EXPONENTIAL MIXING FOR GENERIC VOLUME-PRESERVING ANOSOV FLOWS IN DIMENSION THREE

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Abstract. Let M be a closed 3-dimensional Riemann manifold and let $3 \le r \le \infty$. We prove that there exists an open dense subset in the space of C^r volume-preserving Anosov flows on M such that all the flows in it are exponentially mixing.

1. Introduction

We consider mixing property of volume-preserving Anosov flows on a 3-dimensional closed C^∞ Riemann manifold M. Let $\mathfrak{F}^r_A$ be the space of C^r Anosov flows on M preserving the Riemann volume m and suppose that it is equipped with the C^r compact-open topology as a subspace of $C^r(M \times \mathbb{R}, M)$. A flow $f^t \in \mathfrak{F}^r_A$ is said to be *exponentially mixing* with respect to the volume m if

$$\int \varphi \cdot (\psi \circ f^t)\, dm \le C_\alpha \|\varphi\|_{C^\alpha} \|\psi\|_{C^\alpha} \exp(-c_\alpha t) \tag{1}$$

for any $\varphi, \psi \in C^\alpha(M)$ with[1] $\alpha > 0$ satisfying $\int \varphi\, dm = 0$, where c_α and $C_\alpha > 0$ are constants independent of φ and ψ. In this paper, we prove the following theorem:

Theorem 1.1. *For $3 \le r \le \infty$, there exists a C^3-open and C^r-dense subset $\mathcal{U} \subset \mathfrak{F}^r_A$ such that all the flows in $\mathcal{U}$ are exponentially mixing. Further, for each flow f^t in $\mathcal{U}$, there exists a C^3-open neighborhood of f^t such that the decay estimate (1) holds true for all the flows in it with uniform constants C_α and c_α.*

By Anosov alternative[1, 15], any volume-preserving Anosov flow is either mixing or topologically conjugate to a suspension flow of an Anosov diffeomorphism with a constant roof function. And the former alternate holds for an open dense subset in the space of volume-preserving Anosov flows. In this paper, we study a related problem: whether exponential mixing is an open dense property for volume-preserving Anosov flows. A few important progresses related this problem (in more general context) were made by Chernov[4] and Dolgopyat[6, 7, 8] in late 1990's. In [6], Dolgopyat proved that a volume-preserving Anosov flow is exponentially mixing if the stable and unstable foliations are C^1 and are not jointly integrable. In particular, it is proved in [6] that the geodesic flows on negatively curved surfaces are exponentially mixing. (Later Liverani[12] extended this result to general contact Anosov flows. See also [18, 19].) In [7] and [8], he also studied exponential and rapid (*i.e.* super-polynomial) mixing for suspension flows of subshifts of

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[1]Once the decay estimate (1) holds for some $\alpha > 0$, we can prove (1) for any $\alpha > 0$ by approximation, possibly with different constants. See [8, p.1046]. It is therefore enough to consider (1) for some fixed $\alpha > 0$, say $\alpha = 1$.

finite type, which abstracts Axiom A flow, and gave several criteria for such flows to be rapid and exponential mixing. Based on the argument in [7], Field, Melbourne and Török proved more recently in [11] that rapid mixing is an open dense property for Axiom A flows and consequently for volume-preserving Anosov flows. However, to the author's knowledge, the problem on exponential mixing mentioned above remains open. The aim of this paper is to study the problem in the simplest possible setting of dimension 3 and present an affirmative answer in Theorem 1.1. Theorem 1.1 also provides an example of a non-empty open set of volume-preserving Anosov flows which stably exhibit exponential mixing. (Rather surprisingly, no such example has been known. But see [2] for such an example for Axiom A flow.)

In the following, we first investigate the geometry of the stable and unstable foliation and introduce the notion of *s-template* which describes how the stable subbundle twists along unstable manifolds. We then formulate, in Definition 2.9, the non-integrability condition $(NI)_\rho$ for $\rho > 0$ in terms of s-templates. We show, in Theorem 2.12, that the condition $(NI)_\rho$ for sufficiently small $\rho > 0$ holds for a C^r dense subset in $\mathfrak{F}^r_A$ for any $r \geq 3$. Then we prove Theorem 1.1 by showing in Theorem 2.13 that, if $f^t_0 \in \mathfrak{F}^3_A$ satisfies the non-integrability condition $(NI)_\rho$ for some $\rho > 0$, then there exists a C^3 open neighborhood $\mathcal{V}$ of f^t_0 in $\mathfrak{F}^3_A$ in which all the flows are exponentially mixing with uniform constants c_α and C_α in the decay estimate (1).

The main novelty in this paper is in the argument related to s-template presented in Section 2. Also the argument in the proof of Theorem 2.12, presented in Section 3, may be somewhat new, where we consider deformation families of a flow with huge number of parameters and apply large deviation argument in the parameter spaces. The outline of the proof of Theorem 2.13 is parallel to those in the previous papers[18, 19], but for a few points where we use the non-integrability condition $(NI)_\rho$ and show quasi-compactness of the transfer operators on some Hilbert space of distributions. It occupies almost two thirds of this paper in length. This is mainly because some objects we consider is not smooth and require careful treatment when we apply analytic tools.

Remark 1.2. The argument presented in this paper depends crucially on the assumption that M is three dimensional and we do not expect that it will extend to more general cases directly. But the author would like to emphasize that our argument is based on the following observation which may be useful in much more general cases of partially hyperbolic dynamical systems: *Twist of the stable subbundle along a piece of unstable manifold viewed in the unit scale will be "random" and "rough" in generic cases and such twist will not be cancelled completely in the process where the flow f^t contracts the piece of unstable manifold to microscopic scale as $t \to -\infty$ (if we view things in an appropriate scaling); this leads to a kind of joint non-integrability between the stable and unstable foliations, which is uniform in microscopic scales and is somewhat stable in perturbation.* See also Remark 2.16.

2. The non-integrability condition

Below we suppose $3 \leq r \leq \infty$ and consider a C^r Anosov flow $f^t : M \to M$ preserving a C^r volume μ on M, which may not be the Riemann volume m. Let v be the C^{r-1} vector field generating the flow f^t. Since the argument below does not depend on the choice of the Riemann metric $\|\cdot\|$ on M essentially, we may and do assume that $\|v\| \equiv 1$ without loss of generality.

In some parts of our argument, we will need to check that some constants can be taken uniformly for the flows in a sufficiently small C^r open neighborhood of f^t that preserve C^r

volume forms sufficiently close to μ. To this end, we put the subscript $*$ to the symbols of them and use C_* as a generic symbol for such constants. Also we will write $\mathcal{O}_*(\cdot)$ for a term which is bounded in absolute value by the quantity inside the parenthesis multiplied by some constant C_*.

2.1. **Anosov flows.** From the definition of Anosov flow, there is an f^t-invariant continuous decomposition of the tangent bundle

$$TM = E_0 \oplus E_s \oplus E_u \quad \text{with} \quad \dim E_0 = \dim E_s = \dim E_u = 1 \tag{2}$$

such that $E_0 = \langle v \rangle$ and, for some positive constants $C_* > 0$ and $\chi_* > 0$,

$$|Df^t_x|_{E_s}| \le C_* e^{-\chi_* t}, \quad |Df^t_x|_{E_u}| \ge C_*^{-1} e^{\chi_* t} \quad \text{for all } t \ge 0. \tag{3}$$

The decomposition dual to (2) is $T^*M = E_0^* \oplus E_s^* \oplus E_u^*$ where

$$E_0^* = (E_s \oplus E_u)^\perp, \quad E_s^* = (E_u \oplus E_0)^\perp, \quad E_u^* = (E_s \oplus E_0)^\perp.$$

The distribution E_0 is C^{r-1}, but E_s and E_u are not even C^1 in general. Note however that we have

$$\angle(E_s(p), E_s(q)) \le C_* |p-q| \cdot \langle \log \langle |p-q| \rangle \rangle \tag{4}$$

in local charts[2]. This is same for E_u and E_0^*.

Remark 2.1. The non-smoothness of E_s and E_u mentioned above is caused by their variation in the flow direction. Indeed, the subbundles $E_u^* = (E_s \oplus E_0)^\perp$ and $E_s^* = (E_u \oplus E_0)^\perp$ are C^1 and we have

$$\angle(E_u^*(p), E_u^*(q)) \le C_* |p-q| \tag{5}$$

in local charts and the same for E_s^*.

2.2. **The intrinsic metric on stable and unstable manifolds.** Let $W^s(p)$ and $W^u(p)$ be the stable and unstable manifolds passing through a point $p \in M$. Below we discuss about twist of the stable subbundle E_s along the unstable manifold $W^u(p)$. Note that we can develop the parallel argument about twist of the unstable subbundle E_u along the stable manifold $W^s(p)$ by considering the time-reversal of the flow f^t.

We define a C^{r-1} metric on $W^u(p)$ by

$$|v|_{W^u(p)} = \lim_{t \to -\infty} \log \frac{\|Df^t_q(v)\|}{\|Df^t_p|_{E_u}\|} \qquad \text{for } v \in T_q W^u(p) \text{ at } q \in W^u(p) \tag{6}$$

where $\|\cdot\|$ denotes the Riemann metric.

Lemma 2.2. *If f^t sends $W^u(p)$ to $W^u(p')$, it brings the metric $|\cdot|_{W^u(p)}$ to $|\cdot|_{W^u(p')}$ up to multiplication by a positive constant. If $f^t(p) = p'$, the multiplier is just $\|Df^t(p)|_{E^u}\|$.*

Let $w_p^u : \mathbb{R} \to M$ be the C^r parametrization of $W^u(p)$ by the arc length with respect to the metric $|\cdot|_{W^u(p)}$ such that $w_p^u(0) = p$. (We do not care about the direction of the parametrization.) For an open interval $J \subset \mathbb{R}$, let $W_J^u(p) := w_p^u(J) \subset W^u(p)$.

[2]It is of course possible to formulate (4) without using local charts by introducing a parallel transport.

2.3. **Sections of normal bundle of stable manifolds.** For a point $p \in M$ and an interval $J \subset \mathbb{R}$, let $\Gamma^u(p, J)$ be the space of continuous sections $\gamma : W^u_J(p) \to T^*M$ of the cotangent bundle $\pi : T^*M \to M$ on $W^u_J(p)$ such that $\gamma(q) \in T^*_q M$ is normal to the tangent space $T_q W^u(p)$ at each $q \in W^u_J(p)$. Let $\Gamma^u_1(p, J) \subset \Gamma^u(p, J)$ be the subset that consists of $\gamma \in \Gamma^u(p, J)$ satisfying $\langle \gamma(q), v(q)\rangle = 1$ at each $q \in W^u_J(p)$.

Let $\gamma^\perp_{p,J} \in \Gamma^u(p, J)$ be either of the two C^{r-1} sections such that

$$\langle \gamma^\perp_{p,J}(q), u\rangle = \pm\mu(v(q), (w^u_p)'(\tau), u) \quad \text{for any } u \in T^*_q M \text{ at each } q = w^u_p(\tau) \in W^u_J(p).$$

Also we tentatively[3] fix a C^{r-1} section $\gamma^0_{p,J} \in \Gamma^u_1(p, J)$. At this moment we assume only that the sections $\gamma^0_{p,J}$ are bounded in C^{r-1} sense uniformly for p and J with $J \subset (-1, 1)$. We may then express each section $\gamma \in \Gamma^u_1(p, J)$ as

$$\gamma(q) = \gamma^0_{p,J}(q) + \psi_\gamma(\tau) \cdot \gamma^\perp_{p,J}(q) \quad \text{for } \tau \in J \text{ with } q = w^u_p(\tau) \tag{7}$$

where $\psi_\gamma : J \to \mathbb{R}$ is a continuous section. The last function ψ_γ is called the representation function of $\gamma \in \Gamma^u_1(p, J)$. We define the (maximum) curvature $\kappa(\gamma)$ of $\gamma \in \Gamma^u_1(p, J)$ by

$$\kappa(\gamma) = \sup\{|\psi''_\gamma(\tau)| \mid \tau \in J\}.$$

This definition depends on the choice of the sections $\gamma^0_{p,J}$ and hence the value of $\kappa(\gamma)$ itself does not make good sense.

For a C^{r-1} section $\gamma \in \Gamma^u_1(p, J)$ and $t \in \mathbb{R}$, there is a unique section $\gamma_t \in \Gamma^u_1(f^t(p), J(t))$ with $J(t) = \pm|Df^t|_{E_u}(p)| \cdot J$ such that $f^t(W^u_J(p)) = W^u_{J(t)}(f^t(p))$ and $\gamma(q) = (Df^t)^*\gamma_t(f^t(q))$. The curvature $\kappa(\gamma_t)$ of γ_t tends to infinity as $t \to -\infty$ in most cases, but may be bounded for some γ.

Definition 2.3. A C^{r-1} section $\gamma \in \Gamma^u_1(p, J)$ is said to be *straight* if $\kappa(\gamma_t)$ is bounded for $\forall t \le 0$.

Notice that this definition does not depend on the choice of the sections $\gamma^0_{p,J}$, by virtue of the boundedness assumption we made on their choice. To describe the space of straight sections, we introduce the following definition.

Definition 2.4. C^{r-1} functions $\psi_0, \psi_1 : J \to \mathbb{R}$ are said to be *A-equivalent* if $(\psi_0 - \psi_1)''(\tau) = 0$ for all $\tau \in J$. C^{r-1} sections $\gamma_0, \gamma_1 \in \Gamma^u_1(p, J)$ are said to be *A-equivalent* if their representation functions ψ_{γ_0} and ψ_{γ_1} (defined above in (7)) are A-equivalent.

Lemma 2.5. *For any point $p \in M$ and any interval $J \subset \mathbb{R}$, there exists a straight section $\gamma_0 \in \Gamma^u_1(p, J)$. A C^{r-1} section $\gamma \in \Gamma^u_1(p, J)$ is straight if and only if it is A-equivalent to γ_0. If a C^{r-1} section $\gamma \in \Gamma^u_1(p, J)$ is straight, then $\gamma_t \in \Gamma^u_1(f^t(p), J(t))$ is again straight.*

Proof. If f^t sends a C^{r-1} section $\gamma \in \Gamma^u_1(p, J)$ to $\gamma_t \in \Gamma^u_1(p(t), J(t))$, the expression functions of γ_t is related to that of γ as

$$\psi_{\gamma_t}(\tau) = a(t)\psi_\gamma(a(t)^{-1}\tau) + \tilde{\psi}_{p,t}(\tau), \qquad a(t) = \pm|Df^t|_{E_u}(p)| \tag{8}$$

where $\tilde{\psi}_{p,t}$ is a C^{r-1} function. For any $t_0 > 0$, the C^{r-1} norm of the function $\tilde{\psi}_{p,t}$ is bounded uniformly for $p \in M$ and $t \in \mathbb{R}$ with $|t| \le t_0$. Differentiating the both sides of (8) with respect to τ twice and changing the variable τ, we obtain

$$\psi''_\gamma(\tau) = a(t)\psi''_{\gamma_t}(a(t)\tau) + a(t)\tilde{\psi}''_{p,t}(a(t)\tau). \tag{9}$$

Note that $a(t) \to 0$ exponentially as $t \to -\infty$. By recursive application of (9) for $0 \le t \le t_0$, we see that the right hand side of (9) converges to a unique C^{r-3} function φ as $t \to -\infty$

[3] Later we will choose it more carefully at the end of this section.

provided that $\kappa(\gamma_t) = \|\psi''_{\gamma_t}\|_\infty$ is bounded for $t \le 0$. Hence a C^{r-1} section $\gamma \in \Gamma^u_1(p, J)$ is straight if and only if $\psi''_\gamma = \varphi$. This implies the former two statements. The last statement is an immediate consequence of the definition. □

Since the choice of the sections $\gamma^0_{p,J}$ of reference was rather arbitrary, we henceforth assume without loss of generality that the sections $\gamma^0_{p,J}$ are straight sections. (This is just for avoiding to introduce a new notation.) Further, in the case $J = (-1, 1)$, we specify $\gamma^0_{p,J}$ as the unique straight section satisfying the condition

$$\gamma^0_{p,(-1,1)}(w^u_p(\tau)) = E^*_0(w^u_p(\tau)) \quad \text{for } \tau = \pm 1. \tag{10}$$

2.4. **The definition of s-templates.** Let $\gamma^s_{p,J} \in \Gamma^u_1(p, J)$ be the unique continuous section such that $\gamma^s_{p,J}(q) \in E^*_0(q)$ for $q \in W^u_J(p)$ and let $\psi^s_{p,J} : J \to \mathbb{R}$ be its representation function. (The superscript s in $\gamma^s_{p,J}$ indicates that it represents the direction of E^s.) Note that $\psi^s_{p,J}$ is not even C^1 in general but satisfies

$$|\psi^s_{p,J}(\tau') - \psi^s_{p,J}(\tau)| \le C_*|\tau' - \tau| \cdot \langle \log|\tau' - \tau|\rangle \quad \text{for } \tau, \tau' \in J \tag{11}$$

as a consequence of (4).

Definition 2.6. The functions $\psi^s_{p,(-1,1)}$ for $p \in M$ are called the *s-templates* for the flow f^t. We write $\mathcal{T} = \mathcal{T}(f^t) = \{\psi^s_{p,(-1,1)} \mid p \in M\}$ for the set of s-templates for the flow f^t.

The reason for the name "template" can be found in the next lemma.

Lemma 2.7. *For any $q \in M$ and any $\delta \in (0, 1)$, there exists $t > 0$ such that*

$$\psi^s_{q,(-\delta,\delta)}(\tau) = \delta \cdot \psi^s_{p,(-1,1)}(\delta^{-1}\tau) + \alpha\tau + \beta \qquad \text{with} \quad p = f^t(q) \tag{12}$$

where $|\alpha| \le C_*(|\log\delta| + 1)$ and $|\beta| \le C_*$. In particular, $\psi^s_{q,(-\delta,\delta)}$ is A-equivalent to the function $\tau \mapsto \delta \cdot \psi^s_{p,(-1,1)}(\delta\tau)$ obtained from $\psi^s_{p,(-1,1)}$ by a scaling.

Proof. Let $\delta < \delta' \le 1$ and take $t > 0$ such that $f^t(W^u_{(-\delta,\delta)}(q)) = W^u_{(-\delta',\delta')}(p)$ with $p = f^t(q)$. Let $\tilde\gamma^0_{q,(-\delta,\delta)}$ be the pull-back of $\gamma^0_{p,(-1,1)}$ by f^t and let $\tilde\psi^0_{q,(-\delta,\delta)}$ be its representation function. Then, from (8), we have

$$\psi^s_{q,(-\delta,\delta)}(\tau) = (\delta/\delta') \cdot \psi^s_{p,(-\delta',\delta')}((\delta'/\delta)\tau) + \tilde\psi^0_{q,(-\delta,\delta)}(\tau). \tag{13}$$

Since $\tilde\gamma^0_{q,(-\delta,\delta)}$ is straight, $\tilde\psi^0_{q,(-\delta,\delta)}(\tau)$ is an affine function of τ. We therefore obtain (12) as the case $\delta' = 1$. The estimates on α and β are obtained by recursive application of (13) and the fact that $\tilde\psi^0_{q,(-\delta,\delta)}$ is bounded if the ratio δ'/δ is bounded. □

Remark 2.8. As we noted in the beginning of Subsection 2.2, we can develop the argument above for the time-reversal of f^t in parallel. The objects corresponding to

$$|\cdot|_{W^u(p)}, \quad W^u_J(p), \quad w^u_p(\cdot), \quad \Gamma^u(p, J), \quad \Gamma^u_1(p, J), \quad \gamma^\perp_{p,J}, \quad \gamma^0_{p,J}, \quad \gamma^s_{p,J}, \quad \psi^s_{p,J}$$

in such argument will be denoted respectively by

$$|\cdot|_{W^s(p)}, \quad W^s_J(p), \quad w^s_p(\cdot), \quad \Gamma^s(p, J), \quad \Gamma^s_1(p, J), \quad \hat\gamma^\perp_{p,J}, \quad \hat\gamma^0_{p,J}, \quad \gamma^u_{p,J}, \quad \psi^u_{p,J}.$$

2.5. **The non-integrability condition.** Now we put the following definition.

Definition 2.9. Let $0 < \rho < 1$. We say that a C^3 Anosov flow f^t on M preserving a smooth volume μ satisfies the non-integrability condition $(NI)_\rho$ if, for sufficiently large $b > 0$, it holds

$$\left| \int_{-1}^{1} \exp\left(ib\left(\psi(\tau) + \alpha\tau\right)\right) d\tau \right| < b^{-\rho} \tag{14}$$

for all s-templates $\psi \in \mathfrak{T}$ and $\alpha \in \mathbb{R}$.

Remark 2.10. From (4), the s-templates $\psi \in \mathfrak{T}$ are Hölder continuous with any exponent $0 < \beta < 1$ and the Hölder coefficients are bounded by some constant $C_{\beta,*}$. Hence, for each $\delta > 0$, the condition (14) holds for free if $\alpha > b^{1+\delta}$ and b is sufficiently large. (For instance, we can check this by using "regularized" integration by parts given in Lemma 6.12.)

Remark 2.11. From Lemma 2.7, we can see that the non-integrability condition $(NI)_\rho$ remains unchanged if we replace the Riemann metric on M by another Riemann metric and the volume μ by its scalar multiple.

The main theorem, Theorem 1.1, follows if we prove the following two theorems.

Theorem 2.12. *Let $3 \le r < \infty$. If we let $0 < \rho < 1$ be sufficiently small depending only on r, the subset of flows that satisfy the non-integrability condition $(NI)_\rho$ is dense in $\mathfrak{F}^r_A$.*

Theorem 2.13. *If $f^t_0 \in \mathfrak{F}^3_A$ satisfies the non-integrability condition $(NI)_\rho$ for some $0 < \rho < 1$, there exists an open neighborhood $\mathcal{V}$ of f^t_0 in $\mathfrak{F}^3_A$ such that all $f^t \in \mathcal{V}$ are exponentially mixing and further that the decay estimate (1) holds for all $f^t \in \mathcal{V}$ with uniform constants C_α and c_α.*

We prove Theorem 2.12 in the next section, Section 3. We prove Theorem 2.13 in Section 6, after preparation in Section 4 and Section 5.

2.6. **Approximate infinitesimal non-integrability.** We finish this section by a discussion on another idea about joint non-integrability of the stable and unstable foliation, which is related closer to the idea of uniform non-integrability condition introduced by Chernov[4]. Let us consider how the flow f^t twists the tangent bundle along local unstable (resp. stable) manifolds. Consider a point $q \in M$ and a positive number $0 < \delta < 1$. Note that we have specified the straight sections $\gamma^0_{q,J}$ when $J = (-1,1)$, but not yet for the case $J = (-\delta,\delta)$ with $0 < \delta < 1$. There are two different but natural ways to choose a straight section in $\Gamma^u_1(q,(-\delta,\delta))$:

(a) we take it as a restriction of $\gamma^0_{q,(-1,1)}$ to $W^u_{(-\delta,\delta)}(q) \subset W^u_{(-1,1)}(q)$, or
(b) recalling Lemma 2.7, we take $t > 0$ such that $f^t(W^u_{(-\delta,\delta)}(q)) = W^u_{(-1,1)}(p)$ with $p = f^t(q)$ and let it be the pull-back of $\gamma^0_{p,(-1,1)} \in \Gamma^u_1(p,(-1,1))$ by f^t.

We denote the straight sections obtained in (a) and (b) by $\gamma^0_{q,(-\delta,\delta)}$ and $\gamma^\dagger_{q,(-\delta,\delta)}$ respectively. They are both straight sections and hence A-equivalent. The difference between their expression functions are affine function and the coefficient of its linear part can be understood as the torsion that f^t (with t in (b) above) makes along $W^u_{(-\delta,\delta)}(q)$. For this reason, let us write

$$\gamma^\dagger_{q,(-\delta,\delta)}(\tau) = \gamma^0_{q,(-\delta,\delta)}(\tau) + \psi^\dagger_{q,(-\delta,\delta)}(\tau) \cdot \gamma^\perp_{q,(-\delta,\delta)}(\tau) \tag{15}$$

with an affine function $\psi^\dagger_{q,\delta}(\tau)$ and set

$$\mathrm{Tor}^s(q,\delta) := (\psi^\dagger_{q,\delta})'(0) = (\psi^\dagger_{q,\delta})'(\tau) \quad \text{for } \tau \in (-\delta,\delta).$$

Applying the parallel argument to the time-reversal of f^t, we introduce the section $\hat{\gamma}^{\dagger}_{q,(-\delta,\delta)}$, the function $\hat{\psi}^{\dagger}_{q,\delta}$ and $\mathrm{Tor}^u(q,\delta)$ which correspond to $\gamma^{\dagger}_{q,(-\delta,\delta)}$, $\psi^{\dagger}_{q,\delta}$ and $\mathrm{Tor}^s(q,\delta)$ respectively. In the next definition, we assume

$$\langle \gamma^{\perp}_{q,(-\delta,\delta)}(0), (w^s_{q,(-\delta,\delta)})'(0)\rangle > 0, \quad \langle \hat{\gamma}^{\perp}_{q,(-\delta,\delta)}(0), (w^u_{q,(-\delta,\delta)})'(0)\rangle > 0.$$

Definition 2.14. For $q \in M$ and $0 < \delta < 1$, we set

$$\Delta(q,\delta) = \mathrm{Tor}^u(q,\delta) - \mathrm{Tor}^s(q,\delta) \tag{16}$$

and call it the *infinitesimal approximate non-integrability* at $q \in M$ in the scale δ.

Lemma 2.15. *For $\sigma = s, u$, $0 < \delta, \delta' < 1$, $q \in M$ and $t \in \mathbb{R}$, we have*

$$|\mathrm{Tor}^\sigma(q,\delta) - \mathrm{Tor}^\sigma(q,\delta')| < C_*\langle \log(\delta'/\delta)\rangle \quad \textit{and hence} \quad |\mathrm{Tor}^\sigma(q,\delta)| < C_*\langle \log\delta\rangle, \tag{17}$$

$$|\mathrm{Tor}^\sigma(f^t(q),\delta) - \mathrm{Tor}^\sigma(q,\delta)| \le C_*\langle t\rangle, \quad \textit{and hence } |\Delta(f^t(q),\delta) - \Delta(q,\delta)| \le C_*\langle t\rangle \tag{18}$$

and

$$|\mathrm{Tor}^\sigma(q',\delta) - \mathrm{Tor}^\sigma(q,\delta)| \le C_* \quad \textit{if } d(q,q') < \delta \tag{19}$$

where (and henceforth) $\langle s\rangle$ denotes some fixed C^∞ function of s such that $\langle s\rangle = |s|$ if $|s| \ge 1$ and $\langle s\rangle \ge 1$ for any s.

