T. In inequality (2.17), without loss of generality, we assume that $\frac{\omega_\varepsilon^n}{\omega_{\varepsilon,j}^n(t)}>1$ and $\frac{C_1}{2}(\frac{\omega_\varepsilon^n}{\omega_{\varepsilon,j}^n(t)})^{\frac{1}{n}}+C_2\log\frac{\omega^n_{\varepsilon,j}(t)}{\omega^n_\varepsilon}\geq0$ at (t_0,x_0). By the maximum principle, we have

$$\omega^n_{\varepsilon,j}(t_0,x_0)\geq C_4t^n\omega^n_\varepsilon(x_0), \tag{2.18}$$

where C_4 independent of ε and j. Then it easily follows that $H^-_{\varepsilon,j}(t)$ is bounded from below by a constant C depending only on $\|\varphi_0\|_{L^\infty(M)}$, n, β, ω_0 and T. □

In the following lemma, we prove the uniform equivalence of the metrics along the twisted Kähler-Ricci flow (2.3).

Lemma 2.3. *For any $T>0$, there exists constant C depending only on $\|\varphi_0\|_{L^\infty(M)}$, n, β, ω_0 and T such that for any $t\in(0,T]$, $\varepsilon>0$ and $j\in\mathbb{N}^+$,*

$$e^{-\frac{C}{t}}\omega_\varepsilon\leq\omega_{\varepsilon,j}(t)\leq e^{\frac{C}{t}}\omega_\varepsilon. \tag{2.19}$$

Proof: Let

$$\Psi_{\varepsilon,\rho}=B\frac{1}{\rho}\int_0^{|s|_h^2}\frac{(\varepsilon^2+r)^\rho-\varepsilon^{2\rho}}{r}dr \tag{2.20}$$

be the uniform bound function introduced by H. Guenancia and M. Păun in [24]. By choosing suitable B and ρ, and following the arguments in [34] (see section 2 in [34]), we have

$$\begin{aligned}&(\frac{\partial}{\partial t}-\Delta_{\varepsilon,j})(t\log tr_{\omega_\varepsilon}\omega_{\varepsilon,j}(t)+t\Psi_{\varepsilon,\rho})\\ \leq\ &\log tr_{\omega_\varepsilon}\omega_{\varepsilon,j}(t)+Ctr_{\omega_{\varepsilon,j}(t)}\omega_\varepsilon+C,\end{aligned} \tag{2.21}$$

where constant C depends only on n, β, ω_0 and T.

Let $H_{\varepsilon,j}(t)=t\log tr_{\omega_\varepsilon}\omega_{\varepsilon,j}(t)+t\Psi_{\varepsilon,\rho}-A\phi_{\varepsilon,j}(t)$, A be a sufficiently large constant and (t_0,x_0) be the maximum point of $H_{\varepsilon,j}(t)$ on $[0,T]\times M$. We need only consider $t_0>0$. By the inequality

$$tr_{\omega_\varepsilon}\omega_{\varepsilon,j}(t)\leq\frac{1}{(n-1)!}(tr_{\omega_{\varepsilon,j}(t)}\omega_\varepsilon)^{n-1}\frac{\omega^n_{\varepsilon,j}(t)}{\omega^n_\varepsilon}, \tag{2.22}$$

we conclude that

$$\begin{aligned}(\frac{\partial}{\partial t}-\Delta_{\varepsilon,j})H_{\varepsilon,j}(t)\ &\leq\ \log tr_{\omega_\varepsilon}\omega_{\varepsilon,j}(t)+Ctr_{\omega_{\varepsilon,j}(t)}\omega_\varepsilon-A\dot{\phi}_{\varepsilon,j}(t)\\ &\quad +C+An-Atr_{\omega_{\varepsilon,j}(t)}\omega_\varepsilon\\ &\leq\ \log tr_{\omega_\varepsilon}\omega_{\varepsilon,j}(t)-\frac{A}{2}tr_{\omega_{\varepsilon,j}(t)}\omega_\varepsilon-A\log\frac{\omega^n_{\varepsilon,j}(t)}{\omega^n_\varepsilon}+C\\ &\leq\ (n-1)\log tr_{\omega_{\varepsilon,j}(t)}\omega_\varepsilon-\frac{A}{2}tr_{\omega_{\varepsilon,j}(t)}\omega_\varepsilon-(A-1)\log\frac{\omega^n_{\varepsilon,j}(t)}{\omega^n_\varepsilon}+C,\end{aligned}$$

where constant C depends only on $\|\varphi_0\|_{L^\infty(M)}$, n, β, ω_0 and T.

Without loss of generality, we assume that $-\frac{A}{4}tr_{\omega_{\varepsilon,j}(t)}\omega_\varepsilon+(n-1)\log tr_{\omega_{\varepsilon,j}(t)}\omega_\varepsilon\leq 0$ at (t_0,x_0). Then at (t_0,x_0), by Lemma 2.2, we have

$$(\frac{\partial}{\partial t}-\Delta_{\varepsilon,j})H_{\varepsilon,j}(t)\leq-\frac{A}{4}tr_{\omega_{\varepsilon,j}(t)}\omega_\varepsilon-C\log t+C. \tag{2.23}$$

By the maximum principle, at (t_0,x_0),

$$tr_{\omega_{\varepsilon,j}(t)}\omega_\varepsilon\leq C\log\frac{1}{t}+C. \tag{2.24}$$

By using inequality (2.22), at (t_0,x_0),

$$tr_{\omega_\varepsilon}\omega_{\varepsilon,j}(t)\leq C(\log\frac{1}{t}+1)^{n-1}e^{\frac{C}{t}}\leq e^{\frac{2C}{t}}, \tag{2.25}$$

where constant C depends only on $\|\varphi_0\|_{L^\infty(M)}$, n, β, ω_0 and T. Hence we have

$$tr_{\omega_\varepsilon}\omega_{\varepsilon,j}(t)\leq e^{\frac{C}{t}} \tag{2.26}$$

for some constant C depending only on $\|\varphi_0\|_{L^\infty(M)}$, n, β, ω_0 and T.

Furthermore, by inequality (2.22) again, we know

$$tr_{\omega_{\varepsilon,j}(t)}\omega_\varepsilon\leq e^{\frac{C}{t}}, \tag{2.27}$$

where constant C depends only on $\|\varphi_0\|_{L^\infty(M)}$, n, β, ω_0 and T. From (2.26) and (2.27), we prove the lemma. □

By Lemma 2.3 and the fact that $\omega_\varepsilon > \gamma\omega_0$ for some uniform constant γ (see inequality (24) in [4]), we have

$$e^{-\frac{C}{t}}\omega_0 \le \omega_{\varepsilon,j}(t) \le C_\varepsilon e^{\frac{C}{t}}\omega_0, \tag{2.28}$$

on $(0,T]\times M$, where C is a uniform constant and C_ε depends on ε. We next prove the Calabi's C^3-estimates. Denote

$$S_{\varepsilon,j} = |\nabla_{\omega_0}\omega_{\varepsilon,j}(t)|^2_{\omega_{\varphi_{\varepsilon,j}(t)}} = g^{i\bar{m}}_{\varepsilon,j}g^{k\bar{l}}_{\varepsilon,j}g^{p\bar{q}}_{\varepsilon,j}\nabla_{0i}(g_{\varepsilon,j})_{k\bar{q}}\overline{\nabla}_{0m}(g_{\varepsilon,j})_{p\bar{l}}. \tag{2.29}$$

Lemma 2.4. *For any $T>0$ and $\varepsilon>0$, there exist constants C_ε and C such that for any $t\in(0,T]$ and $j\in\mathbb{N}^+$,*

$$S_{\varepsilon,j} \le C_\varepsilon e^{\frac{C}{t}}, \tag{2.30}$$

where constant C depends only on $\|\varphi_0\|_{L^\infty(M)}$, n, β, ω_0 and T, and constant C_ε depends in addition on ε.

Proof: By the similar arguments in [33] or [34] and choosing sufficiently large α and β, we have

$$(\frac{\partial}{\partial t} - \Delta_{\varepsilon,j})(e^{-\frac{2\alpha}{t}}S_{\varepsilon,j}) \quad\le\quad C_\varepsilon e^{-\frac{\alpha}{t}}S_{\varepsilon,j} + C_\varepsilon, \tag{2.31}$$

$$(\frac{\partial}{\partial t} - \Delta_{\varepsilon,j})(e^{-\frac{2\gamma}{t}}tr_{\omega_0}\omega_{\varepsilon,j}(t)) \quad\le\quad C_\varepsilon - C_\varepsilon^{-1}e^{-\frac{3\gamma}{t}}S_{\varepsilon,j}. \tag{2.32}$$

By choosing $A_\varepsilon = C_\varepsilon(C_\varepsilon+1)$ and $\alpha = 3\gamma$,

$$(\frac{\partial}{\partial t} - \Delta_{\varepsilon,j})(e^{-\frac{2\alpha}{t}}S_{\varepsilon,j} + A_\varepsilon e^{-\frac{2\gamma}{t}}tr_{\omega_0}\omega_{\varepsilon,j}(t)) \le -e^{-\frac{3\gamma}{t}}S_{\varepsilon,j} + C_\varepsilon. \tag{2.33}$$

By the maximum principle, we have

$$S_{\varepsilon,j} \le C_\varepsilon e^{\frac{C}{t}} \quad on \quad (0,T]\times M \tag{2.34}$$

for some constant C depending only on $\|\varphi_0\|_{L^\infty(M)}$, n, β, ω_0 and T, and constant C_ε depending in addition on ε. □

By using the Schauder regularity theory and equation (2.4), we get the high order estimates of $\varphi_{\varepsilon,j}(t)$.

Proposition 2.5. *For any $0<\delta<T<\infty$, $\varepsilon>0$ and $k\ge 0$, there exists constant $C_{\varepsilon,\delta,T,k}$ depending only on δ, T, ε, k, n, β, ω_0 and $\|\varphi_0\|_{L^\infty(M)}$, such that for any $j\in\mathbb{N}^+$,*

$$\|\varphi_{\varepsilon,j}(t)\|_{C^k\left([\delta,T]\times M\right)} \le C_{\varepsilon,\delta,T,k}. \tag{2.35}$$

By (2.9), for any $T>0$, $\varphi_{\varepsilon,j}(t)$ converges to $\varphi_\varepsilon(t)\in L^\infty([0,T]\times M)$ uniformly in $L^\infty([0,T]\times M)$. For any $0<\delta<T<\infty$ and $\varepsilon>0$, $\varphi_{\varepsilon,j}(t)$ is uniformly bounded (depends on ε) in $C^\infty([\delta,T]\times M)$. Therefore $\varphi_{\varepsilon,j}(t)$ converges to $\varphi_\varepsilon(t)$ in $C^\infty([\delta,T]\times M)$. Hence for any $\varepsilon>0$, $\varphi_\varepsilon(t)\in C^\infty((0,\infty)\times M)$.

Proposition 2.6. *For any $\varepsilon>0$, $\varphi_\varepsilon(t)\in C^0([0,\infty)\times M)$ and*

$$\lim_{t\to 0^+}\|\varphi_\varepsilon(t)-\varphi_0\|_{L^\infty(M)} = 0. \tag{2.36}$$

Proof: For any $(t,z)\in(0,T]\times M$,

$$\begin{aligned}|\varphi_\varepsilon(t,z)-\varphi_0(z)| &\leq |\varphi_\varepsilon(t,z)-\varphi_{\varepsilon,j}(t,z)|+|\varphi_{\varepsilon,j}(t,z)-\varphi_{0,j}(z)|\\ &\quad +|\varphi_{0,j}(z)-\varphi_0(z)|.\end{aligned} \tag{2.37}$$

Since $\varphi_{\varepsilon,j}(t)$ is a Cauchy sequence in $L^\infty([0,T]\times M)$,

$$\lim_{j\to\infty}\|\varphi_\varepsilon(t,z)-\varphi_{\varepsilon,j}(t,z)\|_{L^\infty([0,T]\times M)}=0. \tag{2.38}$$

From (2.2), we have

$$\lim_{j\to\infty}\|\varphi_{0,j}(z)-\varphi_0(z)\|_{L^\infty(M)}=0, \tag{2.39}$$

For any $\epsilon>0$, there exists N such that for any $j>N$,

$$\begin{aligned}\sup_{[0,T]\times M}|\varphi_\varepsilon(t,z)-\varphi_{\varepsilon,j}(t,z)| &< \frac{\epsilon}{3},\\ \sup_M|\varphi_{0,j}(z)-\varphi_0(z)| &< \frac{\epsilon}{3}.\end{aligned}$$

On the other hand, fix such j, there exists $0<\delta<T$ such that

$$\sup_{[0,\delta]\times M}|\varphi_{\varepsilon,j}(t,z)-\varphi_{0,j}|<\frac{\epsilon}{3}. \tag{2.40}$$

Combining the above estimates together, for any $t\in[0,\delta]$ and $z\in M$,

$$|\varphi_\varepsilon(t,z)-\varphi_0(z)|<\epsilon. \tag{2.41}$$

This completes the proof of the lemma. □

Proposition 2.7. *$\varphi_\varepsilon(t)$ is the unique solution to the parabolic Monge-Ampère equation*

$$\begin{cases}\frac{\partial\varphi_\varepsilon(t)}{\partial t}=\log\frac{(\omega_0+\sqrt{-1}\partial\bar\partial\varphi_\varepsilon(t))^n}{\omega_0^n}+F_0\\ \qquad\qquad +\beta\varphi_\varepsilon(t)+\log(\varepsilon^2+|s|_h^2)^{1-\beta}, \qquad (0,\infty)\times M\\ \varphi_\varepsilon(0)=\varphi_0\end{cases} \tag{2.42}$$

in the space of $C^0\big([0,\infty)\times M\big)\cap C^\infty\big((0,\infty)\times M\big)$.

