A CHARACTERIZATION OF THE NORMAL DISTRIBUTION BY THE INDEPENDENCE OF A PAIR OF RANDOM VECTORS

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Abstract. Kagan and Shalaevski [11] have shown that if the random variables $\mathbb{X}_1, \dots, \mathbb{X}_n$ are independent and identically distributed and the distribution of $\sum_{i=1}^{n}(\mathbb{X}_i + a_i)^2$ $a_i \in \mathbb{R}$ depends only on $\sum_{i=1}^{n} a_i^2$, then each $\mathbb{X}_i$ follows the normal distribution $N(0, \sigma)$. Cook [6] generalized this result replacing independence of all $\mathbb{X}_i$ by the independence of $(\mathbb{X}_1, \dots, \mathbb{X}_m)$ and $(\mathbb{X}_{m+1}, \dots, \mathbb{X}_n)$ and removing the requirement that $\mathbb{X}_i$ have the same distribution. In this paper, we will give other characterizations of the normal distribution which are formulated in a similar spirit.

1. Introduction

It will be shown that the formulae are much simplified by the use of cumulative moment functions, or semi-invariants, in place of the crude moments. R.A. Fisher [8].

The original motivation for this paper comes from a desire to understand the results about characterization of normal distribution which were shown in [6] and [11]. They proved, that the characterizations of a normal law are given by a certain invariance of the noncentral chi-square distribution. It is a known fact that if $\mathbb{X}_1, \dots, \mathbb{X}_n$ are i.i.d. and following the normal distribution $N(0, \sigma)$ then the distribution of the statistic $\sum_{i=1}^{n}(\mathbb{X}_i + a_i)^2$, $a_i \in \mathbb{R}$ depends on $\sum_{i=1}^{n} a_i^2$ only (see [4, 14]). Kagan and Shalaevski [11] have shown that if the random variables $\mathbb{X}_1, \mathbb{X}_2, ..., \mathbb{X}_n$ are independent and identically distributed and the distribution of $\sum_{i=1}^{n}(\mathbb{X}_i + a_i)^2$ depends only on $\sum_{i=1}^{n} a_i^2$, then each $\mathbb{X}_i$ is normally distributed as $N(0, \sigma)$. Cook generalized this result replacing independence of all $\mathbb{X}_i$ by the independence of $(\mathbb{X}_1, \dots, \mathbb{X}_m)$ and $(\mathbb{X}_{m+1}, \dots, \mathbb{X}_n)$ and removing the requirement that $\mathbb{X}_i$ have the same distribution. The theorem proved below gives a new look on this subject, i.e. we will show that in the statistic $\sum_{i=1}^{n}(\mathbb{X}_i + a_i)^2 = \sum_{i=1}^{n} \mathbb{X}_i^2 + 2\sum_{i=1}^{n} \mathbb{X}_i a_i + \sum_{i=1}^{n} a_i^2$ only the linear part $\sum_{i=1}^{n} \mathbb{X}_i a_i$ is important. In particular, from the above result we get Cook Theorem from [6], but under the assumption that all moments exist. Note that Cook does not assume any moments, but he gets this result under integrability assumptions imposed on the corresponding random variable. This paper is removing or at least relaxing its integrability assumptions.

The paper is organized as follows. In section 2 we review basic facts about cumulants. Next in the third section we state and prove the main results (proposition). In this section we also discuss the problem.

2. Cumulants and moments

Cumulants were first defined and studied by the Danish scientist T. N. Thiele. He called them semi-invariants. The importance of cumulants comes from the observation that many properties of random variables can be better represented by cumulants than by moments. We refer to Brillinger [2] and Gnedenko and Kolmogorov [9] for further detailed probabilistic aspects of this topic.

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Given a random variable $\mathbb{X}$ with the moment generating function $g(t)$, its ith cumulant r_i is defined as

$$r_i(\mathbb{X}) := r_i(\underbrace{\mathbb{X},\dots,\mathbb{X}}_{i-times}) = \frac{d^i}{dt^i}\Big|_{t=0} log(g(t)).$$

That is,

$$\sum_{i=0}^{\infty}\frac{m_i}{i!}t^i = g(t) = exp\Big(\sum_{i=1}^{\infty}\frac{r_i}{i!}t^i\Big)$$

where m_i is the ith moment of $\mathbb{X}$.

