A characterization of tightly triangulated 3-manifolds

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January 08, 2016

Abstract

For a field $\mathbb{F}$, the notion of $\mathbb{F}$-tightness of simplicial complexes was introduced by Kühnel. Kühnel and Lutz conjectured that any $\mathbb{F}$-tight triangulation of a closed manifold is the most economic of all possible triangulations of the manifold. The boundary of a triangle is the only $\mathbb{F}$-tight triangulation of a closed 1-manifold. A triangulation of a closed 2-manifold is $\mathbb{F}$-tight if and only if it is $\mathbb{F}$-orientable and neighbourly. In this paper we prove that a triangulation of a closed 3-manifold is $\mathbb{F}$-tight if and only if it is $\mathbb{F}$-orientable, neighbourly and stacked. In consequence, the Kühnel-Lutz conjecture is valid in dimension ≤ 3.

MSC 2010: 57Q15, 57R05.

Keywords: Stacked spheres; Stacked manifolds; Triangulations of 3-manifolds; Tight triangulations.

1 Introduction

All simplicial complexes considered in this paper are finite and abstract. The vertex set of a simplicial complex X will be denoted by $V(X)$. For $A \subseteq V(X)$, the induced subcomplex $X[A]$ of X on the vertex set A is defined by $X[A] := \{\alpha \in X : \alpha \subseteq A\}$. For $x \in V(X)$, the subcomplexes $\{\alpha \in X : x \notin \alpha\} = X[V(X) \setminus \{x\}]$ and $\{\alpha \in X : x \notin \alpha, \alpha \sqcup \{x\} \in X\}$ are called the *antistar* and the *link* of x in X, respectively. A simplicial complex X is said to be a *triangulated* (*closed*) *manifold* if it triangulates a (closed) manifold, i.e., if the geometric carrier $|X|$ of X is a (closed) topological manifold. A triangulated closed d-manifold X is said to be $\mathbb{F}$-*orientable* if $H_d(X; \mathbb{F}) \neq 0$. If two triangulated d-manifolds X and Y intersect precisely in a common d-face α then $X\#Y := (X \cup Y) \setminus \{\alpha\}$ triangulates the connected sum $|X|\#|Y|$ and is called the *connected sum of* X *and* Y *along* α.

For our purpose, a *graph* may be defined as a simplicial complex of dimension ≤ 1. For $n \geq 3$, the n-*cycle* C_n is the unique n-vertex connected graph in which each vertex lies on exactly two edges. For $n \geq 1$, the *complete graph* K_n is the n-vertex graph in which any two vertices form an edge. For $m, n \geq 1$, the *complete bipartite graph* $K_{m,n}$ is the graph with $m+n$ vertices and mn edges in which each of the first m vertices forms an edge with each of the last n vertices. Two graphs are said to be *homeomorphic* if their geometric carriers are homeomorphic. A graph is said to be *planar* if it is a subcomplex of a triangulation of the 2-sphere S^2. In this paper, we shall have an occasion to use the easy half of Kuratowski's famous characterization of planar graphs [5]: A graph is planar if and only if it has no homeomorph of K_5 or $K_{3,3}$ as a subgraph.

If $\mathbb{F}$ is a field and X is a simplicial complex then, following Kühnel [9], we say that X is $\mathbb{F}$-tight if (a) X is connected, and (b) the $\mathbb{F}$-linear map $H_*(Y;\mathbb{F}) \to H_*(X;\mathbb{F})$, induced by the inclusion map $Y \hookrightarrow X$, is injective for every induced subcomplex Y of X.

If X is a simplicial complex of dimension d, then its *face vector* $(f_0, \dots, f_d)$ is defined by $f_i = f_i(X) := \#\{\alpha \in X \,:\, \dim(\alpha) = i\}$, $0 \le i \le d$. A simplicial complex X is said to be *neighbourly* if any two of its vertices form an edge, i.e., if $f_1(X) = \binom{f_0(X)}{2}$.

A simplicial complex X is said to be *strongly minimal* if, for every triangulation Y of the geometric carrier $|X|$ of X, we have $f_i(X) \le f_i(Y)$ for all i, $0 \le i \le \dim(X)$. Our interest in the notion of $\mathbb{F}$-tightness mainly stems from the following famous conjecture [10].