Proof. Below we prove the claims in the case $\sigma = s$. We can prove the claims in the case $\sigma = u$ in parallel manner considering the time reversal of the flow f^t. Note first of all that $\mathrm{Tor}^s(q,1) = 0$ by definition. For any $t_0 > 0$, we have

$$|\mathrm{Tor}^s(q,\delta) - \mathrm{Tor}^s(f^t(q), |Df^t_q|_{E_u}|\cdot\delta)| \le C_* \quad \text{for } 0 \le t \le t_0 \text{ and } 0 < \delta < |Df^t_q|_{E_u}|^{-1}.$$

By recursive application of this estimate, we see that, for any $t > 0$,

$$|\mathrm{Tor}^s(q,\delta) - \mathrm{Tor}^s(f^t(q), |Df^t_q|_{E_u}|\cdot\delta)| \le C_*\langle t\rangle \quad \text{for } 0 < \delta < |Df^t_q|_{E_u}|^{-1}. \tag{20}$$

Also we have that, for $0 < \delta \le \delta' \le 1$,

$$|\mathrm{Tor}^s(q,\delta) - \mathrm{Tor}^s(q,\delta')| = |\mathrm{Tor}^s(f^t(q),\delta/\delta') - \mathrm{Tor}^s(f^t(q),1)| = |\mathrm{Tor}^s(f^t(q),\delta/\delta')| \tag{21}$$

where $t \ge 0$ is such that $|Df^t_q|_{E_u}|\cdot\delta' = 1$. The last two estimates yield (17). Then (18) follows from (17) and (20). If $q' \in W^u_{(-\delta,\delta)}(f^t(q))$ for some $t \in (-1,1)$, we have

$$|\mathrm{Tor}^s(q',\delta) - \mathrm{Tor}^s(q,\delta)| = |\mathrm{Tor}^s(f^t(q'),\delta') - \mathrm{Tor}^s(f^t(q),1)| + C_* < C_*$$

where $t > 0$ is such that $|Df^t_q|_{E_u}|\cdot\delta = 1$ and we set $\delta' = |Df^t_{q'}|_{E_u}|\cdot\delta$ so that $C_*^{-1} < \delta'/\delta < C_*$. Hence, for the proof of (19), we may assume that $q' \in W^s_{(-2\delta,2\delta)}(p)$. But, under such assumption, we can prove the claim easily because the distance between $f^t(q)$ and $f^t(q')$ is exponentially small with respect to $t > 0$. □

Remark 2.16. In the case of contact Anosov flows, the set $\mathcal{T}$ of s-templates consists of a single trivial A-equivalence class [0]. Therefore our non-integrability condition $(NI)_\rho$ excludes the case of contact Anosov flows (together with the suspension flows of Anosov diffeomorphism with constant roof function!). For contact Anosov flows, we can set up the straight sections $\gamma^0_{p,J}$ appropriately (and a bit differently from what we have done) so that the approximate infinitesimal non-integrablility $\Delta(q,\delta)$ is constant and bounded away from 0 for all $q \in M$ and $\delta > 0$, and this is sufficient for proving exponential mixing. In our argument below, we will see that, if the absolute value of $\Delta(q,\delta)$ is sufficiently large at any point $q \in M$ and at any small scale $\delta > 0$, the geometry of the stable and unstable foliations is comparable to the case of contact Anosov flows and we can indeed show exponential mixing. However the problem with the quantity $\Delta(q,\delta)$ is that it is not uniform in the scale δ and this makes it difficult to control in perturbation. On the other hand, our non-integrability condition $(NI)_\rho$ is formulated in terms of s-templates, which do not involve the scale δ, and therefore are stable and also can be controlled in perturbation to some extent, as we will see in the proof of Theorem 2.12 in Section 3. Note that we will make use of the non-integrability condition $(NI)_\rho$ only in the situation where the approximate infinitesimal non-integrablity $\Delta(q,\delta)$ is not sufficiently large in absolute value.

3. Proof of Theorem 2.12

In this section, we prove Theorem 2.12. Let $3 \le r < \infty$. We are going to perturb the flows in $\mathfrak{F}^r_A$ by time-changes and deform the s-templates. We fix some $R > 0$ depending only on r and show that we can change the values of s-templates on each of many small disjoint intervals in $(-1,1)$ of size $b^{-1/R}$ almost independently. This enables us to show by the large deviation argument that the condition (14) is violated only with very small possibility. As usual in perturbation argument in dynamical system theory, we will face problems caused by interference between perturbations. But, one because the dynamics is uniformly hyperbolic, the problem is fortunately not too difficult.

3.1. **A probability measure on the space of functions.**

Let $C^r(M)$ be the Banach space of C^r functions. The translation on $C^r(M)$ by $\varphi \in C^r(M)$ is written

$$\tau_\varphi : C^r(M) \to C^r(M), \qquad \tau_\varphi(u) = u + \varphi.$$

In the argument below, we fix a Borel probability measure $\boldsymbol{\mu}$ on $C^r(M)$ such that $\boldsymbol{\mu}(U) > 0$ for any non-empty open subset $U \subset C^r(M)$, that $\boldsymbol{\mu}$ is quasi-invariant with respect to the translation τ_φ for any $\varphi \in C^R(M)$ with some large R and further

$$\exp(-\|\varphi\|_{C^R}) \le \left| \frac{d((\tau_\varphi)_*\boldsymbol{\mu})}{d\boldsymbol{\mu}} \right| \le \exp(\|\varphi\|_{C^R}) \qquad \text{for any } \varphi \in C^R(M). \tag{22}$$

We refer [17, Lemma E] for existence of such measure $\boldsymbol{\mu}$ and $R > r$.

In what follows, we consider an arbitrary $f^t \in \mathfrak{F}^\infty_A$ and write v for the vector field that generates f^t. We suppose that $\mathcal{W}$ is a small neighborhood of the origin 0 in $C^r(M)$ and will let it smaller if necessary. Let f^t_φ for $\varphi \in \mathcal{W}$ be the flow generated by the vector field $v_\varphi = (1+\varphi)\cdot v$. Notice that the flow f^t_φ preserves the volume $m_\varphi = (1+\varphi)^{-1}\cdot m$. Hence we can apply the argument in Section 2 to the flow f^t_φ with setting $\mu = m_\varphi$.

For $p \in M$, $\alpha \in \mathbb{R}$ and $b > 0$, let $X_\rho(p,\alpha;b) \subset \mathcal{W}$ be the set of functions $\varphi \in \mathcal{W}$ such that the condition (14) for this α and b *fails* for the s-template at p for the flow f^t_φ. In the following, we are going to estimate the measure $\boldsymbol{\mu}(X_\rho(p,\alpha;b))$ for $p \in M$, $\alpha \in \mathbb{R}$ and $b > 0$ in order to prove Theorem 2.12.

Because of technical problems caused by interference of perturbations, we treat some set of points $p \in M$ as exceptions. Let $\tau_* > 0$ be constant defined for the flow f^t such that, for every periodic point $w \in M$ of the flow f^t, its prime period $T(w)$ is larger than $10\tau_*$. Then the modulus of hyperbolicity $\|Df^{T(w)}|_{E_u}(w)\|$ is greater than $\lambda_* := e^{10\chi_*\tau_*}$. For $b > 1$ and $p \in M$, we set

$$T(p,b) = \inf\{t \ge 0 \mid |Df^{-t}|_{E_u}(p)| \le b^{-1/(4R)}\}. \tag{23}$$

Then let $E(b) \subset M$ be the open set of points $p \in M$ such that there exits a periodic orbit with prime period less than $T(p,b)$ whose minimum distance from $W^u_{(-c_*,c_*)}(p)$ with $c_* = 2(1-\lambda_*^{-1})^{-1}$ is less than $b^{-1/(2R)}$.

For sufficiently small $\rho > 0$, we show the following proposition.

Proposition 3.1. *For sufficiently large $b > 0$, we have*

$$\boldsymbol{\mu}(X_\rho(p, \alpha; b)) < \exp(-b^\rho)$$

for $p \in M \setminus E(b)$ and $\alpha \in \mathbb{R}$.

The proof of this proposition will be given in the following subsections. Below we deduce Theorem 2.12 from this proposition. Note that $X_\rho(p, \alpha; b) = \emptyset$ for α with $|\alpha| \geq b^2$ from Remark 2.10, provided that b is sufficiently large. Let ρ' be a real number such that $0 < \rho' < \rho$.

Corollary 3.2. *For sufficiently large $b > 0$, we have*

$$\boldsymbol{\mu}\left(\bigcup_{p\in M\setminus E(b)} \bigcup_{\alpha\in\mathbb{R}} X_{\rho'}(p, \alpha; b)\right) < \exp(-b^\rho/2).$$

Proof. If we take a finite but sufficiently dense subset of points $\{(p_i, \alpha_i)\}_{i=1}^I$ in

$$(M \setminus E(b)) \times \{\alpha \in \mathbb{R} \mid |\alpha| < b^2\}$$

depending on b, then, by approximation, the union of the subsets $X_\rho(p_i, \alpha_i; b)$ will cover $\bigcup_{p\in M\setminus E(b)} \bigcup_{\alpha\in\mathbb{R}} X_{\rho'}(p, \alpha; b)$. By crude estimate, we can see that the cardinality I of the finite set necessary for this to be true is bounded by a polynomial order in b. Therefore we obtain the conclusion from Proposition 3.1. ◻

Next we prove the following lemma which tells basically that if the condition (14) holds for all $p \notin E(b)$, it also holds for $p \in E(b)$. We say that a flow f_φ^t satisfies the condition $(NI)_{\rho,b}$ for $b > 0$ if the condition (14) for this b holds for all the s-templates (for f_φ^t) and $\alpha \in \mathbb{R}$. Let us write $W_J^u(p;\varphi)$ and $w_p^u(\tau;\varphi)$ for the (piece of) unstable manifold $W_J^u(p)$ and its intrinsic parametrization $w_p^u(\tau)$ defined for the flow f_φ^t. Let ρ'' be a real number such that

$$0 < \rho'' < \rho'(1 - \rho') < \rho'.$$

Lemma 3.3. *If $b > 0$ is sufficiently large and if $\varphi \in \mathcal{W}$ does not belong to the subset $\bigcup_{p\in M\setminus E(b')} \bigcup_{\alpha\in\mathbb{R}} X_{\rho'}(p, \alpha; b')$ for any integer b' with $b^{1-\rho'} \leq b' \leq \lceil b \rceil$, then the flow f_φ^t satisfies the condition $(NI)_{\rho'',b}$.*

Proof. Suppose that b is a sufficiently large integer. (The case where b is not an integer will be considered at the end.) From the assumption, the condition (14) with ρ replaced by ρ'' (for any α) holds for the s-templates at the points in $M \setminus E(b)$. It is therefore enough to prove the estimate (14) with ρ replaced by ρ'' for the s-templates at $p \in E(b)$ and $\alpha \in \mathbb{R}$ with $|\alpha| < b^2$. To show this, we will use the following simple fact: for $q \in M$ and $0 < \delta < 1$, we take $t > 0$ such that $|Df^t|_{E_u}(q)| = \delta^{-1}$; then we have

$$\frac{1}{2\delta}\left|\int_{-\delta}^{\delta} \exp(ib(\psi^s_{q,(\delta,\delta)}(\tau) + \alpha\tau))d\tau\right| = \frac{1}{2}\left|\int_{-1}^{1} \exp(i\delta b(\psi^s_{p,(-1,1)}(\tau) + \alpha'\tau))d\tau\right| \tag{24}$$

for $p = f^t(q)$ and some $\alpha' \in \mathbb{R}$. Of course this is true also for the flow f_φ^t.

Suppose that $p \in E(b)$. From the definition of the set $E(b)$, there exists a unique periodic orbit γ with prime period less than $T(p, b)$ whose distance from $W^u_{(-c_*,c_*)}(p)$ is less than $b^{-1/(2R)}$. We can take[4] an isolating neighborhood U of γ so that it includes the $b^{-1/(3R)}$-neighborhood of $W^u_{(-c_*,c_*)}(p)$ and that the flow f_φ^t in U exhibits simple hyperbolic behavior

[4] If $\lambda_* > 1$ is close to 1 and c_* is large, the picture of the flow in Figure 1 will be much distorted in reality. But this does not make essential problems provided that b is large. If one likes, one can change the Riemann metric in order to reduce the distortion.

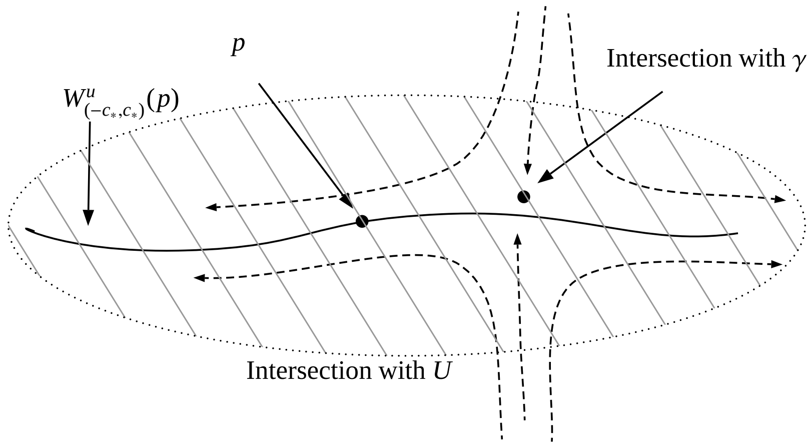

FIGURE 1. A picture of the flow f^t in a section transversal to the flow that contains the unstable manifold $W^u_{(-c_*,c_*)}(p)$. The dashed curve indicate the move of points by the return map to the section. There are a few different cases for the relative position of p to the intersection with γ.

as depicted in Figure 1. In particular, we may and do suppose that γ is the unique periodic orbit of f^t that is contained entirely in U.

We divide $W^u_{[-1,1]}(p;\varphi)$ into finitely many pieces

$$W_k = W^u_{[-\delta(k),\delta(k)]}(q_k;\varphi) \quad \text{with } q_k \in W^u_{(-1,1)}(p;\varphi)$$

for $0 \le k \le k(p)$, and apply (24) to each of the pieces. Let q be the point in $W^u_{[-1,1]}(p;\varphi)$ closest to the periodic orbit γ. We can and do take the pieces W_k so that

- for $k = 0$, we take $b^{-\rho'}/8 \le \delta(0) \le b^{-\rho'}/4$ and q_0 so that $W_0 = W^u_{[-\delta(0),\delta(0)]}(q_0;\varphi)$ contains the $\delta(0)$-neighborhood of the point q in $W^u_{[-1,1]}(p;\varphi)$, and
- for $1 \le k \le k(p)$, we take $b^{-\rho'} \le \delta(k) \le 1$ and q_k so that $\delta(k)b$ is an integer and that $C_*^{-1}d(q_k,q) \le \delta(k) \le C_* d(q_k,q)$.

We regard the piece W_0 as an exception. But this does not make any problem because its length is $2\delta(0) \le b^{-\rho''}/2$. Each of the pieces W_k for $1 \le k \le k(p)$ will eventually goes out of the isolating neighborhood U by the flow f^t_φ (at some positive time) and its length will grow to the unit size. Let us take $t_k > 0$ so that $|Df^{t_k}_\varphi|_{E_u}(q_k)| = \delta(k)^{-1}$, that is,

$$f^{t_k}_\varphi(W^u_{(-\delta(k),\delta(k))}(q_k;\varphi)) = W^u_{(-1,1)}(f^{t_k}_\varphi(q_k);\varphi).$$

We claim that $f^{t_k}_\varphi(q_k)$ for $1 \le k \le k(p)$ does not belong to $E(\delta(k)b)$. We can prove this by contradiction. Suppose that this were not the case. Then, by definition, there would be a periodic orbit γ' with period less than $T(f^{t_k}_\varphi(q_k),\delta(k)b)$ whose distance from $W^u_{(-c_*,c_*)}(f^{t_k}_\varphi(q_k))$ is bounded by $(\delta(k)b)^{-1/(2R)}$. Since we are assuming that ρ is small, we may suppose that $t_k/T(p,b)$ is close to 0 and $T(f^{t_k}_\varphi(q_k),\delta(k)b)/T(p,b)$ is close to 1. Noting that $f^{-t_k}_\varphi$ sends γ' to itself (of course) and that f^t_φ is a time-change of f^t, we see that the distance between γ' and $W^u_{(-c_*,c_*)}(p)$ is less than $b^{-1/(3R)}$. Further, tracing the orbit γ' backward, we see from the definition of $T(p,b)$ that the periodic orbit γ' must be entirely contained in U and therefore $\gamma' = \gamma$. But this is impossible from the construction of the pieces W_k, provided that $\delta(k)$ is sufficiently small according to c_*.

Now we can apply (24) to the pieces W_k for $1 \le k \le k(p)$ and use the assumption of the lemma to bound the integral in (14) for the s-template at p on each W_k. Then we obtain the required estimate (14) with ρ replaced by ρ''.

Finally we consider the case where b is not an integer. From the argument above, we see that f^t_φ satisfies the condition $(NI)_{\tilde{\rho},\lceil b\rceil}$ for some $\rho'' < \tilde{\rho} < \rho'(1-\rho')$. Then we can deduce that f^t_φ satisfies the condition $(NI)_{\rho'',b}$ by using (24). This finishes the proof of Lemma 3.3. □

From Lemma 3.3, we see that, if $\varphi \in \mathcal{W}$ does not belong to the subset

$$\bigcup_L \bigcap_{\ell=L}^{\infty} \left(\bigcup_{p\in M\setminus E(\ell)} \bigcup_{\alpha\in\mathbb{R}} X_\rho(p,\alpha;\ell) \right),$$

the flow f^t_φ satisfies the condition $(NI)_{\rho'}$ for $\rho' < \rho(1-\rho)$. Since the $\boldsymbol{\mu}$-measure of the set above is 0 from Corollary 3.2 and Borel-Cantelli lemma, we can find arbitrarily small $\varphi \in C^r(M)$ such that the flow f^t_φ satisfies the non-integrability condition $(NI)_{\rho/2}$. By a theorem of Moser[14], there is a C^r diffeomorphism $\Phi_\varphi : M \to M$ which transfers the volume $m_\varphi = (1+\varphi)^{-1}m$ to m, and Φ_φ converges to the identity in C^r sense as φ converges to 0. Therefore, taking conjugation of f^t_φ by such diffeomorphism Φ_φ and recalling Remark 2.11, we obtain a C^r flow in $\mathfrak{F}^r_A$ which is arbitrarily close to f^t in the C^r sense and satisfies the non-integrability condition $(NI)_{\rho/2}$. We have finished the proof of Theorem 2.12.

3.2. **Perturbation family.** In this subsection, we explain the scheme of perturbation for the proof of Proposition 3.1. Suppose that $b > 0$ is large and that a point $p \in M \setminus E(b)$ and $\alpha \in \mathbb{R}$ with $|\alpha| < b^2$ are given arbitrarily. Below we set up functions $\varphi_j \in C^\infty(M)$ for $1 \le j \le \lceil b^{1/R} \rceil$ and then, for arbitrary $\varphi_0 \in \mathcal{W}$ and a set $\mathcal{J}$ of integers $1 \le j \le \lceil b^{1/R}\rceil$, we consider the family of vector fields

$$v_{\mathbf{t}} = (1+\varphi_{\mathbf{t}})\cdot v \quad \text{where} \quad \varphi_{\mathbf{t}} = \varphi_0 + \sum_{j\in\mathcal{J}} t_j \cdot \varphi_j \quad \text{and} \quad \mathbf{t} = (t_j) \in [-2,2]^{\mathcal{J}}. \tag{25}$$

Once we fix such family of vector fields, we write $f^t_{\mathbf{t}} = f^t_{\varphi_{\mathbf{t}}}$ for the flow generated by $v_{\mathbf{t}}$.

Let us set $q = f^{4\tau_*}(p)$ and recall the intrinsic parameterization $w^u_q : \mathbb{R} \to M$ of the unstable manifold $W^u(q)$ of $q \in M$ for the flow f^t so that we have $f^{4\tau_*}(w^u_p(\tau)) = w^u_q(\lambda\tau)$ with $\lambda = \|Df^{4\tau_*}|_{E_u}(p)\|$. For $1 \le j \le \lceil b^{1/R}\rceil$, let

$$s(j) = -1 + 2j\cdot b^{-1/R} \in [-1, 1+2b^{-1/R}]$$

and then put

$$p(j) = w^u_p(s(j)), \qquad q(j) = f^{4\tau_*}(p(j)) = w^u_q(\lambda s(j)).$$

Also we take a local coordinate chart

$$\kappa_j = \kappa_{p,j} : U_j \to B(0,r_*)\times[-\tau_*, 6\tau_*],$$

on a neighborhood U_j of $p(j) \in M$, so that it provides flow box coordinates[5] for the flow f^t satisfying $\kappa_j(p(j)) = (0,0)$ and $\kappa_j(q(j)) = (0, 4\tau_*)$. We may and do assume that these coordinates are bounded in C^r sense uniformly in j (and also in b and p), that they transfer the volume m to the standard volume $dxdydz$ on $\mathbb{R}^3$ and further that

$$\kappa_j(w^u_{q(j)}(\lambda\tau)) = \kappa_j(f^{4\tau_*}(w^u_{p(j)}(\tau))) = (\tau,0,0) \quad \text{for } \tau \in [-2b^{-1/R}, 2b^{-1/R}].$$

[5]By "flow box coordinates", we mean a C^r local coordinates (x,y,z) on M in which f^t moves (x,y,z) to $(x,y,z+t)$ when t is small.

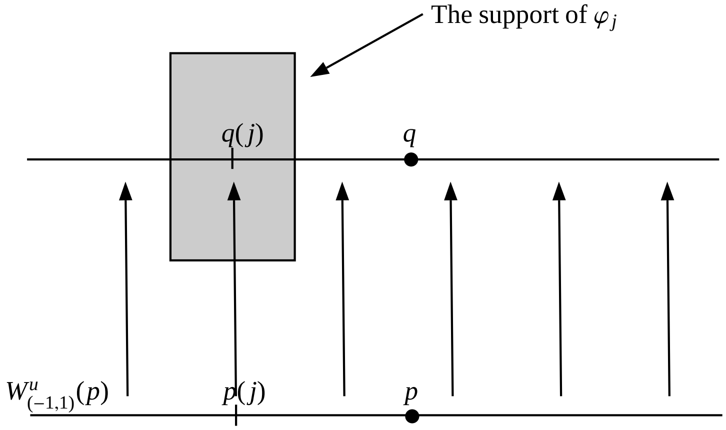

FIGURE 2. A picture of the flow f^t in a section parallel to the flow that contains the unstable manifold $W^u_{(-1,1)}(p)$.

See Figure 2. We take and fix a C^∞ functions $h_0 : \mathbb{R}^2 \to [0,1]$ supported on the disk $|(x,y)| \le 3/2$ such that $h_0(x,y) = y$ if $|(x,y)| \le 1$. Let $\chi : \mathbb{R} \to [0,1]$ be a C^∞ function such that $\chi(s) = 1$ if $|s| \le 1$ and $\chi(s) = 0$ if $|s| \ge 3/2$. Then we define

$$\varphi_j : M \to [0,1] \quad \text{for } 1 \le j \le \lceil b^{1/R} \rceil$$

by

$$\varphi_j(m) = -a_0^{-1} b^{-1-1/R} \cdot \chi((z - 4\tau_*)/\tau_*) \cdot h_0(b^{1/R}x, b^{1/R}y)$$

for $m \in M$ with $\kappa_j(m) = (x,y,z)$, where $a_0 = \pi \int \chi(z/\tau_*) dz$.

Remark 3.4. The motivation for the choice of functions φ_j above is explained as follows. Suppose that $j \in \mathcal{J}$, $\varphi_0 = 0$ and $t_i = 0$ for $i \ne j$ in $\mathbf{t} = (t_i)$. Then, in the local chart κ_j, the vector field $v_\mathbf{t}$ will look $(\kappa_j)_* v_\mathbf{t} = (1 + t_j \varphi_j)\partial_z$ and hence, if $w = (x,y) \in \mathbb{R}^2$ satisfies $|w| \le b^{-1/R}$, the map $f_\mathbf{t}^{-4\tau_*}$ takes a point $(w, 4\tau_*)$ to $(w, b^{-1}\pi y t_j + \mathcal{O}(t_j^2))$. Hence, by changing the parameter t_j, we will be able to rotate the stable subspace $E_s(p(j))$ around the unstable manifold $W^u_{(-1,1)}(p)$ by the rate proportional to $b^{-1}\pi$ provided that $|t_j|$ is sufficiently small. Of course we will have to consider the influence of the perturbation from further future. Also, since φ_0 may not be 0, we will have to introduce a slight adjustment related to the coefficient a_0^{-1}.

The family of functions φ_j satisfies

$$\|D^k \varphi_j\|_\infty \le C_* b^{((k-1)/R)-1} \quad \text{for } 0 \le k \le R. \tag{26}$$

The intersection multiplicity of $\operatorname{supp} \varphi_j$ for $1 \le j \le \lceil b^{-1/R} \rceil$ is bounded by 2. Hence, regardless of the choice of $\mathcal{J}$, we will have

$$\|\varphi_\mathbf{t}\|_{C^R} < C_* b^{-1/R}, \quad \|\partial_\mathbf{t} \varphi_\mathbf{t}\|_{C^1} \le C_* b^{-1}, \quad \|\varphi_\mathbf{t} - \varphi_\mathbf{0}\|_{C^1} < C_* b^{-1} \quad \text{when } \mathbf{t} \in [-2,2]^{\mathcal{J}}. \tag{27}$$

As we noted in Remark 3.4, we would like to modify the s-template $\psi_{p,(-1,1)}$ on the intervals

$$J(j) := [s(j) - b^{-1/R}, s(j) + b^{-1/R}] \cap [-1,1]$$

almost independently by varying the parameter t_j in the family (25). To this end, we have to choose the set $\mathcal{J}$ a little carefully. First we observe that

(I1) $f^t(\operatorname{supp} \varphi_i) \cap \operatorname{supp} \varphi_j = \emptyset$ for $1 \le i, j \le \lceil b^{1/R} \rceil$ and $4\tau_* \le t \le T(p,b) - 4\tau_*$.

Indeed, if (I1) was violated, we could find a periodic orbit γ in the $C_* b^{-1/R}$-neighborhood of $W^s_{(-c_*,c_*)}(p)$ by the pseudo-orbit tracing property and the conditions in the definition of $E(b)$ would hold for this γ.

Next we make the following observation from the fact that the pre-image $f^{-t}(W^u_{(-1,1)}(p))$ of $W^u_{(-1,1)}(p)$ shrink exponentially as t increases:

(I2) There is a subset $\mathcal{E}(p,b)$ of integers $1 \le i \le \lceil b^{1/R} \rceil$ with $\sharp\mathcal{E}(p,b) \le C_* b^{3/(4R)}$ such that, if $1 \le j \le \lceil b^{1/R} \rceil$ does not belong to $\mathcal{E}(p,b)$, the subset $f^{-t}(W^u_{(-1,1)}(p))$ does not meet $\operatorname{supp}\varphi_j$ for $t \ge 0$ satisfying $|Df^{-t}|_{E_u}(p)| \ge b^{-1}$.