Proof: By proposition 2.6, we only need to prove the uniqueness. Suppose there exists another solution $\tilde\varphi_\varepsilon(t)\in C^0\big([0,\infty)\times M\big)\cap C^\infty\big((0,\infty)\times M\big)$ to the Monge-Ampère equation (2.42).

Let $\psi_\varepsilon(t)=\tilde\varphi_\varepsilon(t)-\varphi_\varepsilon(t)$. Then

$$\begin{cases}\frac{\partial\psi_\varepsilon(t)}{\partial t}=\log\frac{\big(\omega_0+\sqrt{-1}\partial\bar\partial\varphi_\varepsilon(t)+\sqrt{-1}\partial\bar\partial\psi_\varepsilon(t)\big)^n}{(\omega_0+\sqrt{-1}\partial\bar\partial\varphi_\varepsilon(t))^n}+\beta\psi_\varepsilon(t).\\ \\ \psi_\varepsilon(0)=0\end{cases} \tag{2.43}$$

For any $T>0$, by the same arguments as that in the proof of Proposition 2.1, we have

$$\|\psi_\varepsilon(t)\|_{L^\infty([0,T]\times M)}\leq e^{\beta T}\|\psi_\varepsilon(0)\|_{L^\infty(M)}=0.$$

Hence $\psi_\varepsilon(t)=0$, that is $\tilde\varphi_\varepsilon(t)=\varphi_\varepsilon(t)$. □

By the similar arguments as that in [41], we prove the uniqueness theorems of the twisted Kähler-Ricci flow.

Theorem 2.8. *Let M be a Fano manifold with complex dimension n, $\omega_0 \in c_1(M)$ be a smooth Kähler metric on M and $\hat{\omega} \in c_1(M)$ be a Kähler current which admits L^p density with respect to ω_0^n for some $p > 1$ and $\int_M \hat{\omega}^n = \int_M \omega_0^n$. Then there exists a unique solution $\omega_\varepsilon(t) \in C^\infty\big((0,\infty)\times M\big)$ to the twisted Kähler-Ricci flow (1.6) with initial data $\hat{\omega}$ in the following sense.*

(1) $\frac{\partial \omega_\varepsilon(t)}{\partial t} = -Ric(\omega_\varepsilon(t)) + \beta\omega_\varepsilon(t) + \theta_\varepsilon$ *on* $(0,\infty)\times M$*;*

(2) *There exists* $\varphi_\varepsilon(t) \in C^0\big([0,\infty)\times M\big) \cap C^\infty\big((0,\infty)\times M\big)$ *such that* $\omega_\varepsilon(t) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon(t)$ *and*

$$\lim_{t\to 0^+} \|\varphi_\varepsilon(t) - \varphi_0\|_{L^\infty(M)} = 0, \tag{2.44}$$

where $\varphi_0 \in \mathcal{E}_p(M,\omega_0)$ is a metric potential of $\hat{\omega}$ with respect to ω_0. In particular, $\omega_\varepsilon(t)$ converges in the sense of distribution to $\hat{\omega}$ as $t \to 0$.

Proof: From Proposition 2.7, we know that there exists a solution $\omega_\varepsilon(t) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon(t)$ to the twisted Kähler-Ricci flow (1.6) with initial data $\hat{\omega}$, where $\varphi_\varepsilon(t) \in C^0\big([0,\infty)\times M\big) \bigcap C^\infty\big((0,\infty)\times M\big)$ satisfies

$$\lim_{t\to 0^+} \|\varphi_\varepsilon(t) - \varphi_0\|_{L^\infty(M)} = 0 \tag{2.45}$$

for some metric potential $\varphi_0 \in \mathcal{E}_p(M,\omega_0)$ of $\hat{\omega}$ with respect to ω_0. Suppose that there is another solution $\tilde{\omega}_\varepsilon(t) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\tilde{\varphi}_\varepsilon(t)$ to the twisted Kähler-Ricci flow (1.6) with initial data $\hat{\omega}$. Then $\tilde{\varphi}_\varepsilon(t) \in C^0\big([0,\infty)\times M\big) \bigcap C^\infty\big((0,\infty)\times M\big)$ satisfies

$$\frac{\partial \tilde{\varphi}_\varepsilon(t)}{\partial t} = \log\frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\tilde{\varphi}_\varepsilon(t))^n}{\omega_0^n} + F_0 + \beta\tilde{\varphi}_\varepsilon(t) + \log(\varepsilon^2 + |s|_h^2)^{1-\beta} + f_\varepsilon(t) \tag{2.46}$$

on $(0,\infty)\times M$ for a smooth function $f_\varepsilon(t)$ on $(0,\infty)$ and

$$\lim_{t\to 0^+} \|\tilde{\varphi}_\varepsilon(t) - \tilde{\varphi}_0\|_{L^\infty(M)} = 0,$$

where $\tilde{\varphi}_0 \in \mathcal{E}_p(M,\omega_0)$ is also a metric potential of $\hat{\omega}$ with respect to ω_0. At the same time, we have $\varphi_0 = \tilde{\varphi}_0 + \tilde{C}$.

Let $\hat{\varphi}(t) = \tilde{\varphi}(t) + \tilde{C}e^{\beta t}$. It is obvious that $\hat{\varphi}_\varepsilon(t) \in C^0\big([0,\infty)\times M\big) \bigcap C^\infty\big((0,\infty)\times M\big)$ is a solution to equation (2.46) and satisfies

$$\lim_{t\to 0^+} \|\hat{\varphi}_\varepsilon(t) - \varphi_0\|_{L^\infty(M)} = 0.$$

Now we consider the function $\psi_\varepsilon(t) = \hat{\varphi}_\varepsilon(t) - \varphi_\varepsilon(t)$.

$$\begin{cases} \frac{\partial \psi_\varepsilon(t)}{\partial t} = \log\frac{\big(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon(t) + \sqrt{-1}\partial\bar{\partial}\psi_\varepsilon(t)\big)^n}{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon(t))^n} + \beta\psi_\varepsilon(t) + f_\varepsilon(t). \\ \\ \psi_\varepsilon(0) = 0 \end{cases} \tag{2.47}$$

For any $0 < t_1 < t_2 < \infty$, by the same arguments as that in the proof of Proposition 2.1, we have

$$\begin{aligned} \sup_M \psi_\varepsilon(t_2) &\leq e^{\beta(t_2-t_1)}\sup_M \psi_\varepsilon(t_1) + \int_{t_1}^{t_2} e^{\beta(t_2-t)} f_\varepsilon(t)dt, \\ \inf_M \psi_\varepsilon(t_2) &\geq e^{\beta(t_2-t_1)}\inf_M \psi_\varepsilon(t_1) + \int_{t_1}^{t_2} e^{\beta(t_2-t)} f_\varepsilon(t)dt. \end{aligned}$$

Therefore, we obtain

$$\inf_M \psi_\varepsilon(t_2) \quad \geq \quad \sup_M \psi_\varepsilon(t_2) - e^{\beta(t_2-t_1)}(\sup_M \psi_\varepsilon(t_1) - \inf_M \psi_\varepsilon(t_1)).$$

Let $t_1 \to 0^+$, we have

$$\inf_M \psi_\varepsilon(t_2) \quad \geq \quad \sup_M \psi_\varepsilon(t_2).$$

By equation (2.47), $\psi_\varepsilon(t) = \int_0^t e^{\beta(t-s)} f_\varepsilon(s) ds$. Hence $\tilde{\omega}_\varepsilon(t) = \omega_\varepsilon(t)$. □

3. The long-time existence of the conical Kähler-Ricci flow with weak initial data

In this section, we study the long-time existence of the conical Kähler-Ricci flow (1.5) by the smooth approximation of the twisted Kähler-Ricci flows. We also prove the uniqueness of the conical Kähler-Ricci flow (1.5).

By Proposition 2.1, Lemma 2.3 and Proposition 2.5, we conclude that for any $T > 0$, there exists constants C_1 and C_2 depending only on $\|\varphi_0\|_{L^\infty(M)}$, β, n, ω_0 and T, such that for any $\varepsilon > 0$,

$$\|\phi_\varepsilon(t)\|_{L^\infty([0,T]\times M)} \leq C_1, \tag{3.1}$$

$$e^{-\frac{C_2}{t}}\omega_\varepsilon \leq \omega_\varepsilon(t) \leq e^{\frac{C_2}{t}}\omega_\varepsilon \quad on \ (0,T]\times M. \tag{3.2}$$

We first prove the local uniform Calabi's C^3-estimate and curvature estimate along the flow (2.3). Our proofs are similar as that in [34] (see section 2 in [34] or section 3 in [40]), but we need some new arguments to handle the weak initial data case.

Lemma 3.1. *For any $T > 0$ and $B_r(p) \subset\subset M \setminus D$, there exist constants C, C' and C'' such that for any $\varepsilon > 0$ and $j \in \mathbb{N}^+$,*

$$S_{\varepsilon,j} \quad \leq \quad \frac{C'}{r^2} e^{\frac{C}{t}},$$
$$|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)} \quad \leq \quad \frac{C''}{r^4} e^{\frac{C}{t}}$$

on $(0,T] \times B_{\frac{r}{2}}(p)$, where constants C, C' and C'' depend only on $\|\varphi_0\|_{L^\infty(M)}$, n, β, T, ω_0 and $dist_{\omega_0}(B_r(p), D)$.

