# A PROOF OF AN OPEN PROBLEM OF YUSUKE NISHIZAWA

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**Abstract**

This paper presents a proof of the following conjecture, stated by Nishizawa in [Appl. Math. Comput. 269, (2015), 146–154.]: for $0 < x < \pi/2$ the inequality $\dfrac{\sin x}{x} > \left(\dfrac{2}{\pi} + \dfrac{\pi-2}{\pi^3}(\pi^2-4x^2)\right)^{\theta(x)}$ holds, where $\theta(x) = -\dfrac{(48-24\pi+\pi^3)x^3}{3(\pi-2)\pi^3} + \dfrac{\pi^3}{24(\pi-2)}$.

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## 1. Introduction

In [1], Nishizawa proved the following exponential inequalities:

**Theorem 1.** *For $0 < x < \pi/2$, we have*

$$\left(\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2)\right)^{\theta} < \frac{\sin x}{x} < \left(\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2)\right)^{\vartheta}$$

*with the best possible constants $\theta = 1$ and $\vartheta = 0$.*

**Theorem 2.** *For $0 < x < \pi/2$, we have*

$$\left(\frac{2}{\pi} + \frac{\pi-2}{\pi^3}(\pi^2 - 4x^2)\right)^{\theta} < \frac{\sin x}{x} < \left(\frac{2}{\pi} + \frac{\pi-2}{\pi^3}(\pi^2 - 4x^2)\right)^{\vartheta}$$

*with the best possible constants $\theta = \pi^3/(24(\pi-2)) \cong 1.13169$ and $\vartheta = 1$.*

**Theorem 3.** *For $0 < x < \pi/2$, we have*

$$\frac{\sin x}{x} < \left(\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2)\right)^{\theta(x)}$$

*where $\theta(x)$ is the function of $x$ and $\theta(x) = 4x^2/\pi^2$.*

Considering the previous theorems, Nishizawa stated the following open problem (Problem 3.1 of [1]):

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For $0 < x < \pi/2$, we have

$$\frac{\sin x}{x} > \left(\frac{2}{\pi} + \frac{\pi-2}{\pi^3}(\pi^2 - 4x^2)\right)^{\theta(x)} \tag{1}$$

where $\theta(x)$ is the function of $x$ and $\theta(x) = -\dfrac{(48-24\pi+\pi^3)x^3}{3(\pi-2)\pi^3} + \dfrac{\pi^3}{24(\pi-2)}$.

This paper provides a proof of Nishizawa's open problem, using approximations and methods from [2] and [4]. Also, the proof makes use of the fact that for the constant $\pi$ and a given rational function $R(x)$, it is possible to determine either $R(\pi) > 0$ or $R(\pi) < 0$. Stated is a consequence of the fact that for an arbitrarily small $\varepsilon > 0$, there exist fractions $p/q$ and $r/s$ such that $p/q > \pi > r/s$ and $p/q - r/s < \varepsilon$. Fractions $p/q$ and $r/s$ can be chosen as two consequential convergents in the continued fractions of $\pi$.

## 2. Proof of the open problem

As $\dfrac{\sin x}{x} > 0$ and $\dfrac{2}{\pi} + \dfrac{\pi-2}{\pi^3}(\pi^2 - 4x^2) > 0$ for $x \in (0, \pi/2)$, inequality (1) is equivalent to the following inequality:

$$\ln\frac{\sin x}{x} > \ln\left(\frac{2}{\pi} + \frac{\pi-2}{\pi^3}(\pi^2 - 4x^2)\right)^{\theta(x)} \tag{2}$$

i.e. to the inequality

$$\ln\sin x - \ln x - \theta(x)\ln\left(\frac{2}{\pi} + \frac{\pi-2}{\pi^3}(\pi^2 - 4x^2)\right) > 0, \tag{3}$$

where $\theta(x) = -\dfrac{(48-24\pi+\pi^3)x^3}{3(\pi-2)\pi^3} + \dfrac{\pi^3}{24(\pi-2)}$.