Indeed, from (I1), the subset $f^{-t}(W^u_{(-1,1)}(p))$ for $0 \le t \le T(p,b) - 4\tau_*$ does not meet $\operatorname{supp}\varphi_j$ for any $1 \le j \le \lceil b^{1/R} \rceil$. Since the length of $f^{-t}(W^u_{(-1,1)}(p))$ is bounded by $C_* b^{-1/(4R)}$ when $t = T(p,b) - 4\tau_*$ and shrink exponentially as t increases, we can find the exceptional set $\mathcal{E}(p,b)$ in (I2).

Below we consider two subsets $\mathcal{J}_{\mathrm{even}}$ and $\mathcal{J}_{\mathrm{odd}}$ as $\mathcal{J}$, that consist of even and odd integers $1 \le j \le \lceil b^{-1/R} \rceil$ respectively, but we exclude those in $\mathcal{E}(p,b)$ in (I2) above and also those j for which $J(j-1) \cup J(j) \cup J(j+1)$ contains either of -1 or 1. Then, for $\mathcal{J} = \mathcal{J}_{\mathrm{even}}, \mathcal{J}_{\mathrm{odd}}$, we have obviously

(I3) $\operatorname{supp}\varphi_i \cap \operatorname{supp}\varphi_j = \emptyset$ if $i \ne j \in \mathcal{J}$.

(I4) $J(j-1) \cap J(j) \cap J(j+1)$ does not contain -1 nor 1 for $j \in \mathcal{J}$.

Note that the number of integers $1 \le i \le \lceil b^{-1/R} \rceil$ that does not belong $\mathcal{J}_{\mathrm{even}} \cup \mathcal{J}_{\mathrm{odd}}$ is bounded by $C_* b^{3/(4R)}$ and hence the Lebesgue measure of the union of $J(i)$ for such i's is bounded by $C_* b^{-1/(4R)}$. So these exceptions are negligible when we consider the estimate (14) in the following subsections.

3.3. **Deformation of s-templates.** In this subsection, we suppose that $\mathcal{J}$ is either of the subsets $\mathcal{J}_{\mathrm{even}}$ and $\mathcal{J}_{\mathrm{odd}}$ and observe how the s-template at the point p is deformed in the perturbation family (25) with arbitrary $\varphi_0 \in \mathcal{W}$.

For each parameter $\mathbf{t} \in [-2,2]^{\mathcal{J}}$, we write $W^u_{(-1,1)}(p;\mathbf{t}) = W^u_{(-1,1)}(p;\varphi_{\mathbf{t}})$ and $w^u_p(\tau;\mathbf{t}) = w^u(\tau;\varphi_{\mathbf{t}})$ for brevity. Let $T_qM = E_0(q;\mathbf{t}) \oplus E_s(q;\mathbf{t}) \oplus E_u(q;\mathbf{t})$ be the hyperbolic decomposition for the flow $f^t_{\mathbf{t}}$ corresponding to (2) for f^t. Let

$$\gamma^0_{p,\mathbf{t}},\ \gamma^\perp_{p,\mathbf{t}},\ \gamma^s_{p,\mathbf{t}}\ : W^u_{(-1,1)}(p;\mathbf{t}) \to T^*M$$

be the sections $\gamma^0_{p,(-1,1)},\ \gamma^\perp_{p,(-1,1)},\ \gamma^s_{p,(-1,1)}$ considered in Section 2 but now defined for the perturbed flow $f^t_{\mathbf{t}}$. Then the s-template for the flow $f^t_{\mathbf{t}}$ at $p \in M$ is the continuous function $\psi^s_{p,(-1,1)}(\cdot;\mathbf{t}) : (-1,1) \to \mathbb{R}$ satisfying

$$\gamma^s_{p,\mathbf{t}}(w) = \psi^s_{p,(-1,1)}(\tau;\mathbf{t}) \cdot \gamma^\perp_{p,\mathbf{t}}(w) + \gamma^0_{p,\mathbf{t}}(w) \qquad \text{for } w = w^u_p(\tau;\mathbf{t}).$$

Actually it is not a very simple task to observe how the function $\psi^s_{p,(-1,1)}(\cdot;\mathbf{t})$ varies, because the frames $\gamma^\perp_{p,\mathbf{t}}$ and $\gamma^0_{p,\mathbf{t}}$ will also vary. In order to simplify the argument, we consider an approximation of $\psi^s_{p,(-1,1)}(\cdot;\mathbf{t})$. Let $\tilde{\gamma}^s_{p,\mathbf{t}} : W^u_{(-1,1)}(p;\mathbf{0}) \to T^*M$ be the unique continuous section in $\Gamma^u_1(p,(-1,1))$ for the flow $f^t_{\mathbf{0}}$ (!) such that $\tilde{\gamma}^s_{p,\mathbf{t}}(q)$ is normal to $E^u(q;\mathbf{0}) \oplus E^s(q;\mathbf{t})$ for $q \in W^u_{(-1,1)}(p;\mathbf{0})$. Then we express $\tilde{\gamma}^s_{p,\mathbf{t}}$ as

$$\tilde{\gamma}^s_{p,\mathbf{t}}(w) = \tilde{\psi}^s_{p,(-1,1)}(\tau;\mathbf{t}) \cdot \gamma^\perp_{p,\mathbf{0}}(w) + \gamma^0_{p,\mathbf{0}}(w) \quad \text{for } w = w^u_p(\tau;\mathbf{0}) \tag{28}$$

using a unique continuous function $\tilde{\psi}^s_{p,(-1,1)}(\cdot;\mathbf{t}) : (-1,1) \to \mathbb{R}$. In the next lemma, we show that the last function $\tilde{\psi}^s_{p,(-1,1)}(\cdot;\mathbf{t})$ is a sufficiently good approximation of $\psi^s_{p,(-1,1)}(\cdot;\mathbf{t})$. Before stating the lemma, we note that there exist functions

$$h : [-2,2]^{\mathcal{J}} \to \mathbb{R} \quad \text{and} \quad \mathcal{T} : (-1,1) \times [-2,2]^{\mathcal{J}} \to \mathbb{R}$$

such that $h(\mathbf{0}) = 0$ and $\mathcal{T}(0, \mathbf{0}) = 0$ and that

$$w^u_p(h(\mathbf{t}) \cdot \tau; \mathbf{t}) = f^{\mathcal{T}(\tau,\mathbf{t})}(w^u(\tau; \mathbf{0})).$$

This is a consequence of the definition of the intrinsic metric (6) and the fact that our perturbation does not change the flow lines.

Lemma 3.5. *For* $\mathbf{t} \in [-2, 2]^I$*, we have* $|h(\mathbf{t}) - 1| \le C_* b^{-2}$ *and*

$$|\tilde{\psi}^s_{p,(-1,1)}(\tau; \mathbf{t}) - \psi^s_{p,(-1,1)}(h(\mathbf{t}) \cdot \tau; \mathbf{t})| < C_* b^{-1-(1/(4R))} \quad \textit{for } \tau \in (-1, 1).$$

Proof. From the observation (I2), it is not difficult to see that the C^1 distance between $W^u_{(-1,1)}(p; \mathbf{t})$ and $W^u_{(-1,1)}(p; \mathbf{0})$ is bounded by $C_* b^{-2}$ and therefore we have

$$|h(\mathbf{t}) - 1| \le C_* b^{-2} \quad \text{and} \quad \|\partial_\tau T(\tau; \mathbf{t})\|_\infty \le C_* b^{-2}. \tag{29}$$

Next we consider the sections $\gamma^s_{p,(-1,1)}(\cdot; \mathbf{t})$, $\gamma^\perp_{p,(-1,1)}(\cdot; \mathbf{t})$ and $\gamma^0_{p,(-1,1)}(\cdot; \mathbf{t})$. For convenience, we look them in flow box coordinates for the flow $f^t_{\mathbf{0}}$ around $W^u_{(-1,1)}(p; \mathbf{0})$, which are also flow box coordinates for $f^t_{\mathbf{t}}$ for $\mathbf{t} \in [-2, 2]^{\mathcal{J}}$. (Accordingly we regard $\gamma^s_{p,(-1,1)}(\cdot; \mathbf{t})$, $\gamma^\perp_{p,(-1,1)}(\cdot; \mathbf{t})$ and $\gamma^0_{p,(-1,1)}(\cdot; \mathbf{t})$ as mappings from $(-1, 1)$ to $\mathbb{R}^3$.) Then, from (29), we have

$$\gamma^\perp_{p,(-1,1)}(w^u_p(h(\mathbf{t}) \cdot \tau; \mathbf{t}); \mathbf{t}) = \gamma^\perp_{p,(-1,1)}(w^u_p(\tau; \mathbf{0}); \mathbf{0}) + \mathcal{O}_*(b^{-2}) \tag{30}$$

and

$$\gamma^s_{p,(-1,1)}(w^u_p(h(\mathbf{t}) \cdot \tau; \mathbf{t}); \mathbf{t}) = \gamma^s_{p,(-1,1)}(w^u_p(\tau; \mathbf{0}); \mathbf{t}) + \mathcal{O}_*(b^{-2}). \tag{31}$$

Recall that $\gamma^0_{p,(-1,1)}(\cdot; \mathbf{t})$ is defined as the unique straight section satisfying (10), that is,

$$\gamma^0_{p,(-1,1)}(w^u_p(\pm 1; \mathbf{t}); \mathbf{t}) = \gamma^s_{p,(-1,1)}(w^u_p(\pm 1; \mathbf{t}); \mathbf{t}).$$

And, for the quantities on the right-hand side, we have

$$\|\gamma^s_{p,(-1,1)}(w^u_p(\pm h(\mathbf{t}); \mathbf{t}); \mathbf{t}) - \gamma^s_{p,(-1,1)}(w^u_p(\pm 1; \mathbf{0}); \mathbf{0})\| < C_* b^{-1-(1/(4R))}.$$

from the condition (I4). Therefore we have

$$\|\gamma^0_{p,(-1,1)}(w^u_p(\pm 1; \mathbf{t}); \mathbf{t}) - \gamma^0_{p,(-1,1)}(w^u_p(\pm 1; \mathbf{0}); \mathbf{0})\| < C_* b^{-1-(1/(4R))}$$

from C^2 boundedness of the sections $\gamma^0_{p,(-1,1)}(\cdot; \mathbf{t})$ and (29). Now we recall from the proof of Lemma 2.5 how the straight sections are determined as a limit in terms of the time evolution along the orbit of $W^u_{(-1,1)}(p; \mathbf{t})$ in the *negative* time direction. It is then easy to see from (I2) and the last estimate above for the end points that

$$\|\gamma^0_{p,(-1,1)}(w^u_p(h(\mathbf{t}) \cdot \tau; \mathbf{t}); \mathbf{t}) - \gamma^0_{p,(-1,1)}(w^u_p(\tau; \mathbf{0}); \mathbf{0})\| < C_* b^{-1-(1/(4R))} \quad \text{for } \tau \in (-1, 1). \tag{32}$$

The latter claim of the lemma follows from (30), (31) and (32). □

From Lemma 3.5, we have

$$\left| \int_I \exp\left(ib\left(\tilde{\psi}^s_{p,(-1,1)}(\tau; \mathbf{t}) + \alpha\tau\right)\right) d\tau - \int_I \exp\left(ib\left(\psi^s_{p,(-1,1)}(\tau; \mathbf{t}) + \alpha\tau\right)\right) d\tau \right| < C_* b^{-1/(4R)}$$

for any Borel subset $I \subset [-1, 1]$. Therefore, in proving (14) for $\psi(\tau) = \psi^s_{p,(-1,1)}(\tau; \mathbf{t})$, we may consider $\tilde{\psi}^s_{p,(-1,1)}(\tau; \mathbf{t})$ in the place of $\psi^s_{p,(-1,1)}(\tau; \mathbf{t})$.

To proceed, we introduce the mapping

$$\Psi_\tau = \Psi_{\tau,p,\alpha,b} : [-2, 2]^{\mathcal{J}} \to \mathbb{R}^{\mathcal{J}}$$

for each $\tau \in (-b^{-1/R}, b^{-1/R})$, defined by

$$\Psi_\tau(\mathbf{t}) = b \cdot \left(\tilde{\psi}^s_{p,(-1,1)}(s(j) + \tau; \mathbf{t}) + \alpha(s(j) + \tau)\right)_{j \in \mathcal{J}}.$$

Let $A(\varphi_0)$ be the diagonal matrix of size $\sharp\mathcal{J}$ with diagonal elements

$$\tilde{a}_j(\varphi_0) = \frac{\int (1+\varphi_0(f^t(q(j))))^{-2} \cdot \chi(t/\tau_*)dt}{\int \chi(t/\tau_*)dt} \quad \text{for } j \in \mathcal{J}.$$

This matrix is introduced for the adjustment mentioned at the end of Remark 3.4 and close to identity for $\varphi_0 \in \mathcal{W}$, provided that we let $\mathcal{W}$ be small. Later we will use the following simple estimate:

$$|\tilde{a}_j(\tilde{\varphi}_0) - \tilde{a}_j(\varphi_0)| < C_* \|\tilde{\varphi}_0 - \varphi_0\|_\infty \quad \text{for } \tilde{\varphi}_0, \varphi_0 \in \mathcal{W}. \tag{33}$$

The next lemma realize the idea in the construction of the perturbation family.

Proposition 3.6. *If $b > 0$ is sufficiently large, we have*

$$\left\| D\Psi_\tau(\mathbf{t}) - \pi \cdot A(\varphi_0) : (\mathbb{R}^{\mathcal{J}}, \|\cdot\|_{\max}) \to (\mathbb{R}^{\mathcal{J}}, \|\cdot\|_{\max}) \right\| \le C_* b^{-(1/(4R))}$$

for $\mathbf{t} \in [-2,2]^{\mathcal{J}}$, $p \in M \setminus E(b)$, $\alpha \in \mathbb{R}$ and $\tau \in (-b^{-1/R}, b^{-1/R})$, where we consider the maximal norm $\|(s_j)\|_{\max} = \max_j |s_j|$ on $\mathbb{R}^{\mathcal{J}}$. Further, for the Jacobian, we have

$$|\log\det D\Psi_\tau(\mathbf{t}) - \log\det(\pi \cdot A(\varphi_0))| \le C_* \cdot \sharp\mathcal{J} \cdot b^{-(1/(4R))}.$$

Proof. Note that the stable subspace at a point $w \in M$ for the flow $f^t_{\mathbf{t}}$ can be expressed as the limit

$$E_s(w;\mathbf{t}) = \lim_{t\to\infty} Df^{-t}_{\mathbf{t}}(E(f^t_{\mathbf{t}}(w)))$$

where $E(w)$ is, for instance, the one dimensional subbundle in T_wM which is orthogonal to $E_u \oplus E_0$ for f^t. Hence, we can compute the differentials of $E_s(w;\mathbf{t})$ and $\tilde{\psi}^s_{p,(-1,1)}(w;\mathbf{t})$ with respect to the parameter $\mathbf{t}$ as an integral of the infinitesimal contribution of the perturbation at time $t \ge 0$. Since our perturbation preserves the flow lines, the differential of $\tilde{\psi}^s_{p,(-1,1)}(w;\mathbf{t})$ is given in the form

$$\partial_{t_j}\tilde{\psi}^s_{p,(-1,1)}(\tau;\mathbf{t}) = \int_0^\infty |Df^t_{\mathbf{0}}|_{E^u}(w^u_p(\tau;\mathbf{0}))|^{-1} \cdot X_j(p,t,\mathbf{t})\,dt$$

where $X_j(p,t,\mathbf{t})$ satisfies $|X_j(p,t,\mathbf{t})| < C_* b^{-1}$ and is non-zero only if $f^t_{\mathbf{t}}(w^u_p(\tau;\mathbf{t})) \in \operatorname{supp}\varphi_j$. (Though we can express $X_j(p,t,\mathbf{t})$ in a explicit form by preparing some notation, this is not necessary.) From the construction of φ_j and the condition (I3), we have that

$$\int_0^{6\tau_*} |Df^t_{\mathbf{0}}|_{E_u}(w^u_p(\tau;\mathbf{0}))|^{-1} \cdot X_j(p,t,\mathbf{t})\,dt = \begin{cases} b^{-1}\pi \cdot \tilde{a}_j(\varphi_0) + \mathcal{O}_*(b^{-2}), & \text{if } \tau \in J(j); \\ 0, & \text{if } \tau \in J(i) \text{ for } j \ne i \in \mathcal{J}. \end{cases}$$

Also from (I3), we have

$$\int_{6\tau_*}^\infty |Df^t_{\mathbf{0}}|_{E^u}(w^u_p(\tau;\mathbf{0}))|^{-1} \cdot X_j(p,t,\mathbf{t})\,dt \le C_* b^{-1-(1/(4R))} \quad \text{for all } \tau \in (-1,1). \tag{34}$$

Further, from (I1) and (I3), we make the following observations:

- each point $w \in M$ belongs to the support of φ_j for at most one $j \in \mathcal{J}$,
- if $w \in \operatorname{supp}\varphi_j$ for some $j \in \mathcal{J}$, the orbit $f^t_{\mathbf{t}}(w)$ for $t \in [4\tau_*, T(p,b) - 4\tau_*]$ does not meet $\bigcup_{i\in\mathcal{J}} \operatorname{supp}\varphi_i$, and
- if $w \in \operatorname{supp}\varphi_j$ and if $f^t_{\mathbf{t}}(w) \in \operatorname{supp}\varphi_{j'}$ for some $j' \in \mathcal{J}$ and $t \ge T(p,b) - 4\tau_*$, we have $|Df^t_{\mathbf{0}}|_{E_u}(w)| \ge C_*^{-1} b^{-1/(4R)}$.

From these, we see that, for $\tau \in \cup_{j\in\mathcal{J}} J(j)$, we have

$$\int_{6\tau_*}^{\infty} |Df_{\mathbf{0}}^t|_{E^u}(w_p^u(\tau;\mathbf{0}))|^{-1} \cdot X_j(p,t,\mathbf{t})dt \le C_* b^{-2} \tag{35}$$

except for a set of $j \in \mathcal{J}$ (depending on τ) whose cardinality is bounded by $5R$. This together with the estimates above implies the first claim of the proposition. Also we can prove the second claim using the formula $\det(I+X) = \exp(\sum_{k=1}^{\infty}(-1)^{k+1}(1/k)\mathrm{Tr}\, X^k)$. □

3.4. **Proof of Proposition 3.1.** Let $\mathcal{J}$ be either of $\mathcal{J}_{\mathrm{even}}$ and $\mathcal{J}_{\mathrm{odd}}$. Below we follow a standard argument in the large deviation theory[5]. First, using the fact that $\exp(s) \le 1 + s + s^2$ when $|s| \ll 1$ and that $\int_{-\pi}^{\pi} \mathrm{Re}\,(\exp(is))ds = 0$, we see

$$\int_{[-\pi,\pi]^{\mathcal{J}}} \exp\left(b^{-1/(8R)} \cdot \mathrm{Re}\left(\sum_{j\in\mathcal{J}} \exp(ix_j)\right)\right) \prod_{j\in\mathcal{J}} dx_j$$
$$= \left(\int_{-\pi}^{\pi} \exp\left(b^{-1/(8R)} \cdot \mathrm{Re}\left(\exp(ix)\right)\right)dx\right)^{\sharp\mathcal{J}} \le (1+\pi b^{-1/(4R)})^{\sharp\mathcal{J}} < \exp(2\pi b^{-1/(4R)} \cdot \sharp\mathcal{J}).$$

From the former assertion of Proposition 3.6, we take a constant $K_* > 0$ so that the subset

$$Y = \{(x_j)_{j\in\mathcal{J}} \in [-2,2]^{\mathcal{J}} \mid \tilde{\alpha}_j(\varphi_0)\cdot|x_j| + K_* b^{-1/(4R)} < 1,\ \forall j \in \mathcal{J}\}$$

satisfies $\Psi_\tau(Y) \subset [-\pi,\pi]^{\mathcal{J}}$ for $\tau \in (-b^{-1/R}, b^{-1/R})$. Also, from the latter assertion, we see

$$\int_Y \exp\left(b^{-1/(8R)} \cdot \mathrm{Re} \sum_{j\in\mathcal{J}} \exp\left(i\,\Psi_\tau(\mathbf{t})_j\right)\right)d\mathbf{t} < \exp\left(C_* \sharp\mathcal{J} \cdot b^{-1/(4R)}\right) \cdot \mathrm{Leb}(Y)$$

where $\Psi_\tau(\mathbf{t})_j$ denotes the j-th component of $\Psi_\tau(\mathbf{t})$. We integrate the both sides with respect to τ on $[-b^{-1/R}, b^{-1/R}]$. Then, noting that $\sharp J \le b^{1/R}$, we find

$$\frac{1}{2b^{-1/R}} \int_{-b^{-1/R}}^{b^{1/R}} \int_Y \exp\left(b^{-1/(8R)} \cdot \sum_{j\in J} \mathrm{Re}\, \exp\left(i(\tilde{\psi}^s_{p,(-1,1)}(s(j)+\tau;\mathbf{t}) + \alpha(s(j)+\tau))\right)d\tau\right)d\mathbf{t}$$
$$< \frac{1}{2b^{-1/R}} \cdot \exp\left(C_* b^{3/(4R)}\right) \cdot \mathrm{Leb}(Y).$$

By Jensen's inequality applied to the exponential function, we deduce

$$\int_Y \exp\left(\frac{b^{7/(8R)}}{2} \int_I \mathrm{Re}\, \exp\left(i(\tilde{\psi}^s_{p,(-1,1)}(\tau;\mathbf{t}) + \alpha\tau)\right)d\tau\right)d\mathbf{t} < \frac{1}{2b^{-1/R}} \exp\left(C_* b^{3/(4R)}\right) \cdot \mathrm{Leb}(Y)$$

where $I = \cup_{j\in\mathcal{J}} J(j)$. This implies

$$\mathrm{Leb}\left\{\mathbf{t} \in Y \,\middle|\, \mathrm{Re} \int_I \exp\left(i(\tilde{\psi}^s_{p,(-1,1)}(\tau;\mathbf{t}) + \alpha\tau)\right)d\tau > b^{-1/(16R)}\right\}$$
$$< b^{1/R} \cdot \exp\left(C_* b^{3/(4R)} - (1/2)b^{13/(16R)})\right) \cdot \mathrm{Leb}(Y) < \exp(-b^{3/(4R)})$$

provided that b is large.

In the argument above, we consider the real part of the function $\tilde{\psi}^s_{p,(-1,1)}(\tau;\mathbf{t}) + \alpha\tau$. But we can consider the imaginary part and also change the sign in it. Therefore we conclude

$$\frac{1}{\mathrm{Leb}(Y)} \cdot \mathrm{Leb}\left\{\ \mathbf{t} \in Y \ \middle|\ \left|\int_I \exp\left(i(\tilde{\psi}^s_{p,(-1,1)}(\tau;\mathbf{t}) + \alpha\tau)\right)d\tau\right| > b^{-1/(16R)}\right\} \le \exp(-b^{-3/(4R)})$$

provided that b is sufficiently large. Finally, from the property (22) of the measure $\boldsymbol{\mu}$ and (33), we can now deduce the estimate

$$\boldsymbol{\mu}\left\{\ \varphi \in \mathcal{W} \ \middle|\ \int_I \exp\left(i(\tilde{\psi}^s_{p,(-1,1)}(\tau;\varphi) + \alpha\tau)\right)d\tau > b^{-1/(16R)}\right\} \le \exp(-b^{-1/(2R)})$$

and obtain the conclusion of Proposition 3.1 from this estimate for the cases $\mathcal{J} = \mathcal{J}_{\text{even}}, \mathcal{J}_{\text{odd}}$ and the remark at the end of Subsection 3.2. We have finished the proof of Theorem 2.12.

4. Local charts

This section and the next are devoted to preparatory argument for the proof of Theorem 2.13. We will consider a flow $f^t \in \mathfrak{F}^3_A$ satisfying the non-integrablity condition $(NI)_\rho$ for some $\rho > 0$ and study the transfer operator (or Perron-Frobenius operator)

$$\mathcal{L}^t : L^2(M) \to L^2(M), \quad \mathcal{L}^t u = u \circ f^{-t}$$

associated to it. To analyze the transfer operator $\mathcal{L}^t$, we will introduce a decomposition of functions on M with respect to the frequency in the flow direction and then consider the action of $\mathcal{L}^t$ on each of the components. Since $\mathcal{L}^t$ virtually preserves the frequency in the flow direction, this method is natural and works efficiently. (See [18, 19].) The analysis of the action of $\mathcal{L}^t$ on high frequency components is particularly important in our argument.

In this subsection, we begin with setting up systems of local charts depending on an integer $\omega \in \mathbb{Z}$, which we will use when we analyze the components of functions that have frequency around ω in the flow direction. Some of the constructions below depend on a large constant $t_\sharp > 0$, which will be specified later. Roughly $t_\sharp > 0$ is the time to wait until hyperbolicity of the flow takes sufficiently strong effect and suppresses non-linearity of the flow. We set

$$\varkappa_\sharp = \exp({t_\sharp}^2)$$

so that $\|Df^{t_\sharp}\|_\infty \le C_* \exp(C_* t_\sharp) \ll \varkappa_\sharp$.

Remark 4.1. We will see that the constant $t_\sharp$ (as well as the constants $\omega_\sharp$ and $m_\sharp$ which will be introduced later) can be taken as the kind of constant that are denoted by symbols with the subscript $*$. (See the beginning of Section 2.) But we use this symbol $t_\sharp$ with the subscript $\sharp$ instead of the subscript $*$, because the choice is made much later.

4.1. **Local charts depending on** $\omega \in \mathbb{Z}$**.** We write $B(w, r)$ for the open disk in $\mathbb{R}^2$ with radius $r > 0$ centered at $w \in \mathbb{R}^2$. Let $r_* > 0$ be a small real number. To begin with, we take a finite system of C^3 local charts

$$\kappa_a : U_a \to B(0, 2r_*) \times (-1, 1) \subset \mathbb{R}^3 \quad \text{for } a \in A \text{ with } \#A < \infty$$

on open subsets $U_a \subset M$, which are flow box coordinates for the flow f^t in the sense that

$$\kappa_a(f^t(p)) = \kappa_a(p) + (0, 0, t) \quad \text{provided that } f^s(p) \in U_a \text{ for all } s \text{ between 0 and } t.$$

Let $\rho_a : \mathbb{R}^3 \to [0, 1]$ for $a \in A$ be C^3 functions such that $\operatorname{supp} \rho_a \subset B(0, r_*) \times (-1, 1)$ and that the family of functions $\rho_a \circ \kappa_a$ for $a \in A$ is a partition of unity on M, *i.e.* $\sum_{a \in A} \rho_a \circ \kappa_a = 1$. Let $\tilde{\rho}_a$ for $a \in A$ be a C^r function such that $\tilde{\rho}_a \equiv 1$ on $\operatorname{supp} \rho_a$ and that $\operatorname{supp} \tilde{\rho}_a \subset B(0, r_*) \times (-1, 1)$. By applying a mollifier along the flow line, we can and do assume that ρ_a are infinitely differentiable with respect to the variable z and each of the partial derivatives $\partial_x^k \partial_y^\ell \partial_z^m \rho_a$ and $\partial_x^k \partial_y^\ell \partial_z^m \tilde{\rho}_a$ with $k + \ell \le 3$ are continuous and therefore bounded.

