TOPOLOGICAL EQUIVALENCE OF HOLOMORPHIC FOLIATION GERMS OF RANK 1 WITH ISOLATED SINGULARITY IN THE POINCARÉ DOMAIN

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Abstract. We show that the topological equivalence class of holomorphic foliation germs with an isolated singularity of Poincaré type is determined by the topological equivalence class of the real intersection foliation of the (suitably normalized) foliation germ with a sphere centered in the singularity. We use this Reconstruction Theorem to completely classify topological equivalence classes of plane holomorphic foliation germs of Poincaré type and discuss a conjecture on the classification in dimension ≥ 3.

0. Introduction

As for isolated singularities of analytic set germs (see [BK86] in the case of plane curve germs) a standard technique to study the topology of holomorphic foliation germs with isolated singularity looks at the intersection of their integral manifolds with spheres centered in the origin (see [LS11] for a more general Morse-theoretic approach). The technique was particularly successful when analyzing holomorphic foliation germs represented by vector fields, that is, foliation germs with 1-dimensional leaves: Guckenheimer [Guc72] and Camacho, Kuiper and Palis [CKP78] (who use polycylinders instead of spheres) classify foliation germs represented by generic linearizable vector fields, whereas Camacho and Sad [CS82] treat resonant cases of plane foliation germs represented by holomorphic vector fields of Siegel type.

In this paper we first prove a reconstruction theorem for holomorphic foliation germs represented by a vector field of Poincaré type, that is, the linear part of the vector field has eigenvalues whose convex hull in $\mathbb{C}$ does not contain $0 \in \mathbb{C}$: The topological equivalence class of such a holomorphic foliation germ is uniquely determined by the real-analytic foliation obtained on a sphere around the singularity when intersecting it with all the leaves of a holomorphically equivalent, normalized foliation germ. For more details see Thm. 2.4 and the preceding discussion in sections 1 and 2.

A similar reconstruction theorem for holomorphic foliation germs represented by vector fields of Siegel type (that is, not of Poincaré type and the linear part has only non-zero eigenvalues) seems possible. The main obstacles to prove such a theorem are missing normal forms and the fact that leaves of such foliation germs may not intersect spheres around the singularity transversally, but tangentially. However, in sufficiently normal situations the intersection of leaves and sphere still combine to

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a real-analytic foliation on the sphere, and the tangential locus is the polar variety of Limón and Seade [LS11] with useful properties.
In sections 3 to 6 we use the Reconstruction Theorem 2.4 to completely classify topological equivalence classes of plane holomorphic foliation germs represented by vector fields of Poincaré type, thus extending Guckenheimer's Stability Theorem [Guc72] in the 2-dimensional case.
Recently, Marín and Mattei presented a classification of topological equivalence classes of plane holomorphic foliation germs satisfying weak genericity assumptions [MM12], by exhibiting an invariant based on the reduction of plane holomorphic foliation singularities and the holonomy around irreducible exceptional components of the reduction. However this classification does not cover the resonant cases discussed in Section 4 because these are not of generic general type, in the terminology of [MM12] (see Rem. 4.5). In any case, Marín and Mattei give no explicit lists of topological equivalent foliation germs. Furthermore, our approach generalizes to higher dimensions, at least in the Poincaré case, as Guckenheimer's proof of stability of generic linearizable vector fields [Guc72] shows. In Section 7 we speculate how to extend the 2-dimensional picture and Guckenheimer's Stability to higher-dimensional foliation germs represented by vector fields of Poincaré type.

1. Preliminaries on holomorphic foliation germs of rank 1

Definition 1.1. *A germ of a holomorphic foliation of rank* 1 *in* $\mathbb{C}^n$ *with an isolated singularity in* $0 \in \mathbb{C}^n$ *is an equivalence class of pairs* $[U, \theta]$ *where* $U \subset \mathbb{C}^n$ *is an open neighborhood of* 0 *with holomorphic coordinates* $z_1, \ldots, z_n$ *and*

$$\theta = f_1 \frac{\partial}{\partial z_1} + \ldots + f_n \frac{\partial}{\partial z_n}$$

is a holomorphic vector field such that $f_1, \ldots, f_n \in \mathcal{O}(U)$ *vanish simultaneously only in* 0.
Two such pairs $[U, \theta]$ *and* $[U', \theta']$ *are equivalent if there exists an open neighborhood* $V \subset U \cap U'$ *of* $0 \in \mathbb{C}^n$ *and a function* $h \in \mathcal{O}^*(V)$ *such that*

$$h \cdot \theta_{|V} = \theta'_{|V}.$$

We denote such holomorphic foliation germs by $\mathcal{F}$.

Proposition 1.2. *Let* $[U, \theta]$ *represent a holomorphic foliation germ* $\mathcal{F}$ *of rank 1 in* $\mathbb{C}^n$ *with an isolated singularity in* $0 \in \mathbb{C}^n$. *Then for all* $p \in U - \{0\}$ *there exists an open neighborhood* $V \subset U$ *of* p *and holomorphic coordinates* $w_1, \ldots, w_n$ *on* V *centred in* p *such that*

$$\theta(w_1) = \ldots = \theta(w_{n-1}) = 0.$$

Proof. This is an immediate consequence of the holomorphic version of Frobenius' theorem on integrability of involutive subbundles of the tangent bundle, see [War83]. □

The foliation charts described in this proposition can be used to define an equivalence relation on $U - \{0\}$: Two points $p, q \in U - \{0\}$ are equivalent if there exists a sequence of points $p = p_0, p_1, \ldots, p_K = q$ and neighborhoods $V_i \subset U - \{0\}$ of p_i with coordinates $w_1^{(i)}, \ldots, w_n^{(i)}$ as in Prop. 1.2 such that the curves $\{w_1^{(i)} = \ldots = w_{n-1}^{(i)} = 0\}$ and $\{w_1^{(i+1)} = \ldots = w_{n-1}^{(i+1)} = 0\}$, $i = 0, \ldots, K-1$, intersect.

Proposition 1.3. *The equivalence classes of this equivalence relation on $U - \{0\}$ have the structure of a holomorphic curve.*

Proof. If V resp. V' and $w_1, \ldots, w_n$ resp. $w'_1, \ldots, w'_n$ are two different open neighborhoods around p with holomorphic coordinates centred in p we have the following commutative diagram:

$$\begin{array}{ccccccccc} \mathbb{C}^n & \hookleftarrow & W & \xleftarrow{\cong} & V \cap V' & \xrightarrow{\cong} & W' & \hookrightarrow & \mathbb{C}^n \\ & & \downarrow & \swarrow_{\pi} & & \searrow^{\pi'} & \downarrow & & \\ & & \mathbb{C}^{n-1} & & & & \mathbb{C}^{n-1} & & \end{array}$$

Here, the horizontal biholomorphisms are given by the coordinates, and the vertical projections project onto the first $n-1$ coordinates. Then two points in $V \cap V'$ are mapped to the same point in $\mathbb{C}^{n-1}$ by π if, and only if the two points are mapped to the same point in $\mathbb{C}^{n-1}$ by π', by Frobenius' Theorem. Consequently, the equivalence class of $p \in U - \{0\}$ coincides with the holomorphic curve $w_1 = \ldots = w_{n-1} = 0$ in a small enough neighborhood around p. □

Definition 1.4. *Let $[U, \theta]$ represent a holomorphic foliation germ $\mathcal{F}$ of rank 1 with an isolated singularity in $0 \in \mathbb{C}^n$. Then the holomorphic curves given by the equivalence classes on $U - \{0\}$ as in Prop. 1.3 are called the leaves of $\mathcal{F}$ in U.*

We are ready to define the topological equivalence relation on holomorphic foliation germs that we want to consider.

Definition 1.5. *Two holomorphic foliation germs $\mathcal{F}$, $\mathcal{F}'$ of rank 1 with an isolated singularity in $0 \in \mathbb{C}^n$ and represented by $[U, \theta]$, $[U', \theta']$ are called topologically equivalent if there exists a homeomorphism $\phi : V \to V'$ of open neighborhoods $V \subset U$, $V' \subset U'$ of 0 such that $\phi(0) = 0$ and the leaves of $\mathcal{F}$ in V are mapped onto the leaves of $\mathcal{F}'$ in V' by ϕ.*

If ϕ is biholomorphic we say that $\mathcal{F}$ and $\mathcal{F}'$ are *holomorphically equivalent.*
We will focus on a special type of holomorphic foliation germs:

Definition 1.6. *A holomorphic foliation germ of rank 1 with an isolated singularity in $0 \in \mathbb{C}^n$ represented by $[U, \theta]$ is said to be of Poincaré type if the eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ of the linear part*

$$A = \left(\frac{\partial f_j}{\partial z_i}(0) \right)_{i,j=1,\ldots,n}$$

of $\theta = \sum_{j=1}^n f_j \frac{\partial}{\partial z_j}$ generate a convex hull not containing $0 \in \mathbb{C}$. Then the tuple of eigenvalues $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ is said to be in the Poincaré domain.

The classical theorems of Poincaré and Poincaré-Dulac (see [Arn83, §24.D and E]) state that all holomorphic foliation germs of rank 1 with an isolated singularity in $0 \in \mathbb{C}^n$ of Poincaré type are even holomorphically equivalent to such germs $\mathcal{F}$ of Poincaré type represented by an open subset $U \subset \mathbb{C}^n$ containing 0 and a holomorphic vector field $\theta = \sum_{i=1}^n f_i(z) \frac{\partial}{\partial z_i}$ for which the following hold:

(i) In U, the $f_i(z)$ can be developed into powers series in the variables $z_1, \ldots, z_n$.
(ii) The linear part $A = \left(\frac{\partial f_j}{\partial z_i}(0) \right)_{i,j=1,\ldots,n}$ of θ is in Jordan normal form.

(iii) If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A appearing with their algebraic multiplicity, then the non-vanishing monomials $z_1^{m_1} \cdots z_n^{m_n}$ in $f_i(z)$, with $m_i \in \mathbb{N}_0$, satisfy
$$\lambda_i = \sum_{j=1}^{n} m_j \lambda_j.$$

Note that condition (ii) implies that the linear term in $f_i(z)$ has the form $\lambda_i z_i$ or $\lambda_i z_i + z_{i+1}$. In the latter case $\lambda_i = \lambda_{i+1}$ by the properties of the Jordan normal form, hence all the non-vanishing monomials in these linear terms satisfy condition (iii). For $m = (m_1, \dots, m_n) \in \mathbb{N}_0^n$ a relation $\lambda_i = \sum_{j=1}^n m_j \lambda_j$ is called a *resonance* of the eigenvalues $\lambda_1, \dots, \lambda_n$, and the monomial $z^m := z_1^{m_1} \cdots z_n^{m_n}$ is called a *resonant monomial* if it appears in $f_i(z)$.

Remark 1.7. If $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ is in the Poincaré domain then there are only finitely many resonances $\lambda_i = \sum_{j=1}^n m_j \lambda_j$. Furthermore, a resonance relation with λ_i on the left hand side is either the *trivial resonance relation* $\lambda_i = \lambda_i$, or λ_i does not appear on the right hand side at all, that is $m_i = 0$. For proofs, see [Arn83, §24.B]. Note finally that we do not require $\sum_i m_i \geq 2$ as in [Arn83] but only distinguish between the trivial resonant monomial z_i in $f_i(z)$ and non-trivial resonant monomials.

Remark 1.8. Let $\mathcal{F}$ be a holomorphic foliation germ satisfying (i), (ii) and (iii). For the tuple $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ in the Poincaré domain there exists a maximal real constant $c > 0$ such that
$$|\sum_{i=1}^{n} \lambda_i t_i| \geq c \cdot \sum_{i=1}^{n} t_i,$$
for all real numbers $t_1, \dots, t_n \geq 0$. The number c can be interpreted as the distance of the convex hull of $\lambda_1, \dots, \lambda_n$ in $\mathbb{C}$ from 0. By separately rescaling the coordinates we can achieve that the entries of the matrix A on the superdiagonal are arbitrarily small. If the entries are $\frac{c}{2n}$ we will call $\mathcal{F}$ *normalized* (see the next section).

Remark 1.9. If $n = 2$ then every normalized holomorphic foliation germ of rank 1 with an isolated singularity in $0 \in \mathbb{C}^n$ of Poincaré type is represented by a vector field of one of the following types:

(1) $\theta = \lambda z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}$, where $\lambda \in \mathbb{C} - \mathbb{R}$,
(2) $\theta = \lambda z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}$, where $\lambda \in \mathbb{R}_{>0}$,
(3) $\theta = (m z_1 + z_2^m) \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}$, where $m \geq 2$, or
(4) $\theta = (z_1 + \frac{1}{4} z_2) \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}$ because the constant c of Rem. 1.8 is 1 in this case.