Denoting the left-hand side of the above inequality by $F(x)$

$$\begin{aligned} F(x) = {} & \ln\sin x - \ln x \\ & - \left(-\frac{(48-24\pi+\pi^3)x^3}{3(\pi-2)\pi^3} + \frac{\pi^3}{24(\pi-2)}\right)\ln\left(\frac{2}{\pi} + \frac{\pi-2}{\pi^3}(\pi^2 - 4x^2)\right) \end{aligned} \tag{4}$$

we have to prove

$$F(x) > 0 \quad \text{for} \quad x \in (0, \pi/2)\,. \tag{5}$$

Consider the following function

$$F_1(x) = \ln\sin x - \ln x - \theta_1(x)\ln\left(\frac{2}{\pi} + \frac{\pi-2}{\pi^3}(\pi^2 - 4x^2)\right) \tag{6}$$

where $\theta_1(x) = -\dfrac{(\pi^3 - 60\pi + 120)}{720(\pi-2)}x^2 + \dfrac{\pi^3}{24(\pi-2)}$.

As

$$\frac{2}{\pi} + \frac{\pi-2}{\pi^3}(\pi^2 - 4x^2) < \frac{2}{\pi} + \frac{\pi-2}{\pi^3}\pi^2 = 1 \tag{7}$$

we can conclude that:

$$
\begin{aligned}
& F(x) - F_1(x) \geq 0 \\
\Longleftrightarrow \quad & \theta(x) - \theta_1(x) \geq 0 \\
\Longleftrightarrow \quad & \frac{1}{3(\pi-2)} \left( \frac{\pi^3 - 60\pi + 120}{240} + \frac{\pi^3 - 24\pi + 48}{\pi^3} x \right) x^2 \; \geq 0
\end{aligned} \tag{8}
$$

for $x \in (0, c]$, where $c = -\dfrac{\big(\pi^3 - 60(\pi-2)\big)\,\pi^3}{240\,(\pi^3 - 24\,(\pi-2))} = 1.342\ldots$.

Thus, we distinguish two cases $x \in (0, c]$ or $x \in (c, \pi/2)$.

*2.1. Case 1: $x \in (0, c\,]$*

It is enough to prove that:

$$F_1(x) > 0 \tag{9}$$

for $x \in (0, c]$. The third derivation of the function $F_1(x)$ is:

$$F_1^{'''}(x) \;=\; \frac{2\,A(x)\,\sin^3 x \;+\; B(x)\cos x}{45\,x^3\,C(x)\,\sin^3 x} \tag{10}$$

where

$$
\begin{aligned}
C(x) \;&=\; -64\,(\pi-2)^3 x^6 \;+\; 48\pi^3(\pi-2)^2 x^4 \;-\; 12\pi^6(\pi-2)x^2 \;+\pi^9 \\
&=\; (\pi^3 - 4(\pi-2)x^2)^3,
\end{aligned} \tag{11}
$$

$$B(x) \;=\; 90x^3\left(\pi^3 \;-\; 4(\pi-2)x^2\right)^3 \;=\; 90\,x^3\,C(x) \tag{12}$$

and

$$
\begin{aligned}
A(x) \;=\; & 8(\pi-2)^2(\pi^3 - 60\pi + 120)x^8 \\
& -6(\pi-2)(\pi^6 - 100\pi^4 + 200\pi^3 - 480(\pi-2)^2)x^6 \\
& +3\pi^3(\,\pi^6 - 720(\pi-2)^2\,)x^4 \\
& +540\pi^6(\pi-2)x^2 \\
& -45\pi^9.
\end{aligned} \tag{13}
$$

Let us determine the sign of the polynomials $C(x)$, $B(x)$ and $A(x)$ for $x \in (0, c]$. By substituting $t = 4(\pi-2)x^2$ for $x \in (0, c]$, the polynomial $C(x)$ can be transformed into the polynomial $C_1(t) = (\pi^3 - t)^3$ for $t \in (0,\, 4(\pi-2)c^2]$. Obviously, the sign of the polynomial $C_1(t)$ coincides with the sign of the polynomial $\pi^3 - t$ for $t \in (0,\, 4(\pi-2)c^2]$.

