arXiv:1601.00064v1 [cond-mat.stat-mech] 1 Jan 2016

Classical particle scattering for power-law two-body potentials

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Abstract

We present a rigorous study of the classical scattering for any two-body inter-particle potential of the form $v(r) = g/r^\gamma$, with $\gamma > 0$, for repulsive ($g > 0$) and attractive ($g < 0$) interactions. We give a derivation of the complete power series of the deflection angle in terms of the impact factor for the weak scattering regime (large impact factors) as well as the asymptotic expressions for the hard scattering regime (small impact factors). We see a very different qualitative and quantitative behavior depending whether the interaction is repulsive or attractive. In the latter case, the families of trajectories depend also strongly on the value of γ. We also study carefully the modifications of the results when a regularization is introduced in the potential at small scales. We check and illustrate all the results with the exact integration of the equations of motion.

PACS numbers: 03.65.Nk,04.40.-b, 05.70.Ln, 05.70.-a

I. INTRODUCTION

Scattering of particles are present in many physical processes in a broad area of Physics, as atomic (e.g. [1]), plasma (e.g. [2]), astrophysics (e.g. [3]), active matter (e.g. [4]), etc. A seminal paper was published by Ernest Rutherford in 1911 [5], in which he studied the deflection of α and β particles by an atom. He calculated analytically the angle of deflection of the (positively charged) incident particles with the (charged) nucleus. His calculations, compared to experimental data (see [5] for references), permitted to conclude that the atom is basically "empty" with a charge concentrated in the center, surrounded by the electron cloud, which lead to the "planetary" model of the atom. These two-body collisions plays also a central role in the collisional relaxation of Coulomb plasmas (see e.g. [2]) and self-gravitating systems (or more generally of systems of particles with long range interactions), as pointed out by Chandrasekhar in a seminal paper [6]. When studying the relaxation of system of particles interacting with generalized power-law interaction (see e.g. [7, 8]), it is necessary to generalize the Chandrasekhar approach.

This paper is devoted to the rigorous mathematical study of the generalization of the classical scattering of two particles interacting with the generic power-law interaction

$$v(r) = \frac{g}{r^\gamma}. \tag{1}$$

Such process is well known only on the qualitative level or in particular cases (see e.g. [1, 9–13]). For example, in the case of a pure repulsive interaction, the angle of deflection χ (defined in Fig. 1) is always well defined for any value γ and impact factor b (see also Fig. 1), and varies between $\chi = 0$ (the particle comes back in its original direction with opposite velocity) and $\chi = \pi$ (the trajectory of the particle suffers no perturbation). In the case of attractive interactions, the angle of deflection varies in the interval $\chi \in [\pi, \infty[$. In this case two different situations arise: if $\gamma < 2$, the angular momentum — which scales with the distance as $1/r^2$ — produces an effective repulsive interaction (the so-called centrifugal potential barrier) which always dominates the potential for $r \to 0$ and the angle of deflection is finite for any b. However, for $\gamma > 2$, the attractive interaction is stronger at small distances than the centrifugal barrier and particles can crash for values of the impact factor smaller than a critical quantity. For impact factors larger than this critical one, particles can make an arbitrarily large number of revolutions one around the other. This phenomenon is called in the literature *orbiting* (see e.g. [13] for a general discussion).

In this paper, we will consider interacting potentials of the form (1) with $\gamma > 0$. The coupling constant g is positive for repulsive interactions and negative for attractive ones. Analytical simple calculations are not possible except for some particular cases for integer γ in term of circular functions (see [14]). In the other cases, only asymptotic expansions can be performed. For this reason, we will derive the asymptotic expressions for the angle of deflection of the particles for the two limiting cases which are determined by the value of the impact factor b (defined in Fig. 1): the regime of *soft scattering*, in which the trajectories of the particles are weakly perturbed, and the regime of *strong scattering*, in which the particles suffer a large deflection. Moreover, we will study in detail the introduction of a regularization (usually called *softening* in the astrophysical literature) at small scales in the potential. This is of primarily interest when studying the relaxation in systems of particles with long range interaction, in which the effect of the regularization at small scales can play a key role (see [7]).

The paper is organized as follows: in the next section, we will review definitions and standard formulas of the interaction of two particles in a central force field. In the subsequent section, we will explain the analytically tractable $\gamma = 1$ (Coulomb or gravitational) case. Then, we will study mathematically the case $\gamma \neq 1$. We will then first explain our general approach with the already known (see e.g. [15, 16]) *soft scattering regime*, for which we extend the domain of validity to arbitrary $\gamma > 0$. Then, we will derive expressions for the *hard scattering regime*, in which we will obtain different classes of solution as a function of γ. In the subsequent section we will explain the physical implications of the mathematical re-

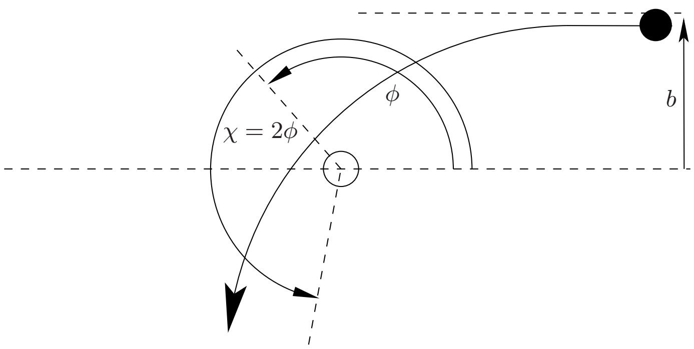

FIG. 1: Collision in the center of mass frame. The black dot represents the fictitious (reduced) particle, and the white dot the center of mass of the particles, which is at rest.

sults, compare them with the exact numerical integration of the equation of motion and show typical trajectories for the different regimes. Then, we will study how the trajectories change when introducing a regularization at small scales in the potential. We conclude the paper with a summary of the results, conclusions and perspectives.

II. PRELIMINARIES

Let us consider the scattering of two isolated particles. It is convenient to use the center of mass frame to transform the two-particle problem in a one-particle one. Let us consider that particles have masses m_1 and m_2 and their position $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively. We define their relative position as

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{2}$$

and fix the origin of the frame at the center of mass, i.e.,

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = \mathbf{0}. \tag{3}$$

The relation between the position of the particles in the center of mass frame $\mathbf{r}$ and in the laboratory frame is, using Eqs. (2) and (3):

$$\mathbf{r}_1 = \frac{m}{m_1}\mathbf{r} \tag{4a}$$

$$\mathbf{r}_2 = -\frac{m}{m_2}\mathbf{r}, \tag{4b}$$

where we have defined the reduced mass

$$m = \frac{m_1 m_2}{m_1 + m_2}. \tag{5}$$

In the center of mass frame, the collision occurs as depicted in Fig. 1, in which appears the definition of the impact factor b, the angle of closest approach ϕ and the angle of deflection χ, which is $\chi = 2\phi$. In order to define the angles with the usual mathematical signs, the incident particle comes from $+\infty$. This picture assumes that the two particles are far away from each other for $t \to -\infty$ and for $t \to +\infty$. The angle ϕ can be calculated, as a function of the impact factor b, using the classical formula [16]

$$\phi(b) = \int_{r_{min}}^{\infty} \frac{(b/r^2)dr}{\sqrt{1-(b/r)^2 - 2v(r)/(mu^2)}}, \tag{6}$$

where u is the asymptotic velocity of the incident particle at $+\infty$ ($u = |\dot{\mathbf{r}}|$). The quantity r_{min} is the largest positive root of the denominator, i.e., of

$$W(r) = 1 - (b/r)^2 - 2v(r)/mu^2. \tag{7}$$

We consider the pure power law pair potential,

$$v(r) = \frac{g}{r^\gamma}, \qquad 0 < \gamma < d, \tag{8}$$

with $g \neq 0$, where $g > 0$ corresponds to a repulsive interaction and $g < 0$ to an attractive one. We introduce the characteristic scale

$$b_0 = \left(\frac{|g|}{mu^2}\right)^{1/\gamma}, \tag{9}$$

which allows us to rewrite Eq. (6) as

$$\phi(b) = \int_{r_{min}}^{\infty} \frac{(b/r^2)dr}{\sqrt{1-(b/r)^2 \mp 2(b_0/r)^\gamma}}. \tag{10}$$

Now, the "minus" sign in the denominator corresponds to a repulsive interaction while the "plus" sign to an attractive one. By using the change of variables $r = b/x$ it is possible to rewrite Eq. (10) in the following form:

$$\phi(b/b_0) = \int_0^{x_{max}} \frac{dx}{\sqrt{1-x^2 \mp 2(b_0/b)^\gamma x^\gamma}}, \tag{11}$$

where x_{max} is the smallest positive root of the denominator. Since x_{max} is a function of b/b_0 depending only on γ, Eq. (11) shows explicitly that ϕ is also a function of b/b_0 depending only on γ. Equation (10) can be solved analytically only in few cases (e.g. gravity in $d = 3$ which is given by $\gamma = 1$), for the general case approximations or numerical computation of the integral should be used.

III. $\gamma = 1$ (COULOMB AND GRAVITATIONAL CASE IN $d = 3$)

In this section, we will first review the well-known Coulomb and gravitational case, which is analytically solvable. It will give us some insight for the general solution for $\gamma \neq 1$.

We start from Eq. (10), compute the value of r_{min} and the integral ϕ explicitly. We obtain:

- for the repulsive case, $r_{min} = b_0 + \sqrt{b^2 + b_0^2}$ and

$$\phi(b/b_0) = \arctan\left(\frac{b}{b_0}\right); \tag{12}$$

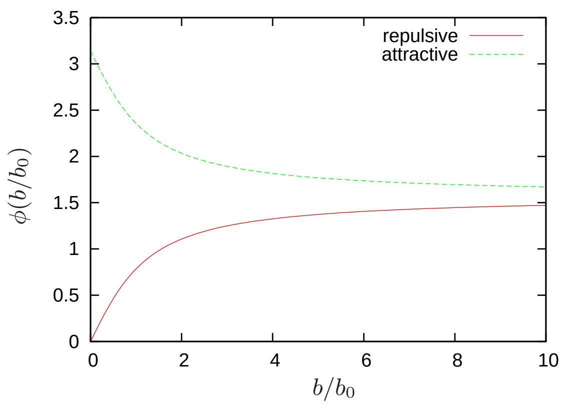

FIG. 2: Graph of the angle ϕ as a function of b/b_0 for $\gamma = 1$ and for repulsive (in red) and attractive (in green) interactions.

- for the attractive case $r_{min} = -b_0 + \sqrt{b^2 + b_0^2}$ and

$$\phi(b/b_0) = \pi - \arctan\left(\frac{b}{b_0}\right). \tag{13}$$

We can identify two regimes in the collision process: the one corresponding to $b/b_0 \gg 1$, which is called the "weak" or "soft" collisions regime, in which the trajectory is weakly perturbed; and the one corresponding to $b/b_0 \ll 1$, which which is called the "strong" or "hard" collision regime, in which the trajectory is strongly modified by the collision. From Eqs. (12) and (13) we obtain the following asymptotic behaviors for the angle ϕ:

- For the repulsive case, for weak collisions ($b/b_0 \gg 1$), we have $\phi(b/b_0) = \pi/2 - b_0/b + \mathcal{O}((b_0/b)^3)$, and for strong ones ($b/b_0 \ll 1$), $\phi(b/b_0) = b/b_0 + \mathcal{O}((b/b_0)^3)$.

- For the attractive case, for weak collisions ($b/b_0 \gg 1$), we have $\phi(b/b_0) = \pi/2 + b_0/b + \mathcal{O}((b_0/b)^3)$, and for strong ones ($b/b_0 \ll 1$), $\phi(b/b_0) = \pi - b/b_0 + \mathcal{O}((b/b_0)^3)$.

In the next sections we will compute the analogous asymptotic behaviors for the generalized case $\gamma \neq 1$.

IV. THE GENERAL CASE: $\gamma \neq 1$

For the general case $\gamma \neq 1$ it is not possible to derive an analytical expression for the angle ϕ as a function of b/b_0, as we did for $\gamma = 1$ in Eqs. (12) and (13). However, it is possible to compute the asymptotic behaviors of ϕ for $b/b_0 \ll 1$ and $b/b_0 \gg 1$, which corresponds to hard and soft scattering respectively.

As a first step, we perform the substitution $r = r_{min}/x$, $0 < x \leq 1$, in Eq. (10), yielding

$$\phi(b/b_0) = \frac{b}{r_{min}} \int_0^1 \frac{dx}{\sqrt{1 - (bx/r_{min})^2 \mp 2(b_0 x/r_{min})^\gamma}}. \tag{14}$$

We recall that the "minus" sign in the denominator corresponds to a repulsive interaction while the "plus" sign to an attractive one. Then will use use the following procedure to compute the two limiting behaviors:

1. Determine, for the considered regime, an approximation for r_{min} in Eq. (14), which is the largest zero of the denominator.

2. Perform an expansion in the appropriately chosen small parameter for each case of the denominator of Eq. (14) and give an expression of the integrals by means of the Γ function.

We will study first the regime of soft collisions (i.e. $b/b_0 \gg 1$) for both attractive and repulsive interactions. Then, we will present in two different subsections (because the mathematical treatment is completely different), the case of hard scattering ($b/b_0 \ll 1$) for repulsive interactions, and then for attractive ones.

A. The regime of *soft* collisions for attractive and repulsive interactions

The regime of *soft* collisions corresponds to the case in which the scale b_0 is small compared to the impact factor b. In this regime the trajectories of the particles are weakly perturbed. In this Subsection, γ is any positive number.

We first give an expansion of r_{min}, which is the positive solution of $1 \mp 2(b_0/r_{min})^\gamma = (b/r_{min})^2$. Recasting this as $b^2 = r_{min}^2 \mp 2b_0^\gamma r_{min}^{2-\gamma}$, we see that if $b \gg b_0$, we must have $r_{min} \gg b_0$, hence $b^2 = r_{min}^2(1 \mp 2(b_0/r_{min})^\gamma) \approx r_{min}^2$ (since $\gamma > 0$) and then $r_{min} \approx b \gg b_0$. We see therefore that the value of r_{min} does not depend, at leading order, on the sign of the interaction nor on the particular value of γ. This is illustrated in Fig. 3.

Expanding further yields

$$\begin{aligned} b/r_{min} &= \sqrt{1 \mp 2(b_0/r_{min})^\gamma} \\ &= 1 \mp (b_0/b)^\gamma + \mathcal{O}((b_0/b)^{2\gamma}). \end{aligned} \tag{15}$$

Next, we introduce the small parameter $\delta = 2(b_0/r_{min})^\gamma = \mp[(b/r_{min})^2 - 1] \approx 2(b_0/b)^\gamma \ll 1$ and obtain

$$\frac{r_{min}}{b}\phi(b/b_0) = \int_0^1 \frac{dx}{\sqrt{1 - x^2 \mp \delta(x^\gamma - x^2)}}.$$

We want an expansion of the above integral using that δ is a small parameter. It is then natural to write it under the form

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1 \mp \delta\frac{x^\gamma - x^2}{1-x^2}}}$$

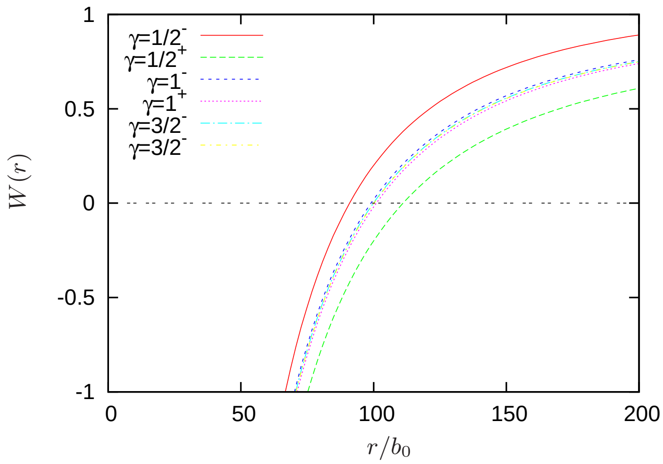

FIG. 3: Graph of W as a function of r/b_0 for $b/b_0 = 100$ and different values of γ for the repulsive (superscript "+") and attractive case (superscript "-"). Observe that in all cases $r_{min} \approx b$.

and to expand the second square root in power series. This is possible since the expression $(x^\gamma - x^2)/(1-x^2)$ is bounded on $[0,1]$ (for $\gamma > 0$) and this implies that $(r_{min}/b)\phi(b/b_0)$ is actually a power series in δ. In particular, we obtain the first order expansion

$$\frac{r_{min}}{b}\phi(b/b_0) = \int_0^1 \frac{dx}{\sqrt{1-x^2}} \pm \frac{\delta}{2}\int_0^1 \frac{x^\gamma - x^2}{(1-x^2)^{3/2}}\,dx + \mathcal{O}(\delta^2). \quad (16)$$

Combining this with Eq. (15) and using that $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \pi/2$ and that

$$\int_0^1 \frac{1-x^\gamma}{(1-x^2)^{3/2}}\,dx = A(\gamma) = \sqrt{\pi}\frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)} \quad (17)$$

(see Appendix A 1), we deduce

$$\phi(b/b_0) = \frac{\pi}{2} \mp A(\gamma)(b_0/b)^\gamma + \mathcal{O}((b_0/b)^{2\gamma}). \quad (18)$$

On the mathematical level, we shall use the above strategy to give expansions with respect to some small parameter δ of integrals of the form

$$\int_0^1 \frac{dx}{\sqrt{F(x)+\delta H(x)}}, \quad (19)$$

where F and H are non-negative functions. If $\int_0^1 \frac{dx}{\sqrt{F(x)}} < +\infty$ and if H/F is bounded on $[0,1]$, then the above integral is an analytic function of δ around $\delta = 0$ and, for $\delta \to 0$,

$$\int_0^1 \frac{dx}{\sqrt{F(x)+\delta H(x)}} = \int_0^1 \frac{dx}{\sqrt{F(x)}} - \frac{\delta}{2}\int_0^1 \frac{H(x)}{F(x)^{3/2}}\,dx + \mathcal{O}(\delta^2).$$

If $H(x)/F(x)$ is not bounded on $[0,1]$, then it may happen (and this is indeed true in some of the cases we shall study) that the above integral is not smooth with respect to δ and thus the correction is possibly not of order δ but much larger.