Conjecture 1.1 (Kühnel-Lutz)**.** *For any field $\mathbb{F}$, every $\mathbb{F}$-tight triangulated closed manifold is strongly minimal.*

Following Walkup [16] and McMullen-Walkup [12], a triangulated ball B is said to be *stacked* if all the faces of B of codimension 2 are contained in the boundary ∂B of B. A triangulated sphere S is said to be *stacked* if there is a stacked ball B such that $S = \partial B$. This notion was extended to triangulated manifolds by Murai and Nevo [14]. Thus, a triangulated manifold Δ with boundary is said to be *stacked* if all its faces of codimension 2 are contained in the boundary $\partial\Delta$ of Δ. A triangulated closed manifold M is said to be *stacked* if there is a stacked triangulated manifold Δ such that $M = \partial\Delta$. A triangulated manifold is said to be *locally stacked* if all its vertex links are stacked spheres or stacked balls. The main result of this paper is the following characterization of $\mathbb{F}$-tight triangulated closed 3-manifolds, for all fields $\mathbb{F}$.

Theorem 1.2. *A triangulated closed 3-manifold M is $\mathbb{F}$-tight if and only if M is $\mathbb{F}$-orientable, neighbourly and stacked.*

The special case of Theorem 1.2, where $\mathrm{char}(\mathbb{F}) \ne 2$, was proved in our previous paper [4]. In this paper we conjectured [4, Conjecture 1.12] the validity of Theorem 1.2 in general.

As a consequence of Theorem 1.2, we show that the Kühnel-Lutz conjecture (Conjecture 1.1) is valid up to dimension 3. Thus,

Corollary 1.3. *If M is an $\mathbb{F}$-tight triangulated closed manifold of dimension ≤ 3, then M is strongly minimal.*

As a second consequence of Theorem 1.2, we show:

Corollary 1.4. *The only closed topological 3-manifolds which may possibly have $\mathbb{F}$-tight triangulations are S^3, $(S^2 \underline{\times} S^1)^{\#k}$ and $(S^2 \times S^1)^{\#k}$, where k is a positive integer such that $80k+1$ is a perfect square.*

Kühnel conjectured that any triangulated closed 3-manifold M satisfies $(f_0(M)-4)\times(f_0(M)-5) \ge 20\beta_1(M;\mathbb{F})$. (This is a part of his Pascal-like triangle of conjectures reported in [11].) This bound was proved by Novic and Swartz in [15]. Burton et al proved in [6] that if the equality holds in this inequality then M is neighbourly and locally stacked. (Actually, these authors stated this result for $\mathbb{F} = \mathbb{Z}_2$, but their argument goes through for all fields $\mathbb{F}$.) In [1], the first author proved that the equality holds in this inequality if and only if M is neighbourly and stacked. In [13], Murai generalized this to all dimensions ≥ 3. Another consequence of Theorem 1.2 is:

Corollary 1.5. *A triangulated closed* 3*-manifold* M *is* $\mathbb{F}$*-tight if and only if* M *is* $\mathbb{F}$*-orientable and* $(f_0(M)-4)(f_0(M)-5) = 20\beta_1(M;\mathbb{F})$.

In [7], $\mathbb{Z}_2$-tight triangulations of $(S^2 \times S^1)^{\#k}$ were constructed for $k = 1, 30, 99, 208, 357$ and 546. However, we do not know any $\mathbb{F}$-tight triangulations of $(S^2 \times S^1)^{\#k}$.

Question 1.6. *Is there any positive integer* k *for which* $(S^2 \times S^1)^{\#k}$ *has an* $\mathbb{F}$*-tight triangulation?*

2 Proofs

The following result is Theorem 3.5 of [4].

Theorem 2.1. *Let* C *be an induced cycle in the link* S *of a vertex* x *in an* $\mathbb{F}$*-tight simplicial complex* X*. Then the induced subcomplex of* X *on the vertex set of the cone* $x * C$ *is a neighbourly triangulated closed* 2*-manifold.*

If, in Theorem 2.1, C is an n-cycle then the triangulated 2-manifold guaranteed by this theorem has $n+1$ vertices, $n(n+1)/2$ edges and hence $n(n+1)/3$ triangles. Thus 3 divides $n(n+1)$, i.e., $n \not\equiv 1 \pmod 3$. Therefore, Theorem 2.1 has the following immediate consequence.

Corollary 2.2. *Let* X *be an* $\mathbb{F}$*-tight simplicial complex. Let* S *be the link of a vertex in* X*. Then* S *has no induced* n*-cycle for* $n \equiv 1 \pmod 3$.

We recall that the Möbius band has a unique 5-vertex triangulation $\mathcal{M}$. The boundary of $\mathcal{M}$ is a 5-cycle C_5. The simplicial complex $\mathcal{M}$ may be uniquely recovered from C_5 as follows. The triangles of $\mathcal{M}$ are $\{x\} \cup e_x$, where, for each vertex x of C_5, e_x is the edge of C_5 opposite to x. We also note the following consequence of Theorem 2.1.