Since $\left(\dfrac{\pi^3-60(\pi-2)}{\pi^3-24(\pi-2)}\right)^2 < 1$, we have

$$\begin{aligned}
\pi^3-4(\pi-2)c^2 &= \pi^3 - 4(\pi-2)\left(\frac{(\pi^3-60(\pi-2))}{240(\pi^3-24(\pi-2))}\pi^3\right)^2 \\
&= \pi^3\left(1-(\pi-2)\frac{1}{14400}\left(\frac{(\pi^3-60(\pi-2))}{(\pi^3-24(\pi-2))}\right)^2\pi^3\right) \\
&> \pi^3\left(1-(\pi-2)\frac{1}{14400}\pi^3\right) \\
&= \pi^3\left(\frac{14400-(\pi-2)\pi^3}{14400}\right) \\
&> 0.
\end{aligned} \tag{14}$$

Therefore, we can conclude that $C_1(t) > 0$ for $t \in (\,0,\, 4(\pi-2)c^2\,] \subset (0,\pi^3)$, i.e. $C(x) > 0$ for $x \in (0,\, c]$ and $B(x) > 0$ for $x \in (0,\, c]$.
Let us prove that

$$A(x) < 0, \tag{15}$$

for $x \in (0,\, c]$. We note that $A(x)$ can be written as

$$A(x) \;=\; 2(\pi-2)x^6\varphi_1(x) \;+\; 3\pi^3x^2\varphi_2(x) \;-\; 45\pi^9, \tag{16}$$

where

$$\varphi_1(x) = 4(\pi-2)(\pi^3-60\pi+120)x^2 - 3\left(\pi^6-100\pi^4+200\pi^3-480(\pi-2)^2\right) \tag{17}$$

and

$$\varphi_2(x) \;=\; (\pi^6-720(\pi-2)^2)x^2 \;+\; 180\pi^3(\pi-2). \tag{18}$$

As $\pi^3-60\pi-120 < 0$ and $\pi^6-720(\pi-2)^2 > 0$, we have the following estimation, for $x \in (0,c]$:

$$A(x) \;\le\; 2(\pi-2)c^6\varphi_1(0) \;+\; 3\pi^3c^2\varphi_2(c) \;-\; 45\pi^9 \;= -138097.851\ldots < 0. \tag{19}$$

In view of all the above, we can conclude that for $x \in (0,\, c]$:

$$C(x) > 0, \quad B(x) > 0 \;\text{ and }\; A(x) < 0. \tag{20}$$

Now we prove that

$$2\,A(x)\,\sin^3 x + B(x)\,\cos x \;>\; 0 \tag{21}$$

for $x \in (0,c]$. Function $2\,A(x)\,\sin^3 x + B(x)\cos x$ is a mixed trigonometric polynomial. To prove the inequality (21) for $x \in (0,c]$, we will use the method from [2] and [4]. In particular, we use the following inequalities from [4]:

$$\cos x > 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}, \qquad \big(x \in (0,c) \subseteq (0,\sqrt{90})\big); \tag{22}$$

and

$$\sin x > x - \frac{x^3}{2!} + \frac{x^5}{5!}, \qquad \big(x \in (0,c) \subseteq (0,\sqrt{72})\big). \tag{23}$$