Proof: By Lemma 2.3, there exists uniform constat C depending only on $\|\varphi_0\|_{L^\infty(M)}$, n, β, T, ω_0 and $dist_{\omega_0}(B_r(p), D)$, such that

$$e^{-\frac{C}{t}}\omega_0 \leq \omega_{\varepsilon,j}(t) \leq e^{\frac{C}{t}}\omega_0, \quad on \ \ B_r(p)\times(0,T]. \tag{3.3}$$

Let $r = r_0 > r_1 > \frac{r}{2}$ and ψ be a nonnegative C^∞ cut-off function that is identically equal to 1 on $\overline{B_{r_1(p)}}$ and vanishes outside $B_r(p)$. We may assume that

$$|\partial\psi|^2_{\omega_0} \leq \frac{C}{r^2} \quad and \quad |\sqrt{-1}\partial\bar{\partial}\psi|_{\omega_0} \leq \frac{C}{r^2}. \tag{3.4}$$

Straightforward calculations show that

$$(\frac{\partial}{\partial t} - \Delta_{\varepsilon,j})(\psi^2 S_{\varepsilon,j}) \leq \frac{C}{r^2} e^{\frac{C}{t}} S_{\varepsilon,j} + Ce^{\frac{C}{t}}. \tag{3.5}$$

By choosing sufficiently large α, γ and A, we get

$$\begin{aligned} &(\frac{\partial}{\partial t}-\Delta_{\varepsilon,j})(e^{-\frac{2\alpha}{t}}\psi^2 S_{\varepsilon,j}+Ae^{-\frac{2\gamma}{t}}tr_{\omega_0}\omega_{\varepsilon,j}(t))\\ \leq\ & \frac{C}{r^2}e^{-\frac{\alpha}{t}}S_{\varepsilon,j}+C-Ae^{-\frac{3\gamma}{t}}S_{\varepsilon,j}+Ae^{-\frac{\gamma}{t}}\\ \leq\ & -\frac{1}{r^2}e^{-\frac{3\gamma}{t}}S_{\varepsilon,j}+\frac{C}{r^2}, \end{aligned}$$

where $\alpha=3\gamma$, $A=\frac{C+1}{r^2}$, constat C depends only on $\|\varphi_0\|_{L^\infty(M)}$, n, β, T ω_0 and $dist_{\omega_0}(B_r(p),D)$. By the maximum principle, we conclude that

$$S_{\varepsilon,j}\quad\leq\quad\frac{C'}{r^2}e^{\frac{6\gamma}{t}}\quad on\quad (0,T]\times B_{\frac{r}{2}}(p).$$

Now we prove that $|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}$ is uniformly bounded. Through computation, there exist uniform constants C such that

$$\begin{aligned} &(\frac{d}{dt}-\Delta_{\varepsilon,j})|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}\\ \leq\ & C|Rm_{\varepsilon,j}|^3_{\omega_{\varepsilon,j}(t)}+Ce^{\frac{C}{t}}|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}+Ce^{\frac{C}{t}}|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}+Ce^{\frac{C}{t}}S^{\frac{1}{2}}_{\varepsilon,j}|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}\\ &+Ce^{\frac{C}{t}}S_{\varepsilon,j}|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}-|\nabla_{\varepsilon,j}Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}-|\overline{\nabla}_{\varepsilon,j}Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}+Ce^{\frac{C}{t}}\\ \leq\ & C(|Rm_{\varepsilon,j}|^3_{\omega_{\varepsilon,j}(t)}+e^{\frac{\tau}{t}}+\frac{1}{r^2}e^{\frac{\tau}{t}}|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)})-|\nabla_{\varepsilon,j}Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}-|\overline{\nabla}_{\varepsilon,j}Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}. \end{aligned}$$

Next, we show that $|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}$ is uniformly bounded. We fix a smaller radius r_2 satisfying $r_1>r_2>\frac{r}{2}$. Let ρ be a cut-off function identically equal to 1 on $\overline{B_{r_2}}(p)$ and identically equal to 0 outside B_{r_1}. We also let ρ satisfy

$$|\partial\rho|^2_{\omega_0},\ |\sqrt{-1}\partial\bar{\partial}\rho|_{\omega_0}\leq\frac{C}{r^2}$$

for some uniform constant C. From the former part we know that $S_{\varepsilon,j}$ is bounded by $\frac{C}{r^2}e^{\frac{\tau}{t}}$ on $B_{r_1}(p)$. Let $K_t=\frac{\hat{C}}{r^2}e^{\frac{k\tau}{t}}$, k and $\hat{C}$ be constants which are large enough such that $\frac{K_t}{2}\leq K_t-S_{\varepsilon,j}\leq K_t$. We consider

$$F_{\varepsilon,j}=\rho^2e^{-\frac{2\delta}{t}}\frac{|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}}{K_t-S_{\varepsilon,j}}+Ae^{-\frac{2\sigma}{t}}S_{\varepsilon,j}. \tag{3.6}$$

By computing, we have

$$\begin{aligned} &(\frac{d}{dt}-\Delta_{\varepsilon,j})F_{\varepsilon,j}\\ =\ & e^{-\frac{2\delta}{t}}\Big((-\triangle_{\varepsilon,j}\rho^2)\frac{|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}}{K_t-S_{\varepsilon,j}}+\rho^2\frac{|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}}{(K_t-S_{\varepsilon,j})^2}(\frac{d}{dt}-\Delta_{\varepsilon,j})S_{\varepsilon,j}+\rho^2\frac{\hat{C}|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}}{r^2(K_t-S_{\varepsilon,j})^2}\frac{k\tau}{t^2}e^{\frac{k\tau}{t}}\\ &+\rho^2\frac{1}{K_t-S_{\varepsilon,j}}(\frac{d}{dt}-\Delta_{\varepsilon,j})|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}-4Re\langle\rho\frac{\nabla_{\varepsilon,j}\rho}{K_t-S_{\varepsilon,j}},\nabla_{\varepsilon,j}|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}\rangle_{\omega_{\varepsilon,j}(t)}\\ &-4Re\langle\rho\frac{|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}}{(K_t-S_{\varepsilon,j})^2}\nabla_{\varepsilon,j}S_{\varepsilon,j},\nabla_{\varepsilon,j}\rho\rangle_{\omega_{\varepsilon,j}(t)}-2\frac{\rho^2|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}}{(K_t-S_{\varepsilon,j})^3}|\nabla_{\varepsilon,j}S|^2_{\omega_{\varepsilon,j}(t)} \end{aligned}$$

$$-2Re\langle\rho^2\frac{\nabla_{\varepsilon,j}S_{\varepsilon,j}}{(K_t-S_{\varepsilon,j})^2},\nabla_{\varepsilon,j}|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}\rangle_{\omega_{\varepsilon,j}(t)}\Big)+Ae^{-\frac{2\sigma}{t}}(\frac{d}{dt}-\triangle_{\omega_{\varepsilon,j}(t)})S_{\varepsilon,j}$$
$$+A\frac{2\sigma}{t^2}e^{-\frac{2\sigma}{t}}S_{\varepsilon,j}+\frac{2\delta}{t^2}e^{-\frac{2\delta}{t}}\rho^2\frac{|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}}{K_t-S_{\varepsilon,j}}.$$

We only consider an inner point (t_0,x_0) which is a maximum point of $F_{\varepsilon,j}$ achieved on $[0,T]\times\overline{B_{r_1}(p)}$. We use the fact that $\nabla_{\varepsilon,j}F_{\varepsilon,j}=0$ at this point, then we get

$$e^{-\frac{2\delta}{t}}\Big(2\rho\nabla_{\varepsilon,j}\rho\frac{|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}}{K_t-S_{\varepsilon,j}}+\rho^2\frac{\nabla_{\varepsilon,j}|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}}{K_t-S_{\varepsilon,j}}+\rho^2\frac{|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}\nabla_{\varepsilon,j}S}{(K_t-S_{\varepsilon,j})^2}\Big)$$
$$+Ae^{-\frac{2\sigma}{t}}\nabla_{\varepsilon,j}S_{\varepsilon,j}=0.$$

Combining the above two equalities, we have

$$\begin{aligned}
&(\frac{d}{dt}-\Delta_{\varepsilon,j})F_{\varepsilon,j}\\
=\;&e^{-\frac{2\delta}{t}}\Big((-\triangle_{\varepsilon,j}\rho^2)\frac{|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}}{K_t-S_{\varepsilon,j}}+\rho^2\frac{|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}}{(K_t-S_{\varepsilon,j})^2}(\frac{d}{dt}-\Delta_{\varepsilon,j})S_{\varepsilon,j}+\rho^2\frac{\hat{C}|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}}{r^2(K_t-S_{\varepsilon,j})^2}\frac{k\tau}{t^2}e^{\frac{k\tau}{t}}\\
&+\rho^2\frac{1}{K_t-S_{\varepsilon,j}}(\frac{d}{dt}-\Delta_{\varepsilon,j})|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}-4Re\langle\rho\frac{\nabla_{\varepsilon,j}\rho}{K_t-S_{\varepsilon,j}},\nabla_{\varepsilon,j}|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}\rangle_{\omega_{\varepsilon,j}(t)}\Big)\\
&+2Ae^{-\frac{2\sigma}{t}}\frac{|\nabla_{\varepsilon,j}S_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}}{K_t-S_{\varepsilon,j}}+Ae^{-\frac{2\sigma}{t}}(\frac{d}{dt}-\triangle_{\omega_{\varepsilon,j}(t)})S_{\varepsilon,j}+A\frac{2\sigma}{t^2}e^{-\frac{2\sigma}{t}}S_{\varepsilon,j}\\
&+\frac{2\delta}{t^2}e^{-\frac{2\delta}{t}}\rho^2\frac{|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}}{K_t-S_{\varepsilon,j}}.
\end{aligned}$$

Our goal is to show that at (t_0,x_0) we have $e^{-\frac{2\delta}{t}}|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}\leq\frac{C}{r^4}$ for some uniform constant C and δ. Without loss of generality, we assume that $|Rm_{\varepsilon,j}|^3_{\omega_{\varepsilon,j}(t)}\geq e^{\frac{\tau}{t}}+\frac{1}{r^2}e^{\frac{\tau}{t}}|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}$ at (t_0,x_0).

$$\begin{aligned}
(\frac{d}{dt}-\triangle_{\omega_{\varepsilon,j}(t)})|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}&\leq C|Rm_{\varepsilon,j}|^3_{\omega_{\varepsilon,j}(t)}-|\nabla_{\varepsilon,j}Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}-|\overline{\nabla}_{\varphi}Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)},\\
(\frac{d}{dt}-\triangle_{\omega_{\varepsilon,j}(t)})S_{\varepsilon,j}&\leq\frac{C}{r^2}e^{\frac{\tau}{t}}-|\nabla_{\varepsilon,j}X|^2_{\omega_{\varepsilon,j}(t)}-|\overline{\nabla}_{\varepsilon,j}X|^2_{\omega_{\varepsilon,j}(t)}
\end{aligned}$$

on $B_{r_1}(p)$. We also note that

$$\begin{aligned}
|\nabla_{\varepsilon,j}|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}|_{\omega_{\varepsilon,j}(t)}&\leq|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}(|\nabla_{\varepsilon,j}Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}+|\overline{\nabla}_{\varepsilon,j}Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}),\\
|\nabla_{\varepsilon,j}S_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}&\leq2S_{\varepsilon,j}(|\nabla_{\varepsilon,j}X|^2_{\omega_{\varepsilon,j}(t)}+|\overline{\nabla}_{\varepsilon,j}X|^2_{\omega_{\varepsilon,j}(t)}).
\end{aligned}$$

By using the above inequalities, at (t_0,x_0), we have

$$\begin{aligned}
&(\frac{d}{dt}-\triangle_{\omega_{\varepsilon,j}(t)})F_{\varepsilon,j}\\
\leq\;&-Ae^{-\frac{2\sigma}{t}}(|\nabla_{\varepsilon,j}X|^2_{\omega_{\varepsilon,j}(t)}+|\overline{\nabla}_{\varepsilon,j}X|^2_{\omega_{\varepsilon,j}(t)})+\frac{AC}{r^2}e^{-\frac{2\sigma}{t}}e^{\frac{\tau}{t}}+e^{-\frac{2\delta}{t}}\Big(\frac{Ce^{\frac{C}{t}}|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}}{K_tr^2}\\
&-\frac{\rho^2|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}(|\nabla_{\varepsilon,j}X|^2_{\omega_{\varepsilon,j}(t)}+|\overline{\nabla}_{\varepsilon,j}X|^2_{\omega_{\varepsilon,j}(t)})}{K_t^2}+\frac{C\rho^2|Rm_{\varepsilon,j}|^3_{\omega_{\varepsilon,j}(t)}}{K_t}
\end{aligned}$$

$$-\frac{\rho^2(|\nabla_{\varepsilon,j}Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}+|\overline{\nabla}_{\varepsilon,j}Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)})}{K_t}+\frac{Ce^{\frac{C}{t}}|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}}{K_tr^2}+\frac{C\rho^2|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}}{K_t^2r^2}e^{\frac{\tau}{t}}$$
$$+\frac{\rho^2(|\nabla_{\varepsilon,j}Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}+|\overline{\nabla}_{\varepsilon,j}Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)})}{K_t}\Big)+\frac{8Ae^{-\frac{2\sigma}{t}}S_{\varepsilon,j}(|\nabla_{\varepsilon,j}X|^2_{\omega_{\varepsilon,j}(t)}+|\overline{\nabla}_{\varepsilon,j}X|^2_{\omega_{\varepsilon,j}(t)})}{K_t}$$
$$+\rho^2e^{-\frac{2\delta}{t}}\frac{2\hat{C}|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}}{r^2K_t^2}\frac{k\tau}{t^2}e^{\frac{k\tau}{t}}+A\frac{2\sigma}{t^2}e^{-\frac{2\sigma}{t}}S_{\varepsilon,j}+\frac{4\delta}{t^2}e^{-\frac{2\delta}{t}}\rho^2\frac{|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}}{K_t}.$$