2. The intersection foliation

In this section $\mathcal{F}$ is always a normalized holomorphic foliation germ of rank 1 with an isolated singularity in $0 \in \mathbb{C}^n$ of Poincaré type, and $[U, \theta]$ represents $\mathcal{F}$, with $\theta = \sum_{i=1}^n f_i(z) \frac{\partial}{\partial z_i}$. Furthermore, $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the linear part of θ appearing with their algebraic multiplicity, and $c > 0$ is the real constant for the tuple $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ in the Poincaré domain introduced in Rem. 1.8.
S_ϵ^{2n-1} denotes the (real) $2n-1$-dimensional sphere in $\mathbb{C}^n$ centered in $0 \in \mathbb{C}^n$ with radius ϵ, and B_ϵ^{2n} denotes the (real) $2n$-dimensional ball in $\mathbb{C}^n$ centered in $0 \in \mathbb{C}^n$ with radius ϵ.

Proposition 2.1. *In each point $p \in S_\epsilon^{2n-1}$, $0 < \epsilon \ll 1$, the leaf of $\mathcal{F}$ through p intersects the sphere S_ϵ^{2n-1} transversally.*

Proof. The holomorphic tangent vector $\theta(p)$ given by $\mathcal{F}$ in a point $p \in S_\epsilon^{2n-1}$ is $\sum_{i=1}^n f_i(p) \frac{\partial}{\partial z_i}$. The leaf of $\mathcal{F}$ through $p = (p_1, \ldots, p_n)$ does not intersect S_ϵ^{2n-1} transversally in p if, and only if $\theta(p)$ is tangent to S_ϵ^{2n-1} in p if, and only if

$$\left(\sum_{i=1}^n \overline{p_i} dz_i + \sum_{i=1}^n p_i d\overline{z_i} \right) \left(\sum_{i=1}^n f_i(p) \frac{\partial}{\partial z_i} \right) = \sum_{i=1}^n \overline{p_i} \cdot f_i(p) = 0,$$

where $\sum_{i=1}^n (\overline{p_i} dz_i + p_i d\overline{z_i})$ is the real differential in p of the equation $\sum_{i=1}^n z_i \overline{z_i} = \epsilon^2$ defining S_ϵ^{2n-1}.
Since $\mathcal{F}$ is normalized $f_i(z) = \lambda_i z_i + g_i(z)$ or $f_i(z) = \lambda_i z_i + \frac{c}{2n} z_{i+1} + g_i(z)$, where $g_i(z)$ is a power series convergent in U of order ≥ 2 (actually, a polynomial by Rem. 1.7). Let $J \subset \{1, \ldots, n\}$ be the subset of indices j such that $f_j(z) = \lambda_j z_j + \frac{c}{2n} z_{j+1} + g_j(z)$. Then we have

$$\left| \sum_{j \in J} \overline{p_j} \frac{c}{2n} p_{j+1} \right| \leq |J| \cdot \frac{c}{2n} \cdot \epsilon^2 \leq \frac{c}{2} \cdot \epsilon^2.$$

Furthermore, since $\sum_{i=1}^n \overline{p_i} g_i(p)$ is a power series in $\overline{p}_i, p_i$, $i = 1, \ldots, n$, convergent in U and of order ≥ 3 there exists a real constant $C > 0$ such that

$$\sup_{p \in S_\epsilon^{2n-1}} |\sum_{i=1}^n \overline{p_i} g_i(p)| \leq C \cdot \epsilon^3.$$

These estimates imply that

$$\begin{aligned} \left| \sum_{i=1}^n \overline{p_i} \cdot f_i(p) \right| &\geq \left| \left| \sum_{i=1}^n \lambda_i p_i \overline{p_i} \right| - \left| \sum_{j \in J} \overline{p_j} \frac{c}{2n} p_{j+1} \right| - \left| \sum_{i=1}^n \overline{p_i} g_i(p) \right| \right| \geq \\ &\geq \left| c \cdot \epsilon^2 - \frac{c}{2} \cdot \epsilon^2 - C \cdot \epsilon^3 \right| > 0, \end{aligned}$$

for $0 < \epsilon \ll 1$. □

Remark 2.2. Since in the Euclidean metric the form $\sum_{i=1}^n (\overline{p_i} dz_i + p_i d\overline{z_i})$ in a point $p \in S_\epsilon^{2n-1}$ has length 2ϵ,

$$\left| \left(\sum_{i=1}^n \overline{p_i} dz_i + \sum_{i=1}^n p_i d\overline{z_i} \right) \left(\sum_{i=1}^n f_i(p) \frac{\partial}{\partial z_i} \right) \right| = \left| \sum_{i=1}^n \overline{p_i} \cdot f_i(p) \right|$$

is 2ϵ times the length of the projection of $\theta(p)$ to the normal direction of S_ϵ^{2n-1} in p. In particular, the length of this projection is $\geq \frac{1}{2\epsilon} \cdot \frac{c}{4} \cdot \epsilon^2 = \frac{c}{8} \cdot \epsilon$ for $0 < \epsilon \ll 1$, by the calculations in the proof above.

From now on, let ϵ be small enough such that the conclusion of Prop. 2.1 and the estimate in Rem. 2.2 hold for $\mathcal{F}$.
Then in each point $p \in S_\epsilon^{2n-1}$ the (real) tangent spaces of the leaf of $\mathcal{F}$ through p and of S_ϵ^{2n-1} intersect in a (real) 1-dimensional subspace. This yields a 1-dimensional distribution on the real $\mathcal{C}^\infty$-manifold S_ϵ^{2n-1} denoted by $\mathcal{F} \cap S_\epsilon^{2n-1}$. This distribution is integrable because it is 1-dimensional, see [War83]. Therefore we obtain a real foliation on S_ϵ^{2n-1} with 1-dimensional leaves , also denoted by $\mathcal{F} \cap S_\epsilon^{2n-1}$ and called the *real intersection foliation* of $\mathcal{F}$.

Its leaves can be canonically oriented: In each point $p \in S^{2n-1}_\epsilon$ choose those vectors in the tangent subspace given by the distribution $\mathcal{F} \cap S^{2n-1}_\epsilon$ in p which together with the tangent vectors of the leaf of $\mathcal{F}$ through p pointing away from $0 \in \mathbb{C}^n$ represent the positive orientation of the complex structure on the leaf. Taking in each point $p \in S^{2n-1}_\epsilon$ a unit vector oriented in that way yields a nowhere vanishing vector field on S^{2n-1}_ϵ whose flow, denoted by $\Phi_{\mathcal{F}}$, has integral curves coinciding with the leaves of $\mathcal{F} \cap S^{2n-1}_\epsilon$.

Definition 2.3. *Two real 1-dimensional foliations $\mathcal{F}, \mathcal{G}$ on the sphere S^{2n-1} are called topologically equivalent if there exists a homeomorphism $\phi : S^{2n-1} \to S^{2n-1}$ mapping the leaves of $\mathcal{F}$ onto the leaves of $\mathcal{G}$.*

Theorem 2.4 (Reconstruction Theorem)**.** *Two normalized holomorphic foliation germs $\mathcal{F}, \mathcal{G}$ with an isolated singularity in $0 \in \mathbb{C}^n$ of Poincaré type are topologically equivalent if, and only if the real intersection foliations $\mathcal{F} \cap S^{2n-1}_\epsilon$ and $\mathcal{G} \cap S^{2n-1}_\epsilon$, $0 < \epsilon \ll 1$, are topologically equivalent.*

Proof. We first show that two normalised germs $\mathcal{F}, \mathcal{G}$ are topologically equivalent if their associated real intersection foliations $\mathcal{F} \cap S^{2n-1}_\epsilon, \mathcal{G} \cap S^{2n-1}_\epsilon$ are topologically equivalent. To this purpose we note that the tangent vectors to leaves of $\mathcal{F}$ orthogonal to the tangent space of $\mathcal{F} \cap S^{2n-1}_\epsilon$ in a point $p \in S^{2n-1}_\epsilon$ pointing towards $0 \in \mathbb{C}^n$ and projecting to a vector of length ϵ in normal direction to S^{2n-1}_ϵ form a real $\mathcal{C}^\infty$ vector field v on the pointed ball $B^{2n}_{\epsilon_0} - \{0\}$, for some fixed $0 < \epsilon_0 \ll 1$. The vector field v is continuously extended to 0 by setting $v(0) = 0$ because by Rem. 2.2 the tangent vector $\theta(p)$ describing $\mathcal{F}$ in a point $p \in S^{2n-1}_\epsilon$ projects to a vector of length $\geq \frac{c}{8} \cdot \epsilon$ in normal direction to S^{2n-1}_ϵ, and the lengths of the tangent vectors $\theta(p)$ uniformly tend to 0 when $\epsilon \to 0$.
Let $\Phi : B^{2n}_{\epsilon_0} \times [0, \infty) \to B^{2n}_{\epsilon_0}$ be the associated flow. By construction, Φ_t maps S^{2n-1}_ϵ homeomorphically onto $S^{2n-1}_{\epsilon e^{-t}}$ and can be continuously extended to $0 \in \mathbb{C}^n$, and Φ_t is a topological equivalence of the real intersection foliations $\mathcal{F} \cap S^{2n-1}_\epsilon$ and $\mathcal{F} \cap S^{2n-1}_{\epsilon e^{-t}}$. This shows in particular that the topological equivalence class of $\mathcal{F} \cap S^{2n-1}_\epsilon$ does not depend on ϵ if ϵ is small enough.
Similarly, the homeomorphisms Ψ_t of the flow

$$\Psi : B^{2n}_{\epsilon_0} \times [0, \infty) \to B^{2n}_{\epsilon_0},\ x \mapsto xe^{-t}$$

associated to the vector field $\sum_{i=1}^n z_i \frac{\partial}{\partial z_i}$ map S^{2n-1}_ϵ homeomorphically onto $S^{2n-1}_{\epsilon e^{-t}}$.
We construct the *foliation cone* of $\mathcal{F} \cap S^{2n-1}_{\epsilon_0}$ as the (not necessarily holomorphic) foliation on $B^{2n}_{\epsilon_0}$ with real 2-dimensional leaves on $B^{2n}_{\epsilon_0} \setminus \{0\}$ given by

$$\Psi(\{\text{leaf of } \mathcal{F} \cap S^{2n-1}_{\epsilon_0}\} \times [0, \infty))$$

and the 0-dimensional leaf $0 \in \mathbb{C}^n$.
Since the foliation cones are constructed from the flow Ψ two topologically equivalent real intersection foliations have topologically equivalent foliation cones, that is, there exists a homeomorphism of $B^{2n}_{\epsilon_0}$ mapping the leaves of one foliation cone to the leaves of the other foliation cone, and in particular fixing $0 \in \mathbb{C}^n$.
On the other hand, the foliation $\mathcal{F}$ on $B^{2n}_{\epsilon_0}$ is topologically equivalent to the foliation cone of $\mathcal{F} \cap S^{2n-1}_{\epsilon_0}$, by the homeomorphism

$$H : B^{2n}_{\epsilon_0} \to B^{2n}_{\epsilon_0},\ q \mapsto \Psi_{t_q}(\Phi^{-1}_{t_q}(q)),\quad q \neq 0,\ \text{and } H(0) = 0,$$

where $t_q = \ln \epsilon_0 - \ln \parallel q \parallel = \ln \frac{\epsilon_0}{\parallel q \parallel}$, implying $\epsilon_0 e^{-t_q} = \parallel q \parallel$.

This finishes the proof of one direction of the theorem.
For the other direction let H be a topological equivalence between the holomorphic foliation germs $\mathcal{F}$ and $\mathcal{G}$, as defined in Def. 1.5. As shown above, $\mathcal{F}$ resp. $\mathcal{G}$ is topologically equivalent to the foliation cone over $\mathcal{F} \cap S_\epsilon^{2n-1}$ resp. $\mathcal{G} \cap S_\epsilon^{2n-1}$, for $\epsilon > 0$ small enough. Thus H defines a topological equivalence H_C between these foliation cones.
By possibly decreasing ϵ to ϵ' we will obtain an embedding $H_C : B_{\epsilon'}^{2n} \hookrightarrow B_\epsilon^{2n}$ such that $H_C(0) = 0$ and leaves of the foliation cone over $\mathcal{F} \cap S_{\epsilon'}^{2n-1}$ are mapped into leaves of the foliation cone over $\mathcal{G} \cap S_\epsilon^{2n-1}$. Composing H_C with the radial projection $r : B_\epsilon^{2n} - \{0\} \to S_\epsilon^{2n-1}$ produces a continuous map

$$h : S_{\epsilon'}^{2n-1} \overset{H_C}{\hookrightarrow} B_\epsilon^{2n} - \{0\} \overset{r}{\to} S_\epsilon^{2n-1}$$

Since H_C may map $S_{\epsilon'}^{2n-1}$ to a topological manifold in B_ϵ^{2n} intersecting the same radial line more than once, h may not be injective, and hence not the wanted homeomorphism. But from h we will be able to construct a homeomorphism $g : S_{\epsilon'}^{2n-1} \to S_\epsilon^{2n-1}$ defining a topological equivalence of $\mathcal{F} \cap S_{\epsilon'}^{2n-1}$ with $\mathcal{G} \cap S_\epsilon^{2n-1}$. Since we have already shown that the topological equivalence class of real intersection foliations do not depend on small enough ϵ this shows the theorem.
Let $L_{\mathcal{F},x}$ denote the leaf of $\mathcal{F} \cap S_{\epsilon'}^{2n-1}$ through $x \in S_{\epsilon'}^{2n-1}$, and $L_{\mathcal{G},y}$ the leaf of $\mathcal{G} \cap S_\epsilon^{2n-1}$ through $y \in S_\epsilon^{2n-1}$. Let $C_{\mathcal{F},x}$ denote the radial cone in $B_{\epsilon'}^{2n}$ with vertex in 0 over the leaf $L_{\mathcal{F},x} \subset S_{\epsilon'}^{2n-1}$, and similarly $C_{\mathcal{G},y}$ the radial cone in B_ϵ^{2n} with vertex in 0 over the leaf $L_{\mathcal{G},y} \subset S_\epsilon^{2n-1}$.