Therefore, for $x \in (0, \pi/2)$, we have:

$$\begin{aligned} 2A(x)\sin^3 x + B(x)\cos x &> 2A(x)\left(x - \frac{x^3}{2!} + \frac{x^5}{5!}\right)^3 \\ &\quad + B(x)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\right) \\ &= x^9 \cdot P_{14}(x), \end{aligned} \tag{24}$$

where $P_{14}(x)$ is the polynomial of the $14^{th}$ degree, as follows:

$$\begin{aligned} P_{14}(x) &= \sum_{i=0}^{14} a_i x^i \\ &= 8\left(\pi^3 - 60\pi + 120\right)(\pi - 2)^2 x^{14} \\ &-6(\pi - 2)\left(\pi^6 - 20\pi^4 + 40\pi^3 - 5280\pi^2 + 21120\pi - 21120\right) x^{12} \\ &+3\left(\pi^9 + 40(\pi - 2)\left(3\pi^6 - 214\pi^4 + 428\pi^3 - 7680\pi^2 + 30720\pi - 30720\right)\right) x^{10} \\ &-20\left(9\pi^9 + (\pi - 2)\left(441\pi^6 - 44320\pi^4 + 88640\pi^3 - 762240\pi^2 + 3048960\pi - 3048960\right)\right) x^8 \\ &+15\left(309\pi^9 + 400(\pi - 2)\left(17\pi^6 - 2552\pi^4 + 5104\pi^3 - 24576\pi^2 + 98304\pi - 98304\right)\right) x^6 \\ &-300\left(215\pi^9 + 72(\pi - 2)\left(13\pi^6 - 6880\pi^4 + 13760\pi^3 - 34560\pi^2 + 138240\pi - 138240\right)\right) x^4 \\ &+5400\left(91\pi^9 - 80(\pi - 2)\left(13\pi^6 + 1744\pi^4 - 3488\pi^3 + 1920\pi^2 - 7680\pi + 7680\right)\right) x^2 \\ &-36000\pi^3\left(47\pi^6 - 1440(\pi - 2)\left(\pi^3 + 20\pi - 40\right)\right). \end{aligned} \tag{25}$$

Therefore, for inequality (21) it is sufficient to prove that

$$P_{14}(x) > 0, \tag{26}$$

for $x \in (0, c]$. It is easy to check that non-zero coefficients $a_i$, $i \in \{14, 12, 10, 8, 6, 4, 2, 0\}$ of the polynomial $P_{14}(x)$ satisfy the following conditions: $a_{14} < 0$, $a_{12} > 0$, $a_{10} < 0$, $a_8 > 0$, $a_6 < 0$, $a_4 > 0$, $a_2 < 0$ and $a_0 > 0$. Thus, for the proof of $P_{14}(x) = x^{12}\left(a_{14}x^2 + a_{12}\right) + x^8\left(a_{10}x^2 + a_8\right) + x^4\left(a_6x^2 + a_4\right) + \left(a_2x^2 + a_0\right) > 0$, for $x \in (0, c]$, it is sufficient and easy to check that the following inequalities hold: $a_{14}c^2 + a_{12} > 0$, $a_{10}c^2 + a_8 > 0$, $a_6c^2 + a_4 > 0$ and $a_2c^2 + a_0 > 0$. Thus we may conclude that $P_{14}(x) > 0$ for $x \in (0, c]$. Hence,

$$F_1^{'''}(x) > 0 \tag{27}$$

for $x \in (0, c]$. This means that, $F_1^{''}(x)$ is a monotonously increasing function for $x \in (0, c]$. From $\lim\limits_{x \to +0} F_1^{''}(x) = 0$ we have $F_1^{''}(x) > 0$ for $x \in (0, c]$. Thus, $F_1^{'}(x)$ monotonously increases for $x \in (0, c]$. Finally, since $\lim\limits_{x \to +0} F_1^{'}(x) = 0$, we have $F_1'(x) > 0$ for $x \in (0, c]$. As a consequence, $F_1(x)$ is monotonously increasing function for $x \in (0, c]$. Finally, as $\lim\limits_{x \to +0} F_1(x) = 0$ we can conclude that

$$F_1(x) > 0 \quad \text{for} \quad x \in (0, c] \tag{28}$$

which also proves that $F(x) > 0$ for $x \in (0, c]$.