Let $\hat{C}$ be sufficiently large so that $\frac{8AS_{\varepsilon,j}Q}{K_t}\leq\frac{AQ}{2}$, where we denote $Q=|\nabla_{\varepsilon,j}X|^2_{\omega_{\varepsilon,j}(t)}+|\overline{\nabla}_{\varepsilon,j}X|^2_{\omega_{\varepsilon,j}(t)}$. Then

$$\begin{aligned}\frac{C\rho^2|Rm_{\varepsilon,j}|^3_{\omega_{\varepsilon,j}(t)}}{K_t} &\leq \frac{\rho^2|Rm_{\varepsilon,j}|^4_{\omega_{\varepsilon,j}(t)}}{2K_t^2}+C\rho^2|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}\\ &\leq \frac{\rho^2|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}Q}{K_t^2}+Ce^{\frac{C}{t}}\rho^2|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}\end{aligned}\tag{3.7}$$

Let $k=1$, $\delta=2\sigma$ and $\tau-2\sigma<0$, where σ is sufficiently large. We conclude that the evolution equation of $F_{\varepsilon,j}$ can be controlled as follows,

$$\begin{aligned}(\frac{d}{dt}-\Delta_{\varepsilon,j})F_{\varepsilon,j} &\leq -\frac{Ae^{-\frac{2\sigma}{t}}Q}{2}+\frac{AC}{r^2}e^{-\frac{2\sigma}{t}}e^{\frac{\tau}{t}}+\frac{AC}{t^2r^2}e^{-\frac{2\sigma}{t}}e^{\frac{\tau}{t}}+Ce^{-\frac{\delta}{t}}|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}\\ &\leq -\frac{Ae^{-\frac{2\sigma}{t}}Q}{2}+\frac{AC}{r^2}e^{-\frac{2\sigma}{t}}e^{\frac{\tau}{t}}+\frac{AC}{t^2r^2}e^{-\frac{2\sigma}{t}}e^{\frac{\tau}{t}}+Ce^{-\frac{\delta}{t}}Q+Ce^{-\frac{\delta}{2t}}.\end{aligned}$$

Now we choose a sufficiently large A such that $A=2(C+1)$ and obtain

$$e^{-\frac{\delta}{t}}Q\leq\frac{C}{r^2}$$

at (t_0,x_0). This implies that $e^{-\frac{2\delta}{t}}|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}\leq\frac{C}{r^2}$ at this point, where C depends only on $\|\varphi_0\|_{L^\infty(M)}$, n, β, T, $dist_{\omega_0}(B_r(p),D)$, $\|\theta\|_{C^2(B_r(p))}$ and ω_0. Following that we conclude that $F_{\varepsilon,j}$ is bounded by $\frac{C}{r^2}$ at (t_0,x_0). Hence on $[0,T]\times\overline{B_{r_2}(p)}$, we obtain

$$|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}\leq\frac{C}{r^4}e^{\frac{2\delta+\tau}{t}},\tag{3.8}$$

where C, δ and τ depend only on $\|\varphi_0\|_{L^\infty(M)}$, n, β, T, $dist_{\omega_0}(B_r(p),D)$ and ω_0. □

By using the standard parabolic Schauder regularity theory [31], we obtain the following proposition.

Proposition 3.2. *For any $0<\delta<T<\infty$, $k\in\mathbb{N}^+$ and $B_r(p)\subset\subset M\setminus D$, there exists constant $C_{\delta,T,k,p,r}$ depends only on $\|\varphi_0\|_{L^\infty(M)}$, n, β, δ, k, T, $dist_{\omega_0}(B_r(p),D)$ and ω_0, such that for any $\varepsilon>0$ and $j\in\mathbb{N}^+$,*

$$\|\varphi_{\varepsilon,j}(t)\|_{C^k\left([\delta,T]\times B_r(p)\right)}\leq C_{\delta,T,k,p,r}.\tag{3.9}$$

Through a further observation to equation (2.42), we prove the monotonicity of $\varphi_\varepsilon(t)$ with respect to ε.

Proposition 3.3. *For any $(t,x)\in[0,T]\times M$, $\varphi_\varepsilon(t,x)$ is monotone decreasing as $\varepsilon\to0$.*

Proof: For any $\varepsilon_1 < \varepsilon_2$, let $\psi_{1,2}(t) = \varphi_{\varepsilon_1}(t) - \varphi_{\varepsilon_2}(t)$. Then

$$\begin{cases} \frac{\partial \psi_{1,2}(t)}{\partial t} = \log \frac{\left(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_{\varepsilon_2}(t) + \sqrt{-1}\partial\bar{\partial}\psi_{1,2}(t)\right)^n}{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_{\varepsilon_2}(t))^n} \\ \qquad\qquad + \beta\psi_{1,2}(t) + (1-\beta)\log\frac{(\varepsilon_1^2 + |s|_h^2)}{(\varepsilon_1^2 + |s|_h^2)}. \\ \psi_{1,2}(0) = 0 \end{cases} \tag{3.10}$$

Since $\log \frac{(\varepsilon_1^2 + |s|_h^2)}{(\varepsilon_1^2 + |s|_h^2)} < 0$, we have

$$\begin{aligned} & \frac{\partial}{\partial t}(e^{-\beta t}\psi_{1,2}(t)) \\ \leq \quad & e^{-\beta t} \log \frac{\left(e^{-\beta t}\omega_0 + \sqrt{-1}\partial\bar{\partial}e^{-\beta t}\varphi_{\varepsilon_2}(t) + \sqrt{-1}\partial\bar{\partial}e^{-\beta t}\psi_{1,2}(t)\right)^n}{(e^{-\beta t}\omega_0 + \sqrt{-1}\partial\bar{\partial}e^{-\beta t}\varphi_{\varepsilon_2}(t))^n}. \end{aligned} \tag{3.11}$$

Let $\tilde{\psi}_{1,2}(t) = e^{-\beta t}\psi_{1,2}(t) - \delta t$ with $\delta > 0$ and (t_0, x_0) be the maximum point of $\tilde{\psi}_{1,2}(t)$ on $[0,T] \times M$. If $t_0 > 0$, by maximum principle, at this point, we have

$$0 \leq \frac{\partial}{\partial t}\tilde{\psi}_{1,2}(t) = \frac{\partial}{\partial t}(e^{-\beta t}\psi_{1,2}(t)) - \delta \leq -\delta \tag{3.12}$$

which is impossible, hence $t_0 = 0$. So for any $(t,x) \in [0,T] \times M$,

$$\psi_{1,2}(t,x) \leq e^{\beta t} \sup_M \psi_{1,2}(0,x) + Te^{\beta T}\delta = Te^{\beta T}\delta. \tag{3.13}$$

Let $\delta \to 0$, we conclude that $\varphi_{\varepsilon_1}(t,x) \leq \varphi_{\varepsilon_2}(t,x)$. □

For any $[\delta,T] \times K \subset\subset (0,\infty) \times M \setminus D$ and $k \geq 0$, $\|\varphi_{\varepsilon,j}(t)\|_{C^k([\delta,T]\times K)}$ is uniformly bounded by Proposition 3.2. Let j approximate to ∞, we obtain that $\|\varphi_\varepsilon(t)\|_{C^k([\delta,T]\times K)}$ is uniformly bounded. Then let δ approximate to 0, T approximate to ∞ and K approximate to $M \setminus D$, by diagonal rule, we get a sequence $\{\varepsilon_i\}$, such that $\varphi_{\varepsilon_i}(t)$ converges in C^∞_{loc} topology on $(0,\infty) \times (M \setminus D)$ to a function $\varphi(t)$ that is smooth on $C^\infty\big((0,\infty) \times (M \setminus D)\big)$ and satisfies equation

$$\frac{\partial \varphi(t)}{\partial t} = \log \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi(t))^n}{\omega_0^n} + F_0 + \beta\varphi(t) + \log|s|_h^{2(1-\beta)} \tag{3.14}$$

on $(0,\infty) \times (M \setminus D)$. Since $\varphi_\varepsilon(t)$ is monotone decreasing as $\varepsilon \to 0$, we conclude that $\varphi_\varepsilon(t)$ converges in C^∞_{loc} topology on $(0,\infty) \times (M \setminus D)$ to $\varphi(t)$. Combining the above arguments with (3.1) and (3.2), for any $T > 0$, we have

$$\|\varphi(t)\|_{L^\infty\left((0,T]\times(M\setminus D)\right)} \leq C_1, \tag{3.15}$$

$$e^{-\frac{C_2}{t}}\omega_\beta \leq \omega(t) \leq e^{\frac{C_2}{t}}\omega_\beta \quad \textit{on } (0,T] \times (M \setminus D), \tag{3.16}$$

where $\omega(t) = \omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi(t)$, constants C_1 and C_2 depend only on $\|\varphi_0\|_{L^\infty(M)}$, β, n, ω_0 and T.

Proposition 3.4. *For any $t > 0$, $\varphi(t)$ is Hölder continuous on M with respect to the metric ω_0.*

Proof: We assume that $t \in [\delta,T]$ for some δ and T satisfying $0 < \delta < T < \infty$. By (3.16) we have

$$C^{-1}\omega_\beta \leq \omega(t) \leq C\omega_\beta \quad \textit{on } [\delta,T] \times (M \setminus D), \tag{3.17}$$

where constant C depends only on $\|\varphi_0\|_{L^\infty(M)}$, β, T, n, ω_0 and δ. Combining this estimate and the fact that $\log\frac{\omega_\beta^n|s|_h^{2(1-\beta)}}{\omega_0^n}$ is bounded uniformly on $M\setminus D$, we obtain

$$\|\log\frac{\omega^n(t)|s|_h^{2(1-\beta)}}{\omega_0^n}\|_{L^\infty\big([\delta,T]\times(M\setminus D)\big)}\leq C \tag{3.18}$$

for some uniform constant C independent of t. Therefore, $\|\frac{\partial\varphi(t)}{\partial t}\|_{L^\infty\big([\delta,T]\times(M\setminus D)\big)}$ is uniformly bounded by equation (3.14) and estimate (3.15). We rewrite equation (3.14) as

$$(\omega_0+\sqrt{-1}\partial\bar{\partial}\varphi(t))^n=e^{\frac{\partial\varphi(t)}{\partial t}-F_0-\beta\varphi(t)}\frac{\omega_0^n}{|s|_h^{2(1-\beta)}}. \tag{3.19}$$

The function on the right side of equation (3.19) is L^p integrable with respect to ω_0^n for some $p>1$. By S. Kolodziej's L^p estimates [29], we know that $\varphi(t)$ is Hölder continuous on M with respect to ω_0 for any $t>0$. □

Next, by using the monotonicity of $\varphi_\varepsilon(t)$ with respect to ε and constructing auxiliary function, we prove the continuity of $\varphi(t)$ as $t\to 0^+$.

Proposition 3.5. *$\varphi(t)\in C^0([0,\infty)\times M)$ and*

$$\lim_{t\to 0^+}\|\varphi(t)-\varphi_0\|_{L^\infty(M)}=0. \tag{3.20}$$

Proof: Through the above arguments, we only need prove limit (3.20). By the monotonicity of $\varphi_\varepsilon(t)$ with respect to ε, for any $(t,z)\in(0,T]\times M$, we have

$$\begin{aligned}\varphi(t,z)-\varphi_0(z) &\leq \varphi_{\varepsilon_1}(t,z)-\varphi_0(z)\\ &\leq |\varphi_{\varepsilon_1}(t,z)-\varphi_{\varepsilon_1,j}(t,z)|+|\varphi_{\varepsilon_1,j}(t,z)-\varphi_{0,j}(z)|\\ &\quad +|\varphi_{0,j}(z)-\varphi_0(z)|.\end{aligned} \tag{3.21}$$

Since $\varphi_{\varepsilon_1,j}(t)$ is a Cauchy sequence in $L^\infty([0,T]\times M)$,

$$\lim_{j\to\infty}\|\varphi_{\varepsilon_1}(t,z)-\varphi_{\varepsilon_1,j}(t,z)\|_{L^\infty([0,T]\times M)}=0. \tag{3.22}$$

From (2.2), we have

$$\lim_{j\to\infty}\|\varphi_{0,j}(z)-\varphi_0(z)\|_{L^\infty(M)}=0, \tag{3.23}$$

For any $\epsilon>0$, there exists N such that for any $j>N$,

$$\begin{aligned}\sup_{[0,T]\times M}|\varphi_{\varepsilon_1}(t,z)-\varphi_{\varepsilon_1,j}(t,z)| &< \frac{\epsilon}{3},\\ \sup_M|\varphi_{0,j}(z)-\varphi_0(z)| &< \frac{\epsilon}{3}.\end{aligned}$$

Fix such j, there exists $0<\delta_1<T$ such that

$$\sup_{[0,\delta_1]\times M}|\varphi_{\varepsilon_1,j}(t,z)-\varphi_{0,j}|<\frac{\epsilon}{3}. \tag{3.24}$$

Combining the above estimates together, for any $t\in(0,\delta_1]$ and $z\in M$,

$$\varphi(t,z)-\varphi_0(z)<\epsilon. \tag{3.25}$$

On the other hand, by S. Kolodziej's results [27], there exists a smooth solution $u_{\varepsilon,j}$ to the equation

$$(\omega_0+\sqrt{-1}\partial\bar{\partial}u_{\varepsilon,j})^n = e^{-F_0-\beta\varphi_{0,j}+\hat{C}}\frac{\omega_0^n}{(\varepsilon^2+|s|_h^2)^{(1-\beta)}}, \tag{3.26}$$

and $u_{\varepsilon,j}$ satisfies

$$\|u_{\varepsilon,j}\|_{L^\infty(M)} \leq C, \tag{3.27}$$

where $\hat{C}$ is a uniform normalization constant, constant C depends only on $\|\varphi_0\|_{L^\infty(M)}$, β and F_0.