The angle ϕ is actually, for b large enough, the sum of a power series in $(b_0/b)^\gamma$, namely

$$\phi(b/b_0) = \sqrt{\pi}\sum_{n=0}^{+\infty} \frac{\Gamma((n\gamma+1)/2)}{2n!\Gamma(1+n(\gamma/2-1))}(\mp 2(b_0/b)^\gamma)^n. \quad (20)$$

This formula has been established in [15] for $\gamma > 2$ for both attractive and repulsive potentials, and converges for $b > \beta b_0$, where

$$\beta = \gamma^{1/\gamma}(1-2/\gamma)^{\frac{2-\gamma}{2\gamma}}. \quad (21)$$

We have been able to extend (see Appendix A 2) this formula for any $\gamma > 0$ and $b > \beta b_0$, where $\beta(\gamma = 2) = \sqrt{2}$ and, for $0 < \gamma < 2$, $\beta = \gamma^{1/\gamma}(2/\gamma - 1)^{\frac{2-\gamma}{2\gamma}}$.

B. The regime of *hard* collisions for repulsive interactions

This corresponds to the minus sign in Eq. (14). In this Subsection again, γ is any positive number. We first give the leading order of r_{min} by writing that $1 = (b/r_{min})^2 + 2(b_0/r_{min})^\gamma$. Thus $b \ll b_0$ implies that $r_{min} \to 2^{1/\gamma}b_0$. Then,

$$b/r_{min} \approx 2^{-1/\gamma}b/b_0,$$

and it follows that

$$b/r_{min} = 2^{-1/\gamma}b/b_0 + \mathcal{O}((b/b_0)^3). \quad (22)$$

In Fig. 4 we illustrate this behavior of r_{min} by plotting W for different values of γ. Here, the small parameter we consider is $\delta = (b/r_{min})^2 \sim (b/b_0)^2 \ll 1$ and substitute $2(b_0/r_{min})^\gamma = 1 - \delta$ to obtain the expression

$$\phi(b/b_0) = \sqrt{\delta}\int_0^1 \frac{dx}{\sqrt{1-x^\gamma+\delta(x^\gamma - x^2)}},$$

which fits the form given in Eq. (19). Since the expression $(x^\gamma - x^2)/(1-x^\gamma)$ is bounded on $[0,1]$, the above integral is here again a power series in δ. In particular, we deduce the expansion

$$\int_0^1 \frac{dx}{\sqrt{1-x^\gamma+\delta(x^\gamma-x^2)}} = \int_0^1 \frac{dx}{\sqrt{1-x^\gamma}} + \mathcal{O}((b/b_0)^2).$$

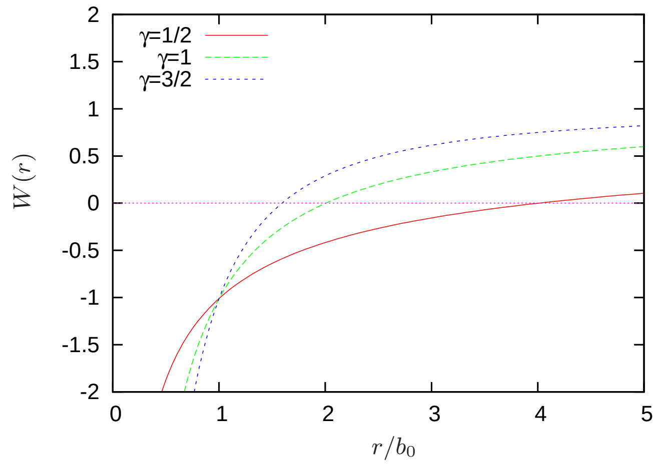

FIG. 4: Graph of W as a function of r/b_0 for $b/b_0 = 1/10$ and different values of γ for the repulsive case. Observe that $r_{min} \sim b_0$.

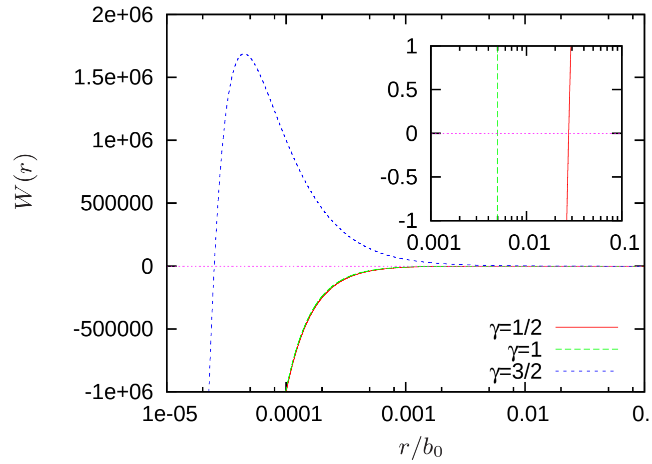

FIG. 5: Graph of W as a function of r/b_0 for $b/b_0 = 1/10$ and different values of γ for the attractive case. Observe that in this case $r_{min} \ll b_0$.

Using the expression

$$\int_0^1 \frac{dx}{\sqrt{1-x^\gamma}} = \frac{\sqrt{\pi}\Gamma\left(1+\frac{1}{\gamma}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{\gamma}\right)} \tag{23}$$

(see Appendix A 3), we infer

$$\phi(b/b_0) = B(\gamma)(b/b_0) + \mathcal{O}((b/b_0)^3), \tag{24}$$

where we have set

$$B(\gamma) = \frac{2^{-1/\gamma}\sqrt{\pi}\Gamma\left(1+\frac{1}{\gamma}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{\gamma}\right)}.$$

C. The regime of *hard* collisions for attractive interactions

We focus now on the plus sign in Eq. (14) in the regime $b \ll b_0$. As we shall see, the situation is drastically different since the qualitative behavior strongly depends on γ. We first give the leading order of r_{min} by writing that $1 + 2(b_0/r_{min})^\gamma = (b/r_{min})^2$. Thus, if $b \ll b_0$, we must have $r_{min} \leq b \ll b_0$ and then $2(b_0/r_{min})^\gamma \approx (b/r_{min})^2$. Consequently, when $\gamma \neq 2$,

$$b/r_{min} \approx (2b_0^\gamma/b^\gamma)^{1/(2-\gamma)} \gg 1. \tag{25}$$

For this regime, we shall consider the small parameter $\delta = (r_{min}/b)^2 \ll 1$ and substitute $2(b_0/r_{min})^\gamma = \delta^{-1} - 1$ to obtain the expression

$$\phi(b/b_0) = \int_0^1 \frac{dx}{\sqrt{x^\gamma - x^2 + \delta(1-x^\gamma)}}, \tag{26}$$

which tends, as $\delta \to 0$, to $\int_0^1 (x^\gamma - x^2)^{-1/2} dx$, which is finite only for $0 < \gamma < 2$. This already leads us to study the case $\gamma \geq 2$ separately (see §. IV C 5). The expression on the right-hand side of Eq. (26) fits the form in Eq. (19), but here, the situation is very different from the cases studied in Subsect. IV A and IV B since now, the expression $(1-x^\gamma)/(x^\gamma - x^2)$ is *unbounded* on $(0, 1]$. Consequently, in the naive expansion of the right-hand side of Eq. (26)

$$\int_0^1 \frac{dx}{\sqrt{x^\gamma - x^2}} - \frac{\delta}{2}\int_0^1 \frac{1-x^\gamma}{(x^\gamma - x^2)^{3/2}}\,dx + \frac{3\delta^2}{8}\int_0^1 \frac{(1-x^\gamma)^2}{(x^\gamma - x^2)^{5/2}}\,dx + \ldots,$$

the first integral converges only for $\gamma < 2$, the second one only for $\gamma < 2/3$, the third one only for $\gamma < 2/5$, etc. This suggests that on the one hand, $\phi(b/b_0)$ is probably not a power series in δ and on the other hand that we should separate the cases $\gamma < 2/3$ (see § IV C 1) and $2/3 < \gamma < 2$ (see § IV C 2).

Before that, we may calculate, when $0 < \gamma < 2$, the leading order in δ of the integral Eq. (26)

$$\alpha(\gamma) = \int_0^1 (x^\gamma - x^2)^{-1/2}\,dx = \frac{\pi}{2-\gamma}, \tag{27}$$

We are going now to study the next order correction in the approximation of Eq. (26) by Eq. (27).

1. $0 < \gamma < 2/3$

If $\gamma < 2/3$, the integral Eq. (26) is indeed of class $\mathcal{C}^1$ with respect to δ (but probably not $\mathcal{C}^2$ when $2/5 < \gamma < 2/3$) and the differentiation under the integral sign is legitimated by the fact that $\int_0^1 \frac{1-x^\gamma}{2(x^\gamma-x^2)^{3/2}}\,dx < \infty$. We then have

$$\phi(b/b_0) = \alpha(\gamma) - \delta \int_0^1 \frac{1-x^\gamma}{2(x^\gamma - x^2)^{3/2}}\,dx + o(\delta).$$

Reporting Eq. (25) and using that

$$\int_0^1 \frac{1-x^\gamma}{2(x^\gamma - x^2)^{3/2}}\,dx = \frac{\gamma}{(2-\gamma)^2} \frac{\sqrt{\pi}\Gamma\left(\frac{2-3\gamma}{2(2-\gamma)}\right)}{\Gamma\left(\frac{2(1-\gamma)}{2-\gamma}\right)} \tag{28}$$

(see Appendix A 4), we deduce

$$\begin{aligned} \phi(b/b_0) = \alpha(\gamma) - C_1(\gamma)(b/b_0)^{2\gamma/(2-\gamma)} \\ + o((b/b_0)^{2\gamma/(2-\gamma)}), \end{aligned} \tag{29}$$

where we have defined

$$C_1(\gamma) = \frac{\gamma}{(2-\gamma)^2} 2^{-2/(2-\gamma)} \frac{\sqrt{\pi}\Gamma\left(\frac{2-3\gamma}{2(2-\gamma)}\right)}{\Gamma\left(\frac{2(1-\gamma)}{2-\gamma}\right)}.$$

2. $2/3 < \gamma < 2$

We now assume $2/3 \leq \gamma < 2$, for which Eq. (26) is no longer expected to be of class $\mathcal{C}^1$ with respect to δ. We then write the correction $\phi(b/b_0) - \alpha(\gamma)$ under the form $\phi(b/b_0) - \alpha(\gamma) = -\delta Q(\delta)$, that is we define

$$\begin{aligned} Q(\delta) &= -\frac{1}{\delta}\left(\phi(b/b_0) - \int_0^1 (x^\gamma - x^2)^{-1/2}\,dx\right) \\ &= \int_0^1 \psi_\delta(x)\,dx, \end{aligned} \tag{30}$$

where we have set

$$\psi_\delta(x) = \frac{(x^\gamma - x^2)^{-1/2}(x^\gamma - x^2 + \delta(1-x^\gamma))^{-1/2}(1-x^\gamma)}{\sqrt{x^\gamma - x^2} + \sqrt{x^\gamma - x^2 + \delta(1-x^\gamma)}}.$$

Clearly, as $\delta \to 0$, $Q(\delta)$ tends to

$$\frac{1}{2}\int_0^1 \frac{1-x^\gamma}{(x^\gamma - x^2)^{3/2}}\,dx = \int_0^1 \psi_0(x)\,dx = +\infty,$$

for $\gamma \geq 2/3$, due to the non integrable singularity at $x = 0$ (hence ϕ is indeed not differentiable with respect to δ at the origin). We then wish to determine the divergence speed in $Q(\delta)$ as $\delta \to 0$, and we shall show that actually $Q(\delta)$ is of order $\delta^{1/\gamma - 3/2}$ when $2/3 < \gamma < 2$ and of order $|\ln \delta|$ if $\gamma = 2/3$.

As a first step, we may get rid of the contribution for $1/2 \leq x \leq 1$ in the integral of ψ_δ since

$$\int_{1/2}^1 \psi_\delta(x)\,dx \to \int_{1/2}^1 \frac{1-x^\gamma}{2(x^\gamma - x^2)^{3/2}}\,dx < +\infty,$$

whereas $Q(\delta) \gg 1$. Therefore,

$$Q(\delta) = \int_0^{1/2} \psi_\delta(x)\,dx + \mathcal{O}(1) \approx \int_0^{1/2} \psi_\delta(x)\,dx.$$

Now, the idea is that the expression $x^\gamma - x^2 + \delta(1-x^\gamma)$ appearing in the denominator of ψ_δ is of order δ if $0 \leq x \leq \delta^{1/\gamma}$ and of order $x^\gamma - x^2 \sim x^\gamma$ if $\delta^{1/\gamma} \leq x \leq 1/2$, which suggests to use the change of variable $y = x/\delta^{1/\gamma}$ in the integral. Therefore,

$$Q(\delta) \approx \int_0^{1/2} \psi_\delta(x)\,dx = \delta^{\frac{1}{\gamma}-\frac{3}{2}} \int_0^{\delta^{-1/\gamma}/2} \Psi_\delta(y)\,dy, \tag{31}$$

where we have set

$$\Psi_\delta(y) = (1-\delta y^\gamma) \times \frac{(y^\gamma - \delta^{2/\gamma-1}y^2)^{-1/2}(y^\gamma - \delta^{2/\gamma-1}y^2 + 1 - \delta y^\gamma)^{-1/2}}{\sqrt{y^\gamma - \delta^{2/\gamma-1}y^2} + \sqrt{y^\gamma - \delta^{2/\gamma-1}y^2 + 1 - \delta y^\gamma}}.$$

As $\delta \to 0$ and for $2/3 \leq \gamma < 2$, one can justify rigorously that

$$\begin{aligned} \int_0^{\delta^{-1/\gamma}/2} \Psi_\delta(y)\,dy \to \int_0^{+\infty} \Psi_0(y)\,dy \\ = \int_0^{+\infty} \frac{y^{-\gamma/2}(y^\gamma+1)^{-1/2}}{y^{\gamma/2} + \sqrt{y^\gamma + 1}}\,dy, \end{aligned}$$

which is finite as soon as $2/3 < \gamma < 2$ since the integrand is $\approx y^{-3\gamma/2}/2$ at infinity and $\approx y^{-\gamma/2}$ near the origin. We obtain finally, in Appendix A 5,

$$\int_0^{+\infty} \Psi_0(y)\,dy = \frac{2^{3-2/\gamma}}{\gamma} \frac{\Gamma\left(\frac{3}{2} - \frac{1}{\gamma}\right)\Gamma\left(\frac{2}{\gamma} - 1\right)}{\Gamma\left(\frac{1}{\gamma} + \frac{1}{2}\right)}, \tag{32}$$

and using Eq. (25) then gives, for $2/3 < \gamma < 2$ and $b \ll b_0$,

$$\phi(b/b_0) = \alpha(\gamma) - C_3(\gamma)(b/b_0) + o(b/b_0), \tag{33}$$

with

$$C_3(\gamma) = \frac{2^{2-3/\gamma}}{\gamma} \frac{\Gamma\left(\frac{3}{2} - \frac{1}{\gamma}\right)\Gamma\left(\frac{2}{\gamma} - 1\right)}{\Gamma\left(\frac{1}{\gamma} + \frac{1}{2}\right)}.$$

3. $\gamma = 2/3$

It remains to study the case $\gamma = 2/3$, for which it is natural to expect from Eq. (31) and the fact that

$$\Psi_0(y) = \frac{1}{y^{1/3}(y^{1/3} + \sqrt{1+y^{2/3}})\sqrt{1+y^{2/3}}} \approx \frac{1}{2y}$$

at infinity that

$$Q(\delta) \approx \int_1^{\delta^{-3/2}/2} \frac{dy}{2y} \simeq \frac{3}{4}|\ln \delta|. \tag{34}$$

The mathematical justification of this result is given in Appendix A 6. Reporting this into Eq. (30) and using that $\delta \approx b/(2\sqrt{2}b_0)$ by Eq. (25) yields

$$\phi(b/b_0) = \frac{3\pi}{4} - C_2(b/b_0)\ln(b_0/b) + o((b/b_0)\ln(b_0/b)), \tag{35}$$

with $C_2 = \dfrac{3}{8\sqrt{2}}$.