Claim 1. h is a surjective map, and $h(L_{\mathcal{F},x}) = L_{\mathcal{G},h(x)}$ for each $x \in S_{\epsilon'}^{2n-1}$.

Proof. There exists $\epsilon'' \ll \epsilon$ such that $B_{\epsilon''}^{2n} \subset H_C(B_{\epsilon'}^{2n})$, as H_C is an embedding fixing 0. Consequently, for every $y \in S_\epsilon^{2n-1}$, the segment $[y, 0] \subset B_\epsilon^{2n}$ intersects $H_C(B_{\epsilon'}^{2n})$ in a point $H_C(x)$, with $x \in S_{\epsilon'}^{2n-1}$. Hence $h(x) = y$, and the surjectivity of h is shown.
The equality $L_{\mathcal{G},h(x)} = h(L_{\mathcal{F},x})$ follows from the fact that by definition, the topological equivalence H_C maps $C_{\mathcal{F},x}$ bijectively onto $C_{\mathcal{G},h(x)} \cap H_C(B_{\epsilon'}^{2n})$. □

We want to relate h to the flows $\Phi_\mathcal{F} : S_{\epsilon'}^{2n-1} \times \mathbb{R} \to S_{\epsilon'}^{2n-1}$ and $\Phi_\mathcal{G} : S_\epsilon^{2n-1} \times \mathbb{R} \to S_\epsilon^{2n-1}$ whose integral curves are the leaves of the real intersection foliations $\mathcal{F} \cap S_{\epsilon'}^{2n-1}$ and $\mathcal{G} \cap S_\epsilon^{2n-1}$. Note that in general, $\Phi_\mathcal{F}$ and $\Phi_\mathcal{G}$ will not commute with h, that is

$$(h \circ \Phi_\mathcal{F})(x, t) \neq \Phi_\mathcal{G}(h(x), t).$$

To obtain the correct relation, we lift $\Phi_\mathcal{F}$ to a flow $\Phi_{\widetilde{\mathcal{F}}}$ on $S_{\epsilon'}^{2n-1} \times \mathbb{R}$, by setting

$$\Phi_{\widetilde{\mathcal{F}}}((x, t'), t) := (\Phi_\mathcal{F}(x, t), t + t'), x \in S_{\epsilon'}^{2n-1}, t, t' \in \mathbb{R}.$$

The integral curves of $\Phi_{\widetilde{\mathcal{F}}}$ define a foliation $\widetilde{\mathcal{F}}$ on $S_{\epsilon'}^{2n-1} \times \mathbb{R}$ whose leaves project onto the leaves of $\mathcal{F}$ on $S_{\epsilon'}^{2n-1}$. Similarly,

$$\Phi_{\widetilde{\mathcal{G}}}((y, s'), s) := (\Phi_\mathcal{G}(y, s), s + s'), y \in S_\epsilon^{2n-1}, s, s' \in \mathbb{R}$$

defines a flow $\Phi_{\widetilde{\mathcal{G}}}$ and a foliation $\widetilde{\mathcal{G}}$ on $S_\epsilon^{2n-1} \times \mathbb{R}$ whose leaves project onto the leaves of $\mathcal{G}$ on S_ϵ^{2n-1}.
Let p_1, p_2 resp. q_1, q_2 denote the projections from $S_{\epsilon'}^{2n-1} \times \mathbb{R}$ resp. $S_\epsilon^{2n-1} \times \mathbb{R}$ to the first and second component. If $U \subset S_{\epsilon'}^{2n-1}$ resp. $V \subset S_\epsilon^{2n-1}$ are foliation

charts of $\mathcal{F}$ resp. $\mathcal{G}$, then $p_1^{-1}(U)$ resp. $q_1^{-1}(V)$ are foliation charts of $\widetilde{\mathcal{F}}$ resp. $\widetilde{\mathcal{G}}$. Consequently, $h : S^{2n-1}_{\epsilon'} \to S^{2n-1}_{\epsilon}$ can be lifted exactly in one way to a continuous map

$$\widetilde{H} : S^{2n-1}_{\epsilon'} \times \mathbb{R} \to S^{2n-1}_{\epsilon} \times \mathbb{R}$$

such that $\widetilde{H}(x, 0) = (h(x), 0)$ for all $x \in S^{2n-1}_{\epsilon'}$ and the leaves of $\widetilde{\mathcal{F}}$ are mapped into the leaves of $\widetilde{\mathcal{G}}$. In particular, $q_1 \circ \widetilde{H} = h \circ p_1$, and since $(\Phi_{\mathcal{F}}(x, t), t) = \Phi_{\widetilde{\mathcal{F}}}((x, 0), t)$ the point $\widetilde{H}(\Phi_{\mathcal{F}}(x, t), t) \in S^{2n-1}_{\epsilon} \times \mathbb{R}$ must be in the same $\widetilde{G}$-leaf as $\widetilde{H}(x, 0) = (h(x), 0)$. Hence there is an $s \in \mathbb{R}$ such that

$$\Phi_{\widetilde{\mathcal{G}}}((h(x), 0), s) = \widetilde{H}(\Phi_{\mathcal{F}}(x, t), t),$$

and the defining equations of $\Phi_{\widetilde{\mathcal{G}}}$ and $\widetilde{H}$ imply that

$$\Phi_{\mathcal{G}}((h(x), s), s) = (h(\Phi_{\mathcal{F}}(x, t)), (q_2 \circ \widetilde{H})(\Phi_{\mathcal{F}}(x, t), t)).$$

Setting $\tau := q_2 \circ \widetilde{H} \circ (\Phi_{\mathcal{F}} \times p_2) : S^{2n-1}_{\epsilon'} \times \mathbb{R} \to \mathbb{R}$ and comparing the second and the first components yield $s = \tau(x, t)$ and

$$h(\Phi_{\mathcal{F}}(x, t)) = \Phi_{\mathcal{G}}((h(x), \tau(x, t)). \tag{1}$$

This is the requested relation between h, $\Phi_{\mathcal{F}}$ and $\Phi_{\mathcal{G}}$.
By construction we have

$$\tau(x, 0) = 0. \tag{2}$$

To obtain further properties of τ we need to investigate the leaves $L_{\mathcal{F},x}$ and $L_{\mathcal{G},y}$ of the real intersection foliations $\mathcal{F} \cap S^{2n-1}_{\epsilon'}$ and $\mathcal{G} \cap S^{2n-1}_{\epsilon}$ in more details. First of all, we must carefully distinguish between the *leaf topology* on $L_{\mathcal{F},x} = \Phi_{\mathcal{F}}(\{x\} \times \mathbb{R})$ and $L_{\mathcal{G},y} = \Phi_{\mathcal{G}}(\{y\} \times \mathbb{R})$ defined as the finest topology such that $\Phi_{\mathcal{F}|\{x\}\times\mathbb{R}}$ resp. $\Phi_{\mathcal{G}|\{y\}\times\mathbb{R}}$ are continuous, and the *inclusion topology* induced by the inclusion in $S^{2n-1}_{\epsilon'}$ resp. S^{2n-1}_{ϵ}. The leaf topology is always finer than the inclusion topology, and the two topologies only coincide if the leaf is locally closed in $S^{2n-1}_{\epsilon'}$ resp. S^{2n-1}_{ϵ}. If $L_{\mathcal{F},x}$ resp. $L_{\mathcal{G},y}$ are not bijective images of $\{x\} \times \mathbb{R}$ resp. $\{y\} \times \mathbb{R}$ under $\Phi_{\mathcal{F}}$ resp. $\Phi_{\mathcal{G}}$ then $\Phi_{\mathcal{F}|\{x\}\times\mathbb{R}}$ resp. $\Phi_{\mathcal{G}|\{y\}\times\mathbb{R}}$ are periodic maps, and the images $L_{\mathcal{F},x}$ resp. $L_{\mathcal{G},y}$ are compact both in leaf topology and inclusion topology. In particular, in that case $L_{\mathcal{F},x}$ resp. $L_{\mathcal{G},y}$ are embedded circles in $S^{2n-1}_{\epsilon'}$ resp. S^{2n-1}_{ϵ}. Note also that $\Phi_{\mathcal{F}|\{x\}\times\mathbb{R}}$ and $\Phi_{\mathcal{G}|\{y\}\times\mathbb{R}}$ are universal coverings of $L_{\mathcal{F},x}$ and $L_{\mathcal{G},y}$ endowed with the leaf topology.

Claim 2. The leaf $L_{\mathcal{F},x} \subset S^{2n-1}_{\epsilon'}$ is an embedded circle if, and only if the leaf $L_{\mathcal{G},h(x)} \subset S^{2n-1}_{\epsilon}$ is an embedded circle.

Proof. Assume that $L_{\mathcal{F},x}$ is an embedded circle, hence compact. Since $h(L_{\mathcal{F},x}) = L_{\mathcal{G},h(x)}$ by Claim 1 and h is continuous in the inclusion topology, $L_{\mathcal{G},h(x)}$ must be compact, hence closed. Then leaf and inclusion topology on $L_{\mathcal{G},h(x)}$ coincide, so $L_{\mathcal{G},h(x)}$ cannot be homeomorphic to $\mathbb{R}$ in leaf topology. Consequently, $L_{\mathcal{G},h(x)}$ is an embedded circle.
On the other hand, if $L_{\mathcal{G},h(x)}$ is an embedded circle then the cone leaf $C_{\mathcal{G},h(x)}$ and hence the intersection $C_{\mathcal{G},h(x)} \cap H_C(S^{2n-1}_{\epsilon'})$ is compact. But the topological equivalence H_C^{-1} maps $C_{\mathcal{G},h(x)} \cap (H_C(S^{2n-1}_{\epsilon'})$ onto $L_{\mathcal{F},x}$. So $L_{\mathcal{F},x}$ is compact in the inclusion topology, hence compact in the coinciding leaf topology, and hence an embedded circle, not a line. □

Using (1) and the functorial property of the flows $\Phi_{\mathcal{F}}$ and $\Phi_{\mathcal{G}}$ we calculate

$$\begin{aligned}\Phi_{\mathcal{G}}(h(x),\tau(x,t)+\tau(\Phi_{\mathcal{F}}(x,t),t')) &= \Phi_{\mathcal{G}}(\Phi_{\mathcal{G}}(h(x),\tau(x,t)),\tau(\Phi_{\mathcal{F}}(x,t),t')) = \\ &= \Phi_{\mathcal{G}}(h(\Phi_{\mathcal{F}}(x,t)),\tau(\Phi_{\mathcal{F}}(x,t),t')) = \\ &= h(\Phi_{\mathcal{F}}(\Phi_{\mathcal{F}}(x,t),t')) = h(\Phi_{\mathcal{F}}(x,t+t')) = \\ &= \Phi_{\mathcal{G}}(h(x),\tau(x,t+t')).\end{aligned}$$

If $\Phi_{\mathcal{G}}(h(x),\cdot)$ is injective this implies

$$\tau(x,t+t') = \tau(x,t)+\tau(\Phi_{\mathcal{F}}(x,t),t'). \tag{3}$$

If $L_{\mathcal{F},x}$ and $L_{\mathcal{G},h(x)}$ are embedded circles then $\Phi_{\mathcal{F}|\{x\}\times\mathbb{R}}$ and $\Phi_{\mathcal{G}|\{h(x)\}\times\mathbb{R}}$ are periodic maps with periods $T_{\mathcal{F},x}$ and $T_{\mathcal{G},h(x)}$. Consequently,

$$\tau(x,t+t') = \tau(x,t)+\tau(\Phi_{\mathcal{F}}(x,t),t')+k(t,t')\cdot T_{\mathcal{G},h(x)},$$

where $k(t,t')$ is an integer continuously depending on t,t', hence a constant k. Setting $t=t'=0$ we obtain $k=k(0,0)=0$ and thus (3).
In this situation, $\tau(x,\cdot):\mathbb{R}\to\mathbb{R}$ is the lifting of $h: L_{\mathcal{F},x}\to L_{\mathcal{G},h(x)}$ to the universal coverings of the leaves along the flows $\Phi_{\mathcal{F}}:\{x\}\times\mathbb{R}\to L_{\mathcal{F},x}$ and $\Phi_{\mathcal{G}}:\{h(x)\}\times\mathbb{R}\to L_{\mathcal{G},h(x)}$. Since liftings preserve fibers of the coverings this implies

$$\tau(x,t+T_{\mathcal{F},x}) = \tau(x,t)+l\cdot T_{\mathcal{G},h(x)}.$$

Since $H_C(L_{\mathcal{F},x})$ is an embedded circle in $C_{\mathcal{G},h(x)}$ with 0 in its interior, $h: L_{\mathcal{F},x}\to L_{\mathcal{G},h(x)}$ is homotopic to a homeomorphism. Since furthermore H_C preserves orientation, we conclude $l=1$ and obtain:

$$\tau(x,t+T_{\mathcal{F},x}) = \tau(x,t)+T_{\mathcal{G},h(x)}. \tag{4}$$

As a last property of τ we show:

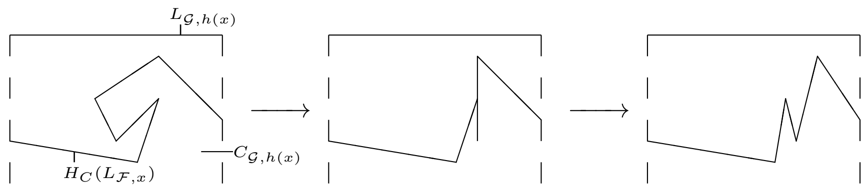
$$\lim_{t\to\infty}\tau(x,t)=\infty \text{ and } \lim_{t\to-\infty}\tau(x,t)=-\infty: \tag{5}$$

If $L_{\mathcal{F},x}$ and $L_{\mathcal{G},h(x)}$ are embedded circles, this follows from (4). Otherwise, both $\Phi_{\mathcal{F}|\{x\}\times\mathbb{R}}$ and $\Phi_{\mathcal{G}|\{h(x)\}\times\mathbb{R}}$ are bijective. In that case, for all $y\in L_{\mathcal{G},h(x)}$ the set of $t\in\mathbb{R}$ such that

$$y = h(\Phi_{\mathcal{F}}(x,t)) = \Phi_{\mathcal{G}}(h(x),\tau(x,t))$$

is bounded because the intersection of the line segment $[y,0]$ with $H_C(L_{\mathcal{F},x})$ equals $[y,0]\cap H_C(S^{2n-1}_{\epsilon'})$, hence is compact. On the other hand, $|\tau(x,t)|$ may be arbitarily large, as $h(L_{\mathcal{F},x})=L_{\mathcal{G},h(x)}$. Both facts together contradict $\lim_{t\to\pm\infty}|\tau(x,t)|\neq\infty$. The signs are again as claimed because H_C preserves orientation.

The aim is now to modify τ to a continuous map $\sigma: S^{2n-1}_{\epsilon'}\times\mathbb{R}\to\mathbb{R}$ which is strictly increasing and surjective for fixed $x\in S^{2n-1}_{\epsilon'}$ but still satisfies a functorial property analogous to (3). We use σ to modify h to a topological equivalence g of $\mathcal{F}\cap S^{2n-1}_{\epsilon'}$ and $\mathcal{G}\cap S^{2n-1}_{\epsilon}$.
The modification of τ to σ and hence from h to g is done in two steps: First, we cut off any "moving backwards" of the image of the leaf $L_{\mathcal{F},x}$ on the leaf $L_{\mathcal{G},h(x)}$ by keeping the map stationary whenever such a backwards move starts. Then we smoothen the stationary intervals to obtain a bijective map. For the image $H_C(L_{\mathcal{F},x})$ in $C_{\mathcal{G},h(x)}$ these steps may locally be visualized as follows:



Continuity of τ and (5) imply that $\mu(x,t) := \max_{t' \le t}\{\tau(x,t')\}$ defines a continuous function $\mu : S^{2n-1}_{\epsilon'} \times \mathbb{R} \to \mathbb{R}$ which is surjective and increasing for fixed x. It holds that

$$\begin{aligned} \mu(x, t+t') &= \max_{t'' \le t+t'}\{\tau(x,t'')\} = \max_{t''' \le t'}\{\tau(x, t'''+t)\} = \\ &= \max_{t''' \le t'}\{\tau(\Phi_{\mathcal{F}}(x,t), t''') + \tau(x,t)\} = \\ &= \mu(\Phi_{\mathcal{F}}(x,t), t') + \tau(x,t). \end{aligned} \tag{6}$$

$\mu(x,\cdot)$ is not necessarily strictly increasing. To modify μ to a strictly increasing function without destroying (6) we introduce the growth function

$$\gamma_\delta(x,t) := \min_{t<t'}\{t' : \tau(x,t') = \tau(x,t) + \delta\} - t > 0,$$

for a fixed $\delta > 0$. It is continuous on $S^{2n-1}_{\epsilon'} \times \mathbb{R}$, hence averaging μ by γ_δ leads to the continuous function

$$\sigma(x,t) := \frac{1}{\gamma_\delta(x,t)} \int_t^{t+\gamma_\delta(x,t)} \mu(x,t')dt'$$

which is strictly increasing and surjective onto $\mathbb{R}$ for fixed x, hence continuously invertible. Using (3) we see that $\gamma_\delta(x, t+t') = \gamma_\delta(\Phi_{\mathcal{F}}(x,t), t')$ and together with (6) this implies

$$\sigma(x, t+t') = \sigma(\Phi_{\mathcal{F}}(x,t), t') + \tau(x,t). \tag{7}$$

Claim 3. The map $g : S^{2n-1}_{\epsilon'} \to S^{2n-1}_{\epsilon}, x \mapsto \Phi_{\mathcal{G}}(h(x), \sigma(x,0))$ defines a homeomorphism inducing a topological equivalence of $\mathcal{F} \cap S^{2n-1}_{\epsilon'}$ and $\mathcal{G} \cap S^{2n-1}_{\epsilon}$.

Proof. If $\Phi_{\mathcal{G}}(h(x), \sigma(x,0)) = \Phi_{\mathcal{G}}(h(y), \sigma(y,0))$ then $h(y)$ is in the same $\mathcal{G}$-leaf as $h(x)$, hence y is in the same $\mathcal{F}$-leaf as x, hence there is $t \in \mathbb{R}$ such that $y = \Phi_{\mathcal{F}}(x,t)$. Using (1), (7) and the functorial properties of the flow $\Phi_{\mathcal{G}}$ we calculate

$$\begin{aligned} \Phi_{\mathcal{G}}(h(x), \sigma(x,0)) &= \Phi_{\mathcal{G}}(h(y), \sigma(y,0)) = \Phi_{\mathcal{G}}(h(\Phi_{\mathcal{F}}(x,t)), \sigma(\Phi_{\mathcal{F}}(x,t),0)) = \\ &= \Phi_{\mathcal{G}}(\Phi_{\mathcal{G}}(h(x), \tau(x,t)), \sigma(\Phi_{\mathcal{F}}(x,t),0)) = \\ &= \Phi_{\mathcal{G}}(h(x), \tau(x,t) + \sigma(\Phi_{\mathcal{F}}(x,t),0)) = \\ &= \Phi_{\mathcal{G}}(h(x), \sigma(x,t)). \end{aligned}$$

If $\Phi_{\mathcal{G}|\{h(x)\}\times\mathbb{R}}$ is bijective, this implies $\sigma(x,0) = \sigma(x,t)$, hence $t = 0$ by injectivity of σ for fixed x, hence $y = \Phi_{\mathcal{F}}(x,0) = x$. If $\Phi_{\mathcal{G}|\{h(x)\}\times\mathbb{R}}$ is periodic with period $T_{\mathcal{G},h(x)}$ and hence $\Phi_{\mathcal{F}|\{x\}\times\mathbb{R}}$ is periodic with period $T_{\mathcal{F},x}$ then for some $k \in \mathbb{Z}$ we have

$$\sigma(x,t) = \sigma(x,0) + k \cdot T_{\mathcal{G},h(x)} = \sigma(x,0) + k \cdot \tau(x, T_{\mathcal{F},x}) = \sigma(x, k \cdot T_{\mathcal{F},x}),$$

by (4) and (7). Injectivity of σ implies $t = k \cdot T_{\mathcal{F},x}$, hence $y = \Phi_{\mathcal{F}}(x, k \cdot T_{\mathcal{F},x}) = x$. So g is injective.

If $y \in S_\epsilon^{2n-1}$ then there exists $x \in S_{\epsilon'}^{2n-1}$ such that $y = h(x)$, since h is surjective. Then $y = \Phi_{\mathcal{F}}(h(x), 0)$. Since $\sigma(x, \cdot)$ is surjective onto $\mathbb{R}$ there exists $t \in \mathbb{R}$ such that

$$\begin{aligned} y = \Phi_{\mathcal{G}}(h(x), \sigma(x,t)) &= \Phi_{\mathcal{G}}(h(x), \tau(x,t) + \sigma(\Phi_{\mathcal{F}}(x,t), 0)) = \\ &= \Phi_{\mathcal{G}}(\Phi_{\mathcal{G}}(h(x), \tau(x,t)), \sigma(\Phi_{\mathcal{F}}(x,t), 0)) = \\ &= \Phi_{\mathcal{G}}(h(\Phi_{\mathcal{F}}(x,t)), \sigma(\Phi_{\mathcal{F}}(x,t), 0)) = \\ &= g(\Phi_{\mathcal{F}}(x,t)), \end{aligned}$$

by (7), (1) and the functorial property of the flow $\Phi_{\mathcal{G}}$. Hence g is surjective.
As a bijective continuous map from a compact topological space to a Hausdorff space, g is a homeomorphism. g is also mapping leaves of $\mathcal{F} \cap S_{\epsilon'}^{2n-1}$ to leaves of $\mathcal{G} \cap S_\epsilon^{2n-1}$, so g is a topological equivalence of $\mathcal{F} \cap S_{\epsilon'}^{2n-1}$ and $\mathcal{G} \cap S_\epsilon^{2n-1}$. □

This finishes the proof of the theorem. □

3. The case of $\mathbb{R}$-linearly independent eigenvalues in dimension 2

In this section, we only consider holomorphic foliation germs $\mathcal{F}$ around $0 \in \mathbb{C}^2$ represented by vector fields of the form

$$\lambda x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \ \lambda \in \mathbb{C} - \mathbb{R}.$$

These foliation germs are invariant under the maps $\mathbb{C}^2 \to \mathbb{C}^2, (x,y) \mapsto r(x,y)$ for all $r \in \mathbb{R}_{>0}$. Hence there is a real intersection foliation $\mathcal{F} \cap S_\epsilon^3$ for all $\epsilon \in \mathbb{R}_{>0}$ as in section 2, and we assume from now on $\epsilon = 1$.

Lemma 3.1. *Let $S^1 \times S^1$ act on S^3 by $(x,y) \mapsto (xe^{it_1}, ye^{it_2})$. Then the intersection foliation $\mathcal{F} \cap S^3$ is invariant under this action.*

Proof. The 1-form $ydx - \lambda x dy$ corresponding to $\lambda x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ is pulled back to

$$ye^{it_2} d(xe^{it_1}) - \lambda x e^{it_1} d(ye^{it_2}) = e^{i(t_1+t_2)}(ydx - \lambda x dy)$$

by the action of $S^1 \times S^1$. Hence the tangent directions of the intersection foliation $\mathcal{F} \cap S^3$ are not changed, and the foliation is invariant under the action. □

For $0 < \epsilon_x, \epsilon_y < 1$, denote the torus $\{(x,y) \in S^3 : |x| = \epsilon_x\}$ by $T^x_{\epsilon_x}$ and the torus $\{(x,y) \in S^3 : |y| = \epsilon_y\}$ by $T^y_{\epsilon_y}$. Then $T^x_{\epsilon_x} = T^y_{\sqrt{1-\epsilon_x^2}}$.

Lemma 3.2. *$T^x_{\epsilon_x}$ intersects all the leaves of the intersection foliation $\mathcal{F} \cap S^3$ not lying on the coordinate axes exactly once and transversally.*

Proof. The real tangent vectors to the torus $T^x_{\epsilon_x}$ in a point (x,y) are those real tangent vectors that are annihilated by the real differential forms

$$d(x\overline{x}) = \overline{x}dx + xd\overline{x} \text{ and } d(y\overline{y}) = \overline{y}dy + yd\overline{y}.$$

The real tangent vectors to the leaf $L_{(x,y)}$ through $(x,y) \in T^x_{\epsilon_x}$ in (x,y) are the $\mathbb{R}$-linear combinations of the real and imaginary part of $\lambda x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$. Since

$$(\overline{y}dy + yd\overline{y})(\lambda x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}) = \overline{y}y \in \mathbb{R} \text{ and } y \neq 0$$

only the imaginary part of $\lambda x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$ can be tangent to $T^x_{\epsilon_x}$. Since

$$(\overline{x}dx+xd\overline{x})(\lambda x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y})=\lambda\overline{x}x \text{ and } x\neq 0$$

this can only be the case if $\mathrm{Im}(\lambda)=0$. But we assumed $\lambda\in\mathbb{C}-\mathbb{R}$, hence the real tangent spaces of $T^x_{\epsilon_x}$ and $L_{(x,y)}$ intersect transversally, hence the leaf $L_{(x,y)}\cap S^3$ of $\mathcal{F}\cap S^3$ and $T^x_{\epsilon_x}$ intersect transversally.
In particular, on a leaf of $\mathcal{F}\cap S^3$ different from $\{x=0\}$ and $\{y=0\}$ the absolute value of the x-coordinate must always strictly increase or decrease. Consequently, such a leaf intersects $T^x_{\epsilon_x}$ exactly once. □

For $0<\epsilon_x,\epsilon_y<1$ and $t\in\mathbb{R}$ denote the disk $\{(x,\sqrt{1-|x|^2}e^{it})\in S^3:|x|<\epsilon_x\}$ by D^x_{t,ϵ_x} and the disk $\{(\sqrt{1-|y|^2}e^{it},y)\in S^3:|y|<\epsilon_y\}$ by D^y_{t,ϵ_y}.