Based on the system of local charts κ_a, we construct finer systems of local charts and associated partition of unity for each $\omega \in \mathbb{Z}$. The construction is given in two steps as follows. For the first step, we take a finite subset $N(a, \omega) \subset B(0, r_*)$ and, for each $n \in N(\omega)$, we take a neighborhood $V_{a,\omega,n} \subset B(0, 2r_*)$ of n and a C^3 diffeomorphism

$$g_{a,\omega,n} : V_{a,\omega,n} \times \mathbb{R} \to D_{a,\omega,n} \times \mathbb{R} \subset B(0, 2\varkappa_\sharp \langle \omega \rangle^{-1/2}) \times \mathbb{R}$$

of the form

$$g_{a,\omega,n}(x, y, z) = (\hat{g}_{a,\omega,n}(x, y), z + \check{g}_{a,\omega,n}(x, y)).$$

We suppose that $V_{a,\omega,n}$ for $n \in N(a,\omega)$ cover $B(0,r_*)$. For the diffeomorphisms $g_{a,\omega,n}$, we may and do assume the following conditions:

(G0) the C^3 norms of $g_{a,\omega,n}$ and those of their inverses are bounded by C_*,
(G1) $g_{a,\omega,n}(n,0) = (0,0,0)$ and the differential $D(g_{a,\omega,n} \circ \kappa_a)$ at $p_{a,\omega,n} := \kappa_a^{-1}(n,0)$ sends $E_u(p_{a,\omega,n})$, $E_s(p_{a,\omega,n})$, $E_0(p_{a,\omega,n})$ to the x-axis, y-axis, z-axis respectively,
(G2) there exists $\omega_0 > 0$ such that, if $|\omega| \geq \omega_0$, we have $D_{a,\omega,n} = B(0, 2\varkappa_\sharp\langle\omega\rangle^{-1/2})$ and
$$g_{a,\omega,n} \circ \kappa_a(w^s_{p_{a,\omega,n}}(\tau)) = (\tau,0,0), \qquad g_{a,\omega,n} \circ \kappa_a(w^u_{p_{a,\omega,n}}(\tau)) = (0,\tau,0)$$
for $\tau \in [-2\varkappa_\sharp\langle\omega\rangle^{-1/2}, 2\varkappa_\sharp\langle\omega\rangle^{-1/2}]$, where $w^s_{p_{a,\omega,n}}(\cdot)$ and $w^u_{p_{a,\omega,n}}(\cdot)$ are the intrinsic parametrization of the stable and unstable manifolds introduced in Section 2.2.
(G3) $((g_{a,\omega,n} \circ \kappa_a)^{-1})^* \mu = c_{a,\omega,n} \cdot dxdydz$ for some constant $c_{a,\omega,n}$.

For the second step, we let
$$b_{a,\omega,n} : \mathbb{R}^3 \to \mathbb{R}^3, \qquad b_{a,\omega,n}(x,y,z) = (x, y, \beta(a,\omega,n) \cdot xy)$$
where we define
$$\beta(a,\omega,n) = \begin{cases} c_{a,\omega,n} \cdot \mathrm{Tor}^s(p_{a,\omega,n}, \varkappa_\sharp\langle\omega\rangle^{-1/2}), & \text{if } |\omega| > \omega_0; \\ 0, & \text{otherwise.} \end{cases}$$
Then we set $U_{a,\omega,n} = U_a \cap \kappa_a^{-1}(V_{a,\omega,n} \times \mathbb{R})$ and regard
$$\kappa_{a,\omega,n} := b_{a,\omega,n} \circ g_{a,\omega,n} \circ \kappa_a : U_{a,\omega,n} \to D_{a,\omega,n} \times \mathbb{R}, \qquad \text{for} \quad (a,n) \in A \times N(a,\omega)$$
the system of local charts on M defined for $\omega \in \mathbb{Z}$. Note that these gives flow box coordinates satisfying $\kappa_{a,\omega,n}(p_{a,\omega,n}) = 0$ and
$$\kappa_{a,\omega,n}(w^s_{p_{a,\omega,n}}(\tau)) = (\tau,0,0), \qquad \kappa_{a,\omega,n}(w^u_{p_{a,\omega,n}}(\tau)) = (0,\tau,0)$$
and also
$$(\kappa^{-1}_{a,\omega,n})^* \mu = c_{a,\omega,n} \cdot dxdydz$$
from (G2) and (G3) above.

In order to see the meaning of the post-composition of $b_{a,\omega,n}$ in the second step, let us recall the argument in Subsection 2.6. Suppose that $|\omega| \geq \omega_0$. From (G2), we can express the sections $\gamma^\dagger_{p,(-\delta,\delta)}$ and $\hat{\gamma}^\dagger_{p,(-\delta,\delta)}$ for $p = p_{a,\omega,n}$ and $\delta = \varkappa_\sharp\langle\omega\rangle^{-1/2}$ as
$$(\kappa^{-1}_{a,\omega,n})^* \circ \gamma^\dagger_{p,(-\delta,\delta)}(w^u_p(\tau)) = (0, \varphi(\tau), 1), \quad (\kappa^{-1}_{a,\omega,n})^* \circ \gamma^\dagger_{p,(-\delta,\delta)}(w^s_p(\tau)) = (\hat{\varphi}(\tau), 0, 1)$$
using C^2 functions $\varphi, \hat{\varphi} : (-\delta,\delta) \to \mathbb{R}$. Also note that we have
$$(\kappa^{-1}_{a,\omega,n})^* \circ \gamma^\perp_{p,(-\delta,\delta)}(w^u_p(\tau)) = (0, c_{a,\omega,n}, 1), \quad (\kappa^{-1}_{a,\omega,n})^* \circ \hat{\gamma}^\dagger_{p,(-\delta,\delta)}(w^s_p(\tau)) = (c_{a,\omega,n}, 0, 1).$$
If we did not have the post-composition of $b_{a,\omega,n}$ in the second step of the construction of $\kappa_{a,\omega,n}$, we would have
$$|\varphi'(\tau) - c_{a,\omega,n} \cdot \mathrm{Tor}^s(p, \varkappa_\sharp\langle\omega\rangle^{-1/2})| < C_*, \quad |\hat{\varphi}'(\tau) - c_{a,\omega,n} \cdot \mathrm{Tor}^u(p, \varkappa_\sharp\langle\omega\rangle^{-1/2})| < C_*$$
because the C^2 norm of $\gamma^0_{p,(-\delta,\delta)}$ is bounded by a uniform constant C_*. Hence, with the post-composition of $b_{a,\omega,n}$, we have actually
$$|\varphi'(\tau)| < C_* \quad \text{and} \quad |\hat{\varphi}'(\tau) - c_{a,\omega,n} \cdot \Delta(p, \varkappa_\sharp\langle\omega\rangle^{-1/2})| < C_*. \tag{36}$$
That is, by the post-composition of $b_{a,\omega,n}$, the subbundle E_s along $W^u_{(-\delta,\delta)}(p)$ will look stabilized in the local chart $\kappa_{a,\omega,n}$ and, instead, E_u will look rotating along $W^s_{(-\delta,\delta)}(p)$ by a rate proportional to $\Delta(p, \varkappa_\sharp\langle\omega\rangle^{-1/2})$.

We next construct partitions of unity associated to the systems of local charts $\{\kappa_{a,\omega,n}\}_{a,n}$ for $\omega \in \mathbb{Z}$. Let $\varrho_{a,\omega,n}, \tilde{\varrho}_{a,\omega,n} : \mathbb{R}^2 \to [0,1]$ for $n \in N(\omega)$ be C^3 functions such that

(1) $\operatorname{supp} \varrho_{a,\omega,n} \subset \operatorname{supp} \tilde{\varrho}_{a,\omega,n} \subset V_{a,\omega,n}$,
(2) $\sum_{n\in N(\omega)} \varrho_{a,\omega,n} = 1$ on $B(0, r_*)$, and $\tilde{\varrho}_{a,\omega,n} \equiv 1$ on $\operatorname{supp} \varrho_{a,\omega,n}$,
(3) $\max\{\|\partial^\alpha \varrho_{a,\omega,n}\|_\infty, \|\partial^\alpha \tilde{\varrho}_{a,\omega,n}\|_\infty\} \le C_*(\alpha)(\varkappa_\sharp^{-1}\langle\omega\rangle^{1/2})^{|\alpha|}$ for any multi-index α with $|\alpha| \le 3$.

For each $\omega \in \mathbb{Z}$, we consider the family of functions

$$\rho_{a,\omega,n} = (\rho_a \cdot \varrho_{a,\omega,n}) \circ g_{a,\omega,n}^{-1} \quad \text{for } a \in A, n \in N(\omega)$$

where we regard $\varrho_{\omega,n}$ as functions on $\mathbb{R}^3$ by setting $\varrho_{\omega,n}(x, y, z) := \varrho_{\omega,n}(x, y)$. Similarly we set

$$\tilde{\rho}_{a,\omega,n} = (\tilde{\rho}_a \cdot \tilde{\varrho}_{a,\omega,n}) \circ g_{a,\omega,n}^{-1} \quad \text{for } a \in A, n \in N(\omega).$$

The set of functions $\rho_{a,\omega,n} \circ \kappa_{a,\omega,n}$ for $a \in A$ and $n \in N(\omega)$ is a partition of unity associated to the system of local charts $\{\kappa_{a,\omega,n}\}$ and we have $\tilde{\rho}_{a,\omega,n} \circ \kappa_{a,\omega,n} \equiv 1$ on $\operatorname{supp}(\rho_{a,\omega,n} \circ \kappa_{a,\omega,n})$.

Remark 4.2. We may and do assume without loss of generality that, for each ω, the intersection multiplicity of the subsets

$$\bigcup_{t\in[-1,1]} f^t(\operatorname{supp} \tilde{\rho}_{a,\omega,n}) \quad \text{for } a \in A \text{ and } n \in N(a, \omega)$$

is bounded by an absolute constant.

4.2. **The central bundle E_0^* viewed in the local charts.** In this subsection, we consider how the central subspace $E_0^* = (E_s \oplus E_u)^\perp$ in the cotangent bundle looks in the local charts $\kappa_{a,\omega,n}$. Note that, since E_0^* is invariant with respect to the flow f^t, there is a unique continuous mapping

$$e_{a,\omega,n} : D_{a,\omega,n} \to \mathbb{R}^2, \quad e_{a,\omega,n}(w) = (\theta^u_{a,\omega,n}(w), \theta^s_{a,\omega,n}(w))$$

be the unique continuous function such that

$$(D\kappa_{a,\omega,n})_p^*(e_{a,\omega,n}(w), 1) \in E_0^*(p) \quad \text{when} \quad \kappa_{a,\omega,n}(p) = (w, z) \text{ and } p \in U_{a,\omega,n}.$$

From the assumption (G2) on the choice of $g_{a,\omega,n}$, we have

$$\theta^u_{a,\omega,n}(\tau, 0) = \theta^s_{a,\omega,n}(0, \tau) = 0 \quad \text{for} \quad \tau \in [-2\varkappa_\sharp\langle\omega\rangle^{-1/2}, 2\varkappa_\sharp\langle\omega\rangle^{-1/2}]$$

By slight abuse of notation, we will sometimes regard the functions $e_{a,\omega,n}$, $\theta^u_{a,\omega,n}$ and $\theta^s_{a,\omega,n}$ above as functions on $\mathbb{R}^3$ by letting $e_{a,\omega,n}(x, y, z) = e_{a,\omega,n}(x, y)$ and so on.

Remark 4.3. The function $e_{a,\omega,n}$ is not smooth in general, but satisfies

$$\|e_{a,\omega,n}(w') - e_{a,\omega,n}(w)\| \le C_*|w' - w| \cdot (\langle \log |w' - w|\rangle + \log\langle\omega\rangle) \quad \text{for } w, w' \in D_{a,\omega,n} \tag{37}$$

from (4) and the fact $|\beta(a, \omega, n)| < C_* \log\langle\omega\rangle$. In particular, we have that

$$\|e_{a,\omega,n}(w)\| = \|e_{a,\omega,n}(w) - e_{a,\omega,n}(0)\| \le C_*\varkappa_\sharp\langle\omega\rangle^{-1/2}\langle\log\langle\omega\rangle\rangle \quad \text{for } w \in D_{a,\omega,n}.$$

For a technical reason to be explained in Remark 5.5, we extend the functions $e_{a,\omega,n}$ so that they are defined on $\mathbb{R}^2$. The choice of the extension is rather arbitrary provided that (37) holds.

In the next lemma, we take and fix a constant $0 < \theta_* < 1/2$.

Lemma 4.4. *For $\langle\omega\rangle^{-1+\theta_*} \le h \le \varkappa_\sharp\langle\omega\rangle^{-1/2}$ and a point $w \in D_{a,\omega,n}$, let*

$$\ell, \hat{\ell} : (-h, h) \to \mathbb{R}^2, \quad \ell(\tau) = w + (\tau, 0), \quad \hat{\ell}(\tau) = w + (0, \tau).$$

Then, for $-h \le \tau \le h$, we have

$$(38)\quad \left|\theta^u_{a,\omega,n} \circ \hat{\ell}(\tau) - \theta^u_{a,\omega,n} \circ \hat{\ell}(0) - c_{a,\omega,n}\Delta(p_{a,\omega,n}, \varkappa_\sharp \langle\omega\rangle^{-1/2}) \cdot \tau\right| < C_* h \langle \log(\varkappa_\sharp \langle\omega\rangle^{-1/2}/h)\rangle,$$

$$(39)\quad \left|\theta^s_{a,\omega,n} \circ \hat{\ell}(\tau) - \theta^s_{a,\omega,n} \circ \hat{\ell}(0)\right| < C_* \varkappa_\sharp^2 \langle\omega\rangle^{-1} \cdot \langle \log\langle\omega\rangle\rangle^2$$

and similarly that

$$(40)\quad \left|\theta^s_{a,\omega,n} \circ \ell(\tau) - \theta^s_{a,\omega,n} \circ \ell(0)\right| < C_* h \cdot \langle \log(\varkappa_\sharp \langle\omega\rangle^{-1/2}/h)\rangle,$$

$$(41)\quad \left|(\theta^u_{a,\omega,n} \circ \ell(\tau) - \theta^u_{a,\omega,n} \circ \ell(0))\right| < C_* \varkappa_\sharp^2 \langle\omega\rangle^{-1} \cdot \langle \log\langle\omega\rangle\rangle^2.$$

If the non-integrability condition $(NI)_\rho$ holds, we have, for sufficiently large $b_0 > 0$, that

$$(42)\quad \frac{1}{2h}\left|\int_{-h}^{h} \exp\left(ibh^{-1}\left(\theta^s_{a,\omega,n}(\ell(\tau)) + \alpha\tau\right)\right) d\tau\right| < b^{-\rho/2}$$

for any $b_0 \le b \le \varkappa_\sharp$, $\alpha \in \mathbb{R}$ and any h, w as above, provided that $|\omega|$ is sufficiently large.

Proof. Below we prove (38) and (39). We can prove (40) and (41) similarly by considering the time-reversal of the flow. Let $q = \kappa^{-1}_{a,\omega,n}(w, 0)$. From (4) and (G1), we have

$$(43)\quad |(\kappa_{a,\omega,n} \circ w^s_q)'(\tau) - (0, 1, 0)| \le C_* \varkappa_\sharp \langle\omega\rangle^{-1/2} \langle \log\langle\omega\rangle\rangle \quad \text{for } -h < \tau < h.$$

For continuity of the second derivative of the local stable manifold, we can argue in parallel to the proof of (4) and, using the condition (G2), we obtain

$$(44)\quad |(\kappa_{a,\omega,n} \circ w^s_q)''(\tau)| < C_* \varkappa_\sharp \langle\omega\rangle^{-1/2} \langle \log\langle\omega\rangle\rangle \quad \text{for } -h < \tau < h.$$

From (37) and (43), we see that, for $-h < \tau < h$,

$$\begin{aligned} |e_{a,\omega,n}(\ell(\tau)) - e_{a,\omega,n}(\kappa_{a,\omega,n} \circ w^s_q(\tau))| &\le C_* |\ell(\tau) - \kappa_{a,\omega,n} \circ w^s_q(\tau))| \cdot \langle \log|\ell(\tau) - \kappa_{a,\omega,n} \circ w^s_q(\tau))|\rangle \\ &\le C_* \varkappa_\sharp^2 \langle\omega\rangle^{-1} \langle \log\langle\omega\rangle\rangle^2. \end{aligned}$$

Since the right hand side is small enough, the claims of the lemma follows if we prove them with the term $\hat{\ell}$ replaced by $\kappa_{a,\omega,n} \circ w^s_q$.

Let e_x be the vector field on $U_{a,\omega,n}$ defined by $e_x = (\kappa^{-1}_{a,\omega,n})_*(\partial_x)$. Then, for $-h < \tau < h$,

$$\begin{aligned} (45)\quad \theta^u_{a,\omega,n}(\kappa_{a,\omega,n}(w^s_q(\tau))) &= \langle \gamma^u_{q,(-h,h)}(w^s_q(\tau)), e_x(w^s_q(\tau))\rangle \\ &= \left\langle \left(\hat{\gamma}^\dagger_{q,(-h,h)}(w^s_q(\tau)) - \hat{\gamma}^0_{p,(-h,h)}(w^s_q(\tau))\right) + \hat{\gamma}^0_{q,(-h,h)}(w^s_q(\tau)),\ e_x(w^s_q(\tau)) \right\rangle + \mathcal{O}_*(h) \\ &= \mathrm{Tor}^u(q,h)\tau \cdot \langle \hat{\gamma}^\perp_{q,(-h,h)}(w^s_q(\tau)), e_x(w^s_q(\tau))\rangle + \langle \hat{\gamma}^0_{q,(-h,h)}(w^s_q(\tau)), e_x(w^s_q(\tau))\rangle + \mathcal{O}_*(h) \end{aligned}$$

where the second equality follows from the fact that

$$\|\gamma^u_{q,(-h,h)}(w^s_q(\tau)) - \hat{\gamma}^\dagger_{q,(-h,h)}(w^s_q(\tau))\|_\infty = h \cdot \|\gamma^u_{q',(-1,1)}(w^s_{q'}(\tau)) - \hat{\gamma}^\dagger_{q',(-1,1)}(\tau)\|_\infty < C_* h$$

for $t > 0$ satisfying $|Df^t_q|_{E_u}| = 1/h$ and $q' = f^t(q)$. From the construction of the local charts $\kappa_{a,\omega,n}$, we have

$$(46)\quad \langle \hat{\gamma}^0_{q,(-h,h)}(w^s_q(\tau)), e_x(w^s(\tau))\rangle - \langle \hat{\gamma}^0_{q,(-h,h)}(w^s_q(0)), e_x(w^s(0))\rangle = \beta(a,\omega,n)\tau + \mathcal{O}_*(h).$$

Also, recalling the definition of the section $\hat{\gamma}^\perp$ (or that of $\gamma^\perp$ for the time reversal of the flow), we see, from the condition (G3) and (43), that

$$(47)\quad \langle \hat{\gamma}^\perp_{q,(-h,h)}(w^s_q(\tau)), e_x(w^s_q(\tau))\rangle = c_{a,\omega,n} + \mathcal{O}_*(\varkappa_\sharp \langle\omega\rangle^{-1/2} \langle \log\langle\omega\rangle\rangle).$$

Since we have

$$|\mathrm{Tor}^u(q,h) - \mathrm{Tor}^u(p_{a,\omega,n}, \varkappa_\sharp \langle\omega\rangle^{-1/2})| < C_* \log\langle \varkappa_\sharp \langle\omega\rangle^{-1/2}/h\rangle$$

from Lemma 2.15, we obtain the first claim (38) with $\hat{\ell}$ replaced by $\kappa_{a,\omega,n} \circ w_q^s$ from (45), (46) and (47). The claim (39) is obtained by using (37), (43) and (44) in the relation

$$(\theta^u(\kappa_{a,\omega,n} \circ w_q^s(\tau)), \theta^s(\kappa_{a,\omega,n} \circ w_q^s(\tau)), 1) \cdot (\kappa_{a,\omega,n} \circ w_q^s)'(\tau) \equiv 0.$$

To prove the last claim (42), we note that, from the non-integrability condition $(NI)_\rho$, there exists some $b_0 > 0$ such that the estimate (14) holds for all the templates $\psi \in \mathcal{T}$, for all $\alpha \in \mathbb{R}$ and for all $b \geq b_0$. Then, from (12), we have that

$$\frac{1}{2h}\left|\int_{-h}^{h} \exp(ibh^{-1}(\psi^s_{q,(-h,h)} + \alpha\tau))d\tau\right| < b^{-\rho} \qquad \text{for any } q \in M,\ \alpha \in \mathbb{R} \text{ and } b \geq b_0. \tag{48}$$

Finally we estimate the difference between $\psi^s_{q,(-h,h)}(\tau)$ and $\theta^s_{a,\omega,n}(\ell(\tau))$. Similarly to the argument above, we let $e_y = (\kappa^{-1}_{a,\omega,n})_*(\partial_y)$ and find

$$\begin{aligned}\theta^s_{a,\omega,n}(\kappa_{a,\omega,n}(w^u_q(\tau))) &= \langle \gamma^s_{q,(-h,h)}(w^u_q(\tau)), e_y(w^u_q(\tau))\rangle \\ &= \psi^s_{q,(-h,h)}(\tau) \cdot \langle \gamma^\perp_{q,(-h,h)}(w^u_q(\tau)), e_y(w^u_q(\tau))\rangle + \langle \gamma^0_{q,(-h,h)}(w^u_q(\tau)), e_y(w^u_q(\tau))\rangle.\end{aligned}$$

Applying the approximation parallel to (47) to the first term and the approximation of $\langle \gamma^0_{q,(-h,h)}(w^u_q(\tau)), e_y(w^u_q(\tau))\rangle$ by its linear part $\tau \mapsto \alpha'\tau + \beta'$ to the second term, we see that

$$|\theta^s_{a,\omega,n}(\ell(\tau)) - c_{a,\omega,n} \cdot \psi^s_{q,(-h,h)}(\tau) - \alpha'\tau - \beta'| < C_* \varkappa_\sharp^2 \langle\omega\rangle^{-1}\langle \log\langle\omega\rangle\rangle.$$

The right hand side is small enough to conclude the third claim (42) from (48). □

4.3. **The flow f^t viewed in the local charts.** In this subsection, we consider how the flow f^t looks in the local charts $\kappa_{a,\omega,n}$. Suppose that $t_\sharp \leq t \leq 2t_\sharp$ and that $U = f^t(U_{a,\omega,n}) \cap U_{a',\omega',n'} \neq \emptyset$. Then the flow f^t viewed in the local charts $\kappa_{a,\omega,n}$ in the source and $\kappa_{a',\omega',n'}$ in the target will be

$$f := \kappa_{a',\omega',n'} \circ f^t \circ \kappa_{a,\omega,n} : V \times \mathbb{R} \to V' \times \mathbb{R}$$

where

$$V := V_{a,\omega,n} = \pi \circ \kappa_{a,\omega,n}(U), \qquad V' := V_{a',\omega',n'} = \pi \circ \kappa_{a',\omega,n}(f(U)).$$

Note that we dropped $a, \omega, n, a', \omega', n'$ from the notation above for brevity. The diffeomorphism f is written in the form

$$f(x,y) = (\hat{f}(x,y), z + \check{f}(x,y)). \tag{49}$$

Letting $t_\sharp$ be large, letting χ_* in (3) be slightly smaller and also choosing the local charts κ_a a little more carefully, we may and do assume that the diffeomorphism f given as above is uniformly hyperbolic in the sense that

$$Df^*_p(\mathbf{C}(2)) \subset \mathbf{C}(1/2) \quad \text{for } p \in V \times \mathbb{R}$$

where $\mathbf{C}(\theta) = \{(\xi_x, \xi_y, 0) \in \mathbb{R}^3 \mid |\xi_x| \leq \theta|\xi_y|\}$ and that

$$|Df^*_p(v)| \geq e^{\chi_* t}|v| \quad \text{if } v \in \mathbf{C}(2) \quad \text{and} \quad |(Df^{-1})^*_{f(p)}(v)| \geq e^{\chi_* t}|v| \quad \text{if } v \notin \mathbf{C}(1/2).$$

For the higher order derivatives, we have a crude estimate

$$\|D^k f\|_\infty \leq C_* \exp(C_* t_\sharp) \cdot \langle \log \max\{\langle\omega\rangle, \langle\omega'\rangle\}\rangle^2 \quad \text{for } k = 2, 3 \tag{50}$$

where the last factor stems from the post-composition of $b_{a,\omega,n}$ and $b_{a',\omega',n'}$ in the construction of the local charts $\kappa_{a,\omega,n}$ and $\kappa_{a',\omega',n'}$.

If $|\omega|$ and $|\omega'|$ are large, the domain V of f is small in the directions transversal to the flow and therefore f is well approximated by its linearization at least in such directions.

In the next lemma, we make this idea more precise. Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be the mappings defined by[6]

$$A(x, y, z) = (\lambda x, \tilde{\lambda} y, z + b \cdot (x, y) + \beta xy) + f(0, 0, 0). \tag{51}$$

where

(52)
$$\lambda = \pm\|D\hat{f}(\partial_x)\|, \qquad \tilde{\lambda} = \pm\|D\hat{f}(\partial_y)\|, \qquad b = (\partial_x \check{f}(0, 0), \partial_y \check{f}(0, 0)), \qquad \beta = \partial_{xy} \check{f}(0, 0)$$

and the signs of λ and $\tilde{\lambda}$ are chosen so that A approximate f better. Then, setting $G = A^{-1} \circ f$, we write the diffeomorphism f as the composition

$$f = A \circ G.$$

The diffeomorphism G is again written in the form

$$G(x, y, z) = (\hat{G}(x, y), z + \check{G}(x, y)).$$

In the next lemma, we suppose that $0 < \theta_* < 1/2$ is a constant that we can take and fix arbitrarily, as in the last lemma.