Generally, if σ denotes the standard deviation, then

$$r_1 = m_1, \qquad r_2 = m_2 - m_1^2 = \sigma, \qquad r_3 = m_3 - 3m_2m_1 + 2m_1^3.$$

The joint cumulant of several random variables $\mathbb{X}_1,\dots\mathbb{X}_n$ of order $(i_1,\dots,i_n)$, where i_j are nonnegative integers, is defined by a similar generating function $g(t_1,\dots,t_n) = E\big(e^{\sum_{i=1}^n t_i\mathbb{X}_i}\big)$

$$r_{i_1+\dots+i_n}(\underbrace{\mathbb{X}_1,\dots,\mathbb{X}_1}_{i_1-times},\dots,\underbrace{\mathbb{X}_n,\dots,\mathbb{X}_n}_{i_n-times}) = \frac{d^{i_1+\dots+i_n}}{dt_1^{i_1}\dots dt_n^{i_n}}\Big|_{t=0} log(g(t_1,\dots,t_n)),$$

where $t = (t_1,\dots,t_n)$.

Random variables $\mathbb{X}_1,\dots,\mathbb{X}_n$ are independent if and only if, for every $n\geq 1$ and every non-constant choice of $\mathbb{Y}_i \in \{\mathbb{X}_1,\dots,\mathbb{X}_n\}$, where $i\in\{1,\dots,k\}$ (for some positive integer $k\geq 2$) we get $r_k(\mathbb{Y}_1,\dots,\mathbb{Y}_k) = 0$.

Cumulants of some important and familiar random distributions are listed as follows:

- The Gaussian distribution $N(\mu,\sigma)$ possesses the simplest list of cumulants: $r_1 = \mu$, $r_2 = \sigma$ and $r_n = 0$ for $n\geq 3$,
- for the Poisson distribution with mean λ we have $r_n = \lambda$.

These classical examples clearly demonstrate the simplicity and efficiency of cumulants for describing random variables. Apparently, it is not accidental that cumulants encode the most important information of the associated random variables. The underlying reason may well reside in the following three important properties (which are in fact related to each other):

- (Translation Invariance) For any constant c, $r_1(\mathbb{X}+c) = c + r_1(\mathbb{X})$ and $r_n(\mathbb{X}+c) = r_n(\mathbb{X})$, $n\geq 2$.
- (Additivity) Let $\mathbb{X}_1,\dots,\mathbb{X}_m$ be any independent random variables. Then, $r_n(\mathbb{X}_1+\dots+\mathbb{X}_m) = r_n(\mathbb{X}_1)+\dots+r_n(\mathbb{X}_m)$, $n\geq 1$.
- (Commutative property) $r_n(\mathbb{X}_1,\dots,\mathbb{X}_n) = r_n(\mathbb{X}_{\sigma(1)},\dots,\mathbb{X}_{\sigma(n)})$ for any permutation $\sigma\in S_n$.
- (Multilinearity) r_k are the k-linear maps.

For more details about cumulants and probability theory, the reader can consult [13] or [16] .

3. The Characterization theorem

The main result of this paper is the following characterization of normal distribution in terms of independent random vectors.

Proposition 3.1. *Suppose vectors $(\mathbb{S}_1,\mathbb{Y})$ and $(\mathbb{S}_2,\mathbb{Z})$ with all moments are independent and $\mathbb{S}_1,\mathbb{S}_2$ are nondegenerate. If for every $a,b\in\mathbb{R}$ the linear combination $a\mathbb{S}_1+\mathbb{Y}+b\mathbb{S}_2+\mathbb{Z}$ has the law that depends on (a,b) through a^2+b^2 only, then random variables $\mathbb{S}_1,\mathbb{S}_2$ have the same normal distribution and $cov(\mathbb{S}_1,\mathbb{Y}) = cov(\mathbb{S}_2,\mathbb{Z}) = 0$.*