4. $\gamma = 2$

The case $\gamma = 2$ allows explicit computation and we see that it is a case where the attractive term is strong enough to form pairs when b is small. Of course, this will be also the case when $\gamma > 2$. This implies that the function W in Eq. (7) may have no positive zero. Actually, when $\gamma = 2$, the behavior of the expression

$$W(r) = 1 - \frac{b^2}{r^2} + 2\frac{b_0^2}{r^2} = 1 - \frac{b^2 - 2b_0^2}{r^2}$$

depends whether $b > b_0\sqrt{2}$ or $b < b_0\sqrt{2}$. If $b > b_0\sqrt{2}$, then W possesses $r_{min} = \sqrt{b^2 - 2b_0^2}$ as unique positive zero, and we have the exact value

$$\phi(b/b_0) = \int_{r_{min}}^{+\infty} \frac{(b/r^2)\,dr}{\sqrt{1 - r_{min}^2/r^2}} = \frac{b\pi}{2r_{min}} = \frac{\pi}{2\sqrt{1 - 2b_0^2/b^2}}. \tag{36}$$

If $b \le b_0\sqrt{2}$, then $W \ge 1$ has no zero. This means that the two particles will crash one onto the other in finite time with a spiraling motion. The integral in the right-hand side of Eq. (10) is then equal to $+\infty$, but the angle ϕ has then no geometrical meaning and the picture given in Fig. 1 is then no longer the good one. There exists then a threshold $b_0\sqrt{2}$ with the property that particles crash as soon as $b \le b_0\sqrt{2}$.

5. $\gamma > 2$

If $\gamma > 2$, the attractive term is strong enough to form pairs for sufficiently small b, and we shall explicit the threshold. Notice first that when $\gamma > 2$, the function $W(r) = 1 - b^2/r^2 + 2b_0^\gamma/r^\gamma$ decreases on $(0, r_*(b)]$ and increases on $[r_*(b), +\infty)$, with

$$r_*(b) = \left(\frac{\gamma b_0^\gamma}{b^2}\right)^{\frac{1}{\gamma-2}}.$$

Since $W(r_*(b)) = 1 - b^2/r_*^2(b) + 2b_0^\gamma/r_*^\gamma(b) = 1 - (b_0/b)^{-\frac{2\gamma}{\gamma-2}}[1-2/\gamma]\gamma^{-\frac{2}{\gamma-2}}$, we may then easily check that if

$$b > \beta b_0, \tag{37}$$

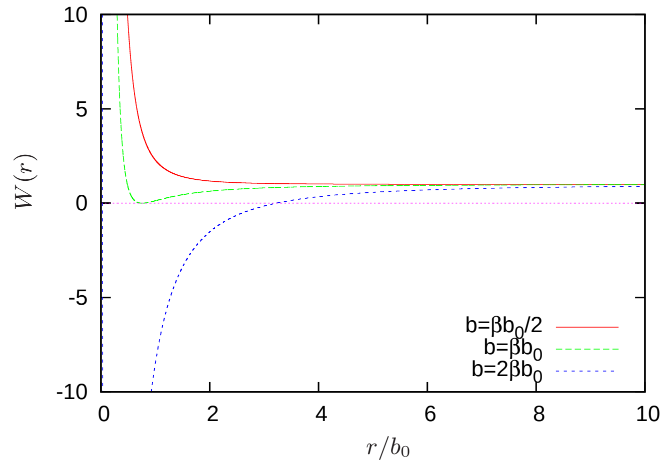

FIG. 6: Graph of W as a function of r/b_0 for different values of b for $\gamma = 5/2$ for the attractive case. Observe that for $b = \beta b_0/2$ there is no root, $b = \beta b_0$ is the limiting case with a double root and for $b = 2\beta b_0$ there is one root.

where

$$\beta = \gamma^{1/\gamma}\left(1 - \frac{2}{\gamma}\right)^{\frac{2-\gamma}{2\gamma}}, \tag{38}$$

which coincides with the expression Eq. (21) appearing in the convergence radius of Eq. (20). Then W has a larger positive zero r_{min}, whereas if $b < \beta b_0$, the expression W is positive on $(0, +\infty)$, and if $b = \beta b_0$, the expression W has a double root at $r = r_*(\beta b_0) = b_0(\gamma - 2)^{1/\gamma} > 0$, where $W(r_*(\beta b_0)) = 0$. These three behaviors are illustrated in Fig. 6.

When $\gamma \to 2^+$, we have, as expected, $\beta = \gamma^{1/\gamma}(1 - 2/\gamma)^{\frac{2-\gamma}{2\gamma}} = \gamma^{1/\gamma}\exp((1/2)(1 - 2/\gamma)\ln(1 - 2/\gamma)) \to \sqrt{2}$. If $b < \beta b_0$, the particles crash in finite time and ϕ has here again no physical or geometrical meaning, despite the fact that the integral

$$\int_0^{+\infty} \frac{(b/r^2)\,dr}{\sqrt{1 - \frac{b^2}{r^2} + 2\frac{b_0^\gamma}{r^\gamma}}} = \int_0^{+\infty} \frac{dx}{\sqrt{1 - x^2 + 2(b_0/b)^\gamma x^\gamma}},$$

where r_{min} has been replaced by 0, converges.

When $b = \beta b_0$, the reduced particle remain asymptotically trapped on a circular orbit of radius $r_*(\beta b_0) > 0$. This phenomenon is called in the atomic physics literature *orbiting* (see e.g. [13]). The angle ϕ has once again no physical or geometrical meaning, and

$$\int_{r_*(\beta b_0)}^{+\infty} \frac{(b/r^2)\,dr}{\sqrt{1 - \frac{b^2}{r^2} + 2\frac{b_0^\gamma}{r^\gamma}}} = +\infty$$

in view of the fact that $1 - b^2/r^2 + 2b_0^\gamma/r^\gamma \sim (r - r_*(\beta b_0))^2$ for r close to $r_*(\beta b_0)$.

Let us now consider the situation where we take $\gamma > 2$ and b slightly larger than βb_0, so that one expect a divergence in the integral ϕ. We have

$$\phi(b/b_0) = \int_{r_{min}}^{+\infty} \frac{b\,dr}{r^2\sqrt{W_b(r)}},$$

with $W_b(r) = 1 - b^2/r^2 + 2b_0^\gamma/r^\gamma$ (we have stressed the dependency on b since we are interested in the limit $b \to \beta b_0$). We set $R = r_*(\beta b_0) = b_0(\gamma-2)^{1/\gamma} > 0$. As b approaches βb_0, we have both $r_*(b) \to R$ ($r_*(b)$ is the minimum for W_b) and $r_{min} \to R$ (r_{min} is the largest zero of W_b). In the integral ϕ, the contributions for r close to R will make the integral diverge since we shall have $W_b(r) \sim (r-R)^2$ (we have a double root when $b = \beta b_0$), whereas the contributions for r much larger than R will remain of order one. As a consequence, for any small length parameter $\ell > 0$, we have

$$\phi(b/b_0) \approx \int_{r_{min}}^{r_{min}+\ell} \frac{b\,dr}{r^2\sqrt{W_b(r)}},$$

and we may then replace $W_b(r)$ by its second order Taylor expansion near $r_*(b)$:

$$\begin{aligned} W_b(r) &= W_b(r_*(b)) + (r - r_*(b))W_b'(r_*(b)) \\ &\quad + \frac{1}{2}(r-r_*(b))^2 W_b''(r_*(b)) + \mathcal{O}((r-r_*(b))^3). \end{aligned}$$

Since $W_b'(r_*(b)) = 0$ and

$$W_b''(r_*(b)) = \frac{2\gamma(\gamma+1)b_0^\gamma}{r_*^{\gamma+2}(b)} - \frac{6b^2}{r_*^4(b)} \approx \frac{2(\gamma-2)b^2}{R^4} > 0, \tag{39}$$

this yields

$$\phi(b/b_0) \approx \int_{r_{min}}^{r_{min}+\ell} br^{-2}\,dr/ \sqrt{W_b(r_*(b)) + (r-r_*(b))^2(W_b''(r_*(b))/2 + \mathcal{O}(r-r_*(b)))}.$$

We have $W_b(r_*(b)) < 0 < W_b''(r_*(b))$ with $W_b(r_*(b))$ small but $W_b''(r_*(b))$ of order one. The idea is then to use the substitution

$$z\sqrt{-W_b(r_*(b))} = (r - r_*(b))\sqrt{W_b''(r_*(b))/2 + \mathcal{O}(r-r_*(b))},$$

so that the expression in the square root in the integral becomes simply $-W_b(r_*(b))(z^2-1)$. This yields

$$\phi(b/b_0) \approx \frac{b}{\sqrt{-W_b(r_*(b))}} \int_1^{z_{max}} \frac{r(z)^{-2}\,dr/dz}{\sqrt{z^2-1}}\,dz, \tag{40}$$

where $z_{min} = 1$ and $z_{max} \approx Cte(\ell)/\sqrt{-W_b(r_*(b))} \gg 1$ are the corresponding values to r_{min} and $r_{min}+\ell$. The idea is now that, roughly speaking, $r(z) \approx r_*(b) \approx R$ and $dr/dz \approx \sqrt{-2W_b(r_*(b))/W_b''(r_*(b))}$, which implies

$$\begin{aligned} \phi(b/b_0) &\approx \frac{b}{R^2}\sqrt{\frac{2}{W_b''(r_*(b))}} \int_1^{z_{max}} \frac{dz}{\sqrt{z^2-1}} \\ &\approx \sqrt{\frac{2b^2}{R^4 W_b''(R)}}\ln(z_{max}) \approx -\frac{\ln|W_b(r_*(b))|}{2\sqrt{\gamma-2}}, \end{aligned} \tag{41}$$

in view of Eq. (39) and the fact that $z_{max} \approx Cte(\ell)/\sqrt{-W_b(r_*(b))} \gg 1$. Finally, $W_b(r_*(b)) = 1 - (\beta b_0/b)^{-\frac{2\gamma}{\gamma-2}}$, and we end up with

$$\phi(b/b_0) \approx -\frac{\ln(1-\beta b_0/b)}{2\sqrt{\gamma-2}}. \tag{42}$$

For the sake of simplicity, we have included the mathematical details leading to Eq. (41) in Appendix A 7.

V. PHYSICAL DISCUSSION

In this section we give a summary, a physical discussion and a numerical checking of the mathematical results derived in the previous section.

A. Summary of the results and numerical checking

We have summarized the results obtained in Sect. IV in table I. In the first column, we make the difference between the regime of soft collisions, for which the angle $\phi \approx \pi/2$ (and hence the angle of deflection $\chi \approx \pi$) — which means that the trajectories are weakly perturbed — and the regime of strong collisions, in which the angle ϕ is far from $\pi/2$, i.e., the trajectory is strongly perturbed. In the case of attractive potentials, different cases arise depending on the value of γ:

- For $0 < \gamma < 2$, the leading order value of ϕ is $\pi/(2-\gamma)$. The exponent of the first order correction depends whether γ is smaller than $2/3$ or not. If we expand to higher order, we will see that the exponent of the second order term depends whether γ is smaller than $2/5$ or not, the exponent of the third order term whether γ is smaller than $2/7$ or not, and so on.
- For $\gamma > 2$, we have formation of pairs for impact factors smaller than a critical one. For impact factors exactly at the critical one there is the phenomena of *orbiting* and for larger impact factors the collision is well behaved. We discuss these phenomena in the subsection below.

In addition, for $0 < \gamma < 2$, we have checked numerically the validity of the asymptotic expansions in Sect. IV for both repulsive potentials (Fig. 7) and attractive potentials (Fig. 8), in the soft collision regime ($b/b_0 \gg 1$)

type of collision	repulsive potential	attractive potential	
soft ($b \gg b_0$)	$\phi - \frac{\pi}{2} \sim -(b_0/b)^\gamma$	$\phi - \frac{\pi}{2} \sim +(b_0/b)^\gamma$	
hard ($b \ll b_0$)	$\phi \sim b/b_0$	$\phi - \frac{\pi}{2-\gamma} \sim -\left(\frac{b}{b_0}\right)^{2\gamma/(2-\gamma)}$	$0 < \gamma < 2/3$
		$\phi - \frac{3\pi}{4} \sim -\frac{b}{b_0} \ln\left(\frac{b_0}{b}\right)$	$\gamma = 2/3$
		$\phi - \frac{\pi}{2-\gamma} \sim -\frac{b}{b_0}$	$2/3 < \gamma < 2$
		particles crash when $b \leq \beta b_0$	$2 = \gamma$
		particles crash when $b < \beta b_0$	$2 < \gamma$
		formation of a binary (orbiting) when $b = \beta b_0$	

TABLE I: Summary of the expansions of the angle ϕ.

where the trajectory is weakly perturbed (top of each figure) and in the hard collision regime (bottom of each figure). We see a perfect matching between the numerical calculations and the analytical asymptotic calculations.

Finally, when $\gamma \geq 2$, we illustrate in Fig. 9 the divergence of ϕ when b approaches βb_0 ($b > \beta b_0$) obtained in Eq. (36) for $\gamma = 2$ (top) and in Eq. (42) when $\gamma = 5/2 > 2$ (bottom).

B. Collisions with loops for attractive potentials

Collisions with loops may appear when the angle ϕ becomes large. This happens for attractive potentials in the two following cases:

- when γ is slightly smaller than 2 and $b \ll b_0$, since then $\phi(b/b_0) \approx \alpha(\gamma) = \frac{\pi}{2-\gamma}$ (see Subsect. IV C).
- when $\gamma \geq 2$ and $b \approx \beta b_0$ ($b > \beta b_0$), with the expression for β given in Eq. (38), since (see § IV C 5)

$$\phi\left(\frac{b}{b_0}\right) = \frac{\pi}{2\sqrt{1 - 2b_0^2/b^2}} \qquad \text{if } \gamma = 2, \tag{43a}$$

$$\phi\left(\frac{b}{b_0}\right) \approx -\frac{\ln(1 - \beta b_0/b)}{2\sqrt{\gamma - 2}} \qquad \text{if } \gamma > 2. \tag{43b}$$

It is interesting to study these trajectories, for which it is numerically convenient to use for the first part of the trajectory the implicit relation between the polar angle $\theta \in [0, \phi]$ and the distance to the origin r of the particle (see e.g. [16]):

$$\theta(b, b_0, r) = \int_r^\infty \frac{(b/r'^2)dr'}{\sqrt{1 - (b/r')^2 - 2(b_0/r')^\gamma}}. \tag{44}$$

Note that $\theta(b, b_0, r_{min}) = \phi(b/b_0)$. The first half of the trajectory is therefore

$$x = r\cos\theta$$
$$y = r\sin\theta,$$

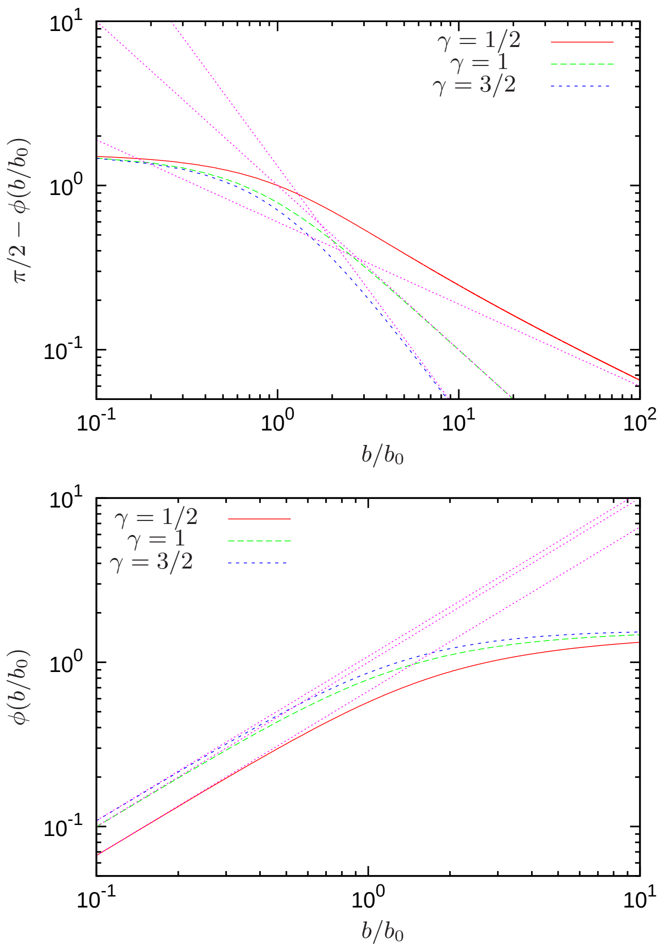

FIG. 7: Numerical computations for repulsive potentials and several values of γ. Top: for soft scattering ($b/b_0 \gg 1$), plot of $\pi/2 - \phi$ (continuous line) and the theoretical predictions (dotted line) Eq. (18) as a function of b/b_0. Bottom: for hard scattering ($b/b_0 \ll 1$), plot of ϕ (continuous line) and the theoretical predictions (dotted line) Eq. (24) as a function of b/b_0.