Lemma 4.5. *There exist constant $\omega_\sharp > 0$, depending on the choice of $t_\sharp$, such that, for any $t_\sharp \le t \le 2t_\sharp$ and any combination of (a, ω, n) and (a', ω', n') as above satisfying $|\omega| > \omega_\sharp$ and $1/2 \le \langle\omega'\rangle/\langle\omega\rangle \le 2$, we have the following estimates:*

(G) *for the diffeomorphism $G : V \to G(V) \subset \mathbb{R}^3$, we have $G(0) = 0$ and*

$$\|\mathrm{Id} - DG\|_\infty < \langle\omega\rangle^{-1/2+\theta_*} \quad \text{and} \quad \|D^k G\|_\infty < \langle\omega\rangle^{\theta_*} \quad \text{for } k = 2, 3, \tag{53}$$

and also

$$\|D\check{G}\|_\infty < \langle\omega\rangle^{-1+\theta_*} \quad \text{and} \quad \|D^2 \check{G}\|_\infty < \langle\omega\rangle^{-1/2+\theta_*}. \tag{54}$$

(A) *for the diffeomorphism A, we have*

$$|b| < \langle\omega\rangle^{-1/2+\theta_*} \quad \text{and} \quad |\beta| < C_* t_\sharp. \tag{55}$$

Consequently we have

$$\|D\check{f}\|_\infty < \langle\omega\rangle^{-1+\theta_*}, \quad \|D^2 \check{f}\|_\infty < C_* t_\sharp \tag{56}$$

Remark 4.6. We will observe the functions on local charts $\kappa_{a,\omega,n}$ in the scale $\langle\omega\rangle^{-1}$ in z-axis while in the scale $\langle\omega\rangle^{-1/2}$ in the xy-plane. In such scale, the diffeomorphism G will look

$$(\tilde{x}, \tilde{y}, \tilde{z}) \mapsto \left(\langle\omega\rangle^{1/2} \hat{G}(\langle\omega\rangle^{-1/2}\tilde{x}, \langle\omega\rangle^{-1/2}\tilde{y}, \langle\omega\rangle^{-1}\tilde{z}), \langle\omega\rangle \check{G}(\langle\omega\rangle^{-1/2}\tilde{x}, \langle\omega\rangle^{-1/2}\tilde{y}\right).$$

The estimates above implies that this rescaled map tends to identity as $|\omega| \to \infty$, uniformly in a and n.

Proof. Note that we will choose large $\omega_\sharp$ depending on θ_* and $t_\sharp$ and also that we will use arbitrariness of θ_* implicitly in a few places below. The claim $\|D^k G\|_\infty < \langle\omega\rangle^{\theta_*}$ for $k = 2, 3$ is obvious from (50). Since the vectors

$$(Df)_0(\partial_x) = ((D\hat{f})_0(1, 0), \partial_x \check{f}(0, 0)), \quad (Df)_0(\partial_y) = ((D\hat{f})_0(0, 1), \partial_y \check{f}(0, 0))$$

points the direction of unstable and stable subspaces at $f(0)$ in the local chart $\kappa_{a',\omega',n'}$ respectively, the estimate on b in (55) follows from (4). Also, from the choice of A, we see that DG_0 preserves the vectors $\partial_x, \partial_y, \partial_y$ up to error terms bounded by $\langle\omega\rangle^{-1/2+\theta_*}$. Hence

[6] Since we will consider the case where $|\omega|$ is large, we may and do suppose that f is actually defined much larger domain than V and therefore $f(0, 0, 0)$ is well defined.

we obtain the former estimate in (53) from the latter (for a slightly smaller θ_*). From (44) and the condition (G2) in the construction of the local charts $\kappa_{a,\omega,n}$, we have

$$|(\kappa_{a,\omega,n}\circ w_p^{\sigma})''(\tau)| < C_*\varkappa_\sharp\langle\omega\rangle^{-1/2}\langle\log\langle\omega\rangle\rangle \quad \text{for } \tau\in[-2\varkappa_\sharp\langle\omega\rangle^{-1/2}, 2\varkappa_\sharp\langle\omega\rangle^{-1/2}]$$

for $p\in U_{a,\omega,n}$ and $\sigma = s, u$. Also we have the same estimates for the local chart $\kappa_{a',\omega',n}$. Since the stable and unstable manifolds are preserved by f, we obtain that

$$\|\partial_{xx}\check{f}\|_\infty < \langle\omega\rangle^{-1/2+\theta_*}, \qquad \|\partial_{yy}\check{f}\|_\infty < \langle\omega\rangle^{-1/2+\theta_*}.$$

This implies the same estimates for $\check{G}$ because $\partial_{xx}A = \partial_{yy}A = 0$. From the choice of β, we have that $\|\partial_{xy}\check{G}\|_\infty < \langle\omega\rangle^{-1/2+\theta_*}$. We have therefore obtained the latter claim in (54). Since $D\check{G}(0,0) = 0$ from the choice of b, the former claim then follows. To prove the last claim on β, we recall the construction of the local chart $\kappa_{a,\omega,n}$. If we did not have the post-composition of $b_{a,\omega,n}$ in the second step of the construction of the local chart $\kappa_{a,\omega,n}$, the claim is obtained by a standard argument using the chain rule. The post-composition of the map $b_{a,\omega,n}$ results in the addition of

$$(\beta(a',\omega',n') - (\lambda\tilde{\lambda})\cdot\beta(a,\omega,n)) = c_{a',\omega',n'}\cdot\left(\frac{\beta(a',\omega',n')}{c_{a',\omega',n'}} - \frac{\beta(a,\omega,n)}{c_{a,\omega,n}}\right)$$

to β up to an error term bounded by C_*. From (18) and (19), this additional term is bounded by $C_* t_\sharp$. We have completed the proof. □

We finish this subsection with the following simple estimate.

Lemma 4.7. *If $0\le t\le 2t_\sharp$, we have, for integers $k,\ell,m\ge 0$ with $k+\ell\le 3$, that*

$$\|\partial_x^k\partial_y^\ell\partial_z^m(\rho_{a,\omega,n}\cdot(\rho_{a',\omega',n'}\circ f))\|_\infty \le C_*\cdot(e^{C_* t_\sharp}\cdot\varkappa_\sharp^{-1}\max\{\langle\omega\rangle,\langle\omega'\rangle\}^{1/2})^{(k+\ell)}.$$

We omit the proof since it is straightforward. But note that, in the case $k=\ell=0$, the right hand side above is just C_*. This is because the function $\varrho_{a,\omega,n}$ in the definition of $\rho_{a,\omega,n}$ is the functions of (x,y) and does not depend on z. (Recall also that ρ_a are C^∞ in the variable z.)

5. The anisotropic Sobolev space

As the next step towards the proof of Theorem 2.13, we introduce the Hilbert space $\mathcal{H}$, called the anisotropic Sobolev space, and consider the action of the transfer operator $\mathcal{L}^t$ on it. The argument in this subsection is a modification of that in the previous papers[9, 19].

5.1. **Partial Bargmann transform.** Our basic idea in analyzing the transfer operator $\mathcal{L}^t$ is to consider its action in the frequency space. But, one because the direction of E_0^* depends on the base point sensitively, we also need to consider the action in the real space. The partial Bargmann transform, introduced below, meets these demands. We refer [19, Section 4-5], [9, Section 4.2-3] and [10, Section 3-4] and the references therein for more detailed accounts on the (partial) Bargmann transform.

For $(w,\xi,\eta)\in\mathbb{R}^{2+2+1}$, define $\phi_{w,\xi,\eta}:\mathbb{R}^3\to\mathbb{C}$ by

$$\phi_{w,\xi,\eta}(w',z') = 2^{-3/2}\pi^{-2}\cdot\langle\eta\rangle^{1/2}\cdot\exp\left(i\eta\cdot z + i\xi\cdot(w'-(w/2)) - \langle\eta\rangle\cdot|w'-w|^2/2\right).$$

The partial Bargmann transform $\mathfrak{B}: L^2(\mathbb{R}^{2+1})\to L^2(\mathbb{R}^{2+2+1})$ is defined by

$$\mathfrak{B}u(w,\xi,\eta) = \int \overline{\phi_{w,\xi,\eta}(w',z')}\cdot u(w',z')\,dw'dz'. \tag{57}$$

Remark 5.1. In the above and also henceforth, we write $\mathbb{R}^{2+1}$ (resp. $\mathbb{R}^{2+2+1}$) for the Euclidean space of dimension 3 (resp. 5) equipped with the standard coordinate $(w, z) = (x, y, z)$ (resp. $(w, \xi, \eta) = (x, y, \xi_x, \xi_y, \eta)$) where $w = (x, y) \in \mathbb{R}^2$ and $\xi = (\xi_x, \xi_y) \in \mathbb{R}^2$. We regard $\xi = (\xi_x, \xi_y)$ and η the dual variables of $w = (x, y)$ and z respectively.

The L^2-adjoint $\mathfrak{B}^* : L^2(\mathbb{R}^{2+2+1}) \to L^2(\mathbb{R}^{2+1})$ of the partial Bargmann transform $\mathfrak{B}$ is

$$\mathfrak{B}^* v(w', z') = \int \phi_{w,\xi,\eta}(w', z') \cdot v(w, \xi, \eta) dw d\xi d\eta. \tag{58}$$

Lemma 5.2. [19, Proposition 5.1] *The partial Bargmann transform $\mathfrak{B}$ is an L^2-isometric injection and $\mathfrak{B}^*$ is a bounded operator such that $\mathfrak{B}^* \circ \mathfrak{B} = \mathrm{Id}$. The composition*

$$\mathfrak{P} := \mathfrak{B} \circ \mathfrak{B}^* : L^2(\mathbb{R}^{2+2+1}) \to L^2(\mathbb{R}^{2+2+1}) \tag{59}$$

is the L^2 orthogonal projection onto the image of $\mathfrak{B}$.

5.2. **Decomposition of functions in the phase space.** We introduce a few C^∞ partitions of unity. Recall the function $\chi : \mathbb{R} \to [0, 1]$ defined in Subsection 3.2.

(1) a partition of unity on the projective space: $\{\chi_\sigma : \mathbb{P}^1 \to [0, 1] \mid \sigma = +, -\}$ such that

$$\chi_+([(x, y)]) = \begin{cases} 1, & \text{if } |x| \geq 2|y|; \\ 0, & \text{if } |y| \geq 2|x|, \end{cases} \qquad \text{and} \qquad \chi_+([(x, y)]) + \chi_-([(x, y)]) \equiv 1.$$

(2) a periodic partition of unity on the real line $\mathbb{R}$: $\{q_\omega : \mathbb{R} \to [0, 1] \mid \omega \in \mathbb{Z}\}$ such that

$$\operatorname{supp} q_\omega \subset [\omega - 1, \omega + 1], \qquad q_\omega(s) = q_0(s - \omega),$$

(3) a Littlewood-Paley type partition of unity: $\{\psi_m : \mathbb{R}^2 \to [0, 1] \mid m \in \mathbb{Z}_+\}$ defined by

$$\chi_m(w) = \begin{cases} \chi(|w|), & \text{if } m = 0; \\ \chi(e^{-m}|w|) - \chi(e^{-m+1}|w|), & \text{if } m > 0. \end{cases}$$

We define also the (anisotropic) partition of unity $\{\psi_m : \mathbb{R}^2 \to [0, 1] \mid m \in \mathbb{Z}\}$ by

$$\chi_m(x, y) = \chi_{\mathrm{sgn}(m)}([(x, y)]) \cdot \chi_{|m|}(x, y)$$

but we ignore the first term on the right-hand side when $m = 0$.

We next introduce partitions of unity on the phase space. For $a \in A$, $\omega \in \mathbb{Z}$, $n \in N(\omega)$ and $m \in \mathbb{Z}$, we define the function $\psi_{a,\omega,n,m} : \mathbb{R}^{2+2+1} \to [0, 1]$ by

$$\psi_{a,\omega,n,m}(w, \xi, \eta) = q_\omega(\eta) \cdot \chi_m\left(\langle\omega\rangle^{-1/2} \cdot \mathbf{\Delta}_{a,\omega,n}^{-1}(\xi - \eta \cdot e_{a,\omega,n}(w))\right) \tag{60}$$

where

$$\mathbf{\Delta}_{a,\omega,n} = \begin{pmatrix} \Delta_{a,\omega,n} & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Delta_{a,\omega,n} := \langle \Delta(p_{a,\omega,n}, \varkappa_\sharp \langle\omega\rangle^{-1/2})\rangle + t_\sharp^2.$$

Then we have, for each $a \in A$ and $n \in \mathcal{N}(a, \omega)$, that

$$\sum_m \psi_{a,\omega,n,m}(w, \xi, \eta) = q_\omega(\eta) \quad \text{and hence} \quad \sum_\omega \sum_m \psi_{a,\omega,n,m}(w, \xi, \eta) \equiv 1.$$

Remark 5.3. Notice that we have the factor $\mathbf{\Delta}_{a,\omega,n}^{-1}$ in (60). (This factor did not appear when we consider contact Anosov flows in [18, 19, 9] in a parallel manner.) In the present case, the factor $\Delta(p_{a,\omega,n}, \varkappa_\sharp\langle\omega\rangle^{-1/2})$ is not uniform in a, ω, n: it can be zero and also can be as large as $\mathcal{O}(\log\langle\omega\rangle)$ in absolute value. As we observed in Lemma 4.4, the function $e_{a,\omega,n}(w)$, which indicates the direction of E_0^* in the local chart $\kappa_{a,\omega,n}$, varies at a rate proportional to $\Delta(p_{a,\omega,n}, \varkappa_\sharp\langle\omega\rangle^{-1/2})$ in the ξ_y direction. We therefore need the factor $\mathbf{\Delta}_{a,\omega,n}$ in (60) in order to do with the case where $\Delta(p_{a,\omega,n}, \varkappa_\sharp\langle\omega\rangle^{-1/2})$ is large.

Remark 5.4. For the argument in the proofs in the next section, we define the family of functions $\tilde{\psi}_{a,\omega,n,m} : \mathbb{R}^{2+2+1} \to [0,1]$ by

$$\tilde{\psi}_{a,\omega,n,m}(w,\xi,\eta) = \tilde{q}_{\omega+1}(\eta) \cdot \tilde{\chi}_m \left(\langle\omega\rangle^{-1/2} \cdot A_m \cdot \mathbf{\Delta}^{-1}_{a,\omega,n}(\xi - \eta \cdot e_{a,\omega,n}(w)) \right) \tag{61}$$

where

$$\tilde{q}_\omega = q_{\omega-1} + q_\omega + q_{\omega+1}, \qquad \tilde{\chi}_m = \chi_{m-1} + \chi_m + \chi_{m+1} \tag{62}$$

and

$$A_m(\xi_x, \xi_y) = \begin{cases} (2\xi_x, \xi_y/2), & \text{if } m > 0; \\ (\xi_x, \xi_y), & \text{if } m = 0; \\ (\xi_x/2, 2\xi_y), & \text{if } m > 0. \end{cases}$$

Then we have $\tilde{\chi}_{a,\omega,n,m} = 1$ on the support of $\chi_{a,\omega,n,m}$.

For each C^r function u on M, we define a family of functions $\hat{u}_{a,\omega,n,m} : \mathbb{R}^{2+2+1} \to \mathbb{C}$ for $a \in A$, $\omega \in \mathbb{Z}$, $n \in N(\omega)$ and $m \in \mathbb{Z}$, by

$$\hat{u}_{a,\omega,n,m}(w,\xi,\eta) = \psi_{a,\omega,n,m}(w,\xi,\eta) \cdot \mathcal{B}(\rho_{a,\omega,n} \cdot (u \circ \kappa^{-1}_{a,\omega,n}))(w,\xi,\eta).$$

We regard this correspondence $u \mapsto (u_{a,\omega,n,m})$ as an operator

$$\mathbf{I} : C^r(M) \to \prod_{a,\omega,n,m} C_0^\infty(\operatorname{supp} \psi_{a,\omega,n,m}), \quad \mathbf{I}(u) = (\hat{u}_{a,\omega,n,m})_{a\in A,\omega\in\mathbb{Z},n\in N(\omega),m\in\mathbb{Z}}.$$

Remark 5.5. The support of $\mathcal{B}(\rho_{a,\omega,n} \cdot (u \circ \kappa^{-1}_{a,\omega,n}))$ must be $\mathbb{R}^2 \oplus \mathbb{R}^3$ unless $u_{a,\omega,n} = 0$ because they are real-analytic. But the functions $\mathcal{B}(\rho_{a,\omega,n} \cdot (u \circ \kappa^{-1}_{a,\omega,n}))$ decays extremely fast on the outside of $\operatorname{supp} \rho_{a,\omega,n} \times \mathbb{R}^3$. More precisely, the decay is exponential with respect to the distance from $\operatorname{supp} \rho_{a,\omega,n} \times \mathbb{R}^3$ in the scale $\langle\omega\rangle^{-1/2}$. Therefore we can basically neglect the part of the function $\mathcal{B}(\rho_{a,\omega,n} \cdot (u \circ \kappa^{-1}_{a,\omega,n}))$ on the outside of $D_{a,\omega,n} \times \mathbb{R}^3$.

The next lemma tells that the operator $\mathbf{I}^* : \bigoplus_{a,\omega,n,m} C_0^\infty(\operatorname{supp} \psi_{a,\omega,n,m}) \to C^r(M)$ defined by

$$\mathbf{I}^*((u_{a,\omega,n,m})_{a\in A,\omega\in\mathbb{Z},n\in N(\omega),m\in\mathbb{Z}}) = \sum_{a,\omega,n,m} (\tilde{\rho}_{a,\omega,n} \cdot \mathcal{B}^* u_{a,\omega,n,m}) \circ \kappa_{a,\omega,n},$$

gives a construction reverse to the decomposition in $\mathbf{I}$.

Lemma 5.6. $\mathbf{I}^* \circ \mathbf{I} = \mathrm{Id}$ *on* $C^r(M)$.

Proof. The claim is not trivial but can be checked by simple computations using (60) and the commutative relation

$$(\mathcal{B} \circ \mathcal{M}(q_\omega) \circ \mathcal{B}^*) \circ \mathcal{M}(\varrho_{a,\omega,n}) = \mathcal{M}(\varrho_{a,\omega,n}) \circ (\mathcal{B} \circ \mathcal{M}(q_\omega) \circ \mathcal{B}^*)$$

where $\mathcal{M}(\varphi)$ denotes the multiplication operator by φ. (The last commutative relation is a consequence of the fact that the operator $\mathcal{B} \circ \mathcal{M}(q_\omega) \circ \mathcal{B}^*$ is a convolution operator that involves only the z-variable, while $\varrho_{a,\omega,n}$ does not depend on z.) We refer [9, Section 4 and Lemma 6.5] for the details. ◻

We can now define the Hilbert space $\mathcal{H}$ of distributions. We henceforth fix $\alpha \in (0, 1/6)$. To simplify notation, we set

$$\mathcal{J} = \{(a,\omega,n,m) \mid a \in A, \omega \in \mathbb{Z}, n \in N(\omega), m \in \mathbb{Z}\}$$

and refer the components of $\mathbf{j} = (a,\omega,n,m) \in \mathcal{J}$ as $a(\mathbf{j}) = a$, $\omega(\mathbf{j}) = \omega$ and so on. Accordingly, we will write

$$\kappa_{\mathbf{j}} := \kappa_{a,\omega,n}, \quad \rho_{\mathbf{j}} := \rho_{a,\omega,n}, \quad \psi_{\mathbf{j}} := \psi_{a,\omega,n,m} \tag{63}$$

and so on. It will be useful to remember that the components a and n are related to the position, ω to the frequency in the flow direction and m to the frequency in the directions normal to the flow direction viewed from the central subspace E_0^*.

Definition 5.7. We define $\mathbb{H}$ as the Hilbert space obtained as the completion of the direct sum $\bigoplus_{\mathbf{j}\in\mathcal{J}} L^2(\operatorname{supp}\psi_{\mathbf{j}})$ with respect to the norm

$$\|(u_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}}\|_{\mathbb{H}} = \left(\sum_{\mathbf{j}\in\mathcal{J}} e^{\alpha\cdot m(\mathbf{j})}\|u_{\mathbf{j}}\|_{L^2}^2\right). \tag{64}$$

We define $\mathcal{H}$ as the Hilbert space of distributions on M that is obtained as the completion of $C^\infty(M)$ with respect to the norm $\|u\|_{\mathcal{H}} = \|\mathbf{I}(u)\|_{\mathbb{H}}$. Then we have

$$C^\alpha(M) \subset \mathcal{H} \subset (C^\alpha(M))' \quad \text{and} \quad C^\alpha(M) \subset \mathcal{H}' \subset (C^\alpha(M))'. \tag{65}$$

By definition, the operator $\mathbf{I}$ extends to an isometric injection $\mathbf{I} : \mathcal{H} \to \mathbb{H}$.

Remark 5.8. In order to check the inclusion (65), we use the characterization of Hölder space $C^\alpha(\mathbb{R}^d)$ in terms of the Littlewood-Paley decomposition. (See [16, Appendix A] for instance.)

We define the operator $\mathbb{L}^t$ formally by $\mathbb{L}^t = \mathbf{I} \circ \mathcal{L}^t \circ \mathbf{I}^*$, so that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\mathbb{L}^t} & \mathbb{H} \\ \mathbf{I}\uparrow & & \mathbf{I}\uparrow \\ \mathcal{H} & \xrightarrow{\mathcal{L}^t} & \mathcal{H} \end{array}$$

Remark 5.9. At this moment, we only know that the operator $\mathbb{L}^t$ is defined as an operator from $\bigoplus_{\mathbf{j}} C_0^\infty(\operatorname{supp}\psi_{\mathbf{j}})$ to $\prod_{\mathbf{j}} C_0^\infty(\operatorname{supp}\psi_{\mathbf{j}})$. We will see that it extends naturally to a bounded operator on $\mathbb{H}$ and consequently that $\mathcal{L}^t$ extends to a bounded operator on $\mathcal{H}$ when t is sufficiently large.

6. Proof of Theorem 2.13

We now give a proof of Theorem 2.13, making use of the definitions and propositions prepared in the previous two sections. We henceforth assume that $f^t \in \mathfrak{F}_A^3$ satisfies the non-integrability condition $(NI)_\rho$ for some $\rho > 0$ and suppose $t_\sharp \le t \le 2t_\sharp$. Below we first show that f^t is exponentially mixing. We will use the non-integrability condition $(NI)_\rho$ in the proof of Proposition 6.7 or, more specifically, in the estimate of the integral (85). Then, in the last subsection, we complete the proof of Theorem 2.13 by examining dependence of the argument on the flow. For this last part of the argument, we emphasize at this moment that, in proving exponential mixing for f^t below, we actually need the estimate (14) in the non-integrability condition $(NI)_\rho$ only for b in some bounded range. (See Remark 6.11.) This is crucial when we prove stability of exponential mixing.

In the following, let $\omega_\sharp > 0$ be the large constant in Lemma 4.5, but we will let it be larger if necessary. We will also introduce a large constant $m_\sharp > 0$ depending on $t_\sharp$ and $\omega_\sharp$. Beware that we will ignore some absolute constants, such as 2π, that appear in Fourier transform, partial Bargmann transform and Gaussian integrals, which are not essential at all.

6.1. **Estimates on the components of $\mathbb{L}^t$.** We write $\mathbb{L}^t_{\mathbf{j}\to\mathbf{j}'} : C_0^\infty(\operatorname{supp}\psi_{\mathbf{j}}) \to C_0^\infty(\operatorname{supp}\psi_{\mathbf{j}'})$ for the component of $\mathbb{L}^t$ that sends the $\mathbf{j}$-component to the $\mathbf{j}'$-component. We can write it as

$$\mathbb{L}^t_{\mathbf{j}\to\mathbf{j}'} u = \psi_{\mathbf{j}'} \cdot \mathcal{B} \circ \mathcal{L}^t_{\mathbf{j}\to\mathbf{j}'} \circ \mathcal{B}^* u \tag{66}$$

where

$$\mathcal{L}^t_{\mathbf{j}\to\mathbf{j}'} v = (\rho^t_{\mathbf{j}\to\mathbf{j}'} \cdot v) \circ (f^t_{\mathbf{j}\to\mathbf{j}'})^{-1} \tag{67}$$

with setting $\rho^t_{\mathbf{j}\to\mathbf{j}'} = (\rho_{\mathbf{j}'} \circ f^t_{\mathbf{j}\to\mathbf{j}'}) \cdot \tilde{\rho}_{\mathbf{j}}$ and $f^t_{\mathbf{j}\to\mathbf{j}'} = \kappa_{\mathbf{j}'} \circ f^t \circ \kappa_{\mathbf{j}}^{-1}$. (Recall (63) and that we have studied $f^t_{\mathbf{j}\to\mathbf{j}'}$ and $\rho^t_{\mathbf{j}\to\mathbf{j}'}$ in Subsection 4.3.) This is an integral operator with smooth kernel

$$K(w,\xi,\eta; w',\xi',\eta') = \psi_{\mathbf{j}'}(w',\xi',\eta') \cdot \int (\rho_{\mathbf{j}\to\mathbf{j}'} \cdot \phi_{w,\xi,\eta}) \circ (f^t_{\mathbf{j}\to\mathbf{j}'})^{-1}(z) \cdot \overline{\phi_{w',\xi',\eta'}(z)}\, dz \tag{68}$$

and in particular defines a compact operator from $L^2(\operatorname{supp}\psi_{\mathbf{j}})$ to $L^2(\operatorname{supp}\psi_{\mathbf{j}'})$.

We suppose that the constants $\omega_\sharp$ and $m_\sharp$ mentioned above are given and let $K : \mathbb{H} \to \mathbb{H}$ be the part of the operator $\mathbb{L}^t$ that consists of the components $\mathbb{L}^t_{\mathbf{j}\to\mathbf{j}'}$ with

$$\max\{|\omega(\mathbf{j})|, \omega(\mathbf{j}')|\} \le \omega_\sharp \quad \text{and} \quad \max\{|m(\mathbf{j})|, |m(\mathbf{j}')|\} \le m_\sharp.$$

This operator K consists of finitely many components and therefore compact regardless of $\omega_\sharp$ and $m_\sharp$. Let $\Pi_\omega : \mathbb{H} \to \mathbb{H}$ be the projection operator that extract the components with $\omega(\mathbf{j}) = \omega$. We are going to prove that the following proposition hold true if we let $t_\sharp$ sufficiently large.

Proposition 6.1. *There exists a constant $c > 0$ (independent of the choice of $t_\sharp$) such that*

$$\|\Pi_{\omega'} \circ (\mathbb{L}^t - K) \circ \Pi_\omega : \mathbb{H} \to \mathbb{H}\| \le \exp(-ct) \cdot \langle \omega' - \omega \rangle^{-1} \quad \textit{for } \omega, \omega' \in \mathbb{Z} \textit{ and } t_\sharp \le t \le 2t_\sharp.$$

This proposition implies that f^t is exponentially mixing. Indeed, from the proposition, we have that $\|\mathbb{L}^t - K : \mathbb{H} \to \mathbb{H}\| < e^{-(c/2)t}$ for $t_\sharp \le t \le 2t_\sharp$, letting $t_\sharp$ be larger if necessary. Since K is compact as we noted above, the essential spectral radius of $\mathbb{L}^t$ is bounded by $e^{-(c/2)t}$ and so is that of $\mathcal{L}^t : \mathcal{H} \to \mathcal{H}$. Since f^t is mixing[7], there is a unique eigenvalue 1 on the region $|z| \ge 1$, which is simple and for which the the spectral projector is the averaging with respect to the volume m, and the other part of the spectrum is contained in the region $|z| < e^{-c't}$ for some $c' > 0$. Therefore, if we set $\mathcal{H}_0 = \{u \in \mathcal{H} \mid \int u\, dm = 0\}$, we have

$$\|\mathcal{L}^t : \mathcal{H}_0 \to \mathcal{H}_0\| \le C e^{-c't} \quad \text{for } t \ge 0.$$

Now, from (65), we conclude

$$\int \varphi \cdot (\psi \circ f^t)\, dm = \int \psi \cdot \mathcal{L}^t \varphi\, dm \le \|\psi\|_{\mathcal{H}'} \cdot \|\mathcal{L}^t \varphi\|_{\mathcal{H}} \le C e^{-c't} \cdot \|\psi\|_{C^\alpha(M)} \cdot \|\varphi\|_{C^\alpha(M)} \tag{69}$$

for $\varphi, \psi \in C^\alpha(M)$ with $\int \varphi\, dm = 0$.