*2.2. Case 2: $x \in (c, \pi/2)$*

In this subsection we prove that $F(x) > 0$ for $x \in (c, \pi/2)$. Let us introduce the substitution $t = \frac{\pi}{2} - x$ in the function $F_1(t)$. ¿From there we obtain the function

$$G(x) = \ln\cos x - \ln\left(\frac{\pi}{2} - x\right) - \omega(x)\ln\left(\frac{2}{\pi} + \frac{\pi - 2}{\pi^3}\left(\pi^2 - 4\left(\frac{\pi}{2} - x\right)^2\right)\right), \quad (29)$$

where $\omega(x) = \theta\big(\frac{\pi}{2} - x\big) = -\frac{(\pi^3 - 24\,\pi + 48)\,(\frac{\pi}{2} - x)^3}{3\,(\pi - 2)\,\pi^3} + \frac{\pi^3}{24\,\pi - 48}$ and $x \in \big(0, \frac{\pi}{2}\big)$. We have to prove the inequality:

$$G(x) > 0 \quad (30)$$

for $x \in (0, c_1)$, where $c_1 = \frac{\pi}{2} - c = 0.228\ldots$. In this aim we define the new function:

$$G_1(x) = \ln\cos x - \ln\left(\frac{\pi}{2} - x\right) - \omega_1(x)\ln\left(\frac{2}{\pi} + \frac{\pi - 2}{\pi^3}\left(\pi^2 - 4\left(\frac{\pi}{2} - x\right)^2\right)\right), \quad (31)$$

where $\omega_1(x) = \frac{x}{5} - 1$ and $x \in \big(0, \frac{\pi}{2}\big)$. Based on the following inequalities

$$\begin{aligned} & \omega(x) > \omega_1(x), \\ \Longleftrightarrow \quad & \frac{x}{(\pi - 2)\,\pi^3}\Big(\big(20\,\pi^3 - 480\,\pi + 960\big)\,x^2 \\ & \qquad + \big(-30\,\pi^4 + 720\,\pi^2 - 1440\,\pi\big)\,x \\ & \qquad + 15\,\pi^5 - 12\,\pi^4 - 336\,\pi^3 + 720\,\pi^2\Big) > 0, \end{aligned} \quad (32)$$

for $x \in (0, c_1)$, we can conclude:

$$G(x) > G_1(x) \quad (33)$$

for $x \in (0, c_1)$. We will now prove $G_1(x) > 0$ for $x \in (0, c_1)$. Let us notice that

$$G_1''(x) = \frac{P(x)\cos^2 x - \sin^2 x\, Q(x)}{Q(x)\cos^2 x} \quad (34)$$

where

$$\begin{aligned} P(x) = {} & \big(-80\,\pi^2 + 320\,\pi - 320\big)\,x^6 \\ & + \big(240\,\pi^3 - 992\,\pi^2 + 1088\,\pi - 128\big)\,x^5 \\ & + \big(-260\,\pi^4 + 1216\,\pi^3 - 1344\,\pi^2 - 576\,\pi + 960\big)\,x^4 \\ & + \big(120\,\pi^5 - 728\,\pi^4 + 720\,\pi^3 + 1472\,\pi^2 - 1920\,\pi\big)\,x^3 \\ & + \big(-20\,\pi^6 + 212\,\pi^5 - 132\,\pi^4 - 1184\,\pi^3 + 1440\,\pi^2\big)\,x^2 \\ & + \big(-24\,\pi^6 - 16\,\pi^5 + 408\,\pi^4 - 480\,\pi^3\big)\,x \\ & + 11\,\pi^6 - 52\,\pi^5 + 60\,\pi^4 \end{aligned} \quad (35)$$

and

$$Q(x) = 5\,(\pi - 2\,x)^2\,\Big((-2\,\pi + 4)\,x^2 + \big(2\,\pi^2 - 4\,\pi\big)\,x + \pi^2\Big)^2. \quad (36)$$