We define function

$$\psi_{\varepsilon,j}(t) = (1-te^{\beta t})\varphi_{0,j} + te^{\beta t}u_{\varepsilon,j} + h(t)e^{\beta t}, \tag{3.28}$$

where

$$\begin{aligned} h(t) &= -t\|\varphi_{0,j}\|_{L^\infty(M)} - t\|u_{\varepsilon,j}\|_{L^\infty(M)} + n(t\log t - t)e^{-\beta t} \\ &\quad +\beta n\int_0^t e^{-\beta s}s\log s ds + \hat{C}t \end{aligned} \tag{3.29}$$

and $h(0)=0$. Straightforward calculations show that

$$\begin{aligned} \frac{\partial}{\partial t}\psi_{\varepsilon,j}(t) - \beta\psi_{\varepsilon,j}(t) &= -\beta\varphi_{0,j} - e^{\beta t}\varphi_{0,j} + e^{\beta t}u_{\varepsilon,j} + e^{\beta t}\frac{\partial}{\partial t}h(t) \\ &= -\beta\varphi_{0,j} - e^{\beta t}\varphi_{0,j} + e^{\beta t}u_{\varepsilon,j} - e^{\beta t}\|\varphi_{0,j}\|_{L^\infty(M)} - e^{\beta t}\|u_{\varepsilon,j}\|_{L^\infty(M)} \\ &\quad + n\log t - \beta n(t\log t - t) + \beta nt\log t + \hat{C} \\ &\leq -\beta\varphi_{0,j} + n\log t + n\beta t + \hat{C}. \end{aligned}$$

Therefore, we have

$$e^{\frac{\partial}{\partial t}\psi_{\varepsilon,j}(t)-\beta\psi_{\varepsilon,j}(t)}\omega_0^n \leq t^n e^{n\beta t}e^{-\beta\varphi_{0,j}+\hat{C}}\omega_0^n.$$

When t is sufficiently small,

$$\begin{aligned} \omega_0+\sqrt{-1}\partial\overline{\partial}\psi_{\varepsilon,j}(t) &= (1-te^{\beta t})(\omega_0+\sqrt{-1}\partial\overline{\partial}\varphi_{0,j}) + te^{\beta t}(\omega_0+\sqrt{-1}\partial\overline{\partial}u_{\varepsilon,j}) \\ &\geq te^{\beta t}(\omega_0+\sqrt{-1}\partial\overline{\partial}u_{\varepsilon,j}). \end{aligned}$$

Combining the above inequalities,

$$\begin{aligned} (\omega_0+\sqrt{-1}\partial\overline{\partial}\psi_{\varepsilon,j}(t))^n &\geq t^n e^{n\beta t}(\omega_0+\sqrt{-1}\partial\overline{\partial}u_{\varepsilon,j})^n \\ &= t^n e^{n\beta t}e^{-F_0-\beta\varphi_{0,j}+\hat{C}}\frac{\omega_0^n}{(\varepsilon^2+|s|_h^2)^{(1-\beta)}} \\ &\geq e^{-F_0+\frac{\partial}{\partial t}\psi_{\varepsilon,j}(t)-\beta\psi_{\varepsilon,j}(t)}\frac{\omega_0^n}{(\varepsilon^2+|s|_h^2)^{(1-\beta)}}. \end{aligned}$$

This equation is equivalent to

$$\begin{cases} \frac{\partial}{\partial t}\psi_{\varepsilon,j}(t) \leq \log\frac{(\omega_0+\sqrt{-1}\partial\overline{\partial}\psi_{\varepsilon,j}(t))^n}{\omega_0^n} + \beta\psi_{\varepsilon,j}(t) \\ \qquad\qquad +F_0 + \log(\varepsilon^2+|s|_h^2)^{(1-\beta)}. \\ \psi_{\varepsilon,j}(0) = \varphi_{0,j} \end{cases} \tag{3.30}$$

Let $\tilde{\psi}_{\varepsilon,j}(t) = \varphi_{\varepsilon,j}(t) - \psi_{\varepsilon,j}(t)$, then

$$\begin{cases} \frac{\partial}{\partial t}\tilde{\psi}_{\varepsilon,j}(t) \geq \log \frac{(\omega_0+\sqrt{-1}\partial\overline{\partial}\psi_{\varepsilon,j}(t)+\sqrt{-1}\partial\overline{\partial}\tilde{\psi}_{\varepsilon,j}(t))^n}{(\omega_0+\sqrt{-1}\partial\overline{\partial}\psi_{\varepsilon,j}(t))^n} + \beta\tilde{\psi}_{\varepsilon,j}(t). \\ \\ \tilde{\psi}_{\varepsilon,j}(0) = 0 \end{cases} \tag{3.31}$$

By the similar arguments as that in the proof of Proposition 3.3, for any $(t,z) \in [0,T] \times M$,

$$\tilde{\psi}_{\varepsilon,j}(t,z) \geq 0, \tag{3.32}$$

That is, for any $(t,z) \in [0,T] \times M$

$$\begin{aligned} \varphi_{\varepsilon,j}(t,z) - \varphi_{0,j}(z) &\geq -te^{\beta t}\varphi_{0,j} + te^{\beta t}u_{\varepsilon,j} + h(t)e^{\beta t} \\ &\geq -Cte^{\beta t} + h(t)e^{\beta t}, \end{aligned} \tag{3.33}$$

where constant constant C depends only on $\|\varphi_0\|_{L^\infty(M)}$, β and F_0. Let $j \to \infty$ and then $\varepsilon \to 0$, we have

$$\varphi(t,z) - \varphi_0(z) \geq -Cte^{\beta t} + h(t)e^{\beta t}. \tag{3.34}$$

There exists δ_2 such that for any $t \in [0,\delta_2]$,

$$-Cte^{\beta t} + h(t)e^{\beta t} > -\epsilon. \tag{3.35}$$

Let $\delta = \min(\delta_1,\delta_2)$, then for any $t \in (0,\delta]$ and $z \in M$,

$$-\epsilon < \varphi(t,z) - \varphi_0(z) < \epsilon. \tag{3.36}$$

This completes the proof of the proposition. □

Theorem 3.6. $\omega_{\varphi(t)} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi(t)$ *is a long-time solution to the conical Kähler-Ricci flow* (1.5).

Proof: We should only prove that $\omega(t)$ satisfies equation (1.5) in the sense of currents on $[0,\infty) \times M$.

Let $\eta = \eta(t,x)$ be a smooth $(n-1,n-1)$-form with compact support in $(0,\infty) \times M$. Without loss of generality, we assume that its compact support included in (δ,T) $(0 < \delta < T < \infty)$. On $[\delta,T] \times M$, by (3.1) and (3.2), $\log \frac{\omega_\varepsilon^n(t)(\varepsilon^2+|s|_h^2)^{1-\beta}}{\omega_0^n}$ and φ_ε are uniformly bounded by constants depending only on $\|\varphi_0\|_{L^\infty(M)}$, n, β, δ and T. On $[\delta,T]$, we have

$$\begin{aligned} &\int_M \frac{\partial\omega_\varepsilon(t)}{\partial t} \wedge \eta = \int_M \sqrt{-1}\partial\bar{\partial}\frac{\partial\varphi_\varepsilon(t)}{\partial t} \wedge \eta \\ = &\int_M \sqrt{-1}\partial\bar{\partial}\big(\log \frac{\omega_\varepsilon^n(\varepsilon^2+|s|_h^2)^{1-\beta}}{\omega_0^n} + F_0 + \beta\varphi_\varepsilon(t)\big) \wedge \eta \\ = &\int_M \log\big(\log \frac{\omega_\varepsilon^n(\varepsilon^2+|s|_h^2)^{1-\beta}}{\omega_0^n} + F_0 + \beta\varphi_\varepsilon(t)\big)\sqrt{-1}\partial\bar{\partial}\eta \\ \xrightarrow{\varepsilon\to 0} &\int_M (\log \frac{\omega_{\varphi(t)}^n}{\omega_0^n} + F_0 + \beta\varphi(t) + \log|s|_h^{2(1-\beta)})\sqrt{-1}\partial\bar{\partial}\eta \\ = &\int_M \sqrt{-1}\partial\bar{\partial}(\log \frac{\omega_{\varphi(t)}^n}{\omega_0^n} + F_0 + \beta\varphi(t) + \log|s|_h^{2(1-\beta)}) \wedge \eta \\ = &\int_M (-Ric(\omega_{\varphi(t)}) + \beta\omega_{\varphi(t)} + 2\pi(1-\beta)[D]) \wedge \eta. \end{aligned} \tag{3.37}$$

At the same time, there also holds

$$\begin{aligned}\int_M \omega_{\varphi_\varepsilon(t)} \wedge \frac{\partial \eta}{\partial t} &= \int_M \omega_0 \wedge \frac{\partial \eta}{\partial t} + \int_M \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon(t) \wedge \frac{\partial \eta}{\partial t} \\ &= \int_M \omega_0 \wedge \frac{\partial \eta}{\partial t} + \int_M \varphi_\varepsilon(t)\sqrt{-1}\partial\bar{\partial}\frac{\partial \eta}{\partial t} \\ &\xrightarrow{\varepsilon_i \to 0} \int_M \omega_0 \wedge \frac{\partial \eta}{\partial t} + \int_M \varphi(t)\sqrt{-1}\partial\bar{\partial}\frac{\partial \eta}{\partial t} \\ &= \int_M \omega_0 \wedge \frac{\partial \eta}{\partial t} + \int_M \sqrt{-1}\partial\bar{\partial}\varphi(t)\frac{\partial \eta}{\partial t} \\ &= \int_M \omega_{\varphi(t)} \wedge \frac{\partial \eta}{\partial t}. \end{aligned} \tag{3.38}$$

On the other hand, $\varphi_\varepsilon(t)$ and $\frac{\partial \varphi_\varepsilon(t)}{\partial t}$ are uniformly bounded on $[\delta, T] \times M$, $\varphi(t)$ and $\frac{\partial \varphi(t)}{\partial t}$ are uniformly bounded on $[\delta, T] \times (M \setminus D)$, therefore

$$\begin{aligned}\frac{\partial}{\partial t}\int_M \omega_{\varphi_\varepsilon(t)} \wedge \eta &= \int_M \varphi_\varepsilon(t)\sqrt{-1}\partial\bar{\partial}\frac{\partial \eta}{\partial t} \\ &\quad + \int_M \frac{\partial \varphi_\varepsilon(t)}{\partial t}\sqrt{-1}\partial\bar{\partial}\eta + \int_M \omega_0 \wedge \frac{\partial \eta}{\partial t} \\ &\xrightarrow{\varepsilon \to 0} \int_M \varphi(t)\sqrt{-1}\partial\bar{\partial}\frac{\partial \eta}{\partial t} \\ &\quad + \int_M \frac{\partial \varphi}{\partial t}\sqrt{-1}\partial\bar{\partial}\eta + \int_M \omega_0 \wedge \frac{\partial \eta}{\partial t} \\ &= \frac{\partial}{\partial t}\int_M \omega_\varphi \wedge \eta. \end{aligned} \tag{3.39}$$