Proof. Let $h_k(a^2+b^2) = r_k(a\mathbb{S}_1+\mathbb{Y}+b\mathbb{S}_2+\mathbb{Z})-r_k(\mathbb{Y}+\mathbb{Z})$. Because of the independence of $(\mathbb{S}_1,\mathbb{Y})$ and $(\mathbb{S}_2,\mathbb{Z})$ we may write

$$h_k(a^2+b^2) = r_k(a\mathbb{S}_1+\mathbb{Y}+b\mathbb{S}_2+\mathbb{Z})-r_k(\mathbb{Y}+\mathbb{Z}) = r_k(a\mathbb{S}_1+\mathbb{Y})-r_k(\mathbb{Y})+r_k(b\mathbb{S}_2+\mathbb{Z})-r_k(\mathbb{Z}). \tag{1}$$

Evaluating (1) first when $b=0$ and then when $a=0$, we get $h_k(a^2) = r_k(a\mathbb{S}_1+\mathbb{Y})-r_k(\mathbb{Y})$ and $h_k(b^2) = r_k(b\mathbb{S}_2+\mathbb{Z})-r_k(\mathbb{Z})$, respectively. Substituting this into (1), we see

$$h_k(a^2+b^2) = h_k(a^2)+h_k(b^2).$$

Note that $h_k(u)$ is continuous in $u\in[0,\infty)$, which implies $h_k(u)=h_k(1)u$ and so we have $h_k(a^2+b^2) = (a^2+b^2)h_k(1) = (a^2+b^2)\varphi_k(\alpha,\beta)$, where

$$\varphi_k(\alpha,\beta) = h_k(\alpha^2+\beta^2) = r_k(\alpha\mathbb{S}_1+\mathbb{Y}+\beta\mathbb{S}_2+\mathbb{Z})-r_k(\mathbb{Y}+\mathbb{Z}),$$

with $\alpha^2+\beta^2=1$. In the next part of the proof, we will compare polynomial, which give us the correct cumulants values. Let's first consider the following equation

$$h_k(a^2) = a^2\varphi_k(1,0),$$

which gives us

$$r_k(a\mathbb{S}_1+\mathbb{Y})-r_k(\mathbb{Y}) = a^2(r_k(\mathbb{S}_1+\mathbb{Y})-r_k(\mathbb{Y})). \tag{2}$$

For $k=1$ we get $E(\mathbb{S}_1)=0$, because $r_1(a\mathbb{S}_1+\mathbb{Y}) = E(a\mathbb{S}_1+\mathbb{Y})$. By putting $k=2$ in (2) and using $r_2(a\mathbb{S}_1+\mathbb{Y}) = Var(a\mathbb{S}_1+\mathbb{Y}) = a^2Var(\mathbb{S}_1)+2acov(\mathbb{S}_1,\mathbb{Y})+Var(\mathbb{Y})$ we see

$$2acov(\mathbb{S}_1,\mathbb{Y})+a^2Var(\mathbb{S}_1) = a^2(2cov(\mathbb{S}_1,\mathbb{Y})+Var(\mathbb{S}_1)),$$

for all $a\in\mathbb{R}$, which implies that $cov(\mathbb{S}_1,\mathbb{Y})=0$. Now by expanding equation (2) (r_k are k-linear maps, we also use independence), we may write

$$\sum_{i=1}^{k} a^i\binom{k}{i} r_k(\underbrace{\mathbb{S}_1,\ldots,\mathbb{S}_1}_{i-times},\underbrace{\mathbb{Y},\ldots,\mathbb{Y}}_{k-i-times}) = a^2(r_k(\mathbb{S}_1+\mathbb{Y})-r_k(\mathbb{Y})), \tag{3}$$

for $k\geq 2$. This gives us $r_k(\mathbb{S}_1)=0$ for $k>2$ and we have actually proved that $\mathbb{S}_1$ have the normal distribution with zero mean. Analogously, we will show that $cov(\mathbb{S}_2,\mathbb{Z})=0$ and normality of $\mathbb{S}_2$.