Obviously, $Q(x) > 0$ for $x \in (0, c_1)$. Let us prove that $P(x) > 0$ for $x \in (0, c_1)$. We note that $P(x)$ can be written as

$$P\,(x) = \phi_1\,(x) + 4\,x^3\,(\pi - 2)\,\big((-20\,\pi + 40)\,x^3 + \phi_2\,(x)\big)\,, \tag{37}$$

where

$$\begin{aligned}\phi_1(x) \,=\, & \big(-20\,\pi^6 + 212\,\pi^5 - 132\,\pi^4 - 1184\,\pi^3 + 1440\,\pi^2\big)\,x^2 \\ & + \big(-24\,\pi^6 - 16\,\pi^5 + 408\,\pi^4 - 480\,\pi^3\big)\,x \\ & + \big(11\,\pi^6 - 52\,\pi^5 + 60\,\pi^4\big)\end{aligned} \tag{38}$$

and

$$\begin{aligned}\phi_2(x) \,=\, & \big(60\,\pi^2 - 128\,\pi + 16\big)\,x^2 \\ & + \big(-65\,\pi^3 + 174\,\pi^2 + 12\,\pi - 120\big)\,x \\ & + \big(30\,\pi^4 - 122\,\pi^3 - 64\,\pi^2 + 240\,\pi\big)\end{aligned} \tag{39}$$

are quadratic trinomials. Let us denote by $y_1$ the minimum of the trinomial $\phi_1(x)$ and by $y_2$ the minimum of the trinomial $\phi_2(x)$ over $[0, c_1]$ respectively. It then becomes possible to verify that $y_1 > 271$ and $y_2 = \phi_2(0) > -739$. Thus, the following inequalities are true:

$$\begin{aligned}P(x) \;&\geq\; y_1 + 4x^3\,(\pi - 2)\,\big((-20\pi + 40)\,x^3 + y_2\big) \\ &>\; 271 + 4x^3\,(\pi - 2)\,\big((-20\pi + 40)\,x^3 - 739\big) \\ &>\; 271 + 4\left(\tfrac{23}{100}\right)^3(\pi - 2)\left((-20\pi + 40)\left(\tfrac{23}{100}\right)^3 - 739\right) \\ &>\; 229 \;>\; 0,\end{aligned} \tag{40}$$

for $x \in \big(0, \frac{23}{100}\big)$. Therfore $P(x) > 0$ for $x \in (0, c_1) \subset \big(0, \frac{23}{100}\big)$.

Now we prove that:

$$\Phi(x) = P(x)\cos^2\!x - Q(x)\sin^2\!x > 0 \tag{41}$$

for $x \in (0, c_1)$. Let us note that $\Phi(x)$ is a mixed trigonometric polynomial, and that the proof of previous inequality will be proved applying the methods from [2] and [4]. We use the following inequalities from [4]:

$$\cos x > 1 - \frac{x^2}{2}, \qquad \big(x \in (0, c_1) \subseteq (0, \sqrt{30})\big), \tag{42}$$

and

$$\sin x < x, \qquad \big(x \in (0, c_1) \subseteq (0, \sqrt{20})\big). \tag{43}$$