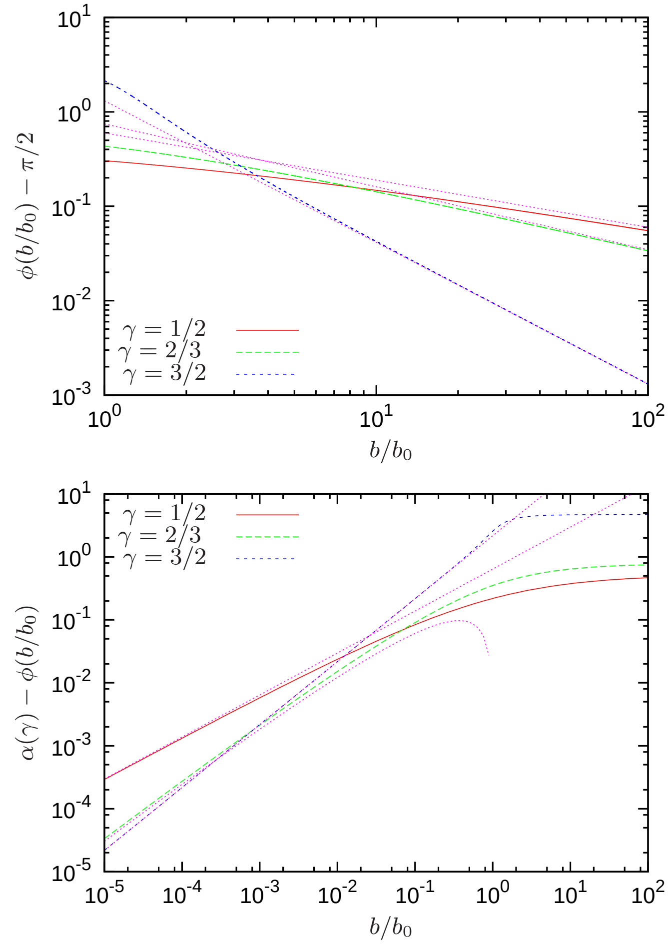

FIG. 8: Numerical computations for attractive potentials and several values of γ. Top: for soft scattering ($b/b_0 \gg 1$), plot of $\pi/2 - \phi$ (continuous line) and the theoretical predictions (dotted line) Eq. (18) as a function of b/b_0. Bottom: for hard scattering ($b/b_0 \ll 1$), plot of ϕ (continuous line) and the theoretical predictions (dotted line) Eq. (29), (33) or (35) (depending on the value of γ) as a function of b/b_0.

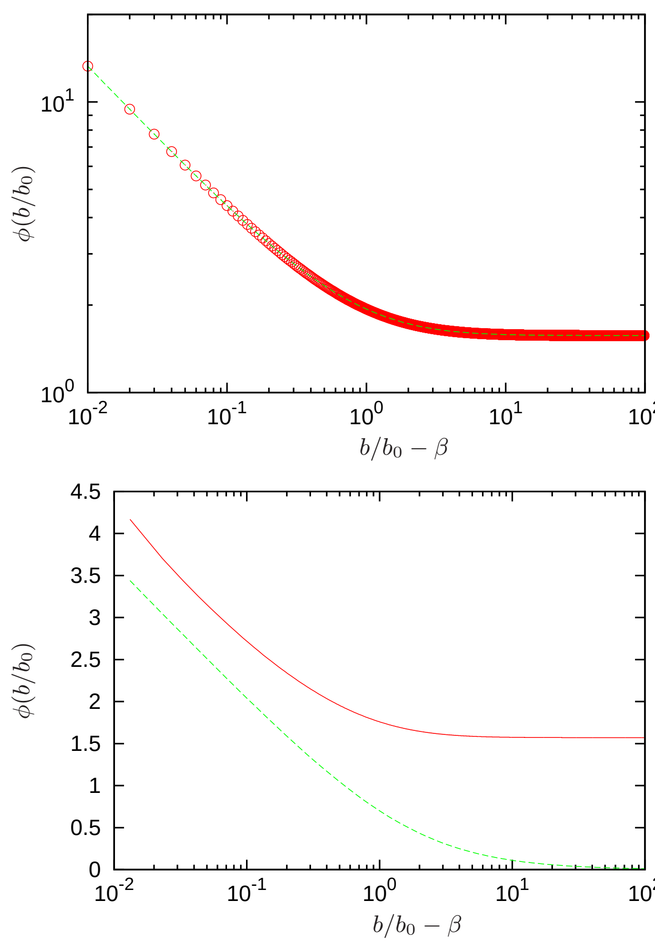

FIG. 9: Divergence of ϕ when $0 < b/b_0 - \beta \ll 1$. Top: Graph of ϕ as a function of b for $\gamma = 2$ (continuous line) given by Eq. (36) and by the numerical calculation (circles). Bottom: same quantity for $\gamma = 5/2$ (continuous red line) and the leading order given in Eq. (42) (dotted green line).

and the second one

$$x = r\cos(2\phi - \theta)$$
$$y = r\sin(2\phi - \theta).$$

We see that the trajectory is symmetric about a straight line which passes by the origin of coordinates (i.e. the center of mass) and the point of closest approach (defined by the angle ϕ). In the plot, the first half part of the trajectory — from $x = +\infty$ to the axis of symmetry — is plotted in red, the other half of the trajectory in green. The points of intersection of the trajectory lie on the axis of symmetry.

1. *The case $\gamma = 2^- < 2$ and $b/b_0 \ll 1$*

In Subsect. IV C we have seen that, for the attractive potential with $\gamma < 2$, we have

$$\lim_{b/b_0 \to 0} \phi\left(\frac{b}{b_0}\right) = \alpha(\gamma) = \frac{\pi}{2 - \gamma} \tag{47}$$

and (see the beginning of that Subsect.)

$$r_{min} \leq b \ll b_0.$$

Therefore, in the limit $\gamma \to 2$, the angle $\phi(0^+)$ diverges. Fixing $\gamma < 2$ but $\gamma \approx 2$ (say $\gamma = 1.95$ for instance), we have, for $b/b_0 \ll 1$, a collision where $r_{min} \ll b_0$ is very small and ϕ very large, which corresponds to many loops in a very small region close to the center of mass of the two particles.

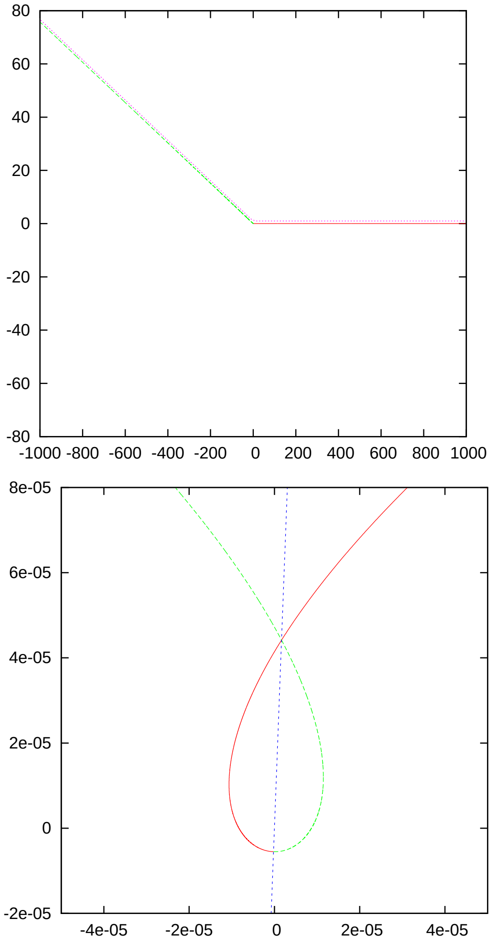

FIG. 10: Top: A trajectory in the center of mass frame for attractive hard interaction and $\gamma = 4/3$ and $b/b_0 = 0.025$. The pink dotted line represents the prediction of Eq. (33) (with a small offset). Bottom: zoom of the plot above, in which a loop is visible. The first half part of the trajectory — from $x = +\infty$ to the axis of symmetry — is plotted in red, the other half of the trajectory in green. The points of intersection of the trajectory lie on the axis of symmetry.

The number of intersections between the first half of the trajectories (red lines in the plots) and the symmetric one (green lines in the plots) depends on the value of ϕ. The number of intersections of the trajectories (and hence the number of loops) is given by the floor function of the angle of closest approach divided by π

$$n_{loops}\left(\frac{b}{b_0}\right) = \text{floor}\left(\frac{\alpha(\gamma)}{\pi}\right) = \text{floor}\left(\frac{1}{2-\gamma}\right). \qquad (48)$$

In the example showed in Fig. 10, $\gamma = 4/3$, $\phi \approx 4.67$, and hence there is one loop. A more complex example appears when choosing $\gamma = 1.95$ and $b/b_0 = 0.8$. In this case $\phi \approx 39.3$ and hence the numbers of loops is, using Eq. (48), $n_{loops} = 12$. This can be shown explicitly in Fig. 11, where we show successive zooms in the trajectory, in which appear smaller and smaller loops.

In Fig. 12, we show $\theta(b, b_0, r)$ (in Eq. (44)) as a function of r. Each horizontal line corresponds to $\phi = n\pi$, in which n is an integer and with n such that $\phi(b/b_0) > n\pi$.

2. *The case $\gamma \geq 2$ and $b/b_0 = \beta^+$*

We have seen in Subsect. IV C 5 that the angle ϕ diverges for $b/b_0 \approx \beta$, with the expression for β given in Eq. (38), see the formulas recalled in Eq. (43).

When $\gamma > 2$, the angle ϕ diverges logarithmically for b approaching βb_0. If we compare to the case ($\gamma = 2^+$ and $b/b_0 \ll 1$) previously studied, we see that the main difference is that now, the distance of closest approach r_{min} is no longer small but of order one since $r_{min}(b) \approx r_{min}(\beta b_0) = R = b_0(2-\gamma)^{1/\gamma} > 0$ (see § IV C 5). The shape of the trajectories is therefore very different from the case ($\gamma = 2^+$ and $b/b_0 \ll 1$) since then, the particle remains in a (close to circular) orbit of positive radius. For the particular value $b = \beta b_0$, the particle remains asymptotically trapped on a closed circular orbit, that is we have the formation of a binary. We illustrate this behavior in Fig. 13.

VI. EFFECT OF A SHORT-SCALE REGULARIZATION IN THE POTENTIAL

In many physical situations, the potential is not a pure power-law as in Eq. (1) but there is a regularization at small scales, which is commonly called *softening* e.g. in the astrophysical literature. This is for example the case in the dark–matter collisionless N-body cosmological simulations, in which a softening is introduced to minimize as much as possible collisional effects. From a more fundamental point of view, we are interested in answering the following questions:

1. Does a regularization in the potential modify the results above presented?

2. If yes, up to what scale and how?

3. Is the formation of pairs (which appears for $\gamma > 2$ and $b/b_0 < \beta$) suppressed when a regularization is introduced, and in the affirmative case is there a minimal softening case needed?

In this section we will answer these questions. In order to be able to make explicit calculations, we will consider two popular regularization used commonly in the astrophysical literature (see e.g. [17, 18]), the *Plummer potential*

$$v^{\text{Pl}}(r, \epsilon) = \frac{g}{(r^2 + \epsilon^2)^{\gamma/2}} \qquad (49)$$

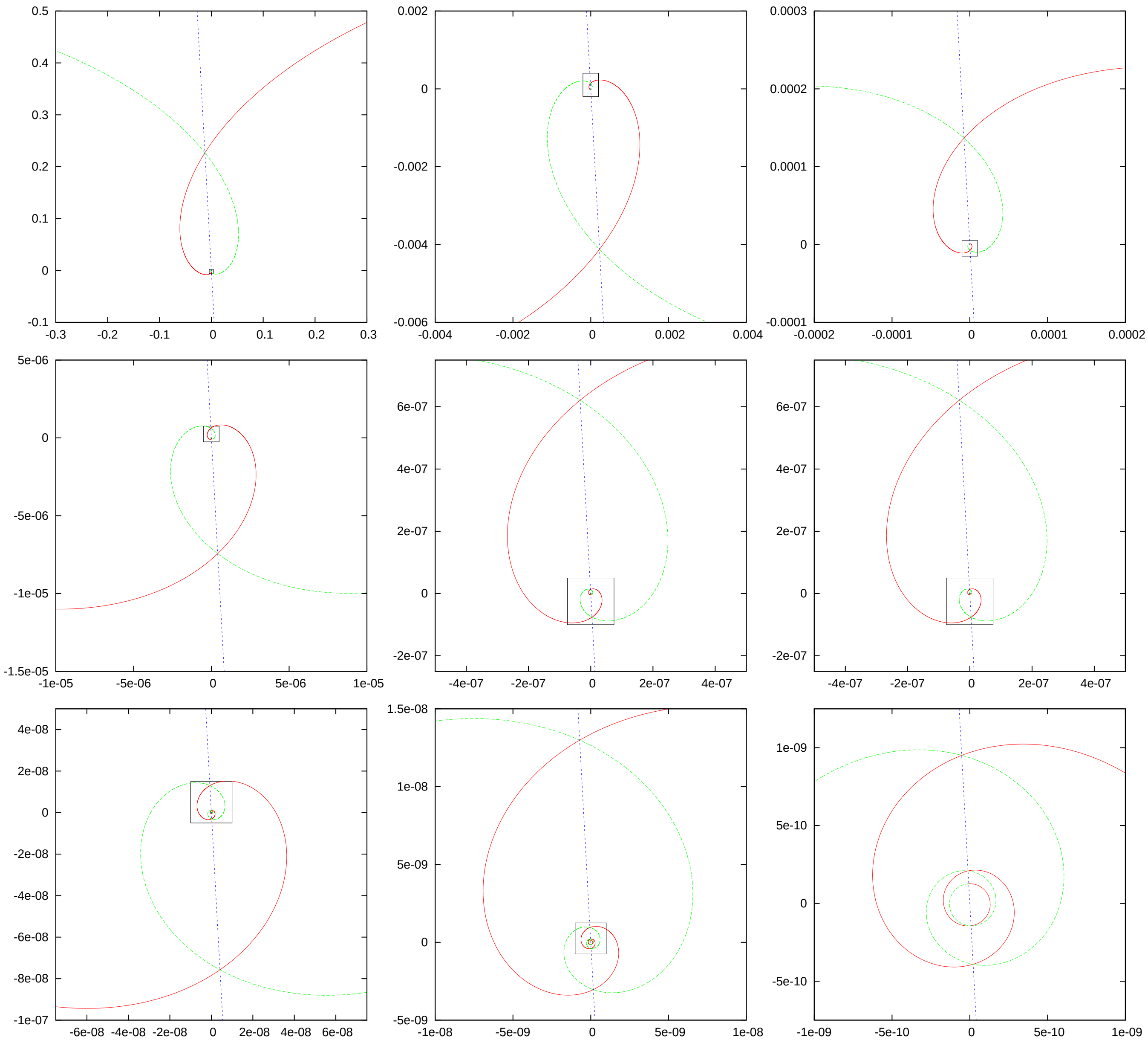

FIG. 11: Collision in the center of mass frame for $\gamma = 1.95$ and $b/b_0 = 0.8$. The dotted line is the axis of symmetry of the trajectory. The square in each plot represents the frame of the next plot (which have to be read from left to right and top to down). The first half part of the trajectory — from $x = +\infty$ to the axis of symmetry — is plotted in red, the other half of the trajectory in green. The points of intersection of the trajectory lie on the axis of symmetry.

and the compact softening

$$v^{\rm co}(r, \epsilon) = \begin{cases} \dfrac{g}{r^\gamma} & \text{if } r \geq \epsilon \\ \dfrac{g}{\epsilon^\gamma} \mathrm{v}\left(r/\epsilon\right) & \text{if } 0 \leq r \leq \epsilon, \end{cases} \tag{50}$$

where v is a function on $[0, 1]$ such that $\mathrm{v}(1) = 1$. The Plummer softening gives rise to an interaction of the same sign than the unsoftened one, whereas it is possible to choose the form of the compact softening (typically v is a polynomial in r/ϵ) in order to be indifferently attractive, repulsive or both. A common feature for both of these potentials is that they fulfill the relation

$$v(r, \epsilon) = \frac{g}{\epsilon^\gamma} \mathcal{V}\left(\frac{r}{\epsilon}\right), \tag{51}$$

with

$$\mathcal{V}^{\rm Pl}(R) = \frac{1}{(R^2 + 1)^{\gamma/2}}$$

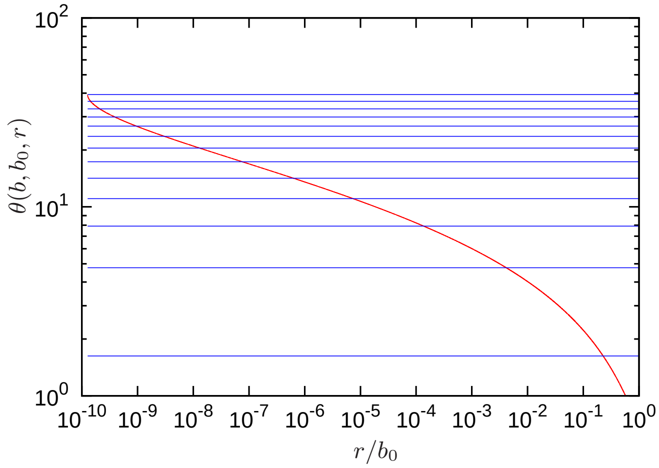

FIG. 12: Graph of $\theta(b,b_0,r)$ as a function r/b_0 for a trajectory with $\gamma = 1.95$ and $b/b_0 = 0.8$. Each horizontal line corresponds to an intersection in the trajectory.

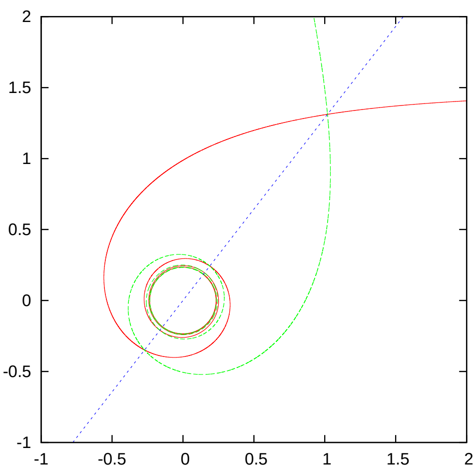

FIG. 13: A trajectory in the center of mass frame for attractive interaction $\gamma = 2.05$ and $b/b_0 = \beta + 10^{-6}$ (only a portion of the trajectory is plotted). The first half part of the trajectory — from $x = +\infty$ to the axis of symmetry — is plotted in red, the other half of the trajectory in green. The points of intersection of the trajectory lie on the axis of symmetry.

and

$$\mathcal{V}^{\rm co}(R) = \begin{cases} \dfrac{1}{R^\gamma} & \text{if } R \geq 1 \\ \mathrm{v}(R) & \text{if } 0 \leq R \leq 1. \end{cases}$$

We will show that the results presented below do not depend qualitatively on the explicit form of the regularization used. In what follow, we will study how the angle ϕ is modified by the regularization in the potential, first for repulsive interactions and then for attractive ones.