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[7] It is easy to see that $(NI)_\rho$ implies joint non-integrability of the stable and unstable foliations and hence f^t is stably mixing. See the argument in the proof of Proposition 6.1 for instance.

6.2. **Estimates on components $\mathbb{L}^t_{\mathbf{j}\to\mathbf{j}'}$.** Below we present a few estimates on the components $\mathbb{L}^t_{\mathbf{j}\to\mathbf{j}'}$ *with respect to the L^2 norm* and prove that Proposition 6.1 follows from them. The proofs of the estimates are deferred to the succeeding two subsections.

To begin with, note that we have

$$\|\mathbb{L}^t_{\mathbf{j}\to\mathbf{j}'}\|_{L^2} \le 1 \quad \text{for any } t \ge 0 \text{ and } \mathbf{j}, \mathbf{j}' \in \mathcal{J} \tag{70}$$

because neither of $\mathcal{B}$, $\mathcal{B}^*$ and $\mathcal{L}^t$ increases the L^2 norm. From the expression (68) of the kernel, we observe that $\mathbb{L}^t_{\mathbf{j}\to\mathbf{j}'}$ is localized in the space. To state a consequence of this observation, let

$$U_{\mathbf{j}\to\mathbf{j}'} = \{w\mathbb{R}^2 \mid \|w - w'\| < \varkappa_\sharp^{1/2}\langle\omega(\mathbf{j})\rangle^{-1/2} \text{ for some } (w', z') \in \operatorname{supp}\rho_{\mathbf{j}\to\mathbf{j}'}\}$$

and similarly

$$\widetilde{U}_{\mathbf{j}\to\mathbf{j}'} = \{w \in \mathbb{R}^2 \mid \|w - w'\| < \varkappa_\sharp^{1/2}\langle\omega(\mathbf{j}')\rangle^{-1/2} \text{ for some } (w', z') \in f^t_{\mathbf{j}\to\mathbf{j}'}(\operatorname{supp}\rho_{\mathbf{j}\to\mathbf{j}'})\}.$$

Also we put

$$V_{\mathbf{j}\to\mathbf{j}'} = U_{\mathbf{j}\to\mathbf{j}'} \times \mathbb{R}^3 = \{(w, \xi, \eta) \in \mathbb{R}^{2+2+1} \mid w \in U_{\mathbf{j}\to\mathbf{j}'}\}, \qquad \widetilde{V}_{\mathbf{j}\to\mathbf{j}'} = \widetilde{U}_{\mathbf{j}\to\mathbf{j}'} \times \mathbb{R}^3$$

(Note that $U_{\mathbf{j}\to\mathbf{j}'}$ and $\widetilde{U}_{\mathbf{j}\to\mathbf{j}'}$ do not depend on $m(\mathbf{j})$ and $m(\mathbf{j}')$.) Then, for the operator

$$\hat{\mathbb{L}}^t_{\mathbf{j}\to\mathbf{j}'} : L^2(\operatorname{supp}\psi_{\mathbf{j}}) \to L^2(\operatorname{supp}\psi_{\mathbf{j}'}), \quad \hat{\mathbb{L}}^t_{\mathbf{j}\to\mathbf{j}'} u = \mathbf{1}_{\widetilde{V}_{\mathbf{j}\to\mathbf{j}'}} \cdot \mathbb{L}^t_{\mathbf{j}\to\mathbf{j}'}(\mathbf{1}_{V_{\mathbf{j}\to\mathbf{j}'}} \cdot u), \tag{71}$$

we have that, for any $\nu > 0$,

$$\|\mathbb{L}^t_{\mathbf{j}\to\mathbf{j}'} - \hat{\mathbb{L}}^t_{\mathbf{j}\to\mathbf{j}'}\|_{L^2} \le C_*(\nu) \cdot \varkappa_\sharp^{-\nu} \quad \text{for } t_\sharp \le t \le 2t_\sharp \text{ and } \mathbf{j}, \mathbf{j}' \in \mathcal{J}. \tag{72}$$

Remark 6.2. From Remark 4.2, we see that, for any $\omega, \omega' \in \mathbb{Z}$ and $m, m' \in \mathbb{Z}$ and for each $\mathbf{j} \in \mathcal{J}$ with $\omega(\mathbf{j}) = \omega$ (resp. $\mathbf{j}' \in \mathcal{J}$ with $\omega(\mathbf{j}') = \omega'$), the intersection multiplicity of

$$\{U_{\mathbf{j}\to\mathbf{j}'} \mid \mathbf{j}' \in \mathcal{J}, \omega(\mathbf{j}') = \omega', m(\mathbf{j}') = m'\} \qquad (\text{ resp. } \{\widetilde{U}_{\mathbf{j}\to\mathbf{j}'} \mid \mathbf{j} \in \mathcal{J}, \omega(\mathbf{j}) = \omega, m(\mathbf{j}) = m\})$$

is bounded by an absolute constant.

We next consider the localized property of the operator $\mathbb{L}^t$ in the phase space. The next lemma is a consequence of the fact that flow f^t is just a translation in each of the flow lines.

Lemma 6.3. *For any $\nu > 0$, there exists a constant $C_*(\nu) > 0$ such that*

$$\|\mathbb{L}^t_{\mathbf{j}\to\mathbf{j}'}\|_{L^2} \le C_*(\nu) \cdot \langle\omega(\mathbf{j}) - \omega(\mathbf{j}')\rangle^{-\nu}$$

and further

$$\|\mathbb{L}^t_{\mathbf{j}\to\mathbf{j}'} - \hat{\mathbb{L}}^t_{\mathbf{j}\to\mathbf{j}'}\|_{L^2} \le C_*(\nu) \cdot \varkappa_\sharp^{-\nu} \cdot \langle\omega(\mathbf{j}) - \omega(\mathbf{j}')\rangle^{-\nu}$$

for $t_\sharp \le t \le 2t_\sharp$ and $\mathbf{j}, \mathbf{j}' \in \mathcal{J}$.

We omit the proof of this lemma because the required estimates are obtained without difficulty if one notes that the partial Bargmann transform $\mathfrak{B}$ is just the Fourier transform in the z variable combined with the Bargmann transform[13] in $w = (x, y)$ with some scaling depending on the frequency in z. (We refer [9, Proof of Lemma 9.8] for more detail, where a slightly different estimates are proved but the argument is completely parallel.)

Below we give another estimate on the localized property of $\mathbb{L}^t$ on the phase space, which is based on hyperbolicity of the flow f^t. We first introduce the following definition. Let $0 < \delta_* < \rho_*$ be small constants, independent of $t_\sharp$, which we will specify later.

Definition 6.4. We write $\mathbf{j} \hookrightarrow^t \mathbf{j}'$ for $\mathbf{j}, \mathbf{j}' \in \mathcal{J}$ and $t \ge 0$ if either

(1) $m(\mathbf{j}) \ge 0$ and $m(\mathbf{j}') \le 0$, or
(2) $m(\mathbf{j}) \cdot m(\mathbf{j}') > 0$ and $m(\mathbf{j}') \le m(\mathbf{j}) - [(\delta_*/2)t] + 10$.

Otherwise we write $\mathbf{j} \not\hookrightarrow^t \mathbf{j}'$.

To understand the meaning of this definition, we make the following observation.

Lemma 6.5. *Suppose that $t_\sharp \le t \le 2t_\sharp$ and that $\mathbf{j} \not\hookrightarrow^t \mathbf{j}'$ for $\mathbf{j}, \mathbf{j}' \in \mathcal{J}$. If*

$$\max\{|m(\mathbf{j})|, |m(\mathbf{j}')|\} \ge \delta_* t_\sharp \tag{73}$$

and

$$\max\{|\omega(\mathbf{j})|, |\omega(\mathbf{j}')|\} \ge \omega_\sharp, \qquad \langle \omega(\mathbf{j}') - \omega(\mathbf{j})\rangle \le \langle \omega(\mathbf{j})\rangle^{1/2}, \tag{74}$$

then $(Df^t_{\mathbf{j}\to\mathbf{j}'})^*(\operatorname{supp}\psi_{\mathbf{j}'})$ is separated from $\operatorname{supp}\psi_{\mathbf{j}}$ in the following sense: for any $(w,\xi,\eta) \in \operatorname{supp}\psi_{\mathbf{j}}$ with $w \in U_{\mathbf{j}\to\mathbf{j}'}$ and for any $(w',\xi',\eta') \in \operatorname{supp}\psi_{\mathbf{j}'}$, we have

$$\langle \langle \omega(\mathbf{j})\rangle^{1/2} |w'-w|\rangle \cdot \langle \langle \omega(\mathbf{j})\rangle^{-1/2} |\mathbf{\Delta}_{\mathbf{j}}^{-1}((D\hat{f}^t_{\mathbf{j}\to\mathbf{j}'})^*_w(\xi') - \xi)|\rangle \ge C_*^{-1} e^{\max\{|m(\mathbf{j})|,|m(\mathbf{j}')|\}/2} \tag{75}$$

where $\hat{f}^t_{\mathbf{j}\to\mathbf{j}'}$ is the diffeomorphism that appears in the expression (49) of $f^t_{\mathbf{j}\to\mathbf{j}'}$.

Proof. From (18), we have $|\Delta_{\mathbf{j}'} - \Delta_{\mathbf{j}}| \le C_* t_\sharp$ and hence the ratio between $\langle \Delta_{\mathbf{j}'}\rangle$ and $\langle \Delta_{\mathbf{j}}\rangle$ is bounded by $C_* t_\sharp$. (Note that this is much smaller than the factor $e^{(\chi_*/2)t} \le e^{(\chi_*/2)t_\sharp}$.) From Lemma 4.4 and the definition of $\mathbf{\Delta}_{\mathbf{j}}$, we have

$$|\mathbf{\Delta}_{\mathbf{j}}^{-1} \cdot e_{\mathbf{j}}(w') - \mathbf{\Delta}_{\mathbf{j}}^{-1} \cdot e_{\mathbf{j}}(w)| \le C_* \langle \omega(\mathbf{j})\rangle^{1/2} \cdot \langle \langle \omega(\mathbf{j})\rangle^{-1/2} |w'-w|\rangle.$$

Hence, provided that $\omega(\mathbf{j}') = \omega(\mathbf{j})$, we obtain the conclusion of the lemma by simple geometric consideration on hyperbolicity of $f^t_{\mathbf{j}\to\mathbf{j}'}$ and the position of the supports of $\psi_{\mathbf{j}}$ and $\psi_{\mathbf{j}'}$. For the case $\omega(\mathbf{j}') \ne \omega(\mathbf{j})$, we note that, if we replace (w',ξ',η') by (w',ξ'',η) with setting

$$\xi'' = (\langle \omega(\mathbf{j})\rangle / \langle \omega(\mathbf{j}')\rangle)^{1/2} (\xi' - \eta' e_{\mathbf{j}'}(w')) + \eta e_{\mathbf{j}'}(w'),$$

we can apply the argument in the case $\omega(\mathbf{j}') = \omega(\mathbf{j})$ to show the required estimate. But, for the difference between ξ'' and ξ', we have

$$|\mathbf{\Delta}_{\mathbf{j}'}^{-1}(\xi''-\xi')| \le \left|1 - (\langle \omega(\mathbf{j})\rangle/\langle \omega(\mathbf{j}')\rangle)^{1/2}\right| \cdot \left|\mathbf{\Delta}_{\mathbf{j}'}^{-1}(\xi' - \eta' \cdot e_{\mathbf{j}'}(w'))\right| + |\eta - \eta'| \cdot |\mathbf{\Delta}_{\mathbf{j}'}^{-1} \cdot e_{\mathbf{j}'}(w')|$$

and, from (74), we see that the right-hand side is much smaller than $e^{|m(\mathbf{j}')|}\langle \omega(\mathbf{j}')\rangle^{1/2}$. Therefore, regarding this as an error term, we obtain the required estimate in the case $\omega(\mathbf{j}') \ne \omega(\mathbf{j})$. □

Since $(D^* f^t_{\mathbf{j}\to\mathbf{j}'})^{-1}$ is the canonical map associated to the transfer operator $\mathcal{L}^t_{\mathbf{j}\to\mathbf{j}'}$ regarded as a Fourier integral operator[8], the last lemma provides intuition to the next lemma.

Lemma 6.6. *For any $\nu > 0$, there exists a constant $C_*(\nu) > 0$ such that, for $t_\sharp \le t \le 2t_\sharp$ and $\mathbf{j}, \mathbf{j}' \in \mathcal{J}$ satisfying $\mathbf{j} \not\hookrightarrow^t \mathbf{j}'$ and (73), we have*

$$\|\mathbb{L}^t_{\mathbf{j}\to\mathbf{j}'}\|_{L^2} \le C_*(\nu) \cdot e^{-\max\{|m(\mathbf{j})|,|m(\mathbf{j}')|\}/2} \cdot \langle \omega(\mathbf{j}') - \omega(\mathbf{j})\rangle^{-\nu}$$

and further

$$\|\mathbb{L}^t_{\mathbf{j}\to\mathbf{j}'} - \hat{\mathbb{L}}^t_{\mathbf{j}\to\mathbf{j}'}\|_{L^2} \le C_*(\nu) \cdot e^{-\max\{|m(\mathbf{j})|,|m(\mathbf{j}')|\}/2} \cdot \varkappa_\sharp^{-\nu} \cdot \max\{\langle \omega(\mathbf{j})\rangle, \langle \omega(\mathbf{j}')\rangle\}^{-\nu}.$$

This lemma together with Lemma 6.3 and the definition of the Hilbert space $\mathcal{H}$ works efficiently for the components $\mathbb{L}^t_{\mathbf{j}\to\mathbf{j}'}$ for which $\max\{|m(\mathbf{j})|, |m(\mathbf{j}')|\}$ is sufficiently large. But our main concern in the proof of Proposition 6.1 is the remaining cases. Such cases are dealt with in the following key proposition.

[8] See [10, Remark 2.5] for an account on this viewpoint.

Proposition 6.7. *There exist constants* $0 < \delta_* < \rho_*$, *which are independent of the choice of* $t_\sharp$ *such that, if* $\mathbf{j}, \mathbf{j}' \in \mathcal{J}$ *satisfy*

$$\max\{|m(\mathbf{j})|, |m(\mathbf{j}')|\} \le \delta_* t_\sharp, \quad |\omega(\mathbf{j})| \ge \omega_\sharp \quad \textit{and} \quad \langle \omega(\mathbf{j}') - \omega(\mathbf{j})\rangle \le \exp(\rho_* t_\sharp/10), \tag{76}$$

we have

$$\|\hat{\mathbb{L}}^t_{\mathbf{j}\to\mathbf{j}'}\|_{L^2} \le \exp(-\rho_* t_\sharp) \quad \textit{for } t_\sharp \le t \le 2t_\sharp.$$

Below we deduce Proposition 6.1 (and hence Theorem 2.13) from Lemma 6.3, Lemma 6.6 and Proposition 6.7.

Proof of Proposition 6.1. Let $0 < \delta_* < \rho_*$ be those constants in Proposition 6.7, which does not depend on the choice of $t_\sharp$. We may and do suppose that δ_* is much smaller than ρ_*. Let $\omega, \omega' \in \mathbb{Z}$ be those in the statement of Proposition 6.1. Below we proceed with the assumption that

$$|\omega| \ge \omega_\sharp/2 \quad \text{and} \quad |\omega' - \omega| < \exp(\alpha\delta_* t_\sharp/10). \tag{77}$$

In the case where this assumption does not hold, the proof is much simpler. We will consider such case at the end of this proof.

Let us take $m, m' \in \mathbb{Z}$ and consider the components $\mathbb{L}^t_{\mathbf{j}\to\mathbf{j}'}$ for $\mathbf{j}, \mathbf{j}' \in \mathcal{J}$ satisfying

$$\omega(\mathbf{j}) = \omega, \quad m(\mathbf{j}) = m, \quad \omega(\mathbf{j}') = \omega', \quad m(\mathbf{j}') = m'. \tag{78}$$

Note at this moment that, since the operator $\Pi_{\omega'} \circ (\mathbb{L}^t - K) \circ \Pi_\omega$ decompose one component of $\mathbb{H}$ into many parts and send each of them to different component. Also parts of many components are sent to one component. Indeed, even with the restriction (78), for each $\mathbf{j}$ (resp. $\mathbf{j}' \in \mathcal{J}$), the cardinality of the set

$$\{\mathbf{j}' \mid \rho^t_{\mathbf{j}\to\mathbf{j}'} \neq 0\} \qquad (\text{ resp. } \{\mathbf{j} \mid \rho^t_{\mathbf{j}\to\mathbf{j}'} \neq 0\}\ ) \tag{79}$$

may be large (*i.e.* grow exponentially with respect to t) and this is not be negligible. We therefore face the problem that the decomposition and superposition of functions may increase the L^2 norm. Our idea to do with this problem is that

- if we consider the operator $\hat{\mathbb{L}}^t_{\mathbf{j}\to\mathbf{j}'}$ defined in (71) instead of $\mathbb{L}^t_{\mathbf{j}\to\mathbf{j}'}$, we will not have this problem by virtue of the property noted in Remark 6.2, and
- the norm $\|\hat{\mathbb{L}}^t_{\mathbf{j}\to\mathbf{j}'} - \mathbb{L}^t_{\mathbf{j}\to\mathbf{j}'}\|_{L^2}$ of the difference is very small and dominates the cardinality of (79) which is bounded by $C_* \exp(C_* t_\sharp)$ uniformly in ω, ω', m, m'.

Below we proceed with this idea in mind. We consider the following three cases for the combination $(m, m') \in \mathbb{Z}^2$:

(i) those satisfying $\max\{|m|, |m'|\} \le \delta_* t_\sharp$,
(ii) those not in (i), but satisfying the assumption of Lemma 6.6,
(iii) those not either in (i) and (ii).

Let us first consider the case (i). If we consider $\hat{\mathbb{L}}^t_{\mathbf{j}\to\mathbf{j}'}$ in the place of $\mathbb{L}^t_{\mathbf{j}\to\mathbf{j}'}$, then, by Proposition 6.7 and the idea mentioned above, the operator norm (with respect to the norm on $\mathbb{H}$) of the totality of components satisfying (78) is bounded by $C_* e^{2\alpha\delta_* t_\sharp} \cdot e^{-\rho_* t}$, where the first factor $e^{2\alpha\delta_* t_\sharp}$ appears because of the weight in the definition of $\mathbb{H}$. For the differences between $\hat{\mathbb{L}}^t_{\mathbf{j}\to\mathbf{j}'}$ and $\mathbb{L}^t_{\mathbf{j}\to\mathbf{j}'}$, we can apply the second claim of Lemma 6.3 to see that they are indeed negligible.

Next we consider the case (ii). Again, if we consider $\hat{\mathbb{L}}^t_{\mathbf{j}\to\mathbf{j}'}$ in the place of $\mathbb{L}^t_{\mathbf{j}\to\mathbf{j}'}$, then, from the first claim of Lemma 6.6 and the idea mentioned above, the operator norm of the totality of components satisfying (78) is bounded by $C_* e^{-((1/2)-2\alpha)\max\{|m|,|m'|\}}$. For the differences between $\hat{\mathbb{L}}^t_{\mathbf{j}\to\mathbf{j}'}$ and $\mathbb{L}^t_{\mathbf{j}\to\mathbf{j}'}$, we can apply the second claim of Lemma 6.6 to see that they are negligible.

Finally we consider the case (iii). The weight in the definition on the Hilbert space $\mathbb{H}$ plays its roll in this case. Like the two cases above, we first suppose that $\mathbb{L}^t_{\mathbf{j}\to\mathbf{j}'}$ are replaced by $\hat{\mathbb{L}}^t_{\mathbf{j}\to\mathbf{j}'}$. Then we could apply the first claim of Lemma 6.3 to each component. By the idea mentioned above (again), we see that the the operator norm (on $\mathbb{H}$) of the totality of components satisfying (78) is bounded by $C_* e^{\alpha(m'-m)}$. For the differences between $\hat{\mathbb{L}}^t_{\mathbf{j}\to\mathbf{j}'}$ and $\mathbb{L}^t_{\mathbf{j}\to\mathbf{j}'}$, we apply the second claim of Lemma 6.3 to see that they are negligible. Note that, from the definition of the relation $\mathbf{j} \hookrightarrow^t \mathbf{j}'$, we have that $e^{\alpha(m'-m)}$ is bounded by $C_* \max\{e^{-\alpha(\chi_*/2)}, e^{-\alpha\delta_* t_\sharp}\}$ in this case.

Collecting the estimates in the cases (i), (ii) and (iii) above and taking sum with respect to the combinations $(m,m') \in \mathbb{Z}^2$, we obtain that the operator norm of the operator $\Pi_{\omega'} \circ (\mathbb{L}^t - K) \circ \Pi_\omega$ on $\mathbb{H}$ is bounded by

$$C_*(\alpha) \max\{\delta_* t_\sharp e^{(2\alpha\delta_* - \rho_*)t_\sharp},\ e^{-((1/2)-2\alpha)\delta_* t_\sharp},\ \delta_* t_\sharp e^{-\alpha\delta_* t_\sharp},\ e^{-\alpha(\chi_*/2)t_\sharp}\}.$$

Since we are assuming $|\omega' - \omega| < \exp(\alpha\delta_* t_\sharp/10)$, this gives the conclusion of Proposition 6.1, provided that δ_* is sufficiently small and $t_\sharp$ is sufficiently large.

In the case where the assumption (77) does not hold, the proof is parallel to the argument above but it becomes much simpler. Indeed,

- In the case where $|\omega| \le \omega_\sharp/2$ and $|\omega'| \le \omega_\sharp$, we may assume $\max\{|m|,|m'|\} \ge m_\sharp$ since we subtract the compact part K from $\mathbb{L}^t$. Since we can choose large $m_\sharp$ depending on $t_\sharp$ and $\omega_\sharp$, we need not consider the case (i). Then the proof goes as well as the argument above.
- In the case where $|\omega| \le \omega_\sharp/2$ and $|\omega'| \ge \omega_\sharp$, we have $|\omega' - \omega| \ge \omega_\sharp/2$. In this case and also in the case $|\omega' - \omega| \ge e^{\alpha\delta_* t/10}$, we may suppose that the factors $\langle\omega' - \omega\rangle^{-\nu}$ that appear in Lemma 6.3 and Lemma 6.6 are small enough by letting $\omega_\sharp$ and ν large. Then we need not distinguish the case (i) from the others and we can go through the argument above.

In particular, we do not have to invoke Proposition 6.7 in either of these remaining cases. We have finished the proof of Proposition 6.1. □

In the following subsections, we prove Lemma 6.3, Lemma 6.6 and Proposition 6.7. We present the proof of Proposition 6.7 first in the next subsection, since this is the most important.

6.3. **Proof of Proposition 6.7.** Let us consider the operator $\hat{\mathbb{L}}^t_{\mathbf{j}\to\mathbf{j}'}$ for $\mathbf{j},\mathbf{j}' \in \mathcal{J}$ satisfying (76). We use the notation (78) for brevity. By Lemma 4.5, we write $f^t_{\mathbf{j}\to\mathbf{j}'} = A^t_{\mathbf{j}\to\mathbf{j}'} \circ G^t_{\mathbf{j}\to\mathbf{j}'}$ and then write

$$\hat{\mathbb{L}}^t_{\mathbf{j}\to\mathbf{j}'} = (\mathbf{1}_{\widetilde{V}_{\mathbf{j}\to\mathbf{j}'}} \cdot \psi_{\mathbf{j}}) \cdot \mathbb{A} \circ \mathbb{G}$$

with setting

$$\mathbb{G} : L^2(\operatorname{supp}\psi_{\mathbf{j}}) \to L^2(\mathbb{R}^{2+2+1}), \quad \mathbb{G}u = \mathfrak{B}\left((\rho^t_{\mathbf{j}\to\mathbf{j}'} \cdot \mathfrak{B}^*(\mathbf{1}_{V_{\mathbf{j}\to\mathbf{j}'}} \cdot u)) \circ (G^t_{\mathbf{j}\to\mathbf{j}'})^{-1}\right)$$

and

$$\mathbb{A} : L^2(\mathbb{R}^{2+2+1}) \to L^2(\mathbb{R}^{2+2+1}), \quad \mathbb{A}u = \mathfrak{B}((\mathfrak{B}^* u) \circ (A^t_{\mathbf{j}\to\mathbf{j}'})^{-1}).$$

Remark 6.8. Since $A^t_{\mathbf{j}\to\mathbf{j}'}$ is of a special form, we can compute the kernel of $\mathbb{A}$ explicitly. See [13, Chapter 3] for instance. But we will not use this in the following.

Remark 6.9. Since we have multiplication by $\rho^t_{\mathbf{j}\to\mathbf{j}'}$ in $\mathbb{G}$, it is clear that the term $\mathbf{1}_{\widetilde{V}_{\mathbf{j}\to\mathbf{j}'}}$ in the definition of $\hat{\mathbb{L}}^t_{\mathbf{j}\to\mathbf{j}'}$ hardly do harm in estimate the operator norm of $\hat{\mathbb{L}}^t_{\mathbf{j}\to\mathbf{j}'}$. For this reason, we will ignore this term in some places below to avoid tedious detailed argument.

To proceed, let us set

$$e_{\mathbf{j}}(w) = (\theta^u(w), \theta^s(w)) := e_{a(\mathbf{j}),\omega(\mathbf{j}),n(\mathbf{j})}(w)$$

and define

$$\Psi_{\mathbf{j}} : \mathbb{R}^{2+2+1} \to \mathbb{R}, \qquad \Psi_{\mathbf{j}}(w,\xi,\eta) = \tilde{q}_{\omega(\mathbf{j})}(\eta) \cdot \chi\Big(e^{-\delta_* t_\sharp}\langle\omega\rangle^{-1/2} \cdot \mathbf{\Delta}_{\mathbf{j}}^{-1}(\xi - \eta \cdot e_{\mathbf{j}}(w))\Big).$$

Remark 6.10. The function $\Psi_{\mathbf{j}}$ is of similar nature as $\psi_{\mathbf{j}}$, though its support is much larger in ξ direction. Below we consider $\Psi_{\mathbf{j}}$ instead of $\psi_{\mathbf{j}}$ because we will later consider the pre-composition of $\mathbb{G}$.