Corollary 2.3. *Let* S *be the link of a vertex* x *in an* $\mathbb{F}$*-tight simplicial complex* X*. Let* C *be an induced cycle in* S*.*

(a) *If* C *is a* 3*-cycle, then it bounds a triangle of* X*.*
(b) *If* C *is a* 5*-cycle then it bounds an induced subcomplex of* X *isomorphic to the* 5*-vertex Möbius band.*

Proof. If C is a 3-cycle, then the induced subcomplex of X on the vertex set of $x * C$ is a neighbourly, 4-vertex, triangulated closed 2-manifold, which must be the boundary complex $\mathcal{T}$ of the tetrahedron. But all four possible triangles occur in $\mathcal{T}$, and C bounds one of them. If C is a 5-cycle then the induced subcomplex $X[V(x * C)]$ of X is a neighbourly, 6-vertex, triangulated closed 2-manifold, which must be the unique 6-vertex triangulation $\mathbb{RP}^2_6$ of the real projective plane. Therefore, the induced subcomplex $X[V(C)]$ of X is the antistar of the vertex x in $\mathbb{RP}^2_6$, which is the 5-vertex Möbius band. □

Let $\mathcal{T}$ and $\mathcal{I}$ denote the boundary complexes of the tetrahedron and the icosahedron, respectively. Thus the faces of $\mathcal{T}$ are all the proper subsets of a set of four vertices. Up to isomorphism, the 20 triangles of $\mathcal{I}$ are as follows:

$$012, 015, 023, 034, 045, 124', 153', 13'4', 235', 24'5', 341',$$
$$31'5', 452', 41'2', 52'3', 0'1'2', 0'1'5', 0'2'3', 0'3'4', 0'4'5'. \qquad (1)$$

The following is Corollary 5.5 of [4].

Theorem 2.4. *Let S be a triangulated 2-sphere which has no induced n-cycle for any $n \equiv 1$ (mod 3). Then S is a connected sum of finitely many copies of $\mathcal{T}$ and $\mathcal{I}$ (in some order).*

As an immediate consequence of Corollary 2.2 and Theorem 2.4, we have:

Corollary 2.5. *Let S be the link of a vertex in an $\mathbb{F}$-tight triangulated closed 3-manifold M. Then S is a connected sum of finitely many copies of $\mathcal{T}$ and $\mathcal{I}$ (in some order).*

Proof of Theorem 1.2. Let M be an $\mathbb{F}$-orientable, neighbourly, stacked, triangulated closed 3-manifold. Then M is $\mathbb{F}$-tight by the case $k = 1$ of Theorem 2.24 in [2]. This proves the "if part". Conversely, let M be $\mathbb{F}$-tight. Since any $\mathbb{F}$-tight triangulated closed manifold is neighbourly and $\mathbb{F}$-orientable (Lemmas 2.2 and 2.5 in [4]), it follows that M is $\mathbb{F}$-orientable and neighbourly. To complete the proof of the "only if" part, it suffices to show that M must be stacked. By [2, Theorem 2.24], every locally stacked, $\mathbb{F}$-tight, triangulated closed 3-manifold is automatically stacked. So, it is enough to show that if S is the link of an arbitrary vertex of M, then S is a stacked 2-sphere. By Corollary 2.5, $S = S_1 \# \cdots \# S_m$, where each S_i is either $\mathcal{T}$ or $\mathcal{I}$. Since any connected sum of copies of $\mathcal{T}$ is stacked (as may be seen by an easy induction on the number of summands), it suffices to show that no S_i can be $\mathcal{I}$.

Suppose, on the contrary, that $S_i = \mathcal{I}$ for some index i. We may take the triangles of $\mathcal{I}$ to be as given in (1). Note that each triangle of S_i is either a triangle of S or its boundary is an induced 3-cycle of S. Since $S \subseteq M$, Corollary 2.3 (a) implies that each triangle of S_i is a triangle of M. Thus, $\mathcal{I} \subseteq M$. In particular, 012 and 023 are triangles of M. Also, we have the following induced 5-cycles (among others) in $\mathcal{I}$, and hence in S.

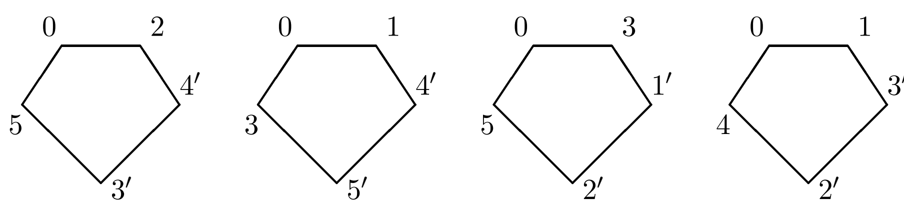

Figure 1 : Some induced 5-cycles in $\mathcal{I}$

Hence Corollary 2.3 (b) gives us eight more triangles of M through the vertex 0, namely, $023'$, $03'4'$, $015'$, $034'$, $04'5'$, $032'$, $012'$, $02'3'$. Thus, if S' is the link of the vertex 0 in M, then we have the graph of Fig. 2 as a subcomplex of S'.