We introduce the angle ϕ_ϵ corresponding to the regularized potential:

$$\phi_\epsilon(b,b_0) = \frac{b}{r_{min}} \int_0^1 \frac{dx}{\sqrt{1 - (\frac{bx}{r_{min}})^2 \pm \frac{2b_0^\gamma}{\epsilon^\gamma} \mathcal{V}(\frac{r_{min}}{\epsilon x})}}. \quad (52)$$

A. Repulsive interactions with Plummer softening

Here $\mathcal{V}(R) = \mathcal{V}^{\rm Pl}(R) = (R^2+1)^{-\gamma/2}$. Then, the function $r \mapsto 1 - b^2/r^2 - 2b_0^\gamma / \left(r^2+\epsilon^2\right)^{\gamma/2}$ increases from $-\infty$ to 1 as r increases from 0^+ to $+\infty$, hence has a single positive zero r_{min}. It is easily checked that r_{min} is an increasing function of b and that the function $r \mapsto 1 - 2b_0^\gamma / \left(r^2+\epsilon^2\right)^{\gamma/2}$ possesses a positive zero if and only if $\epsilon < b_0 2^{1/\gamma}$. Therefore, for small b,

$$r_{min} \approx r_0 = b_0 2^{1/\gamma}\sqrt{1-\hat\epsilon^2}, \qquad \text{where } \hat\epsilon = \frac{\epsilon}{2^{1/\gamma} b_0},$$

if $\hat\epsilon \leq 1$, and

$$r_{min} \approx \frac{b}{\sqrt{1-\hat\epsilon^{-\gamma}}}$$

if $\hat\epsilon > 1$. This naturally leads us to distinguish the case $\epsilon < b_0 2^{1/\gamma}$ and the case $\epsilon > b_0 2^{1/\gamma}$.

1. *The case $\epsilon < b_0 2^{1/\gamma}$*

We assume $\hat\epsilon < 1$, so that $r_0 > 0$, $r_{min} = r_0(1 + \mathcal{O}((b/b_0)^2))$, and consider here again the small parameter $\delta = (b/r_{min})^2 \ll 1$. Substituting

$$\frac{2b_0^\gamma}{\epsilon^\gamma} = \frac{1 - b^2/r_{min}^2}{\mathcal{V}(r_{min}/\epsilon)} = \frac{1-\delta}{\mathcal{V}(r_{min}/\epsilon)}$$

yields

$$\phi_\epsilon(b,b_0) = \sqrt{\delta}\int_0^1 \frac{dx}{\sqrt{1 - \delta x^2 - (1-\delta)\frac{\mathcal{V}(r_{min}/(\epsilon x))}{\mathcal{V}(r_{min}/\epsilon)}}}$$
$$= \sqrt{\delta}\int_0^1 \frac{dx}{\sqrt{F(x,r_{min}/\epsilon) + \delta(1-x^2-F(x,r_{min}/\epsilon))}},$$

where we have set

$$F(x,r_{min}/\epsilon) = 1 - \frac{\mathcal{V}(r_{min}/(\epsilon x))}{\mathcal{V}(r_{min}/\epsilon)}.$$

We prove in App. A 8 that the function $x \mapsto \frac{1-x^2}{F(x,r_{min}/\epsilon)}$ is bounded on $[0,1]$ independently of b. This shows that we may apply the argument for Eq. (19) and write

$$\phi_\epsilon(b,b_0) = \sqrt{\delta}\int_0^1 \frac{dx}{\sqrt{F(x,r_{min}/\epsilon)}\sqrt{1+\delta(\frac{1-x^2}{F(x,r_{min}/\epsilon)}-1)}}$$
$$= \sqrt{\delta}\int_0^1 \frac{dx}{\sqrt{F(x,r_{min}/\epsilon)}} + \mathcal{O}(\delta^{3/2}).$$

At this stage, since $r_{min} = r_0(1+\mathcal{O}((b/b_0)^2))$, one could legitimate the expansion

$$\int_0^1 \frac{dx}{\sqrt{F(x, r_{min}/\epsilon)}} = \int_0^1 \frac{dx}{\sqrt{F(x, r_0/\epsilon)}} + \mathcal{O}((b/b_0)^2).$$

Since $r_{min} = r_0(1+\mathcal{O}((b/b_0)^2))$, $\sqrt{\delta} = b/r_{min} = b/r_0(1+\mathcal{O}((b/b_0)^2))$, and thus, when $\hat{\epsilon} < 1$,

$$\phi_\epsilon^{\mathrm{Pl}}(b, b_0) = B_{\hat{\epsilon}}^{\mathrm{Pl}}(\gamma)(b/b_0) + \mathcal{O}((b/b_0)^3), \tag{53}$$

where

$$B_{\hat{\epsilon}}^{\mathrm{Pl}}(\gamma) = \frac{2^{-1/\gamma}}{\sqrt{1-\hat{\epsilon}^2}} \int_0^1 \frac{dx}{\sqrt{1 - \frac{x^\gamma}{(1-\hat{\epsilon}^2(1-x^2))^{\gamma/2}}}}.$$

Comparing Eq. (53) with the expression Eq. (24) of the angle of closest approach without softening we observe that the linear dependence of ϕ with respect to b/b_0 is not modified, only the pre-factor changes. It is also easy to check that in the limit $\epsilon \to 0$ we have, as expected, $B_{\hat{\epsilon}}^{\mathrm{Pl}}(\gamma) \to B(\gamma)$. As expected, the new introduced scale is ϵ.

2. *The case $\epsilon > b_0 2^{1/\gamma}$*

In the case $\hat{\epsilon} > 1$, that is $\epsilon > 2^{1/\gamma} b_0$, we recall that

$$r_{min} \approx b/\sqrt{1-\hat{\epsilon}^{-\gamma}} \tag{54}$$

and that

$$\phi_\epsilon(b, b_0) = \frac{b}{r_{min}} \int_0^1 \frac{dx}{\sqrt{1 - (\frac{bx}{r_{min}})^2 - 2\frac{b_0^\gamma}{\epsilon^\gamma}\mathcal{V}(\frac{r_{min}}{\epsilon x})}}.$$

Substituting $1 = b^2/r_{min}^2 + \hat{\epsilon}^{-\gamma}\mathcal{V}(r_{min}/\epsilon)$ in the integral and considering the small parameter $\delta = r_{min}^2/\epsilon^2 \sim b^2/\epsilon^2$ gives

$$\phi_\epsilon(b, b_0) = \int_0^1 \frac{dx}{\sqrt{G_b(x)}},$$

where

$$G_b(x) = 1 - x^2 - \frac{r_{min}^2}{b^2\hat{\epsilon}^\gamma}\left(\mathcal{V}(\sqrt{\delta}/x) - \mathcal{V}(\sqrt{\delta})\right).$$

In view of the fact that $r_{min}^2 \approx b^2/(1+\hat{\epsilon}^{-\gamma})$ and $b_0 \lesssim \epsilon$, we expect

$$\phi_\epsilon(b, b_0) \approx \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}.$$

We also see that the situation is similar to the form given in Eq. (19), but the dependency on the small parameter δ is more intricate. Actually, for the Plummer potential, we have $\mathcal{V}^{\mathrm{Pl}}(R) = (R^2+1)^{-\gamma/2}$, thus, for small R, $\mathcal{V}^{\mathrm{Pl}}(R) = 1 - \gamma R^2/2 + \mathcal{O}(R^4)$. Therefore, for fixed x and small δ, we obtain

$$G_b(x) = 1 - x^2 - \frac{\gamma\delta}{2(\hat{\epsilon}^\gamma - 1)}\left(\frac{1}{x^2} - 1\right) + \mathcal{O}(\delta^2),$$

which is a situation very similar to the form given in Eq. (19), but unfortunately the function $x \mapsto (1/x^2 - 1)/(1-x^2) = -1/x^2$ being too singular near the origin, the power series expansion trick used for Eq. (19) as in Subsect. IV A and IV B breaks down.

Following the approach used in § IV C 2, we divide the correction $\phi_\epsilon(b, b_0) - \pi/2$ by δ and write it under the form

$$\begin{aligned} -\frac{1}{\delta}\left(\phi_\epsilon(b, b_0) - \frac{\pi}{2}\right) &= 2\frac{r_{min}^2 b_0^\gamma \epsilon^2}{b^2}\int_0^1 g(x)\,dx \\ &\approx \frac{1}{\hat{\epsilon}^\gamma - 1}\int_0^1 g(x)\,dx \end{aligned}$$

by Eq. (54) and with

$$g(x) = \frac{\mathcal{V}(\sqrt{\delta}) - \mathcal{V}(\sqrt{\delta}/x)}{\delta\sqrt{G_b(x)}\sqrt{1-x^2}[\sqrt{G_b(x)} + \sqrt{1-x^2}]} \geq 0.$$

Clearly, as b/ϵ goes to 0, $\delta \ll 1$, $G_b(x) \approx 1 - x^2$ and we have

$$\int_0^1 g(x)\,dx \to \frac{\gamma}{4}\int_0^1 \frac{\frac{1}{x^2} - 1}{(1-x^2)^{3/2}}\,dx = +\infty,$$

due to the non integrable singularity at the origin. We shall prove that actually $\int_0^1 g(x)\,dx \sim \delta^{-1/2}$. As a first step, as in § IV C 2, we get rid of the contribution for $1/2 \leq x \leq 1$. Indeed, $\int_0^1 g(x)\,dx \to +\infty$ whereas

$$\int_{1/2}^1 g(x)\,dx \to \frac{\gamma}{4}\int_{1/2}^1 \frac{\frac{1}{x^2} - 1}{(1-x^2)^{3/2}}\,dx < +\infty.$$

As a consequence, using the natural substitution $y = \sqrt{\delta}/x$,

$$\begin{aligned} \int_0^1 g(x)\,dx &\approx \int_0^{1/2} g(x)\,dx \\ &= \frac{1}{\sqrt{\delta}}\int_{2\sqrt{\delta}}^{+\infty} \frac{\mathcal{V}(\sqrt{\delta}) - \mathcal{V}(y)}{D_b(y)}\,dy \end{aligned}$$

where we have denoted

$$\begin{aligned} D_b(y) = y^2 &\sqrt{G_b\left(\frac{\sqrt{\delta}}{y}\right)\left(1 - \frac{\delta}{y^2}\right)} \\ &\times \left[\sqrt{G_b\left(\frac{\sqrt{\delta}}{y}\right)} + \sqrt{1 - \frac{\delta}{y^2}}\right]. \end{aligned}$$

When $\delta \to 0$, we have

$$G_b(\sqrt{\delta}/y) \to G_{\hat{\epsilon}}^-(y) = 1 - \frac{1}{\hat{\epsilon}^\gamma - 1}\left(\mathcal{V}(y) - \mathcal{V}(0)\right)$$

and one could rigorously justify that

$$\int_0^1 g(x)\,dx \approx \frac{1}{\sqrt{\delta}}\int_0^{+\infty}\frac{\mathcal{V}(0)-\mathcal{V}(y)}{y^2\sqrt{G^-_{\hat\epsilon}(y)}[\sqrt{G^-_{\hat\epsilon}(y)}+1]}\,dy$$
$$= \frac{\hat\epsilon^\gamma-1}{\sqrt{\delta}}\int_0^{+\infty}\left(1-\frac{1}{\sqrt{1+\frac{1}{\hat\epsilon^\gamma-1}\left(\mathcal{V}(0)-\mathcal{V}(y)\right)}}\right)\frac{dy}{y^2}.$$

The last integral is indeed convergent since: for large y, $\mathcal{V}(y)\to 0$, thus the integrand is $\sim 1/y^2$; for small y, $\mathcal{V}^{\rm Pl}(0)-\mathcal{V}^{\rm Pl}(y) = 1-(1+y^2)^{-\gamma/2}\approx \gamma/(2y^2)$, thus the integrand is continuous at the origin. It then follows that, for $b \ll \epsilon$:

$$\phi_\epsilon(b,b_0) = \frac{\pi}{2} - \tilde B^{\rm Pl}_{\hat\epsilon}(\gamma)b/\epsilon + o(b/\epsilon), \tag{55}$$

with

$$\tilde B^{\rm Pl}_{\hat\epsilon}(\gamma) = \frac{1}{\sqrt{1-\hat\epsilon^{-\gamma}}}$$
$$\times\int_0^{+\infty}\left(1-\frac{1}{\sqrt{1+\frac{1}{\hat\epsilon^\gamma-1}\left(\mathcal{V}^{\rm Pl}(0)-\mathcal{V}^{\rm Pl}(y)\right)}}\right)\frac{dy}{y^2} > 0.$$

If $\epsilon \gg b_0$, that is $\hat\epsilon \gg 1$, we justify in Appendix A 9 that

$$\tilde B^{\rm Pl}_{\hat\epsilon}(\gamma) \approx \hat\epsilon^{-\gamma}\sqrt{\pi}\frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{4\Gamma\left(\frac{\gamma}{2}\right)}. \tag{56}$$

We see here that, because $\epsilon \gg b_0$, the value of ϕ is completely different compared to the case $\epsilon \to 0$. As expected, in the limit $b \to 0$, $\phi \to \pi/2$, which means that the particle trajectory is unperturbed.

B. Repulsive interactions with compact softening

In this Subsection, we give the few modifications appearing in the asymptotic expansions when we consider a compact softening Eq. (50). The formula we shall obtain are qualitatively comparable to those in Subsect. VI A for the Plummer softening. The first step is to determine the asymptotic behavior of r_{min}, and here again, we shall distinguish the cases where $\hat\epsilon = \epsilon/(2^{1/\gamma}b_0)$ is small or large.

1. The case $\epsilon < b_0 2^{1/\gamma}$

Assume that $\epsilon < b_0 2^{1/\gamma}$, that is $\hat\epsilon = \epsilon/(b_0 2^{1/\gamma}) < 1$. Then, the function $r \mapsto 1 - b^2/r^2 - 2b_0^\gamma/r^\gamma$ is increasing on $[\epsilon,+\infty)$. It follows that this function has a unique zero r_{min} on $[\epsilon,+\infty)$, which satisfies, for $b/b_0 \ll 1$,

$$r_{min} \approx b_0 2^{1/\gamma} > \epsilon.$$

In view of the fact that $r_{min} \approx b_0 2^{1/\gamma} > \epsilon$, the trajectory never enters in the region $\{r \le \epsilon\}$ where the softening has an effect, hence we obtain the same asymptotics as in the case without softening (see Eq. (24)), namely

$$\phi_\epsilon(b,b_0) = B(\gamma)(b/b_0) + \mathcal{O}((b/b_0)^3), \tag{57}$$

where $B(\gamma)$ is the same as in Eq. (24).

2. The case $\epsilon > b_0 2^{1/\gamma}(\max_{\mathbb{R}}\mathcal{V})^{1/\gamma}$

Assume now that $\epsilon > b_0 2^{1/\gamma}(\max_{\mathbb{R}}\mathcal{V})^{1/\gamma}$, that is $\hat\epsilon^\gamma = \epsilon^\gamma/(2b_0^\gamma) > \max_{\mathbb{R}}\mathcal{V} = \max_{[0,1]}\mathcal{V} \ge 1$. The function $r \mapsto 1 - b^2/r^2 - 2b_0^\gamma/r^\gamma$ is then increasing on $[\epsilon,+\infty)$ from $1 - b^2/\epsilon^2 - \hat\epsilon^{-\gamma}$ to 1. Since $\hat\epsilon > 1$, we have, for $b \ll \epsilon$, $1 - b^2/\epsilon^2 - \hat\epsilon^{-\gamma} \approx 1 - \hat\epsilon^{-\gamma} > 0$, hence $1 - b^2/r^2 - 2b_0^\gamma/r^\gamma$ is positive on $[\epsilon,+\infty)$. On $[0,\epsilon]$, the function $r \mapsto 1 - b^2/r^2 - \hat\epsilon^{-\gamma}\mathcal{V}(r/\epsilon)$ is >0 for $r=\epsilon$ and tends to $-\infty$ for $r\to 0$, thus has a largest root $r_{min} \le \epsilon$. Moreover, since $b^2/r^2_{min} = 1-\mathcal{V}(r_{min}/\epsilon)\hat\epsilon^{-\gamma} \ge 1-\hat\epsilon^{-\gamma}\max_{\mathbb{R}}\mathcal{V} > 0$ by our hypothesis, we have $r_{min} \lesssim b \ll \epsilon$, hence

$$r_{min} = \frac{b}{\sqrt{1-\mathcal{V}(r_{min}/\epsilon)\hat\epsilon^{-\gamma}}} \approx \frac{b}{\sqrt{1-\mathcal{V}(0)\hat\epsilon^{-\gamma}}}$$

that is close to Eq. (54). We may then carry out computations very similar to those leading to Eq. (55), provided v is $\mathcal{C}^2$ on $[0,1]$, positive on $(0,1]$ and $\mathrm{v}'(0)=0$. This yields

$$\phi_\epsilon(b,b_0) = \frac{\pi}{2} - \tilde B^{\rm co}_{\hat\epsilon}(\gamma)b/\epsilon + o(b/\epsilon), \tag{58}$$

with

$$\tilde B^{\rm co}_{\hat\epsilon}(\gamma) = \frac{1}{\sqrt{1-\mathcal{V}^{\rm co}(0)\hat\epsilon^{-\gamma}}}$$
$$\times\int_0^{+\infty}\left(1-\frac{1}{\sqrt{1+\frac{1}{\hat\epsilon^\gamma-\mathcal{V}^{\rm co}(0)}\left(\mathcal{V}^{\rm co}(0)-\mathcal{V}^{\rm co}(y)\right)}}\right)\frac{dy}{y^2}.$$

Here, we do not claim that $\tilde B^{\rm co}_{\hat\epsilon}(\gamma)$ is a positive constant. For instance, if $\mathrm{v}(0)=0$, then $\tilde B^{\rm co}_{\hat\epsilon}(\gamma) < 0$, whereas if $\mathrm{v}(x) = \mathrm{v}(0)$ on $[0,1]$, then $\tilde B^{\rm co}_{\hat\epsilon}(\gamma) > 0$. For a general function v on $[0,1]$, it may happen exceptionally that $\tilde B^{\rm co}_{\hat\epsilon}(\gamma)$ vanishes, and in this case, the correction $\phi_\epsilon - \pi/2$ is not of order b/ϵ but smaller. This however does not happen for generic functions v.