Therefore:

$$\Phi(x) > T_{10}(x) = P(x)\left(1 - \frac{x^2}{2}\right)^2 - Q(x)\,x^2 \tag{44}$$

for $x \in (0, c_1)$ and it is enough to prove

$$T_{10}(x) > 0, \tag{45}$$

for $x \in (0, c_1)$. For the polynomial

$$
\begin{aligned}
T_{10}(x) = {} & \left(-20\,\pi^2+80\,\pi-80\right) x^{10} \\
& + \left(60\,\pi^3-248\,\pi^2+272\,\pi-32\right) x^9 \\
& + \left(-65\,\pi^4+304\,\pi^3-336\,\pi^2-144\,\pi + 240\right) x^8 \\
& + \left(30\,\pi^5-182\,\pi^4+180\,\pi^3+400\,\pi^2 - 608\,\pi + 128\right) x^7 \\
& + \left(-5\,\pi^6+53\,\pi^5-33\,\pi^4-392\,\pi^3 + 424\,\pi^2+896\,\pi-1280\right) x^6 \\
& + \left(-6\,\pi^6-4\,\pi^5+190\,\pi^4+200\,\pi^3 - 2464\,\pi^2+3008\,\pi-128\right) x^5 \\
& + \left(960 + 2400\,\pi^3-2784\,\pi^2-576\,\pi-413\,\pi^4+11/4\,\pi^6-45\,\pi^5\right) x^4 \\
& + \left(4\,\pi^6 + 196\,\pi^5 - 1136\,\pi^4 + 1200\,\pi^3 + 1472\,\pi^2 - 1920\,\pi\right) x^3 \\
& + \left(-36\,\pi^6 + 264\,\pi^5 - 192\,\pi^4 - 1184\,\pi^3 + 1440\,\pi^2\right) x^2 \\
& + \left(-24\,\pi^6 - 16\,\pi^5 + 408\,\pi^4 - 480\,\pi^3\right) x \\
& + 11\,\pi^6 - 52\,\pi^5 + 60\,\pi^4
\end{aligned} \tag{46}
$$

we form the polynomials:

$$
\begin{aligned}
\psi_1(x) &= \Big((-20\,\pi^2+80\,\pi-80)x + (60\,\pi^3-248\,\pi^2+272\,\pi-32)\Big)x^9, \\
\psi_2(x) &= \Big((-65\,\pi^4+304\,\pi^3-336\,\pi^2-144\,\pi + 240)x^2 \\
&\quad +(30\,\pi^5-182\,\pi^4+180\,\pi^3+400\,\pi^2 - 608\,\pi + 128)x \\
&\quad +(-5\,\pi^6+53\,\pi^5-33\,\pi^4-392\,\pi^3 + 424\,\pi^2+896\,\pi-1280)\Big)x^6, \\
\psi_3(x) &= \Big((-6\,\pi^6-4\,\pi^5+190\,\pi^4+200\,\pi^3 - 2464\,\pi^2+3008\,\pi-128)x^2 \\
&\quad +(960 + 2400\,\pi^3-2784\,\pi^2-576\,\pi-413\,\pi^4+\tfrac{11}{4}\,\pi^6-45\,\pi^5)x, \\
&\quad +(4\,\pi^6 + 196\,\pi^5 - 1136\,\pi^4 + 1200\,\pi^3 + 1472\,\pi^2 - 1920\,\pi)\Big)x^3 \\
\psi_4(x) &= \Big((-36\,\pi^6 + 264\,\pi^5 - 192\,\pi^4 - 1184\,\pi^3 + 1440\,\pi^2)x^2 \\
&\quad +(-24\,\pi^6 - 16\,\pi^5 + 408\,\pi^4 - 480\,\pi^3)x \\
&\quad +(11\,\pi^6 - 52\,\pi^5 + 60\,\pi^4)\Big).
\end{aligned} \tag{47}
$$