The next example presents an analogous construction for $h_k(a^2) = a^2\varphi_k^{0,1}(1)$, which involves the element $\varphi_k^{0,1}(1)$ instead of $\varphi_k^{1,0}(1)$ which leads to $2acov(\mathbb{S}_1,\mathbb{Y})+a^2Var(\mathbb{S}_1) = a^2(2cov(\mathbb{S}_2,\mathbb{Z})+Var(\mathbb{S}_2))$. But in the previous paragraph we calculated that $cov(\mathbb{S}_1,\mathbb{Y}) = cov(\mathbb{S}_2,\mathbb{Z}) = 0$ which means that $Var(\mathbb{S}_1)=Var(\mathbb{S}_2)$ (we have common variance), i.e. we have the same distribution.

□

As a corollary we get the following theorem.

Theorem 3.2. *Let $(\mathbb{X}_1,\ldots,\mathbb{X}_m,\mathbb{Y})$ and $(\mathbb{X}_{m+1},\ldots,\mathbb{X}_n,\mathbb{Z})$ be independent random vectors with all moments, where $\mathbb{X}_i$ are nondegenerate, and let statistic $\sum_{i=1}^n a_i\mathbb{X}_i+\mathbb{Y}+\mathbb{Z}$ have a distribution which depends only on $\sum_{i=1}^n a_i^2$, where $a_i\in\mathbb{R}$ and $1\leq m<n$. Then $\mathbb{X}_i$ are independent and have the same normal distribution with zero means and $cov(\mathbb{X}_i,\mathbb{Y}) = cov(\mathbb{X}_i,\mathbb{Z}) = 0$ for $i\in\{1,\ldots,n\}$.*

Proof. Without loss of generality, we may assume that $m\geq 2$. If we put

$$\mathbb{S}_1 = \frac{\sum_{i=1}^m a_i\mathbb{X}_i}{\sqrt{\sum_{i=1}^m a_i^2}} \text{ and } \mathbb{S}_2 = \frac{\sum_{i=m+1}^n a_i\mathbb{X}_i}{\sqrt{\sum_{i=m+1}^n a_i^2}} \tag{4}$$

and $a=\sqrt{\sum_{i=1}^m a_i^2}$, $b=\sqrt{\sum_{i=m+1}^n a_i^2}$, in Proposition 3.1 then we get that the distribution of

$$\mathbb{S}_1 a+\mathbb{Y}+\mathbb{S}_2 b+\mathbb{Z}=\sum_{i=1}^m a_i\mathbb{X}_i+\mathbb{Y}+\sum_{i=m+1}^n a_i\mathbb{X}_i+\mathbb{Z},$$

depends only on $a^2+b^2=\sum_{i=1}^n a_i^2$,which by Proposition 3.1 implies that $\sum_{i=1}^m a_i\mathbb{X}_i$ have the normal distribution and $cov(\sum_{i=1}^m a_i\mathbb{X}_i,\mathbb{Y})=0$ for all $a_i\in\mathbb{R}$. Now, we once again use Proposition 3.1 with $\mathbb{S}_1=\mathbb{X}_1,\ \mathbb{S}_2=\mathbb{X}_{m+1}$, then we see from assumption that the distribution of

$$a_1\mathbb{X}_1+\mathbb{X}_2+\mathbb{Y}+a_{m+1}\mathbb{X}_{m+1}+\mathbb{Z}$$

(where $\mathbb{X}_2+\mathbb{Y}$ play a role of $\mathbb{Y}$ from Proposition 3.1), depends only on $a_1^2+a_{m+1}^2$ $(a_1^2+a_{m+1}^2+1)$. This gives us $cov(\mathbb{X}_1,\mathbb{X}_2+\mathbb{Y})=0$, but we know that $cov(\mathbb{X}_1,\mathbb{Y})=0$ which implies $cov(\mathbb{X}_1,\mathbb{X}_2)=0$. Similarly, we show that $cov(\mathbb{X}_i,\mathbb{X}_j)=0$ for $i\neq j$. Now we use well known facts from the general theory of probability that if a random vector has a multivariate normal distribution (joint normality), then any two or more of its components that are uncorrelated, are independent. This implies that any two or more of its components that are pairwise independent are independent. Normality of linear combinations $\sum_{i=1}^m a_i\mathbb{X}_i$ for all $a_i\in\mathbb{R}$, means joint normality of $(\mathbb{X}_1,\dots,\mathbb{X}_m)$ (see e.g. the definition of multivariate normal law in Billingsley [1]) and taking into account that random variables $\mathbb{X}_1,\dots,\mathbb{X}_m$ are pairwise uncorrelated, we obtain independence of $\mathbb{X}_1,\dots,\mathbb{X}_m$. □

The above theorem gives us the main result by Cook [6] but under the additional assumption that all moments exist. Note that Cook does not assume any moments but assumes integrability.