C. Attractive interactions with a softening

The function $r \mapsto 1 - b^2/r^2 + \hat\epsilon^{-\gamma}\mathcal{V}(r/\epsilon)$ tends to 1 at infinity and to $-\infty$ at 0^+, hence possesses a larger zero r_{min}, but there may exist several zeros in general. We shall prove that independently whether ϵ/b_0 is small or not, we have

$$r_{min} \approx \frac{b}{\sqrt{1+\mathcal{V}(0)\hat\epsilon^{-\gamma}}} \tag{59}$$

(whereas without softening, we had $r_{min} \sim b^{2/(2-\gamma)}$), and

$$\phi_\epsilon(b, b_0) = \frac{\pi}{2} + C_{\hat{\epsilon}}(\gamma)\frac{b}{\epsilon} + o(b/\epsilon), \tag{60}$$

where

$$C_{\hat{\epsilon}}(\gamma) = \frac{1}{\sqrt{1+\mathcal{V}(0)\hat{\epsilon}^{-\gamma}}} \times \int_0^{+\infty} \left(\frac{1}{\sqrt{1 - \frac{1}{\hat{\epsilon}^\gamma + \mathcal{V}(0)}(\mathcal{V}(0) - \mathcal{V}(y))}} - 1 \right) \frac{dy}{y^2}.$$

If $\epsilon \gg b_0$, that is $\hat{\epsilon} \gg 1$, we can show (as we have done for Eq. (56)) that

$$C_{\hat{\epsilon}}(\gamma) \approx \hat{\epsilon}^{-\gamma}\sqrt{\pi}\frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{4\Gamma\left(\frac{\gamma}{2}\right)}. \tag{61}$$

On the other hand, if $\gamma < 2$ and $\epsilon \ll b_0$, that is $\hat{\epsilon} \ll 1$, we can show that

$$C_{\hat{\epsilon}}(\gamma) \approx \frac{\hat{\epsilon}^{\gamma/2}}{\sqrt{\mathcal{V}(0)}} \int_0^{+\infty} \left(\sqrt{\frac{\mathcal{V}(0)}{\mathcal{V}(y)}} - 1 \right) \frac{dy}{y^2}.$$

We have then a big difference with the case of repulsive interactions studied in Sect. VI B (and also in Sect. VI A), where $\phi_\epsilon \sim b/\max(\epsilon, 2^{1/\gamma}b_0)$, displaying the characteristic length ϵ or b_0 depending which one is the largest one. Here, for attractive interactions, only the softening characteristic length ϵ appears in the first order term $\phi_\epsilon - \pi/2 \sim b/\epsilon$ in Eq. (60). This point is not completely surprising in view of the different cases appearing in Subsect. IV C (for $\gamma < 2/3$, $\gamma = 2/3$, $2/3 < \gamma < 2$, etc), which correspond very roughly to the case $\epsilon = 0$.

Since $1 \leq 1 + \hat{\epsilon}^{-\gamma}\mathcal{V}(r_{min}/\epsilon) = b^2/r_{min}^2$, we must have $r_{min} \leq b \ll \epsilon$, and this in turn implies Eq. (59).

Our small parameter here will be $\delta = r_{min}^2/\epsilon^2 \ll 1$ (by Eq. (59)). Substituting $1 = b^2/r_{min}^2 - \hat{\epsilon}^{-\gamma}\mathcal{V}(r_{min}/\epsilon)$ in the integral gives

$$\phi_\epsilon(b, b_0) = \int_0^1 \frac{dx}{\sqrt{G_b(x)}},$$

where

$$G_b(x) = 1 - x^2 + \frac{r_{min}^2}{b^2\hat{\epsilon}^\gamma}\left(\mathcal{V}(\sqrt{\delta}/x) - \mathcal{V}(\sqrt{\delta})\right).$$

Comparing with § VI A 2, the only difference is a change of sign. Therefore, similar computations to those in that paragraph yield Eq. (60).

D. Computation of a threshold in ϵ for attractive potentials with $\gamma > 2$

When $\gamma > 2$ and without softening in the potential (formally, $\epsilon = 0$), the deflection angle ϕ diverges logarithmically to $+\infty$ when $b > \beta b_0$ approaches βb_0 (see Eq. (42)). This divergence is due to the fact that $r_* \approx R = b_0(2-\gamma)^{1/\gamma}$ becomes a double root of the function W in this limit. The first paragraph of this Subsection is devoted to the proof of the existence of some threshold $\epsilon_*(b_0, \gamma) > 0$, for the Plummer softening, such that if $\epsilon < \epsilon_*(b_0, \gamma)$, then the angle ϕ_ϵ still diverges for some specific value of r (depending on b_0, γ and ϵ), whereas for $\epsilon > \epsilon_*(b_0, \gamma)$, the angle ϕ_ϵ no longer diverges and is a smooth function of b/b_0 for all positive values of b/b_0. This means that in order to remove the divergence in ϕ, one has to use a sufficiently large softening. In the first case, the divergence is here again due to the existence of some positive double root in r for the function

$$W_{b,\epsilon}(r) = 1 - \frac{b^2}{r^2} + \hat{\epsilon}^{-\gamma}\mathcal{V}\left(\frac{r}{\epsilon}\right),$$

whereas for $\epsilon > \epsilon_*(b_0, \gamma)$, the function $W_{b,\epsilon}(r)$ has no double root. In the second paragraph we will discuss the case of the compact softening.

1. *The case of a Plummer softening*

We now consider the Plummer softening $\mathcal{V}(R) = \mathcal{V}^{\mathrm{Pl}}(R) = (1+R^2)^{-\gamma/2}$ and are interested in determining under which condition on ϵ the function $W_{b,\epsilon}$ has a unique zero r_{min} for any $b > 0$. We have

$$W'_{b,\epsilon}(r) = \frac{2\gamma b_0^\gamma}{r^3}\left(\frac{b^2}{\gamma b_0^\gamma} - \frac{r^4}{(r^2+\epsilon^2)^{\gamma/2+1}}\right)$$

and, denoting $r = \epsilon R$,

$$\frac{r^4}{(r^2+\epsilon^2)^{\gamma/2+1}} = \epsilon^{2-\gamma}\frac{R^4}{(R^2+1)^{\gamma/2+1}}.$$

The function $R \mapsto R^4/(R^2+1)^{\gamma/2+1}$ is increasing on $[0, R_{max}]$ and decreasing on $[R_{max}, +\infty)$ (recall $\gamma > 2$), where $R_{max} = \sqrt{4/(\gamma-2)}$; its maximal value is $M(\gamma) = 16(\gamma-2)^{\frac{\gamma}{2}-1}(\gamma+2)^{-\frac{\gamma}{2}-1}$. Therefore, when $b^2/(\gamma b_0^\gamma) < \epsilon^{2-\gamma}M(\gamma)$ (case 1), the function $W_{b,\epsilon}$ is increasing on $(0, r_1]$, decreasing on $[r_1, r_2]$ and increasing on $[r_2, +\infty)$ when $b^2/(\gamma b_0^\gamma) > \epsilon^{2-\gamma}M(\gamma)$ (case 2), the function $W_{b,\epsilon}$ is increasing on $(0, +\infty)$. The two critical points r_1 and r_2 merge for $b^2/(\gamma b_0^\gamma) = \epsilon^{2-\gamma}M(\gamma)$, and we shall see that the threshold is determined by the sign of $W_{b,\epsilon}$ at this merging point $r_1 = r_2$.

Let us now fix $\epsilon > 0$. For b very small, we are in case 1 and the two positive roots r_1 and r_2 of the equation $b^2/(\gamma b_0^\gamma) = r^4/(r^2+\epsilon^2)^{\gamma/2+1}$ are r_1 very small and r_2 very large. The function $W_{b,\epsilon}$ has then a local minimum $W_{b,\epsilon}(r_2) \approx 1$. When b increase, $W_{b,\epsilon}$ decrease, the two critical points r_1 and r_2 merge when $b^2/(\gamma b_0^\gamma) = \epsilon^{2-\gamma}M(\gamma)$, and for larger b, $W_{b,\epsilon}$ is increasing on $(0, +\infty)$.

Let us consider the special value of b_{crit} where $b_{crit}^2/(\gamma b_0^\gamma) = \epsilon^{2-\gamma}M(\gamma)$, for which the two critical points

r_1 and r_2 merge: $r_1 = r_2 = r_{crit} = \epsilon R_{max}$. If $W_{b_{crit},\epsilon}(r_{crit}) > 0$, then by monotonicity in b, for any $b > 0$, the function $W_{b_{crit},\epsilon}$ has a single positive zero r_{min}. If now $W_{b_{crit},\epsilon}(r_{crit}) < 0$, then, still by monotonicity in b, for b smaller, but close to b_{crit}, $W_{b,\epsilon}$ has two critical points $0 < r_1 < r_2$ with $0 > W_{b,\epsilon}(r_1) > W_{b,\epsilon}(r_2)$. As b decreases, the critical value $W_{b,\epsilon}(r_2)$ will be zero for some particular value of $b = b_\sharp$ for which r_2 has become a double root of $W_{b_\sharp,\epsilon}$, yielding a logarithmic divergence in ϕ_ϵ. As a consequence, we simply need to determine the sign of

$$\begin{aligned} W_{b_{crit},\epsilon}(r_{crit}) &= 1 - \frac{b_{crit}^2}{\epsilon^2 R_{max}^2} + \frac{2b_0^\gamma}{(\epsilon^2 R_{max}^2 + \epsilon^2)^{\gamma/2}} \\ &= 1 - \frac{\epsilon^{-\gamma} M(\gamma)\gamma b_0^\gamma}{R_{max}^2} + \frac{2b_0^\gamma \epsilon^{-\gamma}}{(R_{max}^2+1)^{\gamma/2}} \\ &= 1 - (\epsilon_*(b_0,\gamma)/\epsilon)^\gamma, \end{aligned}$$

where the threshold is given by

$$\epsilon_*(b_0,\gamma) = 2^{1/\gamma} b_0 \left(\frac{\gamma-2}{\gamma+2}\right)^{\frac{1}{2}+\frac{1}{\gamma}}. \qquad (62)$$

It follows that if $\epsilon > \epsilon_*(b_0,\gamma)$, then ϕ_ϵ is a smooth function of b see Fig. 17, whereas if $\epsilon < \epsilon_*(b_0,\gamma)$, then ϕ_ϵ diverges as b approaches some value $b_\sharp = b_\sharp(\epsilon)$ corresponding to the case where $W_{b,\epsilon}$ has zero as a local minimum. By computations very similar to those in Sect. IV C, we see that the divergence is logarithmic. One may also check that if $\epsilon = \epsilon_*(b_0,\gamma)$, then ϕ_ϵ is a diverging function of b for some $b_\sharp = b_\sharp(\epsilon)$. In other words, in order to regularize the divergence in the case $\gamma > 2$, we have to use a sufficiently large softening parameter, namely $\epsilon > \epsilon_*(b_0,\gamma)$.

Let us finally consider the case $\gamma = 2$. Notice that formally, $\epsilon_*(b_0,\gamma) \to 0$ as $\gamma \to 2$, hence we may think that ϕ_ϵ is a smooth function of b for any $\epsilon > 0$, and this is indeed the case. Actually, in the case $\gamma = 2$, the function $R \mapsto R^4/(R^2+1)^2$ is increasing on $[0,+\infty)$, and tends to 1 at infinity. Therefore, either $b^2/2b_0^2 < 1$ and then the function $W_{b,\epsilon}$ is increasing on $(0, r_1]$ and decreasing on $[r_1,+\infty)$; either $b^2/(2b_0^2) \geq 1$ and then the function $W_{b,\epsilon}$ is increasing on $(0,+\infty)$. In any case $W_{b,\epsilon}$ has a single zero r_{min} and we never have a double root. It follows that ϕ_ϵ is a smooth function of b.

2. *The case of a compact softening*

For a general compact softening $\mathcal{V} = \mathcal{V}^{\mathrm{co}}$, computations are much less explicit. We first have

$$W'_{b,\epsilon}(r = \epsilon R) = \frac{2b_0^\gamma}{R^3 \epsilon^{\gamma+1}} \left(\frac{b^2 \epsilon^{\gamma-2}}{b_0^\gamma} + R^3 \mathcal{V}'(R)\right),$$

and we then need to know the behavior of the function $R \mapsto -R^3\mathcal{V}'(R)$, which certainly has a positive maximum $M = M(\mathrm{v})$ attained at some $0 < R_{max} \leq 1$ since $\gamma > 2$. If the function $R \mapsto -R^3\mathcal{V}'(R)$ is increasing on $[0, R_{max}]$ and then decreasing on $[R_{max},+\infty)$, the behavior is the same as the one previously described for the Plummer softening. Since

$$\begin{aligned} W_{b_{crit},\epsilon}(r_{crit}) &= 1 - \frac{b_0^\gamma M(\mathrm{v})}{\epsilon^\gamma R_{max}^2} + \frac{2b_0^\gamma}{\epsilon^\gamma}\mathcal{V}(R_{max}) \\ &= 1 - \frac{b_0^\gamma}{\epsilon^\gamma}\left(\frac{M(\mathrm{v})}{R_{max}^2} - 2\mathcal{V}(R_{max})\right) \\ &= 1 + \frac{b_0^\gamma}{\epsilon^\gamma}\left(R_{max}\mathcal{V}'(R_{max}) + 2\mathcal{V}(R_{max})\right), \end{aligned}$$

there exists a threshold if and only if $M(\mathrm{v})/R_{max}^2 = -R_{max}\mathcal{V}'(R_{max}) > 2\mathcal{V}(R_{max})$, and otherwise, we never have a double root for $W_{b,\epsilon}$ hence no divergence in ϕ_ϵ. The example below illustrate the first case.

If $\gamma = 3$ and $\mathrm{v}(R) = 21R^2 - 35R^3 + 15R^4$ for $0 \leq R \leq 1$, then $R \mapsto -R^3\mathcal{V}'(R)$ is decreasing and negative on $[0, \approx 0.474]$, increasing on $[\approx 0.474, \approx 0.984]$ and decreasing on $[\approx 0.984, +\infty)$, hence has maximum value $M(\mathrm{v}) \approx 3.023$ attained at $R_{max} \approx 0.984$. Moreover, $M(\mathrm{v})/R_{max}^2 - 2\mathcal{V}(R_{max}) \approx 1.023 > 0$, thus the variations of $W_{b,\epsilon}$ are the same as for the Plummer softening, with a threshold given by

$$\epsilon_*(b_0,\gamma) = b_0\left(\frac{M(\mathrm{v})}{R_{max}^2} - 2\mathcal{V}(R_{max})\right)^{1/3} \approx 1.0077 b_0.$$

E. Summary of the results and numerical checking

We summarize in table II the results obtained in this section. We have shown that the effect of the softening does not depend strongly on the form of the softening, obtaining the same qualitative results for the two softening considered — Plummer and compact one. There is an exception for repulsive interactions and $\epsilon < b_0 2^{1/\gamma}$, in which case the compact softening does not modify the trajectory of the particles because they do not visit the region in which the potential is regularized.

In the case of repulsive interactions, we have seen that two different behaviours are predicted depending whether ϵ/b_0 is larger than $2^{1/\gamma}$ or not. In the case $\epsilon/b_0 < 2^{1/\gamma}$, the softening does not modify strongly the angle ϕ: it behaves linearly for $b \ll b_0$, only its slope is modified with ϵ. In the case in which $\epsilon/b_0 > 2^{1/\gamma}$, hard collisions are radically modified, obtaining $\lim_{b/b_0\to 0}\phi = \pi/2$. The change of behaviour occurs sharply at $\epsilon/b_0 = 2^{1/\gamma}$ as we show in Fig. 14, in which ϕ is plotted as a function of ϵ at fixed b, for some values of γ. The range of validity in b of the linear correction is given by the *largest* value of b_0 and ϵ. In Fig. 15 we show the comparison between the numerical integration of ϕ_ϵ in Eq. (52) with the asymptotic predictions Eqs. (53) and (55). We see a very good matching between the curves.