Combining equality

$$\frac{\partial}{\partial t}\int_M \omega_{\varphi_\varepsilon(t)} \wedge \eta = \int_M \frac{\partial \omega_{\varphi_\varepsilon(t)}}{\partial t} \wedge \eta + \int_M \omega_{\varphi_\varepsilon(t)} \wedge \frac{\partial \eta}{\partial t}$$

with equalities (3.37)-(3.39), on $[\delta, T]$, we have

$$\begin{aligned}\frac{\partial}{\partial t}\int_M \omega_{\varphi(t)} \wedge \eta &= \int_M \big(-Ric(\omega_{\varphi(t)}) + \beta\omega_{\varphi(t)} + 2\pi(1-\beta)[D]\big) \wedge \eta \\ &\quad + \int_M \omega_{\varphi(t)} \wedge \frac{\partial \eta}{\partial t}. \end{aligned} \tag{3.40}$$

Integrating form 0 to ∞ on both sides,

$$\begin{aligned}\int_{M\times(0,\infty)} \frac{\partial \omega_{\varphi(t)}}{\partial t} \wedge \eta \, dt &= -\int_{M\times(0,\infty)} \omega_{\varphi(t)} \wedge \frac{\partial \eta}{\partial t}\, dt = -\int_0^\infty \int_M \omega_{\varphi(t)} \wedge \frac{\partial \eta}{\partial t}\, dt \\ &= \int_0^\infty \int_M \big(-Ric(\omega_{\varphi(t)}) + \beta\omega_{\varphi(t)} + 2\pi(1-\beta)[D]\big) \wedge \eta \, dt \\ &= \int_{M\times(0,\infty)} \big(-Ric(\omega_{\varphi(t)}) + \beta\omega_{\varphi(t)} + 2\pi(1-\beta)[D]\big) \wedge \eta \, dt. \end{aligned}$$

By the arbitrariness of η, we prove that $\omega_{\varphi(t)}$ satisfies the conical Kähler-Ricci flow (1.5) in the sense of currents on $(0, \infty) \times M$. □

Now we are ready to prove the uniqueness of the parabolic Monge-Ampère equation (3.14) starting with $\varphi_0 \in \mathcal{E}_p(M, \omega_0)$ for some $p > 1$.

Theorem 3.7. *Let* $\varphi_i(t)\in C^0\big([0,\infty)\times M\big)\bigcap C^\infty\big((0,\infty)\times(M\setminus D)\big)$ $(i=1,2)$ *be two long-time solutions to the parabolic Monge-Ampère equation*

$$\frac{\partial\varphi_i(t)}{\partial t}=\log\frac{(\omega_0+\sqrt{-1}\partial\bar\partial\varphi_i(t))^n}{\omega_0^n}+F_0+\beta\varphi_i(t)+\log|s|_h^{2(1-\beta)} \tag{3.41}$$

on $(0,\infty)\times(M\setminus D)$*. If* φ_i $(i=1,2)$ *satisfy*

- *For any* $0<\delta<T<\infty$*, there exists uniform constant* C *such that*
$$C^{-1}\omega_\beta\le\omega_0+\sqrt{-1}\partial\bar\partial\varphi_i(t)\le C\omega_\beta$$
on $[\delta,T]\times(M\setminus D)$*;*
- *On* $[\delta,T]$*, there exist constant* $\alpha>0$ *and* C^* *such that* $\varphi_i(t)$ *is* C^α *on* M *with respect to* ω_0 *and* $\|\frac{\partial\varphi_i(t)}{\partial t}\|_{L^\infty(M\setminus D)}\leqslant C^*$*;*
- $\lim\limits_{t\to0^+}\|\varphi_i(t)-\varphi_0\|_{L^\infty(M)}=0.$

Then $\varphi_1=\varphi_2$*.*

Proof: We apply Jeffres' trick [25] in the parabolic case. For any $0<t_1<T<\infty$ and $a>0$. Let $\phi_1(t)=\varphi_1(t)+a|s|_h^{2q}$, where $0<q<1$ is determined later. The evolution of ϕ_1 is

$$\frac{\partial\phi_1(t)}{\partial t}=\log\frac{(\omega_0+\sqrt{-1}\partial\bar\partial\varphi_1(t))^n}{\omega_0^n}+F_0+\beta\phi_1(t)-a\beta|s|_h^{2q}+\log|s|_h^{2(1-\beta)}.$$

Denote $\psi(t)=\phi_1(t)-\varphi_2(t)$ and $\hat\Delta=\int_0^1 g^{i\bar j}_{s\varphi_1+(1-s)\varphi_2}\frac{\partial^2}{\partial z^i\partial\bar z^j}ds$, $\psi(t)$ evolves along the following equation

$$\frac{\partial\psi(t)}{\partial t}=\hat\Delta\psi(t)-a\hat\Delta|s|_h^{2q}+\beta\psi(t)-a\beta|s|_h^{2q}.$$

By the equivalence of the metrics and the equation

$$\sqrt{-1}\partial\overline\partial|s|_h^{2q}=q^2|s|_h^{2q}\sqrt{-1}\partial\log|s|_h^2\wedge\overline\partial\log|s|_h^2+q|s|_h^{2q}\sqrt{-1}\partial\overline\partial\log|s|_h^2,$$

we obtain the estimate

$$\begin{aligned}\hat\Delta|s|_h^{2q}&\ge q|s|_h^{2q}g^{i\bar j}_{s\varphi_1+(1-s)\varphi_2}(\frac{\partial^2}{\partial z^i\partial\bar z^j}\log|s|_h^2)\\&=-q|s|_h^{2q}g^{i\bar j}_{s\varphi_1+(1-s)\varphi_2}g_{0,i\bar j}\\&\ge -Cq|s|_h^{2q}g^{i\bar j}_\beta g_{0,i\bar j}\\&\ge -C\end{aligned}$$

on $M\setminus D$, where constant C independent of a, and we apply the fact that $\omega_\beta\ge\gamma\omega_0$ on $M\setminus D$ for some constant γ. Then we obtain

$$\frac{\partial\psi(t)}{\partial t}\le\hat\Delta\psi(t)+\beta\psi(t)+aC.$$

Let $\tilde\psi=e^{-\beta(t-t_1)}\psi+\frac{aC}{\beta}e^{-\beta(t-t_1)}-\epsilon(t-t_1)$. By choosing suitable $0<q<1$, we can assume that the space maximum of $\tilde\psi$ on $[t_1,T]\times M$ is attained away from D. Let (t_0,x_0) be the maximum point. If $t_0>t_1$, by the maximum principle, at (t_0,x_0), we have

$$0\le(\frac{\partial}{\partial t}-\hat\Delta)\tilde\psi(t)\le-\epsilon,$$

which is impossible, hence $t_0 = t_1$. Then for $(t, x) \in [t_1, T] \times M$, we obtain

$$\begin{aligned}\psi(t,x) &\leq e^{\beta T}\|\varphi_1(t_1,x)-\varphi_2(t_1,x)\|_{L^\infty(M)}\\ &\quad +aCe^{\beta T}+\epsilon Te^{\beta T}\end{aligned}$$

Let $a \to 0$ and then $t_1 \to 0^+$, we get

$$\varphi_1(t) - \varphi_2(t) \leq \epsilon T e^{\beta T}.$$

It shows that $\varphi_1(t) \leq \varphi_2(t)$ after we let $\epsilon \to 0$. By the same reason we have $\varphi_2(t) \leq \varphi_1(t)$, then we prove that $\varphi_1(t) = \varphi_2(t)$. □

Theorem 3.8. $\omega_{\varphi(t)} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi(t)$ *is the unique long-time solution to the conical Kähler-Ricci flow* (1.5).

Proof: Suppose there is another solution $\omega_{\phi(t)} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi(t)$ to the conical Kählre-Ricci flow (1.5). It is easy to see that

$$(3.42)\quad \frac{\partial\phi(t)}{\partial t} = \log\frac{(\omega_0+\sqrt{-1}\partial\bar{\partial}\phi(t))^n}{\omega_0^n} + F_0 + \beta\phi(t) + \log|s|_h^{2(1-\beta)} + f(t)$$

on $(0,\infty)\times(M\setminus D)$ for a smooth function $f(t)$ defined on $(0,\infty)$, and $\phi(t) \in C^0\big([0,\infty)\times M\big)\bigcap C^\infty\big((0,\infty)\times(M\setminus D)\big)$ satisfies

- For any $0 < \delta < T < \infty$, there exists uniform constant C such that
  $$C^{-1}\omega_\beta \leq \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi(t) \leq C\omega_\beta \quad on \quad [\delta, T]\times(M\setminus D);$$
- On $[\delta, T]$, there exist constant $\alpha > 0$ and C such that $\phi(t)$ is C^α on M with respect to ω_0 and $\|\frac{\partial\phi(t)}{\partial t}\|_{L^\infty(M\setminus D)} \leqslant C$;
- $\lim\limits_{t\to 0^+}\|\phi(t)-\varphi_0\|_{L^\infty(M)} = 0$.

For any $0 < t_1 < T < \infty$ and $a > 0$. Let $\psi(t) = \phi(t) + a|s|_h^{2q} - \varphi(t)$, where $0 < q < 1$ is determined later. Then

$$\frac{\partial\psi(t)}{\partial t} = \hat{\Delta}\psi(t) - a\hat{\Delta}|s|_h^{2q} + \beta\psi(t) - a\beta|s|_h^{2q} + f(t).$$

By the same arguments as that in the proof of Proposition 3.7, for any $(t, x) \in [t_1, T] \times M$, we have

$$\begin{aligned}\psi(t,x) &\leq e^{\beta(t-t_1)}\|\phi(t_1,x)-\varphi(t_1,x)\|_{L^\infty(M)}\\ &\quad +aCe^{\beta(t-t_1)}+\epsilon(t-t_1)e^{\beta(t-t_1)}\\ &\quad +e^{\beta(t-t_1)}\int_{t_1}^t e^{-\beta(s-t_1)}f(s)ds\end{aligned}$$

Let $a \to 0$, we obtain

$$\begin{aligned}\phi(t)-\varphi(t) &\leq e^{\beta(t-t_1)}\|\phi(t_1,x)-\varphi(t_1,x)\|_{L^\infty(M)}\\ &\quad +\epsilon(t-t_1)e^{\beta(t-t_1)}+e^{\beta(t-t_1)}\int_{t_1}^t e^{-\beta(s-t_1)}f(s)ds.\end{aligned}$$

By the similar arguments, we can obtain

$$\begin{aligned}\varphi(t)-\phi(t) &\leq e^{\beta(t-t_1)}\|\phi(t_1,x)-\varphi(t_1,x)\|_{L^\infty(M)}\\ &\quad +\epsilon(t-t_1)e^{\beta(t-t_1)}-e^{\beta(t-t_1)}\int_{t_1}^t e^{-\beta(s-t_1)}f(s)ds.\end{aligned}$$

Therefore, for any $t > t_1 > 0$, we have

$$\begin{aligned}\inf_M(\phi(t)-\varphi(t)) &\geq \sup_M(\phi(t)-\varphi(t)) - 2e^{\beta(t-t_1)}\|\phi(t_1,x)-\varphi(t_1,x)\|_{L^\infty(M)}\\ &\quad -2\epsilon(t-t_1)e^{\beta(t-t_1)}\\ &\geq \sup_M(\phi(t)-\varphi(t)) - 2e^{\beta T}\|\phi(t_1,x)-\varphi(t_1,x)\|_{L^\infty(M)}\\ &\quad -2\epsilon Te^{\beta T}.\end{aligned}$$

Let $t_1 \to 0^+$ and then $\epsilon \to 0$, we conclude that $\phi(t) = \varphi(t) + e^{\beta t}\int_0^t e^{-\beta s} f(s)ds$. Then $\omega_{\phi(t)} = \omega_{\varphi(t)}$ on $(0,\infty)\times(M\setminus D)$. □

4. The convergence of the conical Kähler-Ricci flow with weak initial data

In this section, we study the convergence of the conical Kähler-Ricci flow (1.5) with positive twisted first Chern class. Our discussion is very similar as that in [34], but we need new arguments on estimates of the twisted Ricci potential $u_\varepsilon(t)$ and the term $|\dot{\varphi}_\varepsilon|$ when we handle the weak initial data case.