Corollary 3.3. *Let* $(\mathbb{X}_1,\dots,\mathbb{X}_m)$ *and* $(\mathbb{X}_{m+1},\dots,\mathbb{X}_n)$ *be independent random vectors with all moments, where* $\mathbb{X}_i$ *are nondegenerate, and let statistic* $\sum_{i=1}^n(\mathbb{X}_i+a_i)^2$ *have a distribution which depends only on* $\sum_{i=1}^n a_i^2$, $a_i\in\mathbb{R}$ *and* $1\le m<n$. *Then* $\mathbb{X}_i$ *are independent and have the same normal distribution with zero means.*

Proof. If we put $\mathbb{Y}=\sum_{i=1}^m\mathbb{X}_i^2$ and $\mathbb{Z}=\sum_{i=m+1}^n\mathbb{X}_i^2$ in Theorem 3.2 then we get

$$\sum_{i=1}^m a_i\mathbb{X}_i+\mathbb{Y}+\sum_{i=m+1}^n a_i\mathbb{X}_i+\mathbb{Z}=\sum_{i=1}^n(\mathbb{X}_i+a_i/2)^2-\frac{1}{4}\times\sum_{i=1}^n a_i^2.$$

This means that the distribution of $\sum_{i=1}^m a_i\mathbb{X}_i+\mathbb{Y}+\sum_{i=m+1}^n a_i\mathbb{X}_i+\mathbb{Z}$ depends only on $\sum_{i=1}^n a_i^2$, which by Theorem 3.2 implies the statement. □

A simple modification of the above arguments can be applied to get the following proposition. In this proposition we assume a bit more than in Proposition 3.1 and it offers a bit stronger conclusion.

Proposition 3.4. *Let* $(\mathbb{X},\mathbb{Y})$ *and* $(\mathbb{Z},\mathbb{T})$ *be independent and nondegenerate random vectors with all moments and let* $a\mathbb{X}+\mathbb{Y}+b\mathbb{Z}+\mathbb{T}$ *and* $\mathbb{X}+a\mathbb{Y}+\mathbb{Z}+b\mathbb{T}$ *have a distribution which depends only on* a^2+b^2, $a,b\in\mathbb{R}$. *Then* $\mathbb{X},\mathbb{Y},\mathbb{Z},\mathbb{T}$ *are independent and have normal distribution with zero means and* $Var(\mathbb{X})=Var(\mathbb{Z})$, $Var(\mathbb{Y})=Var(\mathbb{T})$.

Proof. From Proposition 3.1 we get that $\mathbb{X},\mathbb{Y},\mathbb{Z},\mathbb{T}$ have normal distribution with zero means and $Var(\mathbb{X})=Var(\mathbb{Z})$, $Var(\mathbb{Y})=Var(\mathbb{T})$. Now we will show that $\mathbb{X}$ and $\mathbb{Y}$ are independent random variables. We proceed analogously to the proof of Proposition 3.1 with $h_k(a^2+b^2)=r_k(a\mathbb{X}+\mathbb{Y}+b\mathbb{Z}+\mathbb{T})-r_k(\mathbb{Y}+\mathbb{T})$, which gives us equality (3), i.e.