For the case of attractive interactions, the range of validity in b of the linear correction is always given by ϵ.

repulsive potential			attractive potential
$\phi_\epsilon \sim b/b_0$ when $b \ll b_0$	if	$\hat{\epsilon} = \epsilon/(2^{1/\gamma} b_0) < 1$	$\phi_\epsilon - \pi/2 \sim b/\epsilon$ when $b \ll \epsilon$
$\phi_\epsilon - \pi/2 \sim -b/\epsilon$ when $b \ll \epsilon$	if	$\hat{\epsilon} = \epsilon/(2^{1/\gamma} b_0) > 1$	

TABLE II: Summary of the expansions of the angle ϕ_ϵ with a Plummer softening in the potential for hard collisions

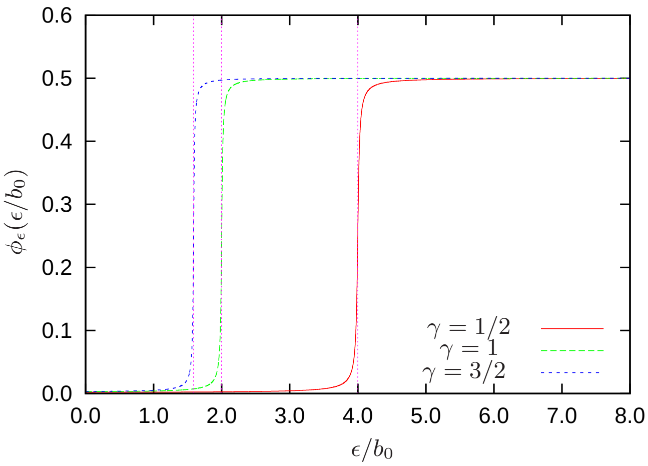

FIG. 14: Value of ϕ for $b/b_0 = 10^{-2}$. The vertical curves correspond to $\epsilon/b_0 = 2^{1/\gamma}$.

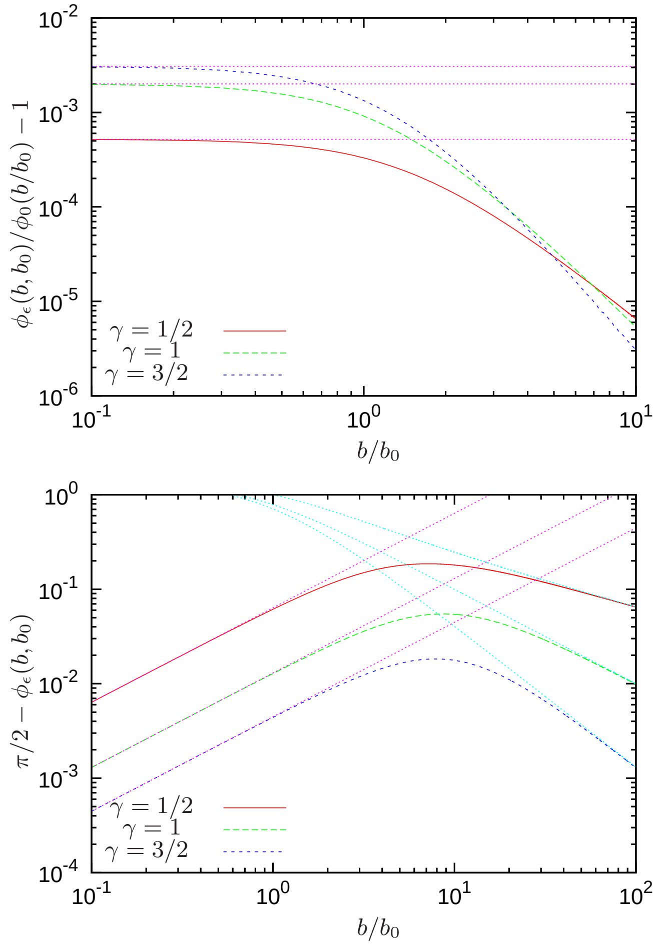

FIG. 15: Numerical computations for repulsive potentials with Plummer softening. Top: Graph of ϕ_ϵ normalized to the angle without softening ϕ_0(continuous line) and of the leading order term (dotted line) given in Eq. (53) as a function of b/b_0 for different values of γ and $\epsilon/b_0 = 1/10$. Bottom: Graph of ϕ_ϵ for $\epsilon/b_0 = 10$ and the leading order expansion given in Eq. (55). The dotted blue lines corresponds to ϕ_0.

In Fig. 16 we show a very good agreement matching between the exact integration Eq. (52) with the asymptotic predictions Eqs. (53) and (60). We have also studied, for $\gamma > 2$, for which value of the softening, there is no formation of pairs for any value of b. We have seen that introducing a softening $\epsilon > 0$ automatically regularizes the angle ϕ for any value b, except one, for which there is *orbiting*, except for some particular softenings, in which the divergences also disappear. It is necessary to introduce a value of the softening larger than a critical value (which we have calculated explicitly) to regularize completely the problem. In Fig. 17 we illustrate this behavior. The continuous red curve corresponds to the case in which $\epsilon > \epsilon_*(b_0, \gamma)$. In this case, ϕ_ϵ is a regular function of b, as it can be seen in the inset. The dashed green curve corresponds to the case in which $\epsilon < \epsilon_*(b_0, \gamma)$, for which ϕ_ϵ diverges for $b = b_\sharp(\epsilon)$.

VII. CONCLUSIONS

In this paper we have studied the scattering of two particles interacting with a central potential $v(r) \sim 1/r^\gamma$. This is a generalization of the Rutherford formula of the scattering of two particles interacting via a Coulomb or gravitational force. Unlike the original case, it is not possible to compute in general the deflection angle of the particles analytically for general $\gamma \neq 1$. We have then calculated the asymptotics of the angle of deflection for the two limiting cases in which we are interested in: the *weak* collisions regime, in which the particles trajectories are weakly perturbed, and the *strong* collision regime, in which they are strongly perturbed. Combining the analytical expressions and the numerical integration of the equation of motion, we have derived the phenomenology we detail as follows.

In the regime of soft collisions, attractive and repulsive

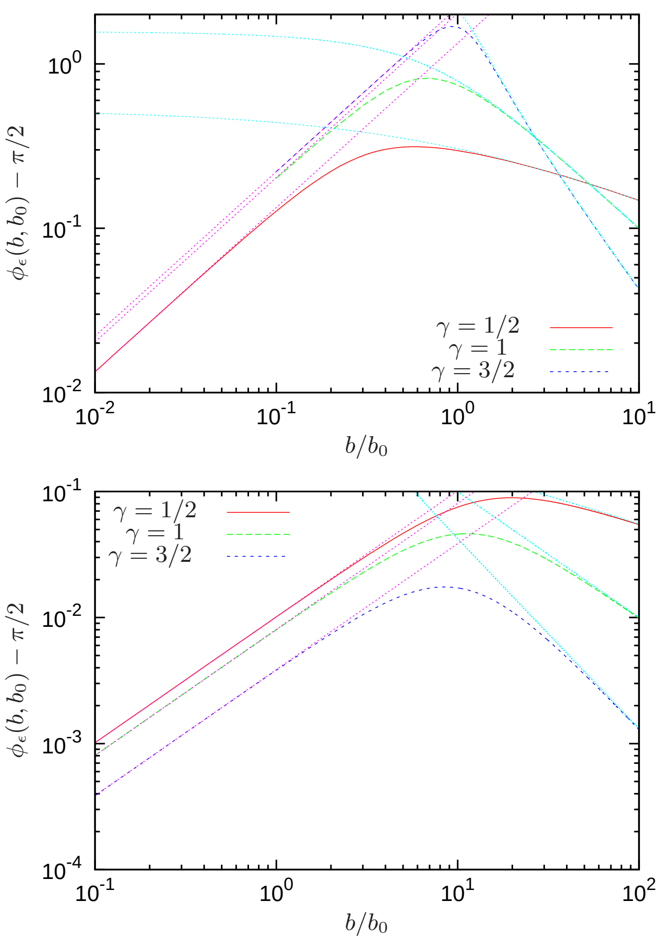

FIG. 16: Numerical computations for attractive potentials with Plummer softening (hard scattering). Top: Graphs of ϕ_ϵ (continuous line) and the theoretical prediction Eq. (53) (dotted lines) as a function of b/b_0 for different values of γ and $\epsilon/b_0 = 1/10$. Bottom: same quantity for $\epsilon/b_0 = 10$ and the theoretical prediction Eq. (55) (dotted lines). The dotted blue lines corresponds to ϕ_0.

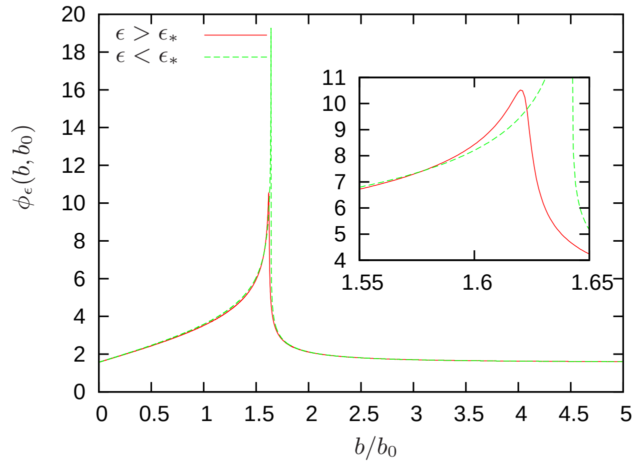

FIG. 17: Plot of ϕ_ϵ as a function of b for $\gamma = 5/2$ and two different values of the softening. The red continuous curve corresponds to a value of ϵ slightly larger than $\epsilon_*(b_0, \gamma)$ and the dashed green one to a value of ϵ slightly smaller than $\epsilon_*(b_0, \gamma)$.

interactions give a very similar result: the angle of closest approach scales as

$$\phi \sim \pi/2 \mp A(\gamma) \left(\frac{b}{b_0}\right)^\gamma, \tag{63}$$

where $A(\gamma) > 0$, b is the impact factor, and b_0 a characteristic scale which depends on the reduced mass, the coupling constant and the relative asymptotic velocity of the particles. The minus sign corresponds to the repulsive interaction and the positive sign to attractive interactions. This is what it is expected: with no interaction, the reduced particle suffers no deflection, and then $\phi = \pi/2$. If the interaction is repulsive, the reduced particle will be deflected in the top left quadrant, which implies that $\phi < \pi/2$. If the interaction is attractive, it will be deflected to the bottom left quadrant, which implies $\phi > \pi/2$ (see Fig. 1).

In the regime of hard collisions, the situation is very different between the repulsive and attractive case: *(i)* for repulsive interactions, the angle of closest approach scales as $\phi \sim b/b_0$. This is what one expects for $b \to 0$: for vanishing impact factor, particle bounce one on each other, coming back in their original directions and opposite sign of the velocity; *(ii)* for attractive interactions with $\gamma < 2$, the leading contribution is

$$\phi \sim \frac{\pi}{2 - \gamma}. \tag{64}$$

We see therefore, that for $b/b_0 \to 0$ the angle of deflection depends on the exponent of the interaction potential γ. Of course, the deflection angle is the same for Coulomb (repulsive) and gravitational interaction ($\gamma = 1$)

Equation (64) implies that, when γ approaches the value of 2, the angle ϕ increases, diverging in the limit $\gamma \to 2$. This is due to the effective potential created by the angular momentum, which scales with the distance as $1/r^2$. When the exponent γ of the attractive potential is larger than 2, the angular momentum term cannot, in general, prevent the system to collapse and the particles crash. Studying the distance of closest approach r_{min} we have found two different behaviors whether γ is smaller or larger than 2:

- If $\gamma < 2$, in the limit $\gamma \to 2^-$ (for any b smaller than some critical value which we have calculated explicitly), the value of r_{min} tends to 0. The trajectories in this limit is a succession of smaller and smaller loops embedded one in the other. An example of such trajectories was given in Fig. 11.

- If $\gamma > 2$, the particles do not crash if the impact factor is larger than some critical value, which we have calculated. For impact factor slightly larger than this critical value, we have trajectories with $r_{min} \sim b_0$. The particles then *orbite* with distance r_{min} forming a binary, which will be destroyed in a finite time. We gave an example of such trajectories in Fig. 13.

We have also studied the effect of introducing a regularization at small scales in the potential. The conclusions are detailed in Subsect. VI E.

A practical application appears naturally in the context of astrophysics or plasma physics, when we are interested in calculating the average change of velocity due to the collisions. It is usual (see e.g. [3]) to decompose the relative velocity of the particles before the collisions $\mathbf{V}$ as the sum of its component along the direction of the initial relative velocity $\mathbf{e}_\parallel$ and the component perpendicular to it $\mathbf{e}_\perp$, i.e.,

$$\mathbf{V} = V_\perp \mathbf{e}_\perp + V_\parallel \mathbf{e}_\parallel. \tag{65}$$

It is possible to compute the average change of velocity $\Delta V_\perp$ and $\Delta V_\parallel$ after a collision has been completed integrating over all the impact factors b:

$$\frac{\Delta V_\perp}{V} = \sin(2\phi) \tag{66a}$$

$$\frac{\Delta V_\parallel}{V} = 1 + \cos(2\phi). \tag{66b}$$

One quantity of interest is the average change velocity square, which can be expressed by the integral over all the impact factors, i.e.,

$$\langle \Delta V_\perp^2 \rangle \sim \int_0^R dbb^{d-2} \sin^2\left(2\phi_\epsilon\left(\frac{b}{b_0}\right)\right) \tag{67a}$$

$$\langle \Delta V_\parallel^2 \rangle \sim \int_0^R dbb^{d-2} \left[1 + \cos\left(2\phi_\epsilon\left(\frac{b}{b_0}\right)\right)\right]^2, \tag{67b}$$

where $d > 1$ is the physical dimension and R the size of the system, which is the maximal impact factor available.

In astrophysical or cosmological N-body simulations, the goal is to simulate *collisionless* dynamics sampling a continuous distribution with macro-particles (see e.g. [19]). The softening used in these simulations is much larger than b_0 (in order to suppress collisional effects), and hence (see Sect. VI), $\phi - \pi/2 \ll 1$. We can therefore write

$$\langle \Delta V_\perp^2 \rangle \sim 4 \int_0^R dbb^{d-2} \left[\phi_\epsilon\left(\frac{b}{b_0}\right) - \frac{\pi}{2}\right]^2 \tag{68}$$

and $\langle \Delta V_\parallel^2 \rangle \ll \langle \Delta V_\perp^2 \rangle$. We can estimate Eq. (68) using the following approximate expression for the angle ϕ_ϵ (we will consider explicitly attractive interactions with Plummer softening to simplify notations, the compact softening or repulsive case is analogous):

$$\phi_\epsilon - \frac{\pi}{2} \simeq \begin{cases} C_\epsilon(\gamma)\frac{b}{\epsilon} & \text{if } b < \epsilon \\ A(\gamma)\left(\frac{b_0}{b}\right)^\gamma & \text{if } b > \epsilon \end{cases} \tag{69}$$

(see e.g. Fig. 16). Using Eq. (69) to compute integral (68), considering softenings such that $b_0 \ll \epsilon \ll R$ we get the scaling, for $\gamma > (d-1)/2$,

$$\langle \Delta V_\perp^2 \rangle \sim b_0^{2\gamma} \epsilon^{d-1-2\gamma} \tag{70}$$

where we have used the asymptotic value of $C_\epsilon(\gamma)$ Eq. (61). Notice that impact factors smaller or larger than ϵ contributes to the scaling (70). In the limiting case $\gamma = (d-1)/2$, we get

$$\langle \Delta V_\perp^2 \rangle \sim b_0^2 \ln\left(\frac{R}{\epsilon}\right). \tag{71}$$

In this case contributions of collisions with $b < \epsilon$ are negligible. For $\gamma < (d-1)/2$, the effect of the softening is negligible because the main contribution to the change of velocity is given by impact factors $b \sim R$.

Acknowledgments

We thank M. Joyce for useful discussions and comments.

Appendix A: Mathematical details

In this appendix we give mathematical details of some derivations given in the paper.

Some of the integrals appearing in the paper may be expressed with the help of the Beta function (also called Euler's integral of the first kind) defined for x, $y > 0$ by

$$\begin{aligned}\mathbb{B}(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}\, dt \\ &= 2\int_0^{\pi/2} \sin^{2x-1}(\vartheta)\cos^{2y-1}(\vartheta)\, d\vartheta \\ &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\end{aligned}$$

where Γ is Euler's function.

1. Expression for the integral Eq. (17)

For the integral Eq. (17), we use the substitution $x = \cos\vartheta$ and integration by parts:

$$\begin{aligned}\int_0^1 \frac{1-x^\gamma}{(1-x^2)^{3/2}}\, dx &= \int_0^{\pi/2} \frac{1-\cos^\gamma(\vartheta)}{\sin^2\vartheta}\, d\vartheta \\ &= \left[\frac{\cos^\gamma(\vartheta)-1}{\tan\vartheta}\right]_0^{\pi/2} + \gamma\int_0^{\pi/2}\cos^\gamma(\vartheta)\, d\vartheta.\end{aligned} \tag{A1}$$

Notice that the bracket term vanishes. The right-hand side of Eq. (A1) may also be expressed as (using that $\Gamma(1+z) = z\Gamma(z)$)

$$\gamma\mathbb{B}\left(\frac{\gamma+1}{2}, \frac{1}{2}\right) = \sqrt{\pi}\frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)}. \tag{A2}$$

2. Extension of the expansion Eq. (20) for $\gamma \in (0,2]$

We proceed in two steps: we first prove that ϕ is a power series in $(b_0/b)^\gamma$ for b sufficiently large, and then identify the coefficients in the expansion.