Lemma 3.3. *D^x_{t,ϵ_x} and D^y_{t,ϵ_y} intersect all the leaves of the intersection foliation $\mathcal{F}\cap S^3$ everywhere transversally.*

Proof. By the $S^1\times S^1$-invariance of the leaves of $\mathcal{F}\cap S^3$ shown in Lemma 3.1 we can assume that $t=0$. Since D^x_{0,ϵ_x} is an open subset of $\{y^2=1-|x|^2\}\subset\mathbb{C}^2$, a smooth manifold for $|x|<1$, the real tangent vectors to D^x_{0,ϵ_x} are exactly those annihilated by the real and the imaginary part of the differential form

$$\omega=d(y^2+|x|^2-1)=2ydy+\overline{x}dx+xd\overline{x}.$$

We have

$$\omega_{\mathrm{Re}}=ydy+\overline{y}d\overline{y}+\overline{x}dx+xd\overline{x} \text{ and } \omega_{\mathrm{Im}}=-i(ydy-\overline{y}d\overline{y}).$$

Let $\theta(x,y)$ denote the complex tangent vector $\lambda x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$ to the leaf $L_{(x,y)}$ through $(x,y)\in D^x_{0,\epsilon_x}$. Then $\omega_{\mathrm{Im}}(\theta(x,y))=-iy^2\in i\mathbb{R}-\{0\}$ since $y\in\mathbb{R}-\{0\}$. But the real part of $\theta(x,y)$ is not tangent to both $\{y^2=1-|x|^2\}$ and $S^3=\{\overline{x}x+\overline{y}y=1\}$ either:

$$\omega_{\mathrm{Re}}(\theta(x,y))=2y^2+\lambda x\overline{x} \text{ and } d(\overline{x}x+\overline{y}y)(\theta(x,y))=\lambda x\overline{x}+y\overline{y},$$

hence the real part of the first number vanishes for $\mathrm{Re}\lambda=-\frac{2y^2}{|x|^2}$, the second for $\mathrm{Re}\lambda=-\frac{y\overline{y}}{|x|^2}$. Since $y\neq 0$ this cannot happen for the same λ. □

Figure 3.1 visualizes the behaviour of leaves of $\mathcal{F}\cap S^3$ in the cut-up solid torus $\bigcup_{0\leq\epsilon_x\leq\epsilon}T^x_{\epsilon_x}$ (resp. $\bigcup_{0\leq\epsilon_y\leq\epsilon}T^y_{\epsilon_y}$) as decribed by Lem. 3.2 and 3.3.

Theorem 3.4. *Let $\lambda_1 x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$ and $\lambda_2 x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$ represent two holomorphic foliation germs $\mathcal{F}_1,\mathcal{F}_2$ in $0\in\mathbb{C}^2$, with $\lambda_1,\lambda_2\in\mathbb{C}-\mathbb{R}$. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are topologically equivalent.*

Proof. We will construct a topological equivalence of the intersection foliations $\mathcal{F}_1\cap S^3$ and $\mathcal{F}_2\cap S^3$. Then the statement follows by the Reconstruction Theorem 2.4. Lemma 3.2 and 3.3 show that every leaf of $\mathcal{F}_i$ in the tubular torus $\{(x,y)\in S^3:0<|x|\leq\frac{1}{2}\}$ is parametrized on the one hand by the absolute value ϵ_x of the x-coordinate, on the other hand by the argument t of the y-coordinate. The parametrisation by ϵ_x yields the homeomorphisms

$$\Phi^{(i)}_x:T_{1/2}\times(0,1/2]\to\{(x,y)\in S^3:0<|x|\leq 1/2\}$$

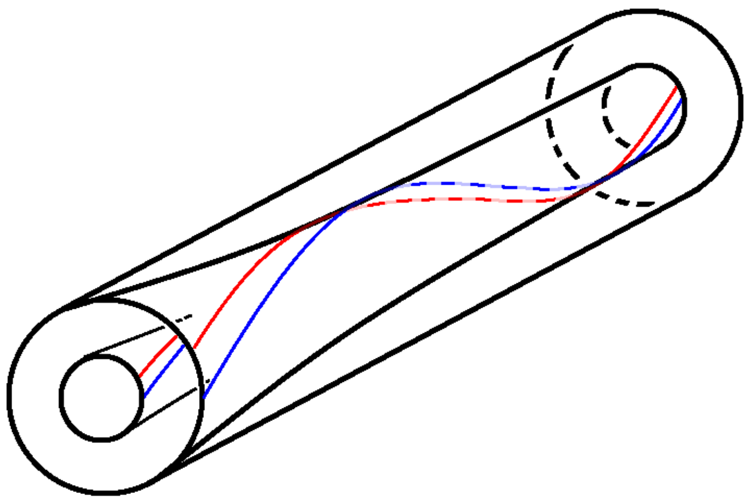


Figure 3.1.

mapping a pair $(x, y) \times \epsilon_x$ to the unique intersection point of the leaf of $\mathcal{F}_i \cap S^3$ through (x, y) with $T^x_{\epsilon_x}$.

Then $\Phi^{(2)}_x \circ (\Phi^{(1)}_x)^{-1}$ is a homeomorphism of the tubular torus $\{(x, y) \in S^3 : 0 < |x| \leq \frac{1}{2}\}$ into itself but might not be extendable to a homeomorphism of the solid torus $\{(x, y) \in S^3 : 0 \leq |x| \leq \frac{1}{2}\}$. To achieve that we reparametrize the ϵ_x-interval $(0, \frac{1}{2}]$ using the second parametrization by the argument of the y-coordinate: Every leaf $L_{(x,y)}$ through a point $(x, y) \in T^x_{\frac{1}{2}}$ defines an invertible function $\phi^{(i)}_x : (0, \frac{1}{2}] \to [0, \infty)$, mapping ϵ_x to $t - t_0$ where t is the argument of the y-coordinate of the intersection point of $L_{(x,y)}$ with $T^x_{\epsilon_x}$ and t_0 is the argument of y. These functions are the same for all such leaves because of the $S^1 \times S^1$-invariance, and we always have $\phi^{(i)}_x(\frac{1}{2}) = 0$. Then

$$\Phi^{(2)}_x \circ \left[\mathrm{id}_{T^x{\frac{1}{2}}} \times ((\phi^{(2)}_x)^{-1} \circ \phi^{(1)}_x)\right] \circ (\Phi^{(1)}_x)^{-1}$$

maps $D^x_{t,\frac{1}{2}}$ and $T^x_{\epsilon_x}$, $0 < \epsilon_x \leq \frac{1}{2}$ onto themselves. This implies that the identity map on $\{x = 0\} \cap S^3$ extends this composition of maps to a homeomorphism Φ_x of the solid torus $\{(x, y) \in S^3 : 0 < |x| \leq \frac{1}{2}\}$ mapping leaves of $\mathcal{F}_1 \cap S^3$ to leaves of $\mathcal{F}_2 \cap S^3$. Furthermore, the restriction of Φ_x to $T^x_{\frac{1}{2}}$ is the identity map.

In the same way we can construct a homeomorphism Φ_y of the solid torus $\{(x, y) \in S^3 : 0 < |y| \leq \frac{1}{2}\}$ mapping leaves of $\mathcal{F}_1 \cap S^3$ to leaves of $\mathcal{F}_2 \cap S^3$. Since again the restriction of Φ_y to $T^x_{\frac{1}{2}}$ is the identity map Φ_x and Φ_y glue to a topological equivalence of the intersection foliations $\mathcal{F}_1 \cap S^3$ and $\mathcal{F}_2 \cap S^3$. □

Remark 3.5. The theorem is Guckenheimer's result in dimension 2 [Guc72]. The proof above yields the construction of an explicit topological equivalence which is missing in Guckenheimer's original argument. Another explicit topological equivalence is constructed in [CKP78] using polycylinders instead of balls.

4. The resonant case in dimension 2

In this section, we only consider holomorphic foliation germs $\mathcal{F}_m$ around $0 \in \mathbb{C}^2$ represented by vector fields of the form

$$(mx + y^m)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},\ m \geq 1.$$

Note that all these foliation germs are equal to the germs in Rem. 1.9.(3) and (4), up to possibly rescaling the x-coordinate.

Lemma 4.1. *The leaves of $\mathcal{F}_m$ intersect all spheres S^3_ϵ, $0 < \epsilon \leq 1$, transversally. In particular the real intersection foliation $\mathcal{F}_m \cap S^3$ on $S^3 = S^3_1$ exists.*

Proof. As in the proof of Prop. 2.1 we have to show that for $(x, y) \in S^3_\epsilon$, $0 < \epsilon \leq 1$ it holds that

$$\overline{x}(mx + y^m) + \overline{y}y = \overline{x}x + \overline{y}y + (m-1)\overline{x}x + \overline{x}y^m \neq 0.$$

But $\overline{x}x + \overline{y}y = \epsilon^2$, $(m-1)\overline{x}x \geq 0$ and

$$|\overline{x}y^m| = |x| \cdot |y|^m < \epsilon^{m+1} \leq \epsilon^2,$$

since $|x|, |y| < \epsilon$ but never $|x| = |y| = \epsilon$. □

Next, we analyse the leaves of the intersection foliations $\mathcal{F}_m \cap S^3$.

Proposition 4.2. *The only closed leaf of $\mathcal{F}_m \cap S^3$ is $\{y = 0\} \cap S^3$. The closure of any leaf $L_{(a,b)}$ through a point $(a, b) \in S^3 - \{y = 0\}$ is $L_{(a,b)} \cup (\{y = 0\} \cap S^3)$. For a certain $\epsilon_y = \epsilon_y(L_{(a,b)})$ with $0 < \epsilon_y \leq 1$ the leaf $L_{(a,b)}$ intersects a torus $T^y_{\epsilon'_y}$*

- *in two distinct points if $0 < \epsilon'_y < \epsilon_y$,*
- *in one point if $\epsilon'_y = \epsilon_y$ and*
- *not at all if $\epsilon'_y > \epsilon_y$.*

Proof. The holomorphic map

$$\lambda_{(a,b)} : \mathbb{C} \to \mathbb{C}^2,\ t \mapsto ((a + b^m t)e^{mt}, be^t)$$

defines the integral curve of the vector field $(mx + y^m)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ through $\lambda_{(a,b)}(0) = (a, b)$, that is the leaf of $\mathcal{F}_m$ through (a, b). If $(a, b) \in S^3$ the leaf of $\mathcal{F}_m \cap S^3$ through (a, b) is the $\lambda_{(a,b)}$-image of the branch through $t = 0$ of the curve in $\mathbb{C}$ implicitely given by

$$1 = e^{m(t+\bar{t})}(a\overline{a} + b^m\overline{a}t + a\overline{b}^m\overline{t} + (b\overline{b})^m t\overline{t}) + b\overline{b}e^{t+\overline{t}}.$$

Decomposing $t = t_R + it_I$ into real and imaginary part and rearranging the equation we obtain

$$(*)\ \ (b\overline{b})^m t_I^2 + 2\mathrm{Im}(a\overline{b}^m)t_I + a\overline{a} + 2\mathrm{Re}(b^m\overline{a})t_R + (b\overline{b})^m t_R^2 + b\overline{b}e^{2(1-m)t_R} - e^{-2mt_R} = 0.$$

This is a quadratic equation in t_I, with coefficients of t_I^2 and t_I independent of t_R.

Claim. For $t_R \leq 0$ the constant term of $(*)$ is increasing with t_R.