$$\sum_{i=1}^k a^i\binom{k}{i}r_k(\underbrace{\mathbb{X},\dots,\mathbb{X}}_{i-times},\underbrace{\mathbb{Y},\dots,\mathbb{Y}}_{k-i-times})=a^2(r_k(\mathbb{X}+\mathbb{Y})-r_k(\mathbb{Y})).\tag{5}$$

for $k \geq 2$. From this we conclude that

$$r_k(\underbrace{\mathbb{X},\dots,\mathbb{X}}_{i-times},\underbrace{\mathbb{Y},\dots,\mathbb{Y}}_{l-times}) = 0 \tag{6}$$

for all $i, l \in \mathbb{N}$, $i \neq 2$. By a similar argument applied to a statistic $\mathbb{X} + a\mathbb{Y} + \mathbb{Z} + b\mathbb{T}$ we get

$$r_k(\underbrace{\mathbb{X},\dots,\mathbb{X}}_{l-times},\underbrace{\mathbb{Y},\dots,\mathbb{Y}}_{i-times}) = 0$$

for all $i, l \in \mathbb{N}$, $i \neq 2$ which together with (6) gives us independence of $\mathbb{X}$ and $\mathbb{Y}$. Independence of $\mathbb{Z}$ and $\mathbb{T}$ follows similarly. □

Open Problem and Remark

Problem 1. In Proposition 3.1 (Theorem 3.2) in this paper we assume that random variables have all moments. I thought it would be interesting to show that we can skip this assumption. A version of Proposition 3.1, with integrability replaced by the assumption that $\mathbb{S}_1, \mathbb{S}_2$ have the same law, can be deduced from known results (note that this version does not imply Theorem 3.2).

Here we sketch the proof of a version of Proposition 3.1 under reduced moment assumptions but we assume additionally that random variables $\mathbb{S}_1, \mathbb{S}_2$ have the same law. Assume that $\mathbb{S}_1, \mathbb{S}_2, \mathbb{Y}, \mathbb{Z}$ have finite moments of some positive order $p \geq 3$. Then, for every $\epsilon > 0$ and any two values of $p_i > 0$, where $i \in \{1, 2\}$, we see that $E(a\mathbb{S}_1 + \epsilon\mathbb{Y} + b\mathbb{S}_2 + \epsilon\mathbb{Z})^{p_i}$ is a function of $a^2 + b^2$. Passing to the limit as $\epsilon \to 0$ and using homogeneity, we deduce that $E(a\mathbb{S}_1 + b\mathbb{S}_2)^{p_i} = K(a^2 + b^2)^{p_i/2}$ with $K = 2^{-p_i/2}E|\mathbb{S}_1 + \mathbb{S}_2|^{p_i} = E|\mathbb{S}_1|^{p_i} = E|\mathbb{S}_2|^{p_i}$. Since this holds for any two odd values $0 < p_1 < p_2 < p$, by Theorem 2 from [3] we see that $\mathbb{S}_1$ is normal (the reason why we assume $p \geq 3$ is that Braverman [3] assumed that p_1, p_2 are odd).

Problem 2. At the end it is worthwhile to mention the most important characterization which is true in noncommutative and classical probability. In free probability Bożejko, Bryc and Ejsmont proved that the first conditional linear moment and conditional quadratic variances characterize free Meixner laws (Bożejko and Bryc [5], Ejsmont [7]). Laha-Lukacs type characterizations of random variables in free probability are also studied by Szpojankowski, Wesołowski [17]. They give a characterization of non-commutative free-Poisson and free-Binomial variables by properties of the first two conditional moments, which mimics Lukacs-type assumptions known from classical probability. The article [12] studies the asymptotic behavior of the Wigner integrals. Authors prove that a normalized sequence of multiple Wigner integrals (in a fixed order of free Wigner chaos) converges in law to the standard semicircular distribution if and only if the corresponding sequence of fourth moments converges to 2, the fourth moment of the semicircular law. This finding extends the recent results by Nualart and Peccati [15] to free probability theory.

At this point it is worth mentioning [10], where the Kagan-Shalaevski characterization for free random variable was shown. It would be worth asking whether the Theorem 3.2 is true in free probability theory. Unfortunately the above proof of Theorem 3.2 doesn't work in free probability theory, because free cumulates are noncommutative. But the Proposition 3.1 is true in free probability, with nearly the same proof (thus we can only get that $\mathbb{X}_i$ has free normal distribution under the assumption of Theorem 3.2).

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