The argument used for Eq. (19) shows that $(r_{min}/b)\phi$ is a power series of the variable δ (with positive radius) provided δ is small enough. Moreover, since

$$\frac{b}{r_{min}} = \sqrt{1 \pm 2(b_0/r_{min})^\gamma} = \sqrt{1 \pm 2(b_0/b)^\gamma (b/r_{min})^\gamma},$$

it is easy to show that b/r_{min}, thus also $\delta = \pm((b/r_{min})^2 - 1)$, is itself a power series of the variable $2(b_0/b)^\gamma$ (with positive radius). By substitution and Cauchy product, ϕ is a power series in $2(b_0/b)^\gamma$ for b sufficiently large, that is there exists some coefficients $\kappa_n(\gamma)$, $n \in \mathbb{N}$, such that, for b large enough,

$$\phi = \sum_{n=0}^{+\infty} \kappa_n(\gamma) \left(2(b_0/b)^\gamma\right)^n .$$

In addition, from the above computation, we know that each coefficient $\kappa_n(\gamma)$ is a finite sum of the type

$$\sum_{k=0}^{n} C(n,k) \int_0^1 \left(\frac{x^\gamma - x^2}{1-x^2}\right)^k \frac{dx}{\sqrt{1-x^2}},$$

the integrals coming from the expansion of the integral $(r_{min}/b)\phi$ in powers of δ, and the coefficients $C(n,k)$ of the Cauchy products and the substitution. In particular, each coefficient $\kappa_n(\gamma)$ is an analytic function of γ in $(0,+\infty)$ (and even in the half-space $\{\mathrm{Re} > 0\}$).

We now identify the coefficients $\kappa_n(\gamma)$ by considering the two expansions valid for $\gamma > 2$ and b large,

$$\begin{aligned}\phi &= \sqrt{\pi}\sum_{n=0}^{+\infty} \frac{\Gamma((n\gamma+1)/2)}{2n!\Gamma(1+n(\gamma/2-1))} (\mp 2(b_0/b)^\gamma)^n \\ &= \sum_{n=0}^{+\infty} \kappa_n(\gamma) \left(2(b_0/b)^\gamma\right)^n .\end{aligned}$$

By uniqueness of the power series expansions, we deduce that if $\gamma > 2$, then for all $n \in \mathbb{N}$,

$$\kappa_n(\gamma) = (\mp 1)^n \sqrt{\pi} \frac{\Gamma((n\gamma+1)/2)}{2n!\Gamma(1+n(\gamma/2-1))}.$$

Since κ_n is an analytic function in $(0,+\infty)$ and both $\gamma \mapsto \Gamma((n\gamma+1)/2)$ and $\gamma \mapsto 1/\Gamma(1+n(\gamma/2-1))$ are analytic in $(0,+\infty)$, we deduce from the principle of permanence for analytic functions that Eq. (20) holds true for any $\gamma > 0$.

We may now compute the radius of convergence. If $\gamma > 2$, this has been carried out in [15] using the generalized Stirling formula $\Gamma(s+1) \approx (s/\mathsf{e})^s\sqrt{2\pi s}$ when $s \to +\infty$. The generalization to $\gamma \le 2$ follows from the same type of computations, combined with Euler's reflection formula $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$.

3. Expression for the integral Eq. (23)

Using the substitution $x^\gamma = \cos^2\vartheta$ provides

$$\begin{aligned}\int_0^1 \frac{dx}{\sqrt{1-x^\gamma}} &= \frac{2}{\gamma}\int_0^{\pi/2} \cos^{\frac{2}{\gamma}-1}\vartheta\, d\vartheta \\ &= \frac{1}{\gamma}\mathbb{B}\left(\frac{1}{\gamma},\frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma\left(1+\frac{1}{\gamma}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{\gamma}\right)},\end{aligned}$$

as claimed.

4. Expression for the integral Eq. (28)

In Eq. (28), we substitute $x^{2-\gamma} = \cos^2(\vartheta)$ and then integrate by parts

$$\begin{aligned}&\int_0^1 \frac{1-x^\gamma}{2(x^\gamma - x^2)^{3/2}}\,dx \\ &= \frac{1}{2-\gamma}\int_0^{\pi/2} \frac{\cos^{-2\gamma/(2-\gamma)}(\vartheta) - 1}{\sin^2\vartheta}\,d\vartheta \\ &= \frac{1}{2-\gamma}\left[\frac{\cos^{-2\gamma/(2-\gamma)}(\vartheta)-1}{\tan\vartheta}\right]_0^{\pi/2} \\ &\quad + \frac{2\gamma}{(2-\gamma)^2}\int_0^{\pi/2}\cos^{-2\gamma/(2-\gamma)}(\vartheta)\,d\vartheta.\end{aligned}$$

Since the bracket vanishes, we then obtain, as wished,

$$\begin{aligned}\int_0^1 \frac{1-x^\gamma}{2(x^\gamma-x^2)^{3/2}}\,dx &= \frac{\gamma}{(2-\gamma)^2}\mathbb{B}\left(\frac{2-3\gamma}{2(2-\gamma)},\frac{1}{2}\right) \\ &= \frac{\gamma}{(2-\gamma)^2}\frac{\sqrt{\pi}\Gamma\left(\frac{2-3\gamma}{2(2-\gamma)}\right)}{\Gamma\left(\frac{2(1-\gamma)}{2-\gamma}\right)}.\end{aligned}$$

5. Expression for the integral Eq. (32)

In the integral $\int_0^{+\infty}\Psi_0(y)\,dy$, we use successively the substitutions $y^\gamma = \sinh^2 u$ and $\mathsf{e}^{-u} = \sin\vartheta$:

$$\begin{aligned}\int_0^{+\infty}\Psi_0(y)\,dy &= \int_0^{+\infty}\frac{y^{-\gamma/2}(1+y^\gamma)^{-1/2}dy}{y^{\gamma/2}+(1+y^\gamma)^{1/2}} \\ &= \frac{2}{\gamma}\int_0^{+\infty}(\sinh u)^{\frac{2}{\gamma}-2}\mathsf{e}^{-u}\,du \\ &= \frac{2^{3-2/\gamma}}{\gamma}\int_0^{\pi/2}(\sin\vartheta)^{2-\frac{2}{\gamma}}(\cos\vartheta)^{\frac{4}{\gamma}-3}\,d\vartheta \\ &= \frac{2^{3-2/\gamma}}{\gamma}\mathbb{B}\left(\frac{3}{2}-\frac{1}{\gamma},\frac{1}{\gamma}-1\right) \\ &= \frac{2^{3-2/\gamma}}{\gamma}\frac{\Gamma\left(\frac{3}{2}-\frac{1}{\gamma}\right)\Gamma\left(\frac{2}{\gamma}-1\right)}{\Gamma\left(\frac{1}{\gamma}+\frac{1}{2}\right)},\end{aligned}$$

which establishes the equality Eq. (32).

6. Justification of Eq. (34)

We may already get rid of the contribution for $0 \le y \le 1$ since we know that $Q(\delta) \to +\infty$ whereas

$$\int_0^1 \Psi_\delta(y)\,dy \to \int_0^1 \Psi_0(y)\,dy < \infty,$$

since the integrand is $\approx y^{-1/3}$ at the origin. Then, Eq. (31) implies

$$Q(\delta) \approx \int_1^{\delta^{-3/2}/2} \Psi_\delta(y)\,dy.$$

To prove the asymptotics Eq. (34) rigorously, we have to pay attention to the y's close to $\delta^{-3/2}/2$. Indeed, when $\gamma = 2/3$,

$$(y^\gamma - \delta^{2/\gamma-1}y^2)^{-1/2} = y^{-1/3}(1-\delta^2 y^{4/3})^{-1/2}$$

but $\delta^2 y^{4/3}$ may be of order one when $y \approx \delta^{-3/2}$. Therefore, we split

$$Q(\delta) \approx Q_1(\delta) + Q_2(\delta) = \int_0^{\delta^{-3/2}/|\ln\delta|} \Psi_\delta(y)\,dy + \int_{\delta^{-3/2}/|\ln\delta|}^{\delta^{-3/2}/2} \Psi_\delta(y)\,dy.$$

For $Q_1(\delta)$, we may write, factorizing the dominant terms,

$$\begin{aligned}(y^\gamma - \delta^{2/\gamma-1}y^2)^{-1/2} &= y^{-1/3}(1-\delta^2 y^{4/3})^{-1/2}\\ &= y^{-1/3}(1+o(1)),\end{aligned} \tag{A3}$$

since $0 \le \delta^2 y^{4/3} \le |\ln\delta|^{-1} = o(1)$ and where $o(1)$ stands for a quantity which is uniformly small for $y \in [0, \delta^{-3/2}/|\ln\delta|]$. Similarly $1 - \delta y^\gamma = 1 - \delta y^{2/3} = 1 + o(1)$ and

$$\begin{aligned}&(y^\gamma - \delta^{2/\gamma-1}y^2 + 1 - \delta y^\gamma)^{-1/2}\\ &= (y^{2/3}+1)^{-1/2}\left(1 - \frac{\delta^2 y^{4/3} + \delta y^{2/3}}{1+y^{2/3}}\right)^{-1/2}\\ &= (y^{2/3}+1)^{-1/2}(1+o(1))\end{aligned}$$

and we then infer

$$\begin{aligned}Q_1(\delta) &= \int_1^{\delta^{-3/2}/|\ln\delta|} (1+o(1))\frac{y^{-1/3}(y^{2/3}+1)^{-1/2}\,dy}{y^{1/3}+\sqrt{y^{2/3}+1}}\\ &\approx \int_1^{\delta^{-3/2}/|\ln\delta|} \frac{dy}{2y} = \frac{1}{2}\ln\left(\delta^{-3/2}/|\ln\delta|\right)\\ &\approx \frac{3}{4}|\ln\delta|,\end{aligned}$$

since the first integrand is $\approx 1/(2y)$ at infinity.

We now consider $Q_2(\delta)$, where $1 \ll \delta^{-3/2}/|\ln\delta| \le y \le \delta^{-3/2}/2$. Then, in Eq. (A3), we no longer have a $o(1)$, but we can write, since $0 \le \delta^2 y^{4/3} \le 1/2$,

$$(y^\gamma - \delta^{2/\gamma-1}y^2)^{-1/2} = y^{-1/3}(1-\delta^2 y^{4/3})^{-1/2} = \mathcal{O}(y^{-1/3})$$

and similarly

$$(y^\gamma - \delta^{2/\gamma-1}y^2 + 1 - \delta y^\gamma)^{-1/2} = \mathcal{O}(y^{-1/3}).$$

As a consequence,

$$Q_2(\delta) = \mathcal{O}\left(\int_{\delta^{-3/2}/|\ln\delta|}^{\delta^{-3/2}/2} \frac{dy}{y}\right) = \mathcal{O}\left(\ln(|\ln\delta|)\right) \ll |\ln\delta|.$$

Combining the estimates for $Q_1(\delta)$ and $Q_2(\delta)$, we have justified Eq. (34).

7. Justification of the leading order expansion Eq. (41)

To completely justify the expansion Eq. (41), we have to pay attention to the z's close to z_{max}. Notice first that

$$dr/dz = \sqrt{-2W_b(r_*)/W_b''(r_*)}(1+\mathcal{O}(z/z_{max}))$$

and that

$$r(z)^{-2} = (r_* + \mathcal{O}(z/z_{max}))^{-2},$$

hence the asymptotics $r(z) \approx r_* \approx R$ and $dr/dz \approx \sqrt{-2W_b(r_*)/W_b''(r_*)}$ are not completely true for $z \sim z_{max}$. We therefore split the right-hand side of Eq. (40) as

$$I_1 + I_2 = \frac{b}{\sqrt{-W_b(r_*)}}\int_1^{z_{max}/\ln(z_{max})} \frac{r(z)^{-2}\,dr/dz}{\sqrt{z^2-1}}\,dz + \frac{b}{\sqrt{-W_b(r_*)}}\int_{z_{max}/\ln(z_{max})}^{z_{max}} \frac{r(z)^{-2}\,dr/dz}{\sqrt{z^2-1}}\,dz.$$

In I_1, we have $0 \le z/z_{max} \le |\ln z_{max}| = o(1)$, thus

$$dr/dz = \sqrt{-2W_b(r_*)/W_b''(r_*)}(1+o(1))$$

and

$$r(z)^{-2} = (r_* + o(1))^{-2} = R^{-2} + o(1),$$

which yields

$$\begin{aligned}I_1 &\approx b\sqrt{\frac{2}{W_b''(R)}}\int_1^{z_{max}/\ln(z_{max})} \frac{R^{-2}\,dz}{\sqrt{z^2-1}}\\ &\approx \sqrt{\frac{2}{R^4 W_b''(R)}}\ln(z_{max}).\end{aligned}$$

Turning back to I_2, where $1 \ll z_{max}/\ln(z_{max}) \le z \le z_{max}$, we simply use that $r(z)^{-2} = \mathcal{O}(1)$ and that $dr/dz = \sqrt{-2W_b(r_*)}\mathcal{O}(1)$, thus

$$\begin{aligned}I_2 &= \mathcal{O}\left(\int_{z_{max}/\ln(z_{max})}^{z_{max}} \frac{dz}{z}\right)\\ &= \mathcal{O}(\ln(\ln z_{max})) \ll \ln(z_{max}).\end{aligned}$$

This concludes the justification of Eq. (41).

8. Bounding the function $\frac{1-x^2}{F(x, r_{min}/\epsilon)}$

We prove here that the function $x \mapsto \frac{1-x^2}{F(x,r_{min}/\epsilon)}$ is bounded on $[0,1]$, independently of $b \ll b_0$ (for the Plummer softening). We recall that for the regime ($\epsilon < b_0 2^{1/\gamma}$ and $b \ll b_0$) we are studying, $r_{min} \approx b_0 2^{1/\gamma}\sqrt{1-\hat{\epsilon}^2}$, thus $r_{min}/\epsilon \approx \hat{\epsilon}^{-1}\sqrt{1-\hat{\epsilon}^2}$.

Let us first work on the interval $[0,1/2]$. Then, $F(x,r_{min}/\epsilon) = 1 - \mathcal{V}^{\rm Pl}(r_{min}/(\epsilon x))/\mathcal{V}^{\rm Pl}(r_{min}/\epsilon)$ is decreasing with respect to x since $\mathcal{V}^{\rm Pl}(R) = (1+R^2)^{-\gamma/2}$ is decreasing on $[0,+\infty)$, hence, for $0 \le x \le 1/2$,

$$0 \le \frac{1-x^2}{F(x,r_{min}/\epsilon)} \le \frac{1}{F(x,r_{min}/\epsilon)} \le \frac{1}{F(1/2,r_{min}/\epsilon)}.$$

The right-hand side does not depend on x and is equal to

$$\left(1 - \frac{\mathcal{V}^{\rm Pl}(2r_{min}/\epsilon)}{\mathcal{V}^{\rm Pl}(r_{min}/\epsilon)}\right)^{-1} \approx \left(1 - \frac{\mathcal{V}^{\rm Pl}(2\hat{\epsilon}^{-1}\sqrt{1-\hat{\epsilon}^2})}{\mathcal{V}^{\rm Pl}(\hat{\epsilon}^{-1}\sqrt{1-\hat{\epsilon}^2})}\right)^{-1},$$

which gives the desired upper bound on $[0,1/2]$.

We now work on $[1/2,1]$, and use that $\frac{d}{dx}\mathcal{V}^{\rm Pl}(r_{min}/(\epsilon x)) = -(r_{min}/(\epsilon x^2))(\mathcal{V}^{\rm Pl})'(r_{min}/(\epsilon x)) \ge m$ for some positive constant $m = m(\hat{\epsilon})$ independent of b, since $\mathcal{V}^{\rm Pl}$ is decreasing on $[0,+\infty)$. As a consequence of the mean value theorem we get

$$0 \le \frac{1-x^2}{F(x,r_{min}/\epsilon)} = \frac{(1+x)(1-x)}{F(x,r_{min}/\epsilon) - F(1,r_{min}/\epsilon)} \le \frac{2}{m}.$$

This concludes the proof of the upper bound on $[0,1/2]$.

9. Justification of the relation Eq. (56)

If $\hat{\epsilon} \gg 1$, we may use for instance the Taylor expansion of the square root to deduce

$$\begin{aligned}
\tilde{B}_{\hat{\epsilon}}(\gamma) &\approx \int_0^{+\infty}\left(1 - \frac{1}{\sqrt{1+\frac{1}{\hat{\epsilon}^\gamma-1}\left(\mathcal{V}^{\rm Pl}(0)-\mathcal{V}^{\rm Pl}(y)\right)}}\right)\frac{dy}{y^2} \\
&\approx \frac{1}{\hat{\epsilon}^\gamma - 1}\int_0^{+\infty}\frac{\mathcal{V}^{\rm Pl}(0)-\mathcal{V}^{\rm Pl}(y)}{2y^2}\,dy \\
&\approx \frac{1}{\hat{\epsilon}^\gamma}\int_0^{+\infty}\frac{1-(1+y^2)^{-\gamma/2}}{2y^2}\,dy \\
&= \frac{\gamma}{4}\int_0^{+\infty}(1+y^2)^{-\gamma/2-1}\,dy \\
&= \frac{\gamma}{4}\int_0^{\pi/2}\cos^\gamma(\vartheta)\,d\vartheta = \sqrt{\pi}\frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{4\Gamma\left(\frac{\gamma}{2}\right)},
\end{aligned}$$

by first integration by parts and then the use of the substitution $y = \tan\vartheta$.

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