As the main step of the proof, we prove

$$\|(\mathbf{1}_{\widetilde{V}_{\mathbf{j}\to\mathbf{j}'}}\Psi_{\mathbf{j}'}) \circ \mathbb{A} : L^2(\operatorname{supp}\Psi_{\mathbf{j}} \cap V_{\mathbf{j}\to\mathbf{j}'}) \to L^2(\operatorname{supp}\Psi_{\mathbf{j}'} \cap \widetilde{V}_{\mathbf{j}\to\mathbf{j}'})\| \le e^{-\rho_* t_\sharp} \tag{80}$$

where $\mathbf{1}_{\widetilde{V}_{\mathbf{j}\to\mathbf{j}'}}\Psi_{\mathbf{j}'}$ denotes the multiplication operator by $\mathbf{1}_{V_{\mathbf{j}\to\mathbf{j}'}}\Psi_{\mathbf{j}'}$. To this end, we are going to estimate the operator norm of

$$\mathbf{1}_{\operatorname{supp}\Psi_{\mathbf{j}}\cap V_{\mathbf{j}\to\mathbf{j}'}} \circ \mathbb{A}^* \circ (\mathbf{1}_{\widetilde{V}_{\mathbf{j}\to\mathbf{j}'}}\Psi_{\mathbf{j}'})^2 \circ \mathbb{A} : L^2(\operatorname{supp}\Psi_{\mathbf{j}} \cap V_{\mathbf{j}\to\mathbf{j}'}) \to L^2(\operatorname{supp}\Psi_{\mathbf{j}} \cap V_{\mathbf{j}\to\mathbf{j}'}), \tag{81}$$

which equals the square of the left-hand side of (80). Let us recall the expression (51) of the diffeomorphism $A^t_{\mathbf{j}\to\mathbf{j}'}$. Below we suppose $A^t_{\mathbf{j}\to\mathbf{j}'}(0) = 0$ by shifting the coordinates, hence

$$A^t_{\mathbf{j}\to\mathbf{j}'}(x,y,z) = (\lambda x, \tilde{\lambda} y, z + b\cdot(x,y) + \beta xy)$$

where λ, $\tilde{\lambda}$, b and σ are those given in (52) with $f = f^t_{\mathbf{j}\to\mathbf{j}'}$. The inverse of $A^t_{\mathbf{j}\to\mathbf{j}'}$ is then written

$$(A^t_{\mathbf{j}\to\mathbf{j}'})^{-1}(x,y,z) = \left(\Lambda^{-1}\begin{pmatrix}x\\y\end{pmatrix}, z - b\cdot\Lambda^{-1}\begin{pmatrix}x\\y\end{pmatrix} - \sigma(x,y)\right)$$

where

$$\Lambda = \begin{pmatrix}\lambda & 0\\ 0 & \tilde{\lambda}\end{pmatrix} \quad \text{and} \quad \sigma(x,y) = \beta\lambda^{-1}\tilde{\lambda}^{-1}xy.$$

We write the operator $\mathbb{A}$ as an integral operator

$$\mathbb{A}u(w'',\xi'',\eta) = \int K_{\mathbb{A}}(w,\xi;w'',\xi'';\eta)\, u(w,\xi,\eta)\, dw d\xi$$

with the kernel

$$K_{\mathbb{A}}(w,\xi;w'',\xi'';\eta) = e^{i\xi w/2 - i\xi'' w''/2}\cdot k_{\mathbb{A}}(w,\xi;w'',\xi'';\eta)$$

where[9]

$$k_{\mathbb{A}}(w,\xi;w'',\xi'';\eta) = e^{-i\xi w/2 + i\xi'' w''/2}\cdot \int \overline{\phi_{w'',\xi'',\eta}(\tilde{w},z)}\cdot \phi_{w,\xi,\eta}((A^t_{\mathbf{j}\to\mathbf{j}'})^{-1}(\tilde{w},z))\, d\tilde{w}. \tag{82}$$

[9]The right hand side of (82) does not depend on z. Also note that we separate the term $e^{-i\xi w/2 + i\xi'' w''/2}$ on purpose.

Using the expression of $(A^t_{\mathbf{j}\to\mathbf{j}'})^{-1}$ above and changing the variable $\tilde{w}$ to $\tilde{w} + w''$, we rewrite the last expression as

$$k_{\mathbb{A}}(w,\xi;w'',\xi'';\eta) =$$
$$\langle\eta\rangle \int d\tilde{w}\ \exp(i(\xi(\Lambda^{-1}(\tilde{w}+w'')-w) - \xi''(\tilde{w}+w'') - \eta b\cdot\Lambda^{-1}(\tilde{w}+w'') - \eta\cdot\sigma(\tilde{w}+w'')))$$
$$\cdot \exp(-\langle\eta\rangle|\Lambda^{-1}(\tilde{w}+w'')-w|^2/2 - \langle\eta\rangle|\tilde{w}|^2/2).$$

Then we can write

$$(\mathbb{A}^*\circ(\mathbf{1}_{\widetilde{V}_{\mathbf{j}\to\mathbf{j}'}}\Psi_{\mathbf{j}'})^2\circ\mathbb{A})u(w',\xi',\eta) = \int e^{-i\xi w/2+i\xi' w'/2}\cdot\mathcal{K}(w',\xi';w,\xi;\eta)\,u(w,\xi,\eta)\,dwd\xi \tag{83}$$

where, introducing the variable $\zeta = \xi'' - \eta e_{\mathbf{j}}(w'')$, we set

$$\mathcal{K}(w',\xi';w,\xi;\eta) = q_\omega(\eta)^2\cdot\int d\zeta\int_{\widetilde{U}_{\mathbf{j}\to\mathbf{j}'}} dw''\cdot\chi\left(e^{-\delta_* t_\sharp}\langle\omega\rangle^{-1/2}\cdot\mathbf{\Delta}_{\mathbf{j}'}^{-1}\zeta\right)^2$$
$$\times k_{\mathbb{A}}(w,\xi;w'',\zeta+\eta e_{\mathbf{j}'}(w'');\eta)\cdot\overline{k_{\mathbb{A}}(w',\xi';w'',\zeta+\eta e_{\mathbf{j}'}(w'');\eta)}. \tag{84}$$

In the integral on the right-hand side of (84), we are going to compute the integration with respect to the variable $w'' = (x'',y'')$. If we extract the part in (84) that is related to the variable $w'' = (x'',y'')$, we find

$$I(w,\xi;w',\xi';\tilde{w},\tilde{w}';\eta) := \tag{85}$$
$$\int_{\widetilde{U}_{\mathbf{j}\to\mathbf{j}'}} dx''dy''\exp(-i\eta(\tilde{y}-\tilde{y}')\cdot\theta^s(x'',y'') + i(\xi_x-\xi'_x)\lambda^{-1}x'' - i\beta\lambda^{-1}\tilde{\lambda}^{-1}\eta(\tilde{y}-\tilde{y}')x'')$$
$$\cdot\exp(-i\eta(\tilde{x}-\tilde{x}')\cdot\theta^u(x'',y''))\times\exp(i(\xi_y-\xi'_y)\tilde{\lambda}^{-1}y'' - i\eta\beta\lambda^{-1}\tilde{\lambda}^{-1}(\tilde{x}-\tilde{x}')y'')$$
$$\cdot\exp(-\langle\eta\rangle|\lambda^{-1}(\tilde{x}+x'')-x|^2/2 - \langle\eta\rangle|\lambda^{-1}(\tilde{x}'+x'')-x'|^2/2)$$
$$\cdot\exp(-\langle\eta\rangle|\tilde{\lambda}^{-1}(\tilde{y}+y'')-y|^2/2 - \langle\eta\rangle|\tilde{\lambda}^{-1}(\tilde{y}'+y'')-y'|^2/2)$$

where we understand that $\tilde{w} = (\tilde{x},\tilde{y})$ is that in (82) and $\tilde{w}' = (\tilde{x}',\tilde{y}')$ is the corresponding one that appears when we express the last term of (84) using (82). And we can write (84) as

$$\mathcal{K}(w',\xi';w,\xi;\eta) = \tag{86}$$
$$q_\omega(\eta)^2\langle\eta\rangle^2\cdot\int d\zeta d\tilde{w}d\tilde{w}'\cdot\chi\left(e^{-\delta_* t_\sharp}\langle\omega\rangle^{-1/2}\cdot\mathbf{\Delta}_{\mathbf{j}'}^{-1}\zeta\right)^2\cdot\exp(-\langle\eta\rangle(|\tilde{w}|^2+|\tilde{w}'|^2)/2)$$
$$\cdot\exp(i(\xi\cdot\Lambda^{-1}(\tilde{w}-\tilde{w}') - \zeta(\tilde{w}-\tilde{w}') - \eta b\Lambda^{-1}(\tilde{w}-\tilde{w}') - \eta(\sigma(\tilde{w})-\sigma(\tilde{w}'))))$$
$$\cdot I(w,\xi;w',\xi';\tilde{w},\tilde{w}';\eta).$$

Below we consider the following two cases separately:

$$\text{(I) } \Delta_{\mathbf{j}} < e^{3\delta_* t_\sharp}, \qquad \text{(II) } \Delta_{\mathbf{j}} \geq e^{3\delta_* t_\sharp}.$$

In the case (I), we use the non-integrability condition $(NI)_\rho$ to deduce the required estimate. In the case (II), we use the fact that the approximate infinitesimal non-integrability $\Delta_{\mathbf{j}} = \Delta(p,\varkappa_\sharp\langle\omega(\mathbf{j})\rangle)$ is sufficiently large. (The argument in the case (II) is somewhat similar to the argument for contact Anosov flows in [18].) In the following, we suppose that $(w,\xi,\eta),(w',\xi',\eta)\in\operatorname{supp}\Psi_{\mathbf{j}}\cap V_{\mathbf{j}\to\mathbf{j}'}$.

Case (I). We consider the integration in (85) with respect to the variable x''. Note that the factor on the second line of (85) is of the form to which we can apply (42) in Lemma 4.4. From the estimate (41), the factor on the third line is almost constant as a function of x''. The factor on the fifth line does not depend on x''. And the derivative the factor on the fourth line with respect to x'' is bounded by $C_*\langle\omega\rangle^{1/2}\lambda^{-1}$ in absolute value. We therefore divide the real line into intervals I with length $\lambda^{1/2}\langle\omega\rangle^{-1/2}$ and apply the estimate (42) to the integral (85) restricted to each of the intervals, approximating the factors on the third to fifth lines by their average. Note that, if $\lambda^{1/2}\langle\omega\rangle^{1/2}|\tilde{y}'-\tilde{y}| \geq 2b_0$, the estimate (42) for $h=\lambda^{-1/2}\langle\omega\rangle^{-1/2}$ and $b=\eta h|\tilde{y}'-\tilde{y}|$ gives

$$\int_I \exp(-i\eta(\tilde{y}-\tilde{y}')\cdot\theta^s(x'',y'')+i(\xi_x-\xi_x')\lambda^{-1}x''-i\beta\lambda^{-1}\tilde{\lambda}^{-1}\eta(\tilde{y}-\tilde{y}')x'')dx'' \leq (\lambda^{1/2}\langle\omega\rangle|\tilde{y}'-\tilde{y}|)^{-\rho}.$$

Therefore, calculating the integration with respect to y'' also, we obtain

$$\begin{aligned}&|I(w,\xi;w',\xi';\tilde{w},\tilde{w}';\eta)|\\ &\leq C_*(\nu)\cdot(\langle\lambda^{1/2}\langle\omega\rangle^{1/2}|\tilde{y}-\tilde{y}'|\rangle^{-\rho}+\lambda^{-1/2})\\ &\quad\cdot\lambda\tilde{\lambda}\langle\omega\rangle^{-1}\cdot\langle\langle\omega\rangle^{1/2}|\lambda^{-1}(\tilde{x}-\tilde{x}')-(x-x')|\rangle^{-\nu}\cdot\langle\langle\omega\rangle^{1/2}|\tilde{\lambda}^{-1}(\tilde{y}-\tilde{y}')-(y-y')|\rangle^{-\nu}\end{aligned}$$

for arbitrarily large ν.

Remark 6.11. Note that, in order to get the estimate above, we actually need the (42) for b (which equals b in the non-integrability condition $(NI)_\rho$) only in a bounded interval, say, $[b_0,\lambda]$, because the factor $\exp(-\langle\eta\rangle|\tilde{w}|^2/2-\langle\eta\rangle|\tilde{w}'|^2/2)$ in (86) becomes very small when $\lambda^{1/2}\langle\omega\rangle^{1/2}|\tilde{y}'-\tilde{y}|\geq\lambda$ and we do not need (42). This is important when we prove local uniformity of exponential mixing in Subsection 6.5.

For the integration with respect to ζ, we can show by integration by parts that

$$\left|\int\chi\left(2^{-1}\langle\omega\rangle^{-1/2}\cdot e^{-\delta_* t_\sharp}\cdot\mathbf{\Delta}_{\mathbf{j}'}^{-1}\zeta\right)^2\exp(-i\zeta(\tilde{w}-\tilde{w}'))d\zeta\right| \leq \frac{C_*(\nu)\cdot e^{2\delta_* t_\sharp}\cdot\Delta_{\mathbf{j}'}\cdot\langle\omega\rangle}{\langle e^{\delta_* t_\sharp}\langle\omega\rangle^{1/2}\mathbf{\Delta}_{\mathbf{j}'}(\tilde{w}'-\tilde{w})\rangle^{\nu}} \tag{87}$$

for arbitrarily large $\nu>0$. Therefore $\mathcal{K}(w',\xi';w,\xi;\eta)$ in (84) is bounded by

$$\begin{aligned}&C_*(\nu)\cdot e^{2\delta_* t_\sharp}\cdot\Delta_{\mathbf{j}'}\cdot\int d\tilde{w}d\tilde{w}'\ (\langle\lambda^{-1/2}\langle\omega\rangle^{1/2}|\tilde{y}-\tilde{y}'|\rangle^{-\rho}+\lambda^{-1/2})\cdot\langle\langle\omega\rangle^{1/2}\tilde{w}\rangle^{-\nu}\cdot\langle\langle\omega\rangle^{1/2}\tilde{w}'\rangle^{-\nu}\\ &\quad\cdot\langle\langle\omega\rangle^{1/2}|(\Lambda^{-1}(\tilde{w}'-\tilde{w})-(w'-w)|\rangle^{-\nu}\cdot\langle e^{\delta_* t_\sharp}\langle\omega\rangle^{1/2}\mathbf{\Delta}_{\mathbf{j}'}(\tilde{w}'-\tilde{w})\rangle^{-\nu}\end{aligned}$$

in absolute value. Inspecting the integration with respect to $\tilde{w}$ and $\tilde{w}'$, we obtain

$$\mathcal{K}(w',\xi';w,\xi;\eta)\leq\frac{C_*(\nu)e^{\delta_* t_\sharp}\lambda^{-1}\cdot(\langle\lambda^{-1/2}\langle\omega\rangle^{1/2}|y-y'|\rangle^{-\rho}+\lambda^{-1/2})}{\langle\langle\omega\rangle^{1/2}(\Lambda^{-2}+1)^{-1/2}(w-w')\rangle^{\nu}}. \tag{88}$$

Finally note that (80) is an operator on $L^2(\operatorname{supp}\Psi_{\mathbf{j}}\cap V_{\mathbf{j}\to\mathbf{j}'})$ and that the $2d$-dimensional Lebesgue measure of $\operatorname{supp}\Psi_{\mathbf{j}}\cap(\{x\}\times\mathbb{R}^2\times\{\eta\})$ is bounded by $C_* e^{2\delta_* t_\sharp}\Delta_{\mathbf{j}}\cdot\langle\omega\rangle$. Hence, by Schur test, we conclude that the operator norm of (81) is bounded by

$$C_* e^{2\delta_* t_\sharp}\Delta_{\mathbf{j}}\cdot\lambda^{-\rho/2}\leq C_* e^{-\rho\chi_* t_\sharp/3}$$

provided that we let δ_* be sufficiently small. This gives the required estimate (80).

Case (II). Before starting the proof in the case (II), we make a preliminary discussion. The key fact in the proof below is that, from (38) in Lemma 4.4, the unstable subspace E_u varies (or rotates) fast along the stable manifolds in the local chart $\kappa_{\mathbf{j}}$ in the case (II). More precisely, we can find a constant $K_*\geq 1$ such that, if $\langle\omega\rangle^{1/2}|y-y'|\geq K_*$, we have

$$|\xi_x-\xi_x'|\geq C_*^{-1}\Delta_{\mathbf{j}}\langle\omega\rangle|y-y'|\quad\text{for }(w,\xi,\eta),(w',\xi',\eta')\in\operatorname{supp}\Psi_{\mathbf{j}}.$$

From this fact, we regard the integral (85) with respect to x'' as an oscillatory integral with the oscillating term $\exp(i(\xi_x - \xi'_x)\lambda^{-1}x'')$ and estimate it by using integration by parts. But, since the function $e_{\mathbf{j}'}(w'') = (\theta^u(w''), \theta^s(w''))$ is not smooth, we have to use the following formula of "regularized" integration by parts.

Lemma 6.12 ([3, p.137]). *Let $\rho : \mathbb{R} \to \mathbb{R}$ be a C^∞ function supported on $[-1, 1]$ such that $\int \rho(s)ds = 1$. If $f \in C^2(\mathbb{R})$ and $g \in C^0_0(\mathbb{R})$ and if $f'(s) \neq 0$ on a neighborhood of $\operatorname{supp} g$, then we have, for sufficiently small $\varepsilon > 0$, that*

$$\int e^{if(s)}g(s)ds = -i\int e^{if(s)} \cdot (g_\varepsilon/f')'(s)ds + \int e^{if(s)}(g(s) - g_\varepsilon(s))ds$$

where $g_\varepsilon = \rho_\varepsilon * g$ with $\rho_\varepsilon(s) = \varepsilon^{-1}\rho(\varepsilon^{-1}s)$.

Proof. The first term on the right-hand side equals $\int e^{if(s)}g_\varepsilon(s)ds$ by integration by parts. □

Now we start the proof in the case (II). Let us set

$$\widetilde{\Psi}_{\mathbf{j}} : \mathbb{R}^{2+2+1} \to \mathbb{R}, \qquad \widetilde{\Psi}_{\mathbf{j}}(w, \xi, \eta) = \tilde{q}_{\omega(\mathbf{j})}(\eta) \cdot \chi\Big(3^{-1}\langle\omega\rangle^{-1/2} \cdot e^{-\delta_* t_\sharp} \cdot \mathbf{\Delta}_{\mathbf{j}}^{-1}(\xi - \eta \cdot e_{\mathbf{j}}(w))\Big),$$

by inserting the factor 3^{-1} in the definition of $\Psi_{\mathbf{j}}$. We are going to prove

Lemma 6.13. *We have*

$$\|\mathfrak{B}^* \circ \widetilde{\Psi}_{\mathbf{j}} \circ \mathbb{A}^* \circ (\mathbf{1}_{\widetilde{V}_{\mathbf{j}\to\mathbf{j}'}}\Psi_{\mathbf{j}'})^2 \circ \mathbb{A} \circ \widetilde{\Psi}_{\mathbf{j}} \circ \mathfrak{B} : L^2(U_{\mathbf{j}\to\mathbf{j}'}) \to L^2(\mathbb{R}^{2+1})\| \le C_* \min\{\lambda, \Delta_{\mathbf{j}}\}^{-1/2} \tag{89}$$

where $\widetilde{\Psi}_{\mathbf{j}}$ denotes the multiplication by the function $\widetilde{\Psi}_{\mathbf{j}}$.

The required estimate (80) follows from this lemma. To see this, let us write

$$\begin{aligned}
&\mathbf{1}_{\operatorname{supp}\Psi_{\mathbf{j}}} \circ \mathbb{A}^* \circ (\mathbf{1}_{\widetilde{V}_{\mathbf{j}\to\mathbf{j}'}}\Psi_{\mathbf{j}'})^2 \circ \mathbb{A} \circ \mathbf{1}_{\operatorname{supp}\Psi_{\mathbf{j}}} \\
&\qquad - \mathbf{1}_{\operatorname{supp}\Psi_{\mathbf{j}}} \circ \mathfrak{B} \circ (\mathfrak{B}^* \circ \widetilde{\Psi}_{\mathbf{j}} \circ \mathbb{A}^* \circ (\mathbf{1}_{\widetilde{V}_{\mathbf{j}\to\mathbf{j}'}}\Psi_{\mathbf{j}'})^2 \circ \mathbb{A} \circ \widetilde{\Psi}_{\mathbf{j}} \circ \mathfrak{B}) \circ \mathfrak{B}^* \circ \mathbf{1}_{\operatorname{supp}\Psi_{\mathbf{j}}} \\
&= D^* \circ \mathbb{A}^* \circ \mathbb{A} \circ \mathbf{1}_{\operatorname{supp}\psi_{\mathbf{j}}} + \mathfrak{B} \circ \mathfrak{B}^* \circ \widetilde{\Psi}_{\mathbf{j}} \circ \mathbb{A}^* \circ \mathbb{A} \circ D
\end{aligned}$$

with setting $D = (1 - \widetilde{\Psi} \circ \mathfrak{B} \circ \mathfrak{B}^*) \circ \mathbf{1}_{\operatorname{supp}\Psi_{\mathbf{j}}}$ and $D^* = \mathbf{1}_{\operatorname{supp}\Psi_{\mathbf{j}}} \circ (1 - \widetilde{\Psi}_{\mathbf{j}} \circ \mathfrak{B} \circ \mathfrak{B}^*)$. Since the kernel of $\mathfrak{B} \circ \mathfrak{B}^*$ is localized in the variable ξ in the scale $\langle\omega\rangle^{1/2}$ on $\operatorname{supp}\Psi_{\mathbf{j}}$, the operator norm of D and D^* is bounded by $C_*(\nu)e^{-\nu\delta_* t_\sharp}$ for arbitrarily large ν and so is the difference above because the operators $\mathbb{A}$, $\mathfrak{B}$ and $\mathfrak{B}^*$ do not increase the L^2 norm. Therefore we obtain the required estimate (80) from (89).

Proof of Lemma 6.13. We are going to estimate kernel of the operator in (89). From now to the last part of this proof, we suppose that $w, w' \in U_{\mathbf{j}\to\mathbf{j}'}$ satisfy

$$\langle\omega\rangle^{1/2}|y - y'| \ge \max\{\lambda/\Delta_{\mathbf{j}}, K_*\}. \tag{90}$$

We fist estimate the integral (85) with respect to x'' by using the formula in Lemma 6.12. Let us assume

$$\frac{1}{2} \le \frac{\tilde{\lambda}^{-1}|\tilde{y} - \tilde{y}'|}{|y - y'|} \le 2 \tag{91}$$

in addition for a while. We set

$$f(x'') = (\xi_x - \xi'_x)\lambda^{-1}x'', \qquad \varepsilon = \frac{1}{\Delta_{\mathbf{j}} \cdot \langle\omega\rangle^{1/2}|y - y'|} \cdot \langle\omega\rangle^{-1/2}$$

and let $g(x'')$ be the integrand of (85) other than the factor $e^{if(x'')} = \exp(i(\xi_x - \xi'_x)\lambda^{-1}x'')$, but we suppose that the factor on the third line of (85) is approximated by a constant, making a negligible error term. Then, from (40) in Lemma 4.4, we have that

$$|g_\varepsilon(x'') - g''(x'')| \le C_* e^{\delta_* t_\sharp} \cdot \varepsilon \quad \text{and} \quad |g'_\varepsilon(x'')| \le C_* e^{\delta_* t_\sharp}$$

provided that $t_\sharp$ is sufficiently large. Hence, under the condition (91), we obtain

$$|I(w,\xi;w',\xi';\tilde{w},\tilde{w}';\eta)| \le \frac{C_*(\nu)e^{\delta_* t_\sharp} \cdot \langle\omega\rangle^{-1} \cdot \max\{\lambda/\Delta_{\mathbf{j}}, K_*\} \cdot \langle\langle\omega\rangle^{1/2}|y-y'|\rangle^{-1}}{\langle\langle\omega\rangle^{-1/2}|\lambda^{-1}(\tilde{x}-\tilde{x}') - (x-x')|\rangle^{\nu} \cdot \langle\langle\omega\rangle^{1/2}|\tilde{\lambda}^{-1}(\tilde{y}-\tilde{y}') - (y-y')|\rangle^{\nu}} \tag{92}$$

for any $\nu > 0$. If the condition (91) does not hold (while (90) holds), we still have

$$|I(w,\xi;w',\xi';\tilde{w},\tilde{w}';\eta)| \le \frac{C_*(\nu) \cdot \langle\omega\rangle^{-1}}{\langle\langle\omega\rangle^{1/2}|\lambda^{-1}(\tilde{x}-\tilde{x}') - (x-x')|\rangle^{\nu} \cdot \langle\langle\omega\rangle^{1/2}|\tilde{\lambda}^{-1}(\tilde{y}-\tilde{y}') - (y-y')|\rangle^{\nu}}$$

by plain estimate without using integration by parts and, on the other hand, the factor $\exp(-\langle\eta\rangle|\tilde{w}|^2/2 - \langle\eta\rangle|\tilde{w}'|^2/2)$ in (86) is very small. Hence, under the assumption (90), we obtain

$$|\mathcal{K}(w',\xi';w,\xi;\eta)| \le C_*(\nu) \cdot e^{\delta_* t_\sharp}\lambda^{-2} \cdot \max\{\lambda/\Delta_{\mathbf{j}}, K_*\} \cdot \langle\langle\omega\rangle^{1/2}|y-y'|\rangle^{-1} \cdot \langle\langle\omega\rangle^{1/2}|(\Lambda^{-2}+1)^{-1/2}(w-w')|\rangle^{-\nu}. \tag{93}$$

Remark 6.14. The estimate above corresponds to (88). But, since $\Delta_{\mathbf{j}}$ is large, we can not follow the last part of the argument in the case (I) with this estimate. Indeed, when we apply Schur test, the factor $\Delta_{\mathbf{j}}$ appears[10] which is not bounded in the case (II).