Without loss of generality, we assume $\lambda = 1$ (i.e. $\mu = \beta$). We first prove the uniform Perelman's estimates along the twisted Kähler-Ricci flow

$$\begin{cases} \frac{\partial \omega_\varepsilon(t)}{\partial t} = -Ric(\omega_\varepsilon(t)) + \beta\omega_\varepsilon(t) + \theta_\varepsilon. \\ \\ \omega_\varepsilon(t)|_{t=0} = \omega_{\varphi_0} \end{cases} \tag{4.1}$$

By the same argument as Proposition 4.1 in [34], we have

Proposition 4.1. *$t^2(R(g_{\varepsilon,j}(t)) - tr_{g_{\varepsilon,j}(t)}\theta_\varepsilon)$ is uniformly bounded from below along the twisted Kähler-Ricci flow (2.3), i.e. there exists a uniform constant C, such that*

$$t^2(R(g_{\varepsilon,j}(t)) - tr_{g_{\varepsilon,j}(t)}\theta_\varepsilon) \geq -C \tag{4.2}$$

for any $t \geqslant 0$, $j \in \mathbb{N}^+$ and $\varepsilon > 0$, while the constant C only depends on β and n. In particular,

$$R(g_{\varepsilon,j}(t)) - tr_{g_{\varepsilon,j}(t)}\theta_\varepsilon \geq -C \tag{4.3}$$

when $t \geq \frac{1}{2}$.

Remark 4.2. *By Proposition 2.5, we know that there exists constant C only depending on β and n, such that*

$$R(g_\varepsilon(t)) - tr_{g_\varepsilon(t)}\theta_\varepsilon \geq -C \tag{4.4}$$

along the twisted Kähler-Ricci flow (4.1) for any $\varepsilon > 0$ when $t \geq \frac{1}{2}$.

Straightforward calculation shows that the twisted Ricci potential $u_\varepsilon(t)$ with respect to $\omega_\varepsilon(t)$ at $t = \frac{1}{2}$ can be written as

$$u_\varepsilon(\frac{1}{2}) = \log\frac{\omega_\varepsilon^n(\frac{1}{2})(\varepsilon^2+|s|_h^2)^{1-\beta}}{\omega_0^n} + F_0 + \beta\varphi_\varepsilon(\frac{1}{2}) + C_{\varepsilon,\frac{1}{2}}, \tag{4.5}$$

where $C_{\varepsilon,\frac{1}{2}}$ is a normalization constant such that $\frac{1}{V}\int_M e^{-u_\varepsilon(\frac{1}{2})}dV_{\varepsilon,\frac{1}{2}} = 1$. By (3.1) and (3.2), we conclude that $C_{\varepsilon,\frac{1}{2}}$ and $u_\varepsilon(\frac{1}{2})$ are uniformly bounded. Let $a_\varepsilon(t) = \frac{\beta}{V}\int_M u_\varepsilon(t)e^{-u_\varepsilon(t)}dV_{\varepsilon,t}$, then by Lemma 4.4 in [34], we have

Lemma 4.3. *There exists a uniform constant C, such that*

$$|a_\varepsilon(t)| \leq C \tag{4.6}$$

for any $t \geq \frac{1}{2}$ and $\varepsilon > 0$.

Now we consider the twisted Kähler-Ricci flows (4.1) starting at $t = \frac{1}{2}$. Using the estimates (4.4), (4.6) and following the arguments in [34] (see section 4), we have the following uniform Perelman's estimates.

Theorem 4.4. *Let $g_\varepsilon(t)$ be a solution of the twisted Kähler Ricci flow, i.e. the corresponding form $\omega_\varepsilon(t)$ satisfies the equation (4.1) with initial metric ω_{φ_0}, $u_\varepsilon(t) \in C^\infty((0,\infty)\times M)$ is the twisted Ricci potential satisfying*

$$-Ric(\omega_\varepsilon(t)) + \beta\omega_\varepsilon(t) + \theta_\varepsilon = \sqrt{-1}\partial\bar{\partial}u_\varepsilon(t) \tag{4.7}$$

and $\frac{1}{V}\int_M e^{-u_\varepsilon(t)}dV_{\varepsilon,t} = 1$, where $\theta_\varepsilon = (1-\beta)(\omega_0 + \sqrt{-1}\partial\overline{\partial}\log(\varepsilon^2 + |s|_h^2))$. Then for any $\beta \in (0,1)$, there exists a uniform constant C, such that

$$|R(g_\varepsilon(t)) - tr_{g_\varepsilon(t)}\theta_\varepsilon| \leq C, \tag{4.8}$$

$$\|u_\varepsilon(t)\|_{C^1(g_\varepsilon(t))} \leq C, \tag{4.9}$$

$$diam(M, g_\varepsilon(t)) \leq C \tag{4.10}$$

hold for any $t \geq 1$ and $\varepsilon > 0$, where $R(g_\varepsilon(t)) - tr_{g_\varepsilon(t)}\theta_\varepsilon$ and $diam(M, g_\varepsilon(t))$ are the twisted scalar curvature and diameter of the manifold respectively with respect to the metric $g_\varepsilon(t)$.

If $\varphi_\varepsilon(t)$ is a solution to the Monge-Ampère equation

$$\frac{\partial\varphi_\varepsilon(t)}{\partial t} = \log\frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon(t))^n}{\omega_0^n} + F_0 + \beta\varphi_\varepsilon(t) + \log(\varepsilon^2 + |s|_h^2)^{1-\beta} \tag{4.11}$$

on $(0,\infty)\times M$ with initial value $\varphi_\varepsilon(0) = \varphi_0$, it is obvious that $\phi_\varepsilon(t) = \varphi_\varepsilon(t) + Ce^{\beta t}$ is a solution to equation (4.11) with initial value $\phi_\varepsilon(0) = \varphi_0 + C$. At the same time, $\omega_{\phi_\varepsilon(t)} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi_\varepsilon(t)$ is also a solution to the twisted Kähler-Ricci flow (4.1) with initial value ω_{φ_0}.

From (3.1), we know that $\varphi_\varepsilon(t)$ is uniformly bounded on $[0,T]\times M$ by a constant C which depends only on $\|\varphi_0\|_{L^\infty(M)}$, β and T. Now, we consider the solution $\psi_\varepsilon(t) = \varphi_\varepsilon(t) + \tilde{C}_{\varepsilon,1}e^{\beta t}$ to the equation

$$\begin{cases} \frac{\partial\psi_\varepsilon(t)}{\partial t} = \log\frac{(\omega_0+\sqrt{-1}\partial\bar{\partial}\psi_\varepsilon(t))^n}{\omega_0^n} + F_0 + \beta\psi_\varepsilon(t) + \log(\varepsilon^2 + |s|_h^2)^{1-\beta} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad on\quad (0,\infty)\times M, \\ \psi_\varepsilon(0) = \varphi_0 + \tilde{C}_{\varepsilon,1} \end{cases} \tag{4.12}$$

where

$$\tilde{C}_{\varepsilon,1} = e^{-\beta}\frac{1}{\beta}\Big(\int_1^{+\infty} e^{-\beta t}\|\nabla u_\varepsilon(t)\|_{L^2}^2 dt - \frac{1}{V}\int_M \big(F_{\varepsilon,1} + \beta\varphi_\varepsilon(1)\big)dV_{\varepsilon,1}\Big),$$

$F_{\varepsilon,1} = F_0 + \log(\frac{\omega_\varepsilon(1)^n}{\omega_0^n}\cdot(\varepsilon^2 + |s|_h^2)^{1-\beta})$ and $dV_{\varepsilon,1} = \frac{\omega_\varepsilon^n(1)}{n!}$. By (3.1), (3.2) and the above uniform Perelman's estimates (4.9), we know that the constant $\tilde{C}_{\varepsilon,1}$ is well-defined and uniformly bounded. Straightforward calculation shows that the twisted Ricci potential $u_\varepsilon(1)$ with respect to $\omega_\varepsilon(1)$ can be written as

$$u_\varepsilon(1) = \log\frac{\omega_\varepsilon^n(1)(\varepsilon^2 + |s|_h^2)^{1-\beta}}{\omega_0^n} + F_0 + \beta\varphi_\varepsilon(1) + C_{\varepsilon,1}, \tag{4.13}$$

where $C_{\varepsilon,1}$ is a normalization constant such that $\frac{1}{V}\int_M e^{-u_\varepsilon(1)}dV_{\varepsilon,1}=1$. Then

$$C_{\varepsilon,1}=\log\Big(\frac{1}{V}\int_M e^{-F_0-\beta\varphi_\varepsilon(1)}\frac{dV_0}{(\varepsilon^2+|s|_h^2)^{1-\beta}}\Big). \tag{4.14}$$

By (3.1) and (3.2), we conclude that $C_{\varepsilon,1}$ and $u_\varepsilon(1)$ are uniformly bounded.

Let $u_\varepsilon(t)=\dot{\psi}_\varepsilon(t)+c_\varepsilon(t)$. By equation (4.12) and equality (4.13), we have

$$c_\varepsilon(1)=C_{\varepsilon,1}-\beta e^\beta\tilde{C}_{\varepsilon,1}. \tag{4.15}$$

Proposition 4.5. *There exists a uniform constant C such that*

$$\|\dot{\psi}_\varepsilon(t)\|_{C^0}\leq C$$

for any $\varepsilon>0$ and $t\geq 1$.

Proof: As in [37], when $t\geqslant 1$, we let

$$\alpha_\varepsilon(t)=\frac{1}{V}\int_M\dot{\psi}_\varepsilon(t)dV_{\varepsilon,t}=\frac{1}{V}\int_M u_\varepsilon(t)dV_{\varepsilon,t}-c_\varepsilon(t). \tag{4.16}$$

Through computing, we have

$$\begin{aligned}\frac{d}{dt}\alpha_\varepsilon(t) &= \beta\alpha_\varepsilon(t)-\|\nabla\dot{\psi}_\varepsilon\|_{L^2}^2,\\ e^{-\beta(t-1)}\alpha_\varepsilon(t) &= \alpha_\varepsilon(1)-\int_1^t e^{-\beta(s-1)}\|\nabla\dot{\psi}_\varepsilon\|_{L^2}^2ds\\ &= \frac{1}{V}\int_M u_\varepsilon(1)dV_{\varepsilon,1}-c_\varepsilon(1)-\int_1^t e^{-\beta(s-1)}\|\nabla\dot{\psi}_\varepsilon\|_{L^2}^2ds. \qquad (4.17)\end{aligned}$$

Putting (4.13) and (4.15) into (4.17), we have

$$\begin{aligned}e^{-\beta(t-1)}\alpha_\varepsilon(t) &= \frac{1}{V}\int_M F_{\varepsilon,1}+\beta\varphi_\varepsilon(1)dV_{\varepsilon,1}+C_{\varepsilon,1}-c_\varepsilon(1)-\int_1^t e^{-\beta(s-1)}\|\nabla\dot{\psi}_\varepsilon\|_{L^2}^2ds\\ &= \frac{1}{V}\int_M F_{\varepsilon,1}+\beta\varphi_\varepsilon(1)dV_{\varepsilon,1}+\beta e^\beta\tilde{C}_{\varepsilon,1}-\int_1^t e^{-\beta(s-1)}\|\nabla\dot{\phi}_\varepsilon\|_{L^2}^2ds\\ &= \frac{1}{V}\int_M F_{\varepsilon,1}+\beta\varphi_\varepsilon(1)dV_{\varepsilon,1}-\int_1^t e^{-\beta(s-1)}\|\nabla\dot{\psi}_\varepsilon\|_{L^2}^2ds\\ &\quad+\int_1^{+\infty}e^{-\beta(t-1)}\|\nabla u_\varepsilon(t)\|_{L^2}^2dt-\frac{1}{V}\int_M\big(F_{\varepsilon,1}+\beta\varphi_\varepsilon(1)\big)dV_{\varepsilon,1}\\ &= \int_t^{+\infty}e^{-\beta(s-1)}\|\nabla\dot{\psi}_\varepsilon\|_{L^2}^2ds.\end{aligned}$$

By Theorem 4.4, we conclude that

$$0\leq\alpha_\varepsilon(t) = \int_t^{+\infty}e^{\beta(t-s)}\|\nabla\dot{\psi}_\varepsilon\|_{L^2}^2ds\leq C. \tag{4.18}$$