Proof. When we derive the constant term with respect to t_R we obtain the gradient

$$2\mathrm{Re}(b^m\overline{a}) + 2(b\overline{b})^m t_R + 2(1-m)b\overline{b}e^{2(1-m)t_R} + 2me^{-2mt_R}$$

which is $> 2e^{-2mt_R} + 2t_R + 2(1-m)e^{2t_R} - 2$ for $t_R \leq 0$ since $(a, b) \in S^3$ implies $|a|, |b| \leq 1$ and $|\mathrm{Re}(b^m\overline{a})| < 1$. But the function $x \mapsto e^{-2mx} + x + (1-m)e^{2x} - 1$

has derivative $-2me^{-2mx} + 2(1-m)e^{2x} + 1 < 0$ for $x \leq 0$, hence the gradient is always > 0 for $t_R \leq 0$, as it is > 0 for $t_R = 0$. $\square$

If $t_R \to \infty$ the constant term also tends to ∞. So we conclude: There exists a $t_0 \geq 0$ such that for all $t_R < t_0$ there are two solutions t_I to the equation $(*)$ symmetric to $-\frac{\mathrm{Im}(a\bar{b}^m)}{(b\bar{b})^m}$, and one solution $t_I = -\frac{\mathrm{Im}(a\bar{b}^m)}{(b\bar{b})^m}$ if $t_R = t_0$.
In particular, if $t_R \to -\infty$ we have $y = be^t \to 0$ which implies the claim on the closure of $L_{(a,b)}$. Since this leaf intersects a torus $T^y_{\epsilon'_y}$ in all points (a', b') of the leaf where $|b'| = \epsilon'_y$ the last claim follows. In particular, the maximal ϵ_y such that $L_{(a,b)} \cap T^y_{\epsilon_y} \neq \emptyset$ is given by $\epsilon_y = |e^{t_0} b|$. $\square$

Corollary 4.3. *All the leaves of $\mathcal{F}_m \cap S^3$ away from $\{y = 0\}$ are uniquely parametrised by the points of the set*

$$\{(a,b) \in S^3 : \mathrm{Im}(a\bar{b}^m) = 0, b \neq 0\}.$$

Proof. Since there is only one point on a leaf $L_{(a,b)}$ with maximal distance $\epsilon_y(L_{(a,b)})$ to $\{y = 0\}$, these points uniquely parametrise all leaves of $\mathcal{F}_m$ away from $\{y = 0\}$. Furthermore, (a, b) is such a point on $L_{(a,b)}$ if for $t = 0$ the linear and constant term of $(*)$ vanish. This is exactly the case when $\mathrm{Im}(a\bar{b}^m) = 0$ since $a\bar{a} + b\bar{b} = 1$. $\square$

Theorem 4.4. *The intersection foliations $\mathcal{F}_m \cap S^3$ are not topologically equivalent for different $m = 1, 2, \ldots$.*

Proof. Assume that $\Phi : S^3 \to S^3$ is a topological equivalence of $\mathcal{F}_{m_1} \cap S^3$ with $\mathcal{F}_{m_2} \cap S^3$. Then Φ maps the only closed leaf of $\mathcal{F}_{m_1}$ to the only closed leaf of $\mathcal{F}_{m_2}$, that is, $\{y = 0\} \cap S^3$ to itself. Hence Φ maps the open complement $U_1 := S^3 - \bigcup_{0 \leq \epsilon_y \leq \epsilon_1} T^y_{\epsilon_y}$ of the solid torus $\bigcup_{0 \leq \epsilon_y \leq \epsilon_1} T^y_{\epsilon_y}$ to an open set $\Phi(U_1)$ in S^3 not intersecting $\{y = 0\} \cap S^3$ but containing $\{x = 0\} \cap S^3$ if ϵ_1 is small enough, by a compactness argument.
Let $\overline{U_1}$ be the union of all leaves of $\mathcal{F}_{m_1} \cap S^3$ intersecting U_1. Then the complement $V_1 := S^3 - (\overline{U_1} \cup \{y = 0\})$ consists of leaves of the foliation $\mathcal{F}_{m_1} \cap S^3$. Cor. 4.3 shows that these leaves are uniquely parametrised by points of $\{\mathrm{Im}(a\bar{b}^{m_1}) = 0\} \cap \bigcup_{0 < \epsilon_y \leq \epsilon_1} T^y_{\epsilon_y}$.
Note that for $0 < \epsilon_y < 1$ the intersection $\{\mathrm{Im}(a\bar{b}^m) = 0\} \cap T^y_{\epsilon_y}$ consists of m connected curves given by $m\arg(b) - \arg(a) \in \pi \cdot \mathbb{Z}$ on the torus $T^y_{\epsilon_y}$, each of them of homology class $(m, 1)$ with respect to the generating cycles $\{\arg(x) = 0\} \cap T^y_{\epsilon_y}$ and $\{\arg(y) = 0\} \cap T^y_{\epsilon_y}$. These curves are visualized in Figure 4.1 when $m = 2$, as the red and the blue curve on the torus $T^y_{\epsilon_y}$ cut up along a disk D^y_t. Hence $\{\mathrm{Im}(a\bar{b}^m) = 0\} \cap \bigcup_{0 < \epsilon_y \leq \epsilon_1} T^y_{\epsilon_y}$ has m connected components, and all of them can be retracted to a curve of homology class $(m, 1)$ in $T^y_{\epsilon_1}$. Since $S^3 - \{y = 0\}$ can be retracted to $S^3 \cap \{x = 0\}$, the homology class of this curve in $S^3 - \{y = 0\}$ is m times the generator represented by $S^3 \cap \{x = 0\}$.
The flow on S^3 associated to $\mathcal{F}_{m_1}$ induces a retraction of V_1 to $\{\mathrm{Im}(a\bar{b}^{m_1}) = 0\} \cap \bigcup_{0 < \epsilon_y \leq \epsilon_1} T^y_{\epsilon_y}$, hence V_1 consists of m_1 connected components $V'_1, \ldots, V_1^{(m_1)}$. These components are visualized in Figure 4.1 when $m_1 = 2$, as the two regions enclosed by the red and the blue surfaces in the cut-up solid torus $\bigcup_{0 \leq \epsilon_y \leq \epsilon_1} T^y_{\epsilon_y}$. By construction, $\Phi(V_1)$ does not intersect the complement of a solid

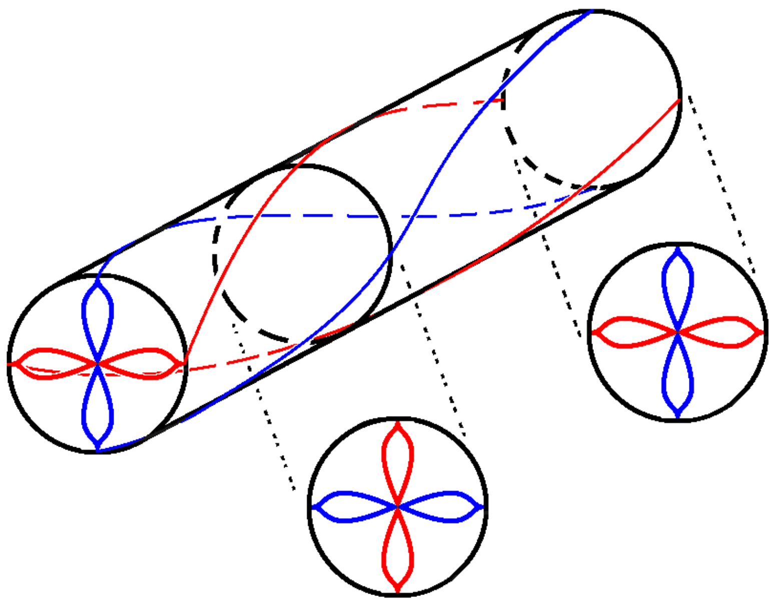


FIGURE 4.1.

torus, $U_2 := S^3 - \bigcup_{0 \leq \epsilon_y \leq \epsilon_2} T^y_{\epsilon_y}$ if ϵ_2 is close enough to 1. Constructing V_2 from U_2 using $\mathcal{F}_{m_2}$ as V_1 was constructed from U_1 using $\mathcal{F}_{m_1}$, this implies $\Phi(V_1) \subset V_2$, and $\Phi(V_1')$ lies in one of the m_2 connected components of V_2, say V_2'.
Consequently, we have a commutative diagram of homeomorphisms and embeddings,

$$\begin{array}{ccccc} V_1' & \stackrel{\Phi}{\to} & \Phi(V_1') & \subset & V_2' \\ \cap & & \cap & & \cap \\ S^3 - \{y = 0\} & \stackrel{\Phi}{\to} & S^3 - \{y = 0\} & = & S^3 - \{y = 0\}. \end{array}$$

This diagram induces the commutative diagram of group homomorphisms of homology group

$$\begin{array}{ccccc} \mathbb{Z} & \stackrel{\cdot \pm 1}{\longrightarrow} & \mathbb{Z} & \to & \mathbb{Z} \\ \cdot m_1 \downarrow & & & & \downarrow \cdot m_2 \\ \mathbb{Z} & = & \mathbb{Z} & = & \mathbb{Z} \end{array}$$

The left and right vertical homomorphism are given by multiplications with m_1 and m_2 because of the retractions constructed above, whereas the upper right homomorphism is given by multiplication with an arbitrary integer n.
Consequently, we obtain $\pm m_1 = \pm n \cdot m_2$, hence $m_1 \geq m_2$. Exchanging the roles of m_1 and m_2 we also obtain $m_1 \leq m_2$ and therefore $m_1 = m_2$. □

Remark 4.5. The holomorphic foliation germs $\mathcal{F}_m$ discussed in this section are not of general type, in the terminology of [MM12]: One feature of plane holomorphic foliation germs of general type is that the singularities of the reduction are represented by vector fields without a linear part with eigenvalue 0. But from

$(mx+y^m)\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$ respectively the holomorphic 1-form $ydx-(mx+y^m)dy$ representing the same holomorphic foliation germ we obtain 1-forms resp. vector fields

$$t(1-m+t^m x^{m-1})dx-(mx+t^m x^m)dt \text{ resp. } (mx+t^m x^m)\frac{\partial}{\partial x}+t(1-m+t^m x^{m-1})\frac{\partial}{\partial t}$$

in the (x,t)-chart with $(x,y)=(x,xt)$ and

$$yds+(s(1-m)+y^{m-1})dy \text{ resp. } y\frac{\partial}{\partial y}+((m-1)s+y^{m-1})\frac{\partial}{\partial s}$$

in the (s,y)-chart with $(x,y)=(sy,y)$, by blowing up $\mathbb{C}^2$ in 0.
If $m=1$ the blown-up foliation in the (x,t)-chart is represented by $x\frac{\partial}{\partial x}+t^2\frac{\partial}{\partial t}$, yielding a reduced singularity in $(x,t)=(0,0)$ but not one of general type.
If $m\geq 2$ the blown-up foliation has a singularity of type $\mathcal{F}_{m-1}$ in $(s,y)=(0,0)$. Thus further reducing this singularity will finally lead to another reduced singularity not of general type.

5. The non-resonant case of $\mathbb{R}$-linearly dependent eigenvalues in dimension 2

In this section, we only consider holomorphic foliation germs $\mathcal{F}_\lambda$ around $0\in\mathbb{C}^2$ represented by vector fields of the form

$$\lambda x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y},\ \lambda\in\mathbb{R}_{>0}.$$

As in Section 3 these foliation germs are invariant under rescaling with positive real constants. Hence it is enough to consider the real intersection foliations $\mathcal{F}_\lambda\cap S^3_1=\mathcal{F}_\lambda\cap S^3$.

Lemma 5.1. *Every leaf of the intersection foliation $\mathcal{F}_\lambda\cap S^3$ lies on a torus $T^x_{\epsilon_x}$, $0\leq\epsilon_x\leq 1$.*

Proof. The flow of the vector field $\lambda x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$ is given by $(a,b,t)\mapsto(ae^{\lambda t},be^t)$. Since $\lambda\in\mathbb{R}_{>0}$ the intersection of the associated integral manifold through a point $(a,b)\in S^3$ with S^3 is parametrised by $t\mapsto(ae^{\lambda it},be^{it})$. Thus the leaf of $\mathcal{F}_\lambda\cap S^3$ through (a,b) lies on the torus $T_x(|a|)$. □

5.1. $\lambda\in\mathbb{Q}_{>0}$.

Assume that $\lambda=\frac{p}{q}$, where $p,q\in\mathbb{N}$ are relatively prime.