It is easy to check

$$
\begin{aligned}
&\psi_1(x) > 0, \\
&\psi_2(x) > 0, \\
&\psi_3(x) > \Big((-6\,\pi^6-4\,\pi^5+190\,\pi^4+200\,\pi^3 - 2464\,\pi^2+3008\,\pi-128)\big(\tfrac{23}{100}\big)^2 \\
&\quad +(960+2400\,\pi^3-2784\,\pi^2-576\,\pi-413\,\pi^4+\tfrac{11}{4}\,\pi^6-45\,\pi^5)\big(\tfrac{23}{100}\big) \\
&\quad +(4\,\pi^6+196\,\pi^5-1136\,\pi^4+1200\,\pi^3+1472\,\pi^2 - 1920\,\pi)\Big)\big(\tfrac{23}{100}\big)^3 \\
&\quad > -27, \\
&\psi_4(x) > \Big((-36\,\pi^6 + 264\,\pi^5 - 192\,\pi^4 - 1184\,\pi^3 + 1440\,\pi^2)\big(\tfrac{23}{100}\big)^2 \\
&\quad +(-24\,\pi^6 - 16\,\pi^5 + 408\,\pi^4 - 480\,\pi^3)\big(\tfrac{23}{100}\big) \\
&\quad +(11\,\pi^6 - 52\,\pi^5 + 60\,\pi^4)\Big)\big(\tfrac{23}{100}\big)^3 \\
&\quad > 54,
\end{aligned} \tag{48}
$$

for $x \in (0, c_1) \subset \left(0, \frac{23}{100}\right)$. Therefore

$$T_{10}(x) = \psi_1(x) + \psi_2(x) + \psi_3(x) + \psi_4(x) > 27 > 0. \tag{49}$$

Thus the inequality

$$G_1''(x) > 0, \tag{50}$$

was proved for $x \in (0, c_1)$. This means that $G_1'(x)$ is a monotonously increasing function for $x \in (0, c_1)$. Therefore, based on

$$\begin{aligned} G_1'(0) &= \tfrac{1}{5}\ln\left(\tfrac{\pi}{2}\right) - \tfrac{2\pi-6}{\pi} = \tfrac{1}{5}\ln\left(1+\left(\tfrac{\pi}{2}-1\right)\right) - \tfrac{2\pi-6}{\pi} \\ &> \tfrac{1}{5}\sum_{k=1}^{4}\left(\frac{\left(\frac{\pi}{2}-1\right)^{2k-1}}{2k-1} - \frac{\left(\frac{\pi}{2}-1\right)^{2k}}{2k}\right) - \frac{2\pi-6}{\pi} \\ &= -\tfrac{\pi^8}{10240} + \tfrac{\pi^7}{560} - \tfrac{7\pi^6}{480} + \tfrac{7\pi^5}{100} - \tfrac{7\pi^4}{32} + \tfrac{7\pi^3}{15} \\ &\quad -\tfrac{7\pi^2}{10} + \tfrac{4\pi}{5} - \tfrac{3561}{1400} + \tfrac{6}{\pi} \\ &> 0, \end{aligned} \tag{51}$$

it follows that $G_1'(x) > 0$ for $x \in (0, c_1)$. Consequently, $G_1(x)$ is a monotonously increasing function for $x \in (0, c_1)$. Based on

$$G_1(0) = 0 \tag{52}$$

it follows that $G_1(x) > 0$ for $x \in (0, c_1)$. This also proves that $G(x) > 0$ for $x \in (0, c_1)$, which in turn proves that $F(x) > 0$ for $x \in \left(c_1, \frac{\pi}{2}\right)$.
Therefore, we can conclude that $F(x) > 0$ for any $x \in \left(0, \frac{\pi}{2}\right)$. The proof of Nishizava's open problem is completed.

## 3. Conclusions and future work

This paper proved an open problem stated by Nishizawa in [1], applying computation method from [2] and [4]. Let us remark that proofs of polynomial inequalities (15), (26), (40) and (45) can be based on reducing (by differentiation) of the corresponding polynomials to polynomials of a degree up to four (as illustrated in papers [3]-[6]), which allows symbolic radical representation of roots. The formulation of a systematic method for proving similar exponential inequalities, as well as refinements of some already proven inequalities with power exponential functions, will be the focus of further work.

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