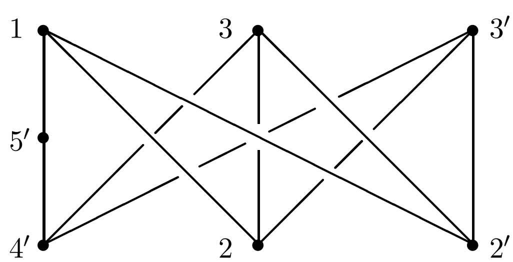

Figure 2 : A homeomorph of $K_{3,3}$

So we have a homeomorph of $K_{3,3}$ as a subcomplex of the triangulated 2-sphere S'. This is a contradiction since $K_{3,3}$ is not a planar graph. □

Proof of Corollary 1.3. Let M be an $\mathbb{F}$-tight triangulated closed d-manifold, $d \leq 3$. By Lemma 2.2 in [4], M is neighbourly. But the boundary complex of the triangle is the only neighbourly triangulated closed 1-manifold. This is trivially the strongly minimal triangulation of S^1. So, we have the result for $d = 1$. Next let $d = 2$. Let N be another triangulation of $|M|$. Let (f_0, f_1, f_2) be the face vector of N. Let χ be the Euler characteristic of M (hence also of N). Then $f_0 - f_1 + f_2 = \chi$ and $2f_1 = 3f_2$. Therefore we get $f_1 = 3(f_0 - \chi)$ and $f_2 = 2(f_0 - \chi)$. Thus, f_1 and f_2 are strictly increasing functions of f_0. So, it is sufficient to show that $f_0 \geq f_0(M)$. Now, trivially, $f_1 \leq \binom{f_0}{2}$, with equality if and only if N is neighbourly. Substituting $f_1 = 3(f_0 - \chi)$ in this inequality, we get that $f_0(f_0 - 7) \geq -6\chi = f_0(M)(f_0(M) - 7)$. This implies that $f_0 \geq f_0(M)$. Thus, M is strongly minimal. So we have the result for $d = 2$. If $d = 3$ then, by Theorem 1.2, M is stacked and hence is locally stacked. But any locally stacked, $\mathbb{F}$-tight triangulated closed manifold is strongly minimal by Corollary 3.13 in [3]. So, we are done when $d = 3$. □

Proof of Corollary 1.4. Let M be a closed 3-manifold which has an $\mathbb{F}$-tight triangulation X. By Theorem 1.2, X is stacked. But, by Corollary 3.13 (case $d = 3$) of [8], any stacked triangulation of a closed 3-manifold can be obtained from a stacked 3-sphere by a finite sequence of elementary handle additions. It is easy to see by an induction on the number k of handles added that X triangulates either S^3 $(k = 0)$ or $(S^2 \times S^1)^{\#k}$ or $(S^2 \mathbin{\underline{\times}} S^1)^{\#k}$ $(k \geq 1)$. Let X be obtained from the stacked 3-sphere S by k elementary handle additions. It follows by induction on k that $f_0(S) = f_0(X) + 4k$ and $f_1(S) = f_1(X) + 6k = \binom{f_0(X)}{2} + 6k$. Since S is a stacked 3-sphere, $f_1(S) = 4f_0(S) - 10$. Thus, $\binom{f_0(X)}{2} + 6k = 4(f_0(X) + 4k) - 10$. This implies $(f_0(X) - 4)(f_0(X) - 5) = 20k$ and hence $f_0(X) = \frac{1}{2}(9 + \sqrt{80k + 1})$. So, $80k + 1$ must be a perfect square. □

Proof of Corollary 1.5. If $(f_0(M) - 4)(f_0(M) - 5) = 20\beta_1(M; \mathbb{F})$ then Theorem 1.3 of [6] says that M must be neighbourly and locally stacked. Therefore, the 'if part' follows from Theorem 2.24 of [2]. The 'if part' also follows from Theorem 1.12 of [1] and Theorem 2.24 of [2]. The 'only if' part follows from the proof of Corollary 1.4 above. □

Acknowledgements: The second and third authors are supported by DIICCSRTE, Australia (project AISRF06660) and DST, India (DST/INT/AUS/P-56/2013(G)) under the Australia-India Strategic Research Fund. The second author is also supported by the UGC Centre for Advanced Studies (F.510/6/CAS/2011 (SAP-I)). The authors thank Wolfgang Kühnel for some useful comments on the earlier version of this paper.

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