Then we conclude that $\dot{\psi}_\varepsilon(t)$ is uniformly bounded by the uniform Perelman's estimates when $t\geqslant 1$. □

We recall Aubin's functionals, Ding's functional and the twisted Mabuchi $\mathcal{K}$-energy functional.

$$(4.19)\qquad \begin{aligned} I_{\omega_0}(\phi) &= \frac{1}{V}\int_M \phi(dV_0 - dV_\phi), \\ J_{\omega_0}(\phi) &= \frac{1}{V}\int_0^1\int_M \dot{\phi}_t(dV_0 - dV_{\phi_t})dt, \end{aligned}$$

where ϕ_t is a path with $\phi_0 = c$, $\phi_1 = \phi$.

$$(4.20)\qquad F^0_{\omega_0}(\phi) = J_{\omega_0}(\phi) - \frac{1}{V}\int_M \phi dV_0,$$

$$(4.21)\qquad F_{\omega_0,\theta}(\phi) = J_{\omega_0}(\phi) - \frac{1}{V}\int_M \phi dV_0 - \frac{1}{\beta}\log(\frac{1}{V}\int_M e^{-u_{\omega_0}-\beta\phi}dV_0),$$

$$(4.22)\qquad \begin{aligned} \mathcal{M}_{\omega_0,\ \theta}(\phi) &= -\beta(I_{\omega_0}(\phi) - J_{\omega_0}(\phi)) - \frac{1}{V}\int_M u_{\omega_0}(dV_0 - dV_\phi) \\ &\quad +\frac{1}{V}\int_M \log\frac{\omega_\phi^n}{\omega_0^n}dV_\phi, \end{aligned}$$

where u_{ω_0} is the twisted Ricci potential of ω_0, *i.e.* $-Ric(\omega_0)+\beta\omega_0+\theta = \sqrt{-1}\partial\bar{\partial}u_{\omega_0}$ and $\frac{1}{V}\int_M e^{-u_{\omega_0}}dV_{\omega_0} = 1$.

Proposition 4.6. *For any $t \geq 1$, the solution $\psi_\varepsilon(t)$ to equation (4.12) satisfies:*

$$(i)\qquad \mathcal{M}_{\omega_0,\ \theta_\varepsilon}(\psi_\varepsilon(t)) - \beta F^0_{\omega_0}(\psi_\varepsilon(t)) - \frac{1}{V}\int_M \dot{\psi}_\varepsilon(t)dV_{\varepsilon,t} = C_\varepsilon,$$

$$(ii)\qquad \mathcal{M}_{\omega_0,\ \theta_\varepsilon}(\psi_\varepsilon(1))\ \textit{is uniformly bounded},$$

where C_ε in (i) can be bounded by a uniform constant C.

Proof: Following the argument in [34], since

$$(4.23)\qquad \frac{d}{dt}(\mathcal{M}_{\omega_0,\ \theta_\varepsilon}(\psi_\varepsilon(t)) - \beta F^0_{\omega_0}(\psi_\varepsilon(t)) - \frac{1}{V}\int_M \dot{\psi}_\varepsilon(t)dV_{\varepsilon,t}) = 0,$$

we obtain that

$$\begin{aligned} &\mathcal{M}_{\omega_0,\ \theta_\varepsilon}(\psi_\varepsilon(t)) - \beta F^0_{\omega_0}(\psi_\varepsilon(t)) - \frac{1}{V}\int_M \dot{\psi}_\varepsilon(t)dV_{\varepsilon,t} \\ =\ &\mathcal{M}_{\omega_1,\ \theta_\varepsilon}(\psi_\varepsilon(1)) - \beta F^0_{\omega_0}(\psi_\varepsilon(1)) - \frac{1}{V}\int_M \dot{\psi}_\varepsilon(1)dV_{\varepsilon,1} \\ =\ &\frac{1}{V}\int_M \log\frac{\omega_\varepsilon^n(1)(|s|_h^2+\varepsilon^2)^{1-\beta}}{e^{-F_0}\omega_0^n}dV_{\varepsilon,1} + \frac{\beta}{V}\int_M \psi_\varepsilon(1)dV_{\varepsilon,1} \\ &-\frac{1}{V}\int_M F_0 + \log(|s|_h^2+\varepsilon^2)^{1-\beta}dV_0 - \frac{1}{V}\int_M \dot{\psi}_\varepsilon(1)dV_{\varepsilon,1}. \end{aligned}$$

where the last equality can be bounded by a uniform constant. This gives a proof of (i). Furthermore, by the definition of $\mathcal{M}_{\omega_0,\ \theta_\varepsilon}$, we have

$$\begin{aligned} \mathcal{M}_{\omega_0,\ \theta_\varepsilon}(\psi_\varepsilon(1)) &= \frac{1}{V}\int_M \log\frac{\omega_\varepsilon^n(1)(|s|_h^2+\varepsilon^2)^{1-\beta}}{e^{-F_0}\omega_0^n}dV_{\varepsilon,1} - \beta I_{\omega_0}(\psi_\varepsilon(1)) + \beta J_{\omega_0}(\psi_\varepsilon(1)) \\ &\quad -\frac{1}{V}\int_M F_0 + \log(|s|_h^2+\varepsilon^2)^{1-\beta}dV_0. \end{aligned}$$

Since $I_{\omega_0}(\psi_\varepsilon(1))$ is uniformly bounded and $\frac{1}{n}J_{\omega_0} \leq \frac{1}{n+1}I_{\omega_0} \leq J_{\omega_0}$, we prove (ii). $\square$

Using Proposition 4.5 and 4.6, by following the arguments in [34] (see section 5), we obtain the following uniform C^0 estimate of $\psi_\varepsilon(t)$ along the equation (4.12) under the assumption that the twisted Mabuchi $\mathcal{K}$-energy functional $\mathcal{M}_{\omega_0,\ \theta_\varepsilon}$ is uniformly proper on the space

$$\mathcal{H}(\omega_0) = \{\phi \in C^\infty(M) |\ \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi > 0\}. \tag{4.24}$$

Theorem 4.7. *Let $\psi_\varepsilon(t)$ be a solution of the flow* (4.12)*. If the twisted Mabuchi $\mathcal{K}$-energy functional $\mathcal{M}_{\omega_0,\ \theta_\varepsilon}$ is uniformly proper on $\mathcal{H}(\omega_0)$, i.e. there exists a uniform function f such that*

$$\mathcal{M}_{\omega_0,\ \theta_\varepsilon}(\phi) \geq f(J_{\omega_0}(\phi)) \tag{4.25}$$

for any ε and $\phi \in \mathcal{H}(\omega_0)$, where $f(t) : \mathbb{R}^+ \to \mathbb{R}$ is some monotone increasing function satisfying $\lim\limits_{t\to+\infty} f(t) = +\infty$, then there exists a uniform constant C such that for any $\varepsilon > 0$ and $t \geq 0$

$$\|\psi_\varepsilon(t)\|_{C^0} \leq C. \tag{4.26}$$

Since we study the flow (4.1) start at $t = \frac{1}{2}$ and obtain

$$C^{-1}\omega_\varepsilon \leqslant \omega_\varepsilon(\frac{1}{2}) \quad \leqslant \quad C\omega_\varepsilon \quad on \quad M, \tag{4.27}$$

$$\|\psi_\varepsilon(\frac{1}{2})\|_{C^k(K)} \quad \leqslant \quad C_{k,K} \quad on \quad K \subset\subset M \setminus D, \tag{4.28}$$

for some uniform constants C and $C_{k,K}$ in section 3, after getting the uniform bound of $\dot{\psi}_\varepsilon(t)$ and $\psi_\varepsilon(t)$, we can prove the uniform Laplacian C^2 estimates and local high order uniform estimates for any $t \geqslant 1$ and $\varepsilon > 0$ by the arguments in [34] (see Proposition 2.1 and 2.3 in [34]). In fact, we prove the following theorem.

Theorem 4.8. *Under the assumption in Theorem 4.7, for any $k \in \mathbb{N}^+$ and $K \subset\subset M \setminus D$, there exists constant $C_{k,K}$ depending only on $\|\varphi_0\|_{L^\infty(M)}$, n, β, k, ω_0 and $dist_{\omega_0}(K, D)$, such that for any $\varepsilon > 0$ and $t \geqslant 1$, we have*

$$\|\psi_\varepsilon(t)\|_{C^k(K)} \leq C_{k,K}. \tag{4.29}$$

Now we assume that there exists a conical Kähler-Einstein metric with cone angle $2\pi\beta$ along D. When $\lambda > 0$ and there is no nontrivial holomorphic field which is tangent to D along D, G.Tian and X.H. Zhu [45] obtained the following Moser-Trudinger type inequality

$$F_{\omega_0,(1-\beta)D}(\phi) \geqslant \delta J_{\omega_0}(\phi) - C, \qquad \forall \phi \in \mathcal{H}(\omega_0) \tag{4.30}$$

for some constants δ and C, where

$$F_{\omega_0,(1-\beta)D}(\phi) = J_{\omega_0}(\phi) - \frac{1}{V}\int_M \phi dV_0 - \frac{1}{\beta}\log\big(\frac{1}{V}\int_M \frac{1}{|s|_h^{2(1-\beta)}} e^{-F_0-\beta\phi} dV_0\big)$$

is defined in [45] (see also [30]).

Remark 4.9. *When $\lambda \geqslant 1$, R. Berman* [1]*, C. Li and S. Sun* [30] *proved that there is no nontrivial holomorphic vector field on M tangent to divisor D, and Li-Sun also proved that the existence of conical Kähler-Einstein metric can deduce the properness of the Log Mabuchi $\mathcal{K}$-energy functional (see also J. Song and X.W. Wang's results in* [42]*).*

By the definition of $F_{\omega_0,\theta_\varepsilon}$ and $F_{\omega_0,(1-\beta)D}$, we have

$$\begin{aligned} F_{\omega_0,\theta_\varepsilon}(\phi) - F_{\omega_0,(1-\beta)D}(\phi) &= \frac{1}{\beta}\log\Big(\frac{1}{V}\int_M e^{-F_0-C_0-\beta\phi}\frac{dV_0}{|s|_h^{2(1-\beta)}}\Big) \\ &\quad -\frac{1}{\beta}\log\Big(\frac{1}{V}\int_M e^{-F_0-C_\varepsilon-\beta\phi}\frac{dV_0}{(\varepsilon^2+|s|_h^2)^{(1-\beta)}}\Big) \\ &\geq -C, \end{aligned} \tag{4.31}$$

where C_0 and C_ε are two normalized constants, and C is a constant independent of ε. So the Ding's functional F_{ω_0} is uniform proper. By the normalization and Jensen's inequality, we have

$$\frac{1}{V}\int_M -u_{\omega_\phi}dV_\phi \leq \log\big(\frac{1}{V}\int_M e^{-u_{\omega_\phi}}dV_\phi\big) = 0. \tag{4.32}$$

Then we have the following inequalities by (4.21), (4.22) and (4.32).

$$\begin{aligned} \mathcal{M}_{\omega_0,\ \theta_\varepsilon}(\phi) &= \beta F_{\omega_0,\theta_\varepsilon}(\phi) + \frac{1}{V}\int_M u_{\omega_\phi}dV_\phi - \frac{1}{V}\int_M u_{\omega_0}dV_0 \\ &\geq \beta F_{\omega_0,\theta_\varepsilon}(\phi) - \frac{1}{V}\int_M F_0 + C_\varepsilon + (1-\beta)\log(\varepsilon^2+|s|_h^2)dV_0 \\ &\geq \beta F_{\omega_0,\theta_\varepsilon}(\phi) - C, \end{aligned} \tag{4.33}$$

where constant C independent of ε. Hence we deduce the uniform properness of the twisted Mabuchi $\mathcal{K}$-energy functional by (4.30), (4.31) and (4.33), i.e.

$$M_{\omega_0,\ \theta_\varepsilon}(\phi) \geqslant C_1 J_{\omega_0}(\phi) - C_2, \qquad \forall\phi\in\mathcal{H}(\omega_0) \tag{4.34}$$

for some uniform constants C_1 and C_2. At the same time, we have the uniqueness theorem of conical Kähler-Einstein metric (proved by B. Berndtsson in [2]) under the assumption that there is no nontrivial holomorphic field which is tangent to D. Using the above C^0 estimate and the uniqueness theorem, we can apply the arguments in [34] (see section 6) to obtain the convergence result of the conical Kähler-Ricci flow (1.5), i.e. Theorem 1.3.

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