Proposition 5.2. *Every leaf of the intersection foliation $\mathcal{F}_\lambda\cap S^3$ is closed. A leaf on the torus $T_x(\epsilon_x)$, $0<\epsilon_x<1$, is a curve of type (p,q), where p describes the winding number of the leaf around $\{x=0\}\cap S^3$ and q the winding number around $\{y=0\}\cap S^3$. The holonomy in a point $(0,e^{it})\in\{x=0\}\cap S^3$ following the leaf in counter-clockwise direction is given by the germ of the map $D^t_x(\epsilon_x)\to D^t_x(\epsilon_x)$, $0<\epsilon_x\ll 1$, multiplying the x-coordinate by $e^{2\pi i\cdot\frac{p}{q}}$. Similarly, the holonomy of the leaf in a point $(e^{it},0)$ following the leaf in counter-clockwise direction is described by the germ of the map $D^t_y(\epsilon_y)\to D^t_y(\epsilon_y)$ multiplying the y-coordinate with $e^{2\pi i\cdot\frac{q}{p}}$. The holonomy in all points of S^3 away from $\{x=0\}\cup\{y=0\}$ is the identity.*

Proof. $\mathcal{F}_\lambda$ is also represented by the vector field $px\frac{\partial}{\partial x}+qy\frac{\partial}{\partial y}$. The flow of this vector field is given by $(a,b,t)\mapsto(ae^{pt},be^{qt})$, and the intersection of the associated integral manifold through (a,b) with S^3 is parametrised as $t\mapsto(ae^{pit},be^{qit})$, $t\in\mathbb{R}$. These parametrisations are periodic, with period $\frac{2\pi}{\gcd(p,q)}=2\pi$. The claims of the proposition follow. □

Corollary 5.3. *Two foliation germs $\mathcal{F}_\lambda$, $\mathcal{F}_\mu$, $\lambda, \mu \in \mathbb{Q}_{>0}$, are topologically equivalent if, and only if $\lambda = \mu$ or $= \frac{1}{\mu}$.*

Proof. By the Reconstruction Theorem 2.4 we only have to decide whether the intersection foliations $\mathcal{F}_\lambda \cap S^3$ and $\mathcal{F}_\mu \cap S^3$ are topologically equivalent or not. Now, the topological types of the holonomy along closed paths on leaves of these real foliation are topologically invariant, in particular the order of the holonomy germ. Consequently, Prop. 5.2 implies that only $\mathcal{F}_{\frac{p}{q}} \cap S^3$ and $\mathcal{F}_{\frac{q}{p}} \cap S^3$ can be topologically equivalent, and in that case the equivalence is given by $(x, y) \mapsto (y, x)$. ☐

5.2. $\lambda \in \mathbb{R}_{>0} - \mathbb{Q}_{>0}$. As in the proof of Lemma 5.1 the leaf of the intersection foliation $\mathcal{F}_\lambda \cap S^3$ through a point $(a, b) \in S^3$ is parametrised by $t \mapsto (ae^{i\lambda t}, be^{it})$, hence lies on $T_x(|a|)$. Since λ is irrational the leaf is not closed but dense on the torus $T_x(|a|)$, for all $a \in \mathbb{C}$ such that $0 < |a| < 1$. Thus we can describe the leaves of $\mathcal{F}_\lambda \cap S^3$ as follows:

Lemma 5.4. *The intersection foliation $\mathcal{F}_\lambda \cap S^3$ has two closed leaves, $\{x = 0\} \cap S^3$ and $\{y = 0\} \cap S^3$, whereas the closure of every other leaf is a torus $T_x(\epsilon_x)$, $0 < \epsilon_x < 1$.* ☐

Next, we consider the continuous map $f : S^3 \to [0, 1], (x, y) \mapsto |x|$. Its fibers are $f^{-1}(\epsilon_x) = T_x(\epsilon_x)$, $0 \leq \epsilon_x \leq 1$. Lemma 5.4 shows that a topological equivalence Φ of $\mathcal{F}_\lambda \cap S^3$ with $\mathcal{F}_\mu \cap S^3$, $\lambda, \mu \in \mathbb{R}_{>0} - \mathbb{Q}_{>0}$, induces a homeomorphism $\phi : [0, 1] \to [0, 1]$ such that $\phi \circ f = f \circ \Phi$, with $\phi(\{0, 1\}) = \{0, 1\}$. Note that $\phi(0) = 0$ and $\phi(1) = 1$ means that Φ maps the closed leaves $\{x = 0\} \cap S^3$ resp. $\{y = 0\} \cap S^3$ onto themselves, whereas $\phi(0) = 1$, $\phi(1) = 0$ indicates that Φ interchanges the closed leaves.
Furthermore, Φ maps the torus $T_x(\epsilon_x)$ homeomorphically onto the torus $T_x(\phi(\epsilon_x))$, $0 < \epsilon_x < 1$. Recall that the (extended) mapping class group of a 2-dimensional torus $T^2 \cong S^1 \times S^1$ is given by $GL(H_1(T^2), \mathbb{Z})$ [FM12, Thm.2.5]. Identifying the tori $T_x(\epsilon_x)$ for different $0 < \epsilon_x < 1$ by rescaling the x- and the y-coordinate the following statement makes sense:

Proposition 5.5. *If $\Phi : S^3 \to S^3$ is a topological equivalence of $\mathcal{F}_\lambda \cap S^3$ with $\mathcal{F}_\mu \cap S^3$, $\lambda, \mu \in \mathbb{R}_{>0} - \mathbb{Q}_{>0}$ then the restriction $\Phi_{|T_x(\epsilon_x)} : T_x(\epsilon_x) \to T_x(\phi(\epsilon_x))$ is of one of the types $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$ in the mapping class group of a 2-dimensional torus, for all $0 < \epsilon_x < 1$.*

Proof. Interchanging the coordinates yields a homeomorphism $\Psi : S^3 \to S^3, (x, y) \mapsto (y, x)$ whose restriction to tori $T_x(\epsilon_x)$ is of type $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the mapping class group of a 2-dimensional torus. Composing Ψ with a topological equivalence Φ of $\mathcal{F}_\lambda \cap S^3$ with $\mathcal{F}_\mu \cap S^3$ such that $\phi(0) = 1$, $\phi(1) = 0$ yields a topological equivalence Φ' of $\mathcal{F}_\lambda \cap S^3$ with $\mathcal{F}_{\frac{1}{\mu}} \cap S^3$ such that $\phi'(0) = 0$, $\phi'(1) = 1$. Hence, from now on we will only consider that case.
For all $0 < \epsilon_0 < 1$ the topological equivalence Φ maps the solid torus $\bigcup_{0 \leq \epsilon_x \leq \epsilon_0} T_x(\epsilon_x)$ homeomorphically onto the solid torus $\bigcup_{0 \leq \epsilon_x \leq \epsilon_0} T_x(\phi(\epsilon_x))$ and $\bigcup_{\epsilon_0 \leq \epsilon_x \leq 1} T_x(\epsilon_x)$ onto $\bigcup_{\epsilon_0 \leq \epsilon_x \leq 1} T_x(\phi(\epsilon_x))$, always mapping the tori $T_x(\epsilon_x)$ onto $T_x(\phi(\epsilon_x))$. The fundamental groups of these solid tori are generated by $L_x := \{x = 0\} \cap S^3$ resp. $L_y := \{y = 0\} \cap S^3$, and a curve of type (p, q) on

the torus $T_x(\epsilon_x)$ (for the notation, see Prop. 5.2) is mapped to the class of $q \cdot L_x$ resp. $p \cdot L_y$ by the inclusion into the solid tori. Consequently, the homeomorphism Φ must map a curve of type (p, q) on $T_x(\epsilon_x)$ to a curve of type $(\pm p, \pm q)$ on $T_x(\phi(\epsilon_x))$. This implies the claim on the isotopy classes of $\Phi_{|T_x(\epsilon_x)}$. □

To finally classify the holomorphic foliation germs $\mathcal{F}_\lambda$, $\lambda \in \mathbb{R}_{>0} - \mathbb{Q}_{>0}$, we consider Kronecker foliations F_λ, $\lambda \in \mathbb{R}_{>0}$, on the 2-dimensional torus $T^2 = S^1 \times S^1$. These foliations are given by the orbits of the flow

$$t \cdot_\lambda (e^{ia}, e^{ib}) = (e^{i(a+\lambda t)}, e^{i(b+t)}),\ t, a, b \in \mathbb{R}.$$

Proposition 5.6. *Two Kronecker foliations F_λ and F_μ, $\lambda, \mu \in \mathbb{R}$, are topologically equivalent if $\mu = \frac{a\lambda+b}{c\lambda+d}$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z})$.*

Proof. Let $Q := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z})$. Then

$$\phi_Q : T^2 \to T^2, (e^{ix}, e^{iy}) \mapsto (e^{i(ax+by)}, e^{i(cx+dy)})$$

is a homeomorphism with inverse map $\phi_{Q^{-1}}$. Since for $s = (c\lambda + d)t$,

$$\begin{aligned} \phi_Q(t \cdot_\lambda (e^{ix}, e^{iy})) &= (e^{i(ax+by+(a\lambda+b)t)}, e^{i(cx+dy+(c\lambda+d)t)}) = \\ &= (e^{i(ax+by+\mu s)}, e^{i(cx+dy+s)}) = \\ &= s \cdot_\mu \phi_Q(e^{ix}, e^{iy}), \end{aligned}$$

ϕ_Q is a topological equivalence of F_λ and F_μ. □

If $\lambda, \mu \in \mathbb{R}_{>0} - \mathbb{Q}_{>0}$ the converse is also true, as the following theorem shows:

Theorem 5.7. *Let $\phi : T^2 \to T^2$ be a topological equivalence of Kronecker foliations F_λ and F_μ, $\lambda, \mu \in \mathbb{R}_{>0} - \mathbb{Q}_{>0}$. If ϕ has the homotopy type $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the mapping class group $GL(2, \mathbb{Z})$ of T^2 then $\mu = \frac{a\lambda+b}{c\lambda+d}$.*

Proof. First of all, we may assume that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, that is, ϕ is isotopic to the identity: If not, Prop. 5.6 shows that F_μ is topologically equivalent to $F_{Q^{-1}\cdot\mu}$ where Q^{-1} is the inverse matrix of $Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Furthermore, the topological equivalence $\phi_{Q^{-1}}$ is of homotopy type $Q^{-1} \in GL(2, \mathbb{Z})$, so the topological equivalence $\phi_{Q^{-1}} \circ \phi$ between F_λ and $F_{Q^{-1}\cdot\mu}$ is of homotopy type $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Consequently, if we show that $\lambda = Q^{-1} \cdot \mu$, then as claimed

$$\mu = Q \cdot \lambda = \frac{a\lambda + b}{c\lambda + d}.$$

For a given $\lambda \in \mathbb{R}_{>0} - \mathbb{Q}_{>0}$ and a point $P = (e^{ia}, e^{ib}) \in T^2$, let

$$L_P^{(\lambda)} := \{(e^{i(a+\lambda t)}, e^{i(b+t)}) | t \in \mathbb{R}\} \subset T^2$$

be the leaf of F_λ through P. Following ideas from ergodic theory we express the "slope" of the leaf $L_P^{(\lambda)}$ as a quotient of its topological intersection numbers with two curves representing generators of $H_1(T^2, \mathbb{Z})$. To this purpose we need arbitrarily long pieces of the leaf $L_P^{(\lambda)}$ starting in P and ending in P' arbitrarily close to P.

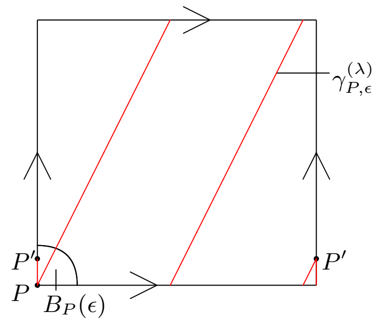


FIGURE 5.1.

Here, we measure distances on T^2 using the metric induced by the Euclidean metric on the universal covering $\mathbb{R}^2$.
So consider the preimage $p^{-1}(B_P(\epsilon))$ of a ball $B_P(\epsilon)$, $0 < \epsilon \ll 1$ under the parametrisation $p : \mathbb{R} \to L^{(\lambda)}_P \subset T^2$ given by $t \mapsto (e^{i(a+\lambda t)}, e^{i(b+t)})$. Since λ is irrational, $L^{(\lambda)}_P$ is dense in T^2, hence $p^{-1}(B_P(\epsilon))$ consists of infinitely many intervals in arbitrarily large distances to $0 \in \mathbb{R}$. One of the intervals in $p^{-1}(B_P(\epsilon))$, say I_0, contains 0, whereas the images of all the other intervals have a non-zero distance to P. In particular, if $\epsilon \to 0$ then the boundaries of all the intervals not containing 0 tend to $\pm\infty$. This observation holds for the intervals in the preimage of an arbitrary neighborhood basis of P.
Let I_1 be the interval in $p^{-1}(B_P(\epsilon))$ closest to the right to I_0. As indicated in Figure 5.1 we can construct a closed path $\gamma^{(\lambda)}_{P,\epsilon} : [0,1] \to T^2$ starting and ending in P, by following the leaf $L^{(\lambda)}_P$ to a point $P' \in p(I_1)$ and connecting P' to P by a path inside $B_P(\epsilon)$.
Note that the homotopy class of $\gamma^{(\lambda)}_{P,\epsilon}$ depends neither on the choice of P' nor on the path connecting P' and P. Hence we can even construct a smoothly embedded path in that way. By construction, this path covers arbitrarily long segments of the leaf $L^{(\lambda)}_P$ if ϵ is small enough.
Next, set $C_1 := \{(e^{it}, 1) : t \in \mathbb{R}\}$ and $C_2 := \{(1, e^{is}) : s \in \mathbb{R}\}$. The closed curves $C_1, C_2 \subset T^2$ represent generators $[C_1], [C_2] \in H_1(T^2, \mathbb{Z})$ intersecting exactly once in the point $(1,1) \in T^2$. Let $[\gamma^{(\lambda)}_{P,\epsilon}] \in H_1(T^2, \mathbb{Z})$ denote the homological 1-class represented by $\gamma^{(\lambda)}_{P,\epsilon}$, and consider the topological intersection numbers $[\gamma^{(\lambda)}_{P,\epsilon}] \cdot [C_i]$ (see [SZ94, 14.6]).
Claim: $\lambda = \lim_{\epsilon \to 0} \frac{[\gamma^{(\lambda)}_{P,\epsilon}] \cdot [C_2]}{[\gamma^{(\lambda)}_{P,\epsilon}] \cdot [C_1]}$.