To proceed, observe that we can actually strengthen the estimate (92) as

$$|\partial_\xi^\alpha \partial_{\xi'}^\beta I(\tilde{w},\tilde{w}',\xi,\xi';\eta)| \le \frac{C_{\alpha,\beta,\gamma,*}(\nu,b_0) \cdot \langle\omega\rangle \left\langle (\max\{\lambda/\Delta_{\mathbf{j}}, K_*\})^{-1} \cdot \langle\langle\omega\rangle^{1/2}|\tilde{y}-\tilde{y}'|\rangle \right\rangle^{-1} \langle|(\Lambda^2+1)^{-1/2}(\xi-\xi')|\rangle^{-|\alpha|-|\beta|}}{\langle\langle\omega\rangle^{-1/2}|\lambda^{-1}(\tilde{x}-\tilde{x}') - (x-x')|\rangle^{\nu} \cdot \langle\langle\omega\rangle^{1/2}|\tilde{\lambda}^{-1}(\tilde{y}-\tilde{y}') - (y-y')|\rangle^{\nu}}$$

for any multi-indices α and β, by examining the result of integration by parts. Note that, from the assumption (91), we have

$$\langle|(\Lambda^2+1)^{-1/2}(\xi-\xi')|\rangle \ge C_*^{-1}\Delta_{\mathbf{j}}\langle\omega\rangle^{1/2}.$$

Hence, the estimate above in (86), we obtain

$$|\partial_\xi^\alpha \partial_{\xi'}^\beta \mathcal{K}(w',\xi';w,\xi;\eta)| \le C_{\alpha,\beta,*}(\nu,b_0) \cdot (\Delta_{\mathbf{j}}\langle\omega\rangle^{1/2})^{-|\alpha|-|\beta|} \cdot \left\langle \max\{\lambda/\Delta_{\mathbf{j}}, K_*\}^{-1} \cdot \langle\langle\omega\rangle^{1/2}|y-y'|\rangle \right\rangle^{-1} \cdot \lambda^{-1}\langle\langle\omega\rangle^{1/2}|(\Lambda^{-2}+1)^{-1/2}(w-w')|\rangle^{-\nu}. \tag{94}$$

Now we write the operator above as an integral operator

$$(\mathfrak{B}^* \circ \widetilde{\Psi}_{\mathbf{j}} \circ \mathbb{A}^* \circ \mathbb{A} \circ \widetilde{\Psi}_{\mathbf{j}} \circ \mathfrak{B})u(w'_\dagger, z') = \int \widetilde{\mathcal{K}}(w'_\dagger, z'; w_\dagger, z)u(w_\dagger, z)dw_\dagger dz$$

[10] But this problem is rather superficial. We would not find this problem if we introduced some additional scaling in the x-variable in the construction of the local charts $\kappa_{\mathbf{j}}$.

with setting

$$\widetilde{\mathcal{K}}(w'_\dagger, z'; w_\dagger, z) = \int dw d\xi dw' d\xi' d\eta \cdot e^{i\xi w/2 - i\xi' w'/2} \cdot \mathcal{K}(w', \xi'; w, \xi; \eta) \cdot \widetilde{\Psi}_{\mathbf{j}}(w, \xi, \eta) \cdot \widetilde{\Psi}_{\mathbf{j}}(w', \xi', \eta') \cdot \overline{\phi_{w,\xi,\eta}(w_\dagger, z)} \cdot \phi_{w',\xi',\eta}(w'_\dagger, z').$$

We estimate the integral with respect to ξ, ξ' and η above in the same manner as (87), but using (94). Then we find

$$|\widetilde{\mathcal{K}}(w'_\dagger, z'; w_\dagger, z)| \le C_*(\nu) \cdot \langle z' - z\rangle^{-\nu} \cdot \int dw dw' \cdot \left\langle \max\{\lambda/\Delta_{\mathbf{j}}, K_*\}^{-1} \cdot \langle\langle\omega\rangle^{1/2} \lfloor y - y' \rfloor\rangle \right\rangle^{-1} \cdot \lambda^{-1} \cdot \langle\langle\omega\rangle^{1/2} |(\Lambda^{-2}+1)^{-1/2}(w - w')|\rangle^{-\nu} \cdot e^{2\delta_* t_\sharp} \cdot \langle\omega\rangle\Delta_{\mathbf{j}} \cdot \langle e^{\delta_* t_\sharp} \cdot \langle\omega\rangle^{1/2} \mathbf{\Delta}_{\mathbf{j}}^{-1}(w_\dagger - w)\rangle^{-\nu} \cdot e^{2\delta_* t_\sharp} \cdot \langle\omega\rangle\Delta_{\mathbf{j}} \cdot \langle e^{\delta_* t_\sharp} \cdot \langle\omega\rangle^{1/2} \mathbf{\Delta}_{\mathbf{j}}^{-1}(w'_\dagger - w')\rangle^{-\nu}.$$

Notice that we have proved this estimate only under the condition (90). But we can check without difficulty that, following the argument above without using integration by parts, we obtain the same estimate without the term $\langle \max\{\lambda/\Delta_{\mathbf{j}}, K_*\}^{-1} \cdot \langle\langle\omega\rangle^{1/2} \lfloor y - y'\rfloor\rangle\rangle^{-1}$ on the right hand side even when (90) does not holds. Therefore we obtain the required estimate (89) by Young inequality. □

We have done with the main step of the proof. To finish the proof of Proposition 6.7, we consider[11] the effect of the pre-composition of the operator $\mathbb{G}$. Let $\hat{\rho}^t_{\mathbf{j}\to\mathbf{j}'} : \mathbb{R}^{2+1} \to \mathbb{C}$ be the function obtained as the Fourier transform of $\rho^t_{\mathbf{j}\to\mathbf{j}'}$ in the variable z, that is, we set

$$\hat{\rho}^t_{\mathbf{j}\to\mathbf{j}'}(w, \eta) = \int e^{-i\eta z} \cdot \rho^t_{\mathbf{j}\to\mathbf{j}'}(w, z) dz.$$

In the next lemma, we compare $\mathbb{G}$ with the operator

$$\mathbb{P} : L^2(\operatorname{supp}\psi_{\mathbf{j}}) \to L^2(\mathbb{R}^{2+2+1}), \quad \mathbb{P}u = \int \hat{\rho}^t_{\mathbf{j}\to\mathbf{j}'}(w, \eta') \cdot \mathfrak{P}u(w, \xi, \eta - \eta') d\eta'$$

where $\mathfrak{P} = \mathfrak{B} \circ \mathfrak{B}^*$ is the projection operator in (59).

Lemma 6.15. $\|\mathbb{G} - \mathbb{P} : L^2(\operatorname{supp}\psi_{\mathbf{j}}) \to L^2(\mathbb{R}^{2+2+1})\| \le \varkappa_\sharp^{-1/2}$.

Proof. As an intermediate approximation, we consider the operator

$$\widetilde{\mathbb{P}} := \mathfrak{B} \circ \rho^t_{\mathbf{j}\to\mathbf{j}'} \circ \mathfrak{B}^* : L^2(\operatorname{supp}\psi_{\mathbf{j}}) \to L^2(\mathbb{R}^{2+2+1}).$$

The operator norm of $\mathbb{G} - \widetilde{\mathbb{P}} : L^2(\operatorname{supp}\psi_{\mathbf{j}}) \to L^2(\mathbb{R}^{2+2+1})$ is bounded by that of

$$\mathfrak{B}^* \circ (\mathbb{G} - \widetilde{\mathbb{P}}) : L^2(\operatorname{supp}\psi_{\mathbf{j}}) \to L^2(\mathbb{R}^{2+1})$$

because $\mathfrak{B}^* \circ \mathfrak{B} = \mathrm{Id}$ and $\mathfrak{B}$ is an isometric embedding with respect to the L^2 norms. We may write this operator as an integral operator with kernel $K(w, \xi, \eta; w', z')$ and find

$$|K(w, \xi, \eta; w', z')| = \left| (\rho^t_{\mathbf{j}\to\mathbf{j}'} \cdot \phi_{w,\xi,\eta})((G^t_{\mathbf{j}\to\mathbf{j}'})^{-1}(w', z')) - (\rho^t_{\mathbf{j}\to\mathbf{j}'} \cdot \phi_{w,\xi,\eta})(w', z') \right| \le C_*(\nu)\langle\omega\rangle^{-1/2+2\theta_*} \cdot \left(\langle\omega\rangle^{1/2} \cdot \langle\langle\omega\rangle^{1/2}(w - w')\rangle^{-\nu}\right) \cdot \langle z'\rangle^{-\nu}$$

from Lemma 4.5 and Lemma 4.7. (See also Remark 4.6.) It is then easy to check that

$$\sup_{w,\xi,\eta} \int |K(w, \xi, \eta; w', z')| dw' dz' < C_* \langle\omega\rangle^{-1+2\theta_*}$$

[11]In the following, we take seemingly a bit roundabout way because the flow is assume to be only C^3.

and that

$$\sup_{w',z'} \int_{\operatorname{supp}\psi_{\mathbf{j}}} |K(w,\xi,\eta;w',z')| dw d\xi d\eta < C_* e^{2|m|} \Delta_{\mathbf{j}} \langle \omega \rangle^{2\theta_*}.$$

Therefore, by Schur test, we see that the operator norm of $\mathfrak{B}^* \circ (\mathbb{G} - \widetilde{\mathbb{P}})$ is bounded by

$$C_* e^{|m|} \Delta_{bj}^{1/2} \langle \omega \rangle^{-1/2+2\theta_*} \le C_* e^{\delta_* t_\sharp} \cdot \Delta_{\mathbf{j}}^{1/2} \cdot \omega_\sharp^{-1/2+2\theta_*} \le C_* e^{\delta_* t_\sharp} \cdot \omega_\sharp^{-1/2+3\theta_*}$$

and consequently so is the operator norm of $\mathbb{G} - \widetilde{\mathbb{P}} : L^2(\operatorname{supp}\psi_{\mathbf{j}}) \to L^2(\mathbb{R}^{2+2+1})$.

Next we consider the difference $\mathbb{P} - \widetilde{\mathbb{P}}$ whose kernel $K'(w,\xi,\eta;w',\xi',\eta')$ is written

$$e^{i(\xi w - \xi' w')/2} \langle \eta \rangle^{1/2} \langle \eta' \rangle^{1/2} \cdot \int e^{i(\xi-\xi')w'' - \langle \eta' \rangle |w'-w''|^2/2 - \langle \eta \rangle |w-w'|^2/2} (\hat{\rho}^t_{\mathbf{j}\to\mathbf{j}'}(w', \eta'-\eta) - \hat{\rho}^t_{\mathbf{j}\to\mathbf{j}'}(w'', \eta'-\eta)) dw''.$$

Estimating the integral above using Lemma 4.7 and integration by parts, we obtain that

$$|K'(w,\xi,\eta;w',\xi',\eta')| \le C_*(\nu) \cdot (e^{C_* t_\sharp} \cdot \varkappa_\sharp^{-1}) \cdot \langle \langle \omega \rangle^{1/2} |w-w'| \rangle^{-\nu} \cdot \langle \langle \omega \rangle^{-1/2} |\xi - \xi'| \rangle^{-\nu} \cdot \langle \eta' - \eta \rangle^{-\nu}$$

for arbitrarily large $\nu > 0$. Therefore the operator norm of $\mathbb{P} - \widetilde{\mathbb{P}} : L^2(\operatorname{supp}\psi_{\mathbf{j}}) \to L^2(\mathbb{R}^{2+2+1})$ is bounded by $C_*(e^{C_* t_\sharp} \cdot \varkappa_\sharp^{-1}) = C_*(e^{C_* t_\sharp} \cdot e^{-t_\sharp^2})$.

From the estimates above on $\mathbb{G} - \widetilde{\mathbb{P}}$ and $\mathbb{P} - \widetilde{\mathbb{P}}$, we obtain the conclusion of the lemma, provided that we take sufficiently large $t_\sharp$ and then take sufficiently large $\omega_\sharp$ according to the choice of $t_\sharp$. □

Since the operator $\mathbb{A}$ does not increase the L^2 norm of functions, the last lemma tells that

$$\|\mathbb{A} \circ \mathbb{G} - \mathbb{A} \circ \mathbb{P} : L^2(\operatorname{supp}\psi_{\mathbf{j}}) \to L^2(\operatorname{supp}\psi_{\mathbf{j}'})\| \le \varkappa_\sharp^{-1/2}.$$

Therefore, for the proof of Proposition 6.1, it is enough to show that

$$\|\mathbb{A} \circ \mathbb{P} : L^2(\operatorname{supp}\psi_{\mathbf{j}}) \to L^2(\operatorname{supp}\psi_{\mathbf{j}'})\| \le e^{-\rho_* t}.$$

But, because $\mathbb{P}$ is a simple operator whose kernel $K_{\mathbb{P}}$ satisfies

$$(95) \quad |K_{\mathbb{P}}(w,\xi,\eta;w',\xi',\eta')| \le C_*(\nu) \cdot \langle \langle \omega \rangle^{1/2} |w-w'| \rangle^{-\nu} \cdot \langle \langle \omega \rangle^{-1/2} |\xi - \xi'| \rangle^{-\nu} \cdot \langle \eta' - \eta \rangle^{-\nu},$$

it is clear that pre-composition of $\mathbb{P}$ hardly affects the argument on the operator $\mathbb{A}$ in the former part of this proof to give the required estimate. (We omit the tedious details about the part of functions which go out of $\operatorname{supp}\psi_{\mathbf{j}}$ by $\mathbb{P}$.)

6.4. **Proof of Lemma 6.3 and Lemma 6.6.** The proofs of Lemma 6.3 and Lemma 6.6 below are based on estimates of the kernel of $\mathbb{L}^t_{\mathbf{j}\to\mathbf{j}'}$ using integration by parts. One because these claims have nothing to do with the non-integrability condition $(NI)_\rho$ and one because the estimates are straightforward, we will omit the detail of the proofs. (We refer [18] for more details.)

Proof of Lemma 6.3. Let us regard the operator $\mathcal{L}^t_{\mathbf{j}\to\mathbf{j}'}$ as the composition of the multiplication operator by $\rho^t_{\mathbf{j}\to\mathbf{j}'}$ with the operator $u \mapsto u \circ f^t_{\mathbf{j}\to\mathbf{j}'}$. If we did not have the latter, we could deduce the claims immediately from the estimates in Lemma 4.7. But, since the latter transfer operator is a unitary operator in L^2 norm and preserves the frequency in the z-direction, it does not do any harm for validity of the conclusion. □

Proof of Lemma 6.6. Let $\mathbf{j}, \mathbf{j}' \in \mathcal{J}$ be those in the statement of Lemma 6.6. For the proof of Lemma 6.6, we may assume

$$|\omega(\mathbf{j}') - \omega(\mathbf{j})| \le e^{\max\{|m(\mathbf{j})|,|m(\mathbf{j}')|\}/10} \cdot \langle \omega(\mathbf{j})\rangle^{1/10}$$

because the claims follows from Lemma 6.3 otherwise. We consider the operators

$$\check{\mathbb{L}}^t_{\mathbf{j}\to\mathbf{j}'} := (\mathfrak{B}^* \circ \tilde{\psi}_{\mathbf{j}'} \circ \mathfrak{B}) \circ \mathcal{L}^t_{\mathbf{i}\to\mathbf{i}'} \circ (\mathfrak{B}^* \circ \tilde{\psi}_{\mathbf{j}} \circ \mathfrak{B}) : L^2(\mathbb{R}^{2+1}) \to L^2(\mathbb{R}^{2+1})$$

where $\tilde{\psi}_{\mathbf{j}}$ and $\tilde{\psi}_{\mathbf{j}'}$ are those in Remark 5.4. We may write its kernel as

$$\begin{aligned}&\langle \eta'\rangle^2\langle\eta\rangle^2 \int dw'' d\tilde{w} d\tilde{\eta} d\tilde{\xi} d\tilde{w}' d\tilde{\eta}' d\tilde{\xi}' \cdot \rho^t_{\mathbf{j}\to\mathbf{j}'}(w'',z'') \cdot \psi_{\mathbf{j}'}(\tilde{w},\tilde{\xi},\tilde{\eta}) \cdot \psi_{\mathbf{j}}(\tilde{w}',\tilde{\xi}',\tilde{\eta}')\\ &\cdot \exp(i(\tilde{\xi}',\tilde{\eta}')\cdot((w',z') - f^t_{\mathbf{j}\to\mathbf{j}'}(w'',z'')) + i(\tilde{\xi},\tilde{\eta})\cdot((w,z)-(w'',z'')))\\ &\cdot \exp(-(1/2)(\langle\eta\rangle(|w'-\tilde{w}|^2 + |\tilde{w} - \hat{f}^t_{\mathbf{j}\to\mathbf{j}'}(w'',z'')|^2) + \langle\eta'\rangle(|w'-\tilde{w}'|^2 + |\tilde{w}'-w''|^2))).\end{aligned}$$

To integral with respect to w'' above, we apply integration by parts using the differential operator

$$L = \frac{1 + i\langle\omega\rangle^{-1}\mathbf{\Delta}_{\mathbf{j}}^{-2}((\xi,\eta) - (Df^t)^*_{w''}(\xi',\eta'))\cdot(\partial_{w''},\partial_{z''})}{1+\langle\omega\rangle^{-1}|\mathbf{\Delta}_{\mathbf{j}}^{-1}(\xi,\eta) - (Df^t)^*_{w''}(\xi',\eta')|^2}$$

once and then integration by parts with respect z for several times. Then, by crude estimate on the resulting terms, we see that

$$\begin{aligned}|K(w,z;w',z')| \le C_*(\nu,t_\sharp)\cdot e^{-\max\{|m(\mathbf{j})|,|m(\mathbf{j}')|\}} \cdot \langle\omega(\mathbf{j}) - \omega(\mathbf{j}')\rangle^{-\nu}\cdot\langle z'-z\rangle^{-\nu}\cdot \int dw''\\ \cdot e^{2m(\mathbf{j}')}\Delta_{\mathbf{j}'}\cdot\langle\omega(\mathbf{j}')\rangle\cdot\langle e^{m(\mathbf{j}')}\langle\omega(\mathbf{i}')\rangle^{1/2}|\mathbf{\Delta}_{\mathbf{j}'}(w' - \hat{f}_{\mathbf{j}\to\mathbf{j}'}(w''))|\rangle^{-\nu}\\ \cdot e^{2m(\mathbf{j})}\Delta_{\mathbf{j}}\cdot\langle\omega(\mathbf{j})\rangle\cdot\langle e^{m(\mathbf{j})}\langle\omega(\mathbf{j})\rangle^{1/2}\cdot\langle|\mathbf{\Delta}_{\mathbf{j}}(w''-w)|\rangle^{-\nu}.\end{aligned}$$

And, in the case where $|\omega(\mathbf{j})| \ge \omega_\sharp$ and $|\omega(\mathbf{j}')| \ge \omega_\sharp$, we use the estimates (56) to check that the constant $C_*(\nu,t_\sharp)$ is actually bounded by $C_*(\nu)t_\sharp$. By Young inequality, we obtain

$$\|\check{\mathbb{L}}^t_{\mathbf{j}\to\mathbf{j}'}\|_{L^2} \le C_*(t_\sharp,\nu)\cdot e^{-\max\{|m(\mathbf{j})|,|m(\mathbf{j}')|\}}\cdot\langle\omega(\mathbf{j}') - \omega(\mathbf{j})\rangle^{-\nu} \tag{96}$$

To finish the proof, note that, from the localized property of the kernel of the Bargmann projector $\mathfrak{B}\circ\mathfrak{B}^*$, we have

$$\left\|(1-\tilde{\psi}_{\mathbf{j}})\circ\mathfrak{B}\circ\mathfrak{B}^*\cdot\mathbf{1}_{\mathrm{supp}\,\psi_{\mathbf{j}}} : L^2(\mathbb{R}^{2+2+1}) \to L^2(\mathbb{R}^{2+2+1})\right\| \le C_*(\nu)\cdot e^{-\nu|m(\mathbf{j})|} \tag{97}$$

and the parallel estimate with $\mathbf{j}$ replaced by $\mathbf{j}'$. Let us write

$$\mathbb{L}^t_{\mathbf{j}\to\mathbf{j}'} = \mathfrak{B}\circ\mathfrak{B}^*\circ((1-\tilde{\psi}_{\mathbf{j}'}) + \tilde{\psi}_{\mathbf{j}'})\circ\mathbb{L}^t_{\mathbf{j}\to\mathbf{j}'}\circ((1-\tilde{\psi}_{\mathbf{j}'}) + \tilde{\psi}_{\mathbf{j}'})\circ\mathfrak{B}\circ\mathfrak{B}^*$$

If $\frac{2}{3} \le |m(\mathbf{j})|/|m(\mathbf{j}')| \le \frac{3}{2}$ and $\min\{|\omega(\mathbf{j})|,|\omega(\mathbf{j}')|\} \ge \omega_\sharp$, the required estimate follows immediately from the estimates above. If $\frac{2}{3} \le |m(\mathbf{j})|/|m(\mathbf{j}')| \le \frac{3}{2}$ and $\min\{|\omega(\mathbf{j})|,|\omega(\mathbf{j}')|\} \le \omega_\sharp$, we may assume that $\min\{|m(\mathbf{j})|,|m(\mathbf{j}')|\} \ge m_\sharp$ and obtain the required estimate by letting $m_\sharp$ large depending on $t_\sharp$. In the case where $\frac{2}{3} \le |m(\mathbf{j})|/|m(\mathbf{j}')| \le \frac{3}{2}$ is not true, we have to modify the argument a little. In the case, from the assumption (73) and the definition of the relation $\hookrightarrow^t$, we have either

$$m(\mathbf{j}) < 0 < m(\mathbf{j}') \quad \text{or} \quad 0 < m(\mathbf{j}) < m(\mathbf{j}') \quad \text{or} \quad m(\mathbf{j}) < m(\mathbf{j}') < 0.$$

If $|m(\mathbf{j})| < |m(\mathbf{j}')|$ (in the first or second case above), we can and do modify $\tilde{\psi}_{\mathbf{j}}$ so that its support is of size comparable with that of $\tilde{\psi}_{\mathbf{j}'}$, that (96) remains true and (97) holds with $e^{-\nu|m(\mathbf{j})|}$ on the right hand side replaced by $e^{-\nu|m(\mathbf{j}')|} = e^{-\nu\max\{|m(\mathbf{j})|,|m(\mathbf{j}')|\}}$. Then we obtain the required estimate in the same manner as above. We can argue similarly in the case $|m(\mathbf{j})| > |m(\mathbf{j}')|$. $\square$

6.5. **Local uniformity of exponential mixing.** Finally we prove the conclusion of Theorem 2.13 to finish the proof. Let us write f_0^t for the flow f^t that we have considered in the argument in the previous subsections, which satisfies the non-integrability condition $(NI)_\rho$. We first show that, if we take a sufficiently small C^3 neighborhood $\mathcal{V}$ of f_0^t in $\mathfrak{F}_A^3$, all the flows in $\mathcal{V}$ are exponentially mixing. To this end, we recall the argument in the previous subsections and check dependence of objects on the flow. Clearly we can construct the local charts $\kappa_{a,\omega,n}$ and the functions $\rho_{a,\omega,n}$ so that *each* of them depend on the flow continuously in C^3 sense. Then we can define the Hilbert space $\mathbb{H}$ and $\mathcal{H}$ and also the operator $\mathbb{L}^t : \mathbb{H} \to \mathbb{H}$ in parallel manner so that each of the components $\mathbb{L}^t_{\mathbf{j}\to\mathbf{j}'}$ depend on the flow continuously. We can check that all the estimates remains valid with uniform constants that are denoted by the symbols with the subscript $*$ and also $t_\sharp$, $\omega_\sharp$, $m_\sharp$. The most important point is that, in the proof of Proposition 6.7, we have used the estimate (14) in the non-integrability condition $(NI)_\rho$ only for b in a bounded interval, as we noted in Remark 6.11. And the condition (14) for α with $|\alpha| > b^2$ follows from (4) as we noted in Remark 2.10. Hence Proposition 6.1 remains true for *each* of the flows in $\mathcal{V}$ (provided that we let $\mathcal{V}$ be sufficiently small). We can therefore conclude that each of the flows in $\mathcal{V}$ are exponentially mixing.

We next consider uniformity of the constants c_α and C_α in the decay estimate (1). For this point, we have to beware that continuity in dependence of the local charts $\kappa_{a,\omega,n}$ and the operators $\mathbb{L}^t_{\mathbf{j}\to\mathbf{j}'}$ on the flow in $\mathcal{V}$ is *not* uniform (especially in ω) and consequently the Hilbert spaces $\mathcal{H}$ and the operator $\mathbb{L}^t$ will *not* depend on the flow in $\mathcal{V}$ continuously. For a flow $\mathbf{f} = \{f^t\} \in \mathcal{V}$, we write $\mathcal{H}(\mathbf{f})$ for the Hilbert space $\mathcal{H}$ constructed for $\mathbf{f}$ and set $\mathcal{H}_0(\mathbf{f}) = \{u \in \mathcal{H}(\mathbf{f}) \mid \int u dm = 0\}$. Also let $\mathcal{L}^t_{\mathbf{f}}$ be the transfer operator $\mathcal{L}^t$ defined for $\mathbf{f} \in \mathcal{V}$. To obtain the conclusion, it is enough to show that, for some $T > 0$ and $\delta > 0$, we have

$$\|\mathcal{L}^T_{\mathbf{f}} : \mathcal{H}_0(\mathbf{f}) \to \mathcal{H}_0(\mathbf{f})\| < 1 - \delta \quad \text{for all } \mathbf{f} \in \mathcal{V}.$$

Suppose that this assertion is not true. Then, for any $T > 0$, we can find a sequence of flows $\mathbf{f}_k$ which converges to $\mathbf{f}_0 = \{f_0^t\}$ in C^3 sense and a sequence of functions $u_k \in \mathcal{H}_0(\mathbf{f}_k)$ such that $\|u_k\|_{\mathcal{H}(\mathbf{f}_k)} = 1$ and $\|\mathcal{L}^T_{\mathbf{f}_k} u_k\|_{\mathcal{H}(\mathbf{f}_k)} \geq 1 - (1/k)$. Now we recall from Proposition 6.1, which is valid uniformly for $\mathbf{f} \in \mathcal{V}$, that, if T is sufficiently large, the operators $\mathcal{L}^T_{\mathbf{f}_k}$ contracts the high frequency part of functions (*i.e.* the components $u_{a,\omega,n,m}$ with $|\omega| \geq \omega_\sharp$ or $|m| \geq m_\sharp$) by a uniform rate. Hence, for the assumption on u_k to be true, the high frequency part of u_k must be relatively small (uniformly in k). Therefore we can find a subsequence $u_{k(\ell)}$ of u_k which converges to some $u_0 \in \mathcal{H}_0(\mathbf{f}_0)$ (as a distribution at least) and see that u_0 satisfies

$$\|u_0\|_{\mathcal{H}(\mathbf{f}_0)} = \lim_{\ell\to\infty} \|u_{k(\ell)}\|_{\mathcal{H}(\mathbf{f}_k)} = 1, \qquad \|\mathcal{L}^T_{\mathbf{f}_0} u_\infty\|_{\mathcal{H}(\mathbf{f}_0)} = \lim_{\ell\to\infty} \|\mathcal{L}^T_{\mathbf{f}_k} u_{k(\ell)}\|_{\mathcal{H}(\mathbf{f}_k)} \geq 1.$$

Clearly this conclusion for arbitrarily large $T > 0$ contradicts what we have proved for $\mathbf{f}_0$.

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