Proof. We calculate the intersection numbers using their differential-topological interpretation, for smoothly embedded paths $\gamma^{(\lambda)}_{P,\epsilon}$ (see [Hir94, 5.2]). Since $L^{(\lambda)}_P$ intersects C_1 and C_2 everywhere with the same orientation, we just need to count the intersection points in $\gamma^{(\lambda)}_{P,\epsilon} \cap C_i$. Assuming for the moment that $P \notin C_1 \cup C_2$, for small enough ϵ we only need to count the intersection points of the part of $\gamma^{(\lambda)}_{P,\epsilon}$ lying on $L^{(\lambda)}_P$ with C_i. This part is the image $p([0, b_\epsilon])$ of an interval $[0, b_\epsilon] \subset \mathbb{R}$

under the parametrisation $p : \mathbb{R} \to L_P^{(\lambda)}$ introduced above. Then

$$\left[\frac{\lambda b_\epsilon}{2\pi}\right] \leq |p([0,b_\epsilon]) \cap C_2| \leq \left[\frac{\lambda b_\epsilon}{2\pi}\right] + 1 \text{ and } \left[\frac{b_\epsilon}{2\pi}\right] \leq |p([0,b_\epsilon]) \cap C_1| \leq \left[\frac{b_\epsilon}{2\pi}\right] + 1,$$

where $[x]$ denotes the maximal integer $\leq x \in \mathbb{R}$ and $|p([0,b_\epsilon]) \cap C_i|$ the number of intersection points of $p([0,b_\epsilon])$ and C_i. As discussed above, $b_\epsilon \to \infty$ if $\epsilon \to 0$, and the claim follows.
If $P \in C_1 \cup C_2$ the path in $\gamma_{P,\epsilon}^{(\lambda)}$ connecting P' with P can be chosen to intersect C_i only in a number of points bounded from above independently of ϵ. Hence the claim also holds in that case. ◻

Now, we calculate:

$$\lambda = \lim_{\epsilon\to 0} \frac{[\gamma_{P,\epsilon}^{(\lambda)}]\cdot[C_2]}{[\gamma_{P,\epsilon}^{(\lambda)}]\cdot[C_1]} = \lim_{\epsilon\to 0} \frac{[\phi(\gamma_{P,\epsilon}^{(\lambda)})]\cdot[\phi(C_2)]}{[\phi(\gamma_{P,\epsilon}^{(\lambda)})]\cdot[\phi(C_1)]} = \lim_{\epsilon\to 0} \frac{[\phi(\gamma_{P,\epsilon}^{(\lambda)})]\cdot[C_2]}{[\phi(\gamma_{P,\epsilon}^{(\lambda)})]\cdot[C_1]},$$

by the Claim and since $\phi : T^2 \to T^2$ is a homeomorphism assumed to be homotopic to the identity. But $\phi(L_P^{(\lambda)}) = L_{\phi(P)}^{(\mu)}$, hence $\phi(\gamma_{P,\epsilon}^{(\lambda)})$ is a path constructed as above for the leaf $L_{\phi(P)}^{(\mu)}$ of F_μ and the neighborhood basis $U_\epsilon := \phi(B_P(\epsilon))$ of $\phi(P)$, so the above limit is equal to

$$\lim_{\epsilon\to 0} \frac{[\gamma_{\phi(P),U_\epsilon}^{(\mu)}]\cdot[C_2]}{[\gamma_{\phi(P),U_\epsilon}^{(\mu)}]\cdot[C_1]} = \mu,$$

once again by the Claim. ◻

Theorem 5.8. *Two holomorphic foliation germs $\mathcal{F}_\lambda, \mathcal{F}_\mu$, $\mu, \lambda \in \mathbb{R}_{>0} - \mathbb{Q}_{>0}$, are topologically equivalent if, and only if $\lambda = \mu$ or $= \frac{1}{\mu}$.*

Proof. By the Reconstruction Theorem 2.4 it is enough to show the statement for the intersection foliations $\mathcal{F}_\lambda \cap S^3$ and $\mathcal{F}_\mu \cap S^3$.
Exchanging the coordinates yields a topological equivalence Φ of $\mathcal{F}_\lambda \cap S^3$ with $\mathcal{F}_{\frac{1}{\lambda}} \cap S^3$. On the other hand, let Φ be a topological equivalence of $\mathcal{F}_\lambda \cap S^3$ with $\mathcal{F}_\mu \cap S^3$. As above, for $0 < \epsilon_x < 1$ the restriction $\Phi_{|T_x(\epsilon_x)}$ maps the torus $T_x(\epsilon_x)$ to another torus $T_x(\epsilon_x')$ and induces a topological equivalence of the Kronecker foliations $F_\lambda = \mathcal{F}_{\lambda|T_x(\epsilon_x)}$ and $F_\mu = \mathcal{F}_{\mu|T_x(\epsilon_x')}$. Prop. 5.5 shows that $\Phi_{|T_x(\epsilon_x)}$ must be of type $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$ in the mapping class group of a 2-dimensional torus. Then Thm. 5.7 implies that $\lambda = \mu$ or $\lambda = \frac{1}{\mu}$. ◻

6. Topological equivalence classes in dimension 2

In each of the sections 3, 4, 5.1 and 5.2 we identified the topological equivalence classes of plane holomorphic foliation germs represented by vector fields of a certain type, and the list in Rem. 1.9 shows that every plane holomorphic foliation germ is of one of these types. Consequently, the classification is completed by the following statement:

Theorem 6.1. *The topological equivalence classes determined in sections 3, 4, 5.1 and 5.2 are pairwise distinct.*

Proof. If the eigenvalues of the linear part of the representing vector field are $\mathbb{R}$-linearly independent then there exists two closed leaves in the intersection foliation, and the closure of any other leaf of the intersection foliation consists of the leaf and these two closed leaves – see the results of Section 3. If the eigenvalues are $\mathbb{R}$-linearly dependent amd have resonances then there is only one closed leaf in the intersection foliation – see the results of Section 4. If the eigenvalues are $\mathbb{Q}$-linearly dependent but the vector field is non-resonant then every leaf in the intersection foliation is closed – see the results of Section 5.1. Finally, if the eigenvalues are $\mathbb{R}$-linearly dependent but $\mathbb{Q}$-linearly independent then all leaves in the intersection foliation besides the two closed leaves have as closure a torus – see the results of Section 5.2.
Thus, in each of the four cases, there exist leaves of the intersection foliation with topological properties not occuring in the other cases. Hence the Reconstruction Theorem 2.4 shows the theorem. □

7. Topological equivalence classes in dimension ≥ 3

Guckenheimer's Stability Theorem generalizes Thm. 3.4 to arbitrary dimensions:

Theorem 7.1 ([Guc72]). *Let $\sum_{i=1}^n \lambda_i z_i \frac{\partial}{\partial z_i}$ and $\sum_{i=1}^n \mu_i z_i \frac{\partial}{\partial z_i}$ represent two holomorphic foliation germs with an isolated singularity in $0 \in \mathbb{C}^n$ such that $\lambda_1, \ldots, \lambda_n$ resp. $\mu_1, \ldots, \mu_n$ are in the Poincaré domain and pairwise $\mathbb{R}$-linearly independent. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are topologically equivalent.*

Guckenheimer also showed that $\mathcal{F}_1$ and $\mathcal{F}_2$ are topologically equivalent if, under the same assumptions on the λ_i, the vector field θ_2 representing $\mathcal{F}_2$ is obtained from $\sum_{i=1}^n \lambda_i z_i \frac{\partial}{\partial z_i}$ representing $\mathcal{F}_1$ by a sufficiently small holomorphic perturbation. This implies the following classification result:

Proposition 7.2. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two holomorphic foliation germs of rank 1 with an isolated singularity in $0 \in \mathbb{C}^n$ represented by $[U_1, \theta_1]$ and $[U_2, \theta_2]$ such that the eigenvalues of the linear parts of the vector fields θ_1 resp. θ_2 are in the Poincaré domain and pairwise $\mathbb{R}$-linearly independent. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are topologically equivalent.*

Proof. Assume that $\theta_1 = \sum_{i=1}^n f_i(z) \frac{\partial}{\partial z_i}$ and $\theta_2 = \sum_{i=1}^n g_i(z) \frac{\partial}{\partial z_i}$. As discussed in Section 1 we can assume that the non-linear terms of the power series $f_i(z)$ and $g_i(z)$ consist of resonant monomials $z_1^{m_1} \cdots z_n^{m_n}$ wrt the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the linear part of θ_1 resp. $z_1^{n_1} \cdots z_n^{n_n}$ wrt the eigenvalues $\mu_1, \ldots, \mu_n$ of the linear part of θ_n, that is, the $\lambda_1, \ldots, \lambda_n$ resp. $\mu_1, \ldots, \mu_n$ satisfy the resonance $\lambda_i = \sum_{j=1}^n m_j \lambda_j$ resp. $\mu_i = \sum_{j=1}^n n_j \mu_j$ for some integers $m_j, n_j \geq 0$. Since $\lambda_1, \ldots, \lambda_n$ resp. $\mu_1, \ldots, \mu_n$ are in the Poincaré domain there are only finitely many of these resonant monomials, hence $f_i(z)$ and $g_i(z)$ are polynomials.
Possibly after a holomorphic coordinate change we can furthermore assume that the real parts of all the λ_i and μ_i are positive and that

$$0 < \mathrm{Re}\lambda_1 < \cdots < \mathrm{Re}\lambda_n \text{ resp. } 0 < \mathrm{Re}\mu_1 < \cdots < \mathrm{Re}\mu_n.$$

Thus, resonances $\lambda_i = \sum_{j=1}^n m_j \lambda_j$ resp. $\mu_i = \sum_{j=1}^n n_j \mu_j$ always satisfy $m_j = n_j = 0$ for $j \geq i$. Consequently, rescaling the ith coordinate z_i by a real factor ϵ_i such that $0 < \epsilon_1 \ll \epsilon_2 \ll \cdots \ll \epsilon_n$ changes the vector fields θ_1, θ_2 to vector fields with non-linear parts arbitarily close to 0.

So Guckenheimer's Stability Theorem implies that $\mathcal{F}_1$ resp. $\mathcal{F}_2$ are topologically equivalent to the foliations represented by the linear parts $\sum_{i=1}^n \lambda_i z_i \frac{\partial}{\partial z_i}$ resp. $\sum_{i=1}^n \mu_i z_i \frac{\partial}{\partial z_i}$ of θ_1 resp. θ_2, and these foliations are topologically equivalent by Thm. 7.1. $\square$

Under the assumptions of the proposition the appearence of resonant monomials involving only $\mathbb{R}$-linearly independent eigenvalues does not influence the topological equivalence class. So for more general situations we introduce the following notion:

Definition 7.3. *Let $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ be a set of complex numbers in the Poincaré domain. A resonance $\lambda_i = \sum_{j=1}^n m_j \lambda_j$ is called inessential if not all the $\lambda_j \in \mathbb{C}$ with $m_j \neq 0$ lie on the same real ray starting in the origin. Otherwise the resonance is called essential.*

The 2-dimensional classification in Sections 3 - 6 shows that $\mathbb{R}$-linear (in)dependence of the two eigenvalues of the linear part of a representing vector field distinguishes the topological equivalence class of holomorphic foliation germs of rank 1 with an isolated singularity in $0 \in \mathbb{C}^2$ of Poincaré type. In higher dimension we extend this dichotomy to the following invariant:

Definition 7.4. *The ray configuration of a tuple $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ is the ordered partition of this set into subsets consisting of those $\lambda_i \in \mathbb{C}$ lying on the same real ray starting in the origin, and the subsets are ordered by increasing angle of this ray with the positive real axis.*
Two ray configurations are called equivalent if the sizes of the partition subsets, in the order of the partition, are equal, or become equal after reversing the order of one of the partitions.

Finally, the 2-dimensional classification shows that topologically equivalent plane holomorphic foliation germs of rank 1 with an isolated singularity in $0 \in \mathbb{C}^2$ of Poincaré type having equivalent ray configurations are also holomorphically equivalent.
Combining all these observations we predict the following behaviour of such foliation germs in arbitrary dimensions:

Conjecture 7.5. *Two holomorphic foliation germs of rank 1 with an isolated singularity in $0 \in \mathbb{C}^n$ of Poincaré type are topologically equivalent if and only if the following two conditions are satisfied:*

1. *The ray configurations of the tuples of eigenvalues of the linear part of a vector field representing the foliation germs are equivalent.*
2. *For every two corresponding partition subsets $\{i_1, \ldots, i_k\}$, $\{j_1, \ldots, j_k\} \subset \{1, \ldots, n\}$ of the two ray configurations, the restrictions of the two foliation germs to the linear subspaces*

$$L_1 := \{z_l = 0 : l \neq i_1, \ldots, i_k\}, L_2 := \{z_m = 0 : m \neq j_1, \ldots, j_k\} \subset \mathbb{C}^n$$

are holomorphically equivalent.

In particular, the conjecture predicts in full generality that the appearence of inessential resonant monomials does not influence the topological equivalence class.

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