MODULAR FORMS OF REAL WEIGHTS
AND GENERALIZED DEDEKIND SYMBOLS

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ABSTRACT. In a previous paper, I have defined non–commutative generalised Dedekind symbols for classical $PSL(2, \mathbf{Z})$–cusp forms using iterated period polynomials. Here I generalise this construction to forms of real weights using their iterated period functions introduced and studied in a recent article by R. Bruggeman and Y. Choie.

1. Introduction: generalized Dedekind symbols

The classical Dedekind symbol encodes an essential part of modular properties of the Dedekind eta–function, and appears in many contexts seemingly unrelated to modular forms (cf. [KiMel], [Mel]). Fukuhara in [Fu1], [Fu2], and others ([Ap], [ChZ], gave an abstract definition of generalized Dedekind symbols with values in an arbitrary commutative group and produced such symbols from period polynomials of $PSL(2, \mathbf{Z})$–modular forms of any even weight.

In the note [Ma5], I have given an abstract definition of generalised Dedekind symbols for the full modular group $PSL(2, \mathbf{Z})$ taking values in arbitrary non–necessarily commutative group and constructed such symbols from iterated versions of period integrals of modular forms of integral weights considered earlier in [Ma3], [Ma4].

In this article, I extend these constructions to cusp forms of real weights, studied in particular in [Kn], [KnMa], [BrChDi]. The essential ingredient here is furnished by the introduction of iterated versions of their period integrals following [BrCh] .

1.1. Dedekind symbols. The Dedekind eta function is a holomorphic function of the complex variable z with positive imaginary part given by

$$\eta(z) = e^{\pi i z/24} \prod_{k=1}^{\infty} (1 - e^{2\pi i k z}).$$

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It is a cusp form of weight $1/2$, and from $PSL(2, \mathbf{Z})$–invariance of $\eta(z)^{24}dz^6$ it follows that for any fractional linear transformation $\gamma \in PSL(2, \mathbf{Z})$,

$$\gamma z := \frac{az + b}{cz + d}$$

with $c > 0$ we can obtain a rational number

$$s(a, c) := -\frac{1}{\pi i}[\log \eta(\gamma z) - \log \eta(z) - \frac{1}{2}\log((cz + d)/i)] + \frac{a + d}{12c}$$

called the (classical) Dedekind symbol.

It satisfies the reciprocity relation (which can be easily extended to all $c \in \mathbf{Z}$)

$$s(a, c) + s(c, a) = a/c + c/a + 1/ac - 3\operatorname{sgn}(a).$$

1.2. Generalised Dedekind symbols with values in an abelian group. Using a slightly different normalisation and terminology of [Fu1], one can define the generalised Dedekind symbol $d(p, q)$ as a function $d : W \to \mathbf{G}$ where W is the set of pairs of co–prime integers (p, q), and $\mathbf{G}$ an abelian group. It can be uniquely reconstructed from the functional equations

$$d(p, q) = d(p, q + p), \quad d(p, -q) = -d(p, q), \tag{1.1}$$

$$d(p, q) - d(q, -p) = \frac{p^2 + q^2 - 3pq + 1}{12pq}. \tag{1.2}$$

Studying other $PSL(2, \mathbf{Z})$–modular forms in place of η, one arrives to *generalised Dedekind symbols*, satisfying similar functional equations, in which the right hand side of (1.2) is replaced by a different *reciprocity function*, which in turn satisfies simpler functional equations and which uniquely defines the respective Dedekind symbol: see [Fu1], [Fu2] and 1.3.2 below.

In particular, let $F(z)$ be a cusp form of even integral weight $k + 2$ for $\Gamma := PSL(2, \mathbf{Z})$. Its period polynomial is the following function of $t \in \mathbf{C}$:

$$P_F(t) := \int_0^{i\infty} F(z)(z - t)^k dz \tag{1.3}$$

Fukuhara has shown that (slightly normalized) values of period polynomials at rational points form a reciprocity function

$$f_F(p,q) := p^k P_F(q/p) = \int_0^{i\infty} F(z)(pz-q)^k dz \tag{1.4}$$

whose respective Dedekind symbol is

$$d_F(p,q) = \int_{p/q}^{i\infty} F(z)(pz-q)^k dz \tag{1.5}$$

1.3. Non–commutative generalised Dedekind symbols. In [Ma5], I introduced *non–commutative generalised Dedekind symbols* with values in a non–necessarily abelian group **G** by the following definition.

1.3.1. Definition. *A **G**–valued reciprocity function is a map $f : W \to \mathbf{G}$ satisfying the following conditions*

$$f(p,-q) = f(-p,q). \tag{1.6}$$

$$f(p,q)f(-q,p) = 1_{\mathbf{G}} \tag{1.7}$$

$$f(p,p+q)f(p+q,q) = f(p,q). \tag{1.8}$$

Applying (1.8) to $p = 1, q = 0$, we get $f(1,1) = 1_{\mathbf{G}}$ where $1_{\mathbf{G}}$ is the identity. From (1.7) we then get $f(-1,1) = 1_{\mathbf{G}}$. Moreover, $f(-p,-q) = f(p,q)$ so that $f(p,q)$ depends only on q/p which obviously can now be an arbitrary point in $\mathbf{P}^1(\mathbf{Q})$ including ∞. (Of course, $i\infty$ in integrals like (1.3)–(1.5) coincides with ∞ of the real projective line).

The function (1.4) taking values in the additive group of complex numbers satisfies equations (1.6)–(1.8) (written additively).

1.3.2. Definition. *Let f be a **G**–valued reciprocity function.*

*A generalized **G**–valued Dedekind symbol $\mathcal{D}$ with reciprocity function f is a map*

$$\mathcal{D} : W \to \mathbf{G} : \ (p,q) \mapsto \mathcal{D}(p,q)$$

satisfying the following conditions (1.9)–(1.11):

$$\mathcal{D}(p, q) = \mathcal{D}(p, q + p), \tag{1.9}$$

$$\mathcal{D}(p, -q) = \mathcal{D}(-p, q), \tag{1.10}$$

so that $\mathcal{D}(-p, -q) = \mathcal{D}(p, q)$. Finally,

$$\mathcal{D}(p, q)\mathcal{D}(q, -p)^{-1} = f(p, q). \tag{1.11}$$

Clearly, knowing $\mathcal{D}$ one can uniquely reconstruct its reciprocity function f. Conversely, any reciprocity function uniquely defines the respective generalised Dedekind symbol ([Ma5], Theorem 1.8).

In [Ma5] I constructed such reciprocity functions using iterated integrals of cusp forms of integral weights. In the main part of this note I will (partly) generalise this construction to cusp forms of real weights.

Period polynomials of cusp forms of integer weights appear in many interesting contexts. Their coefficients are values of certain L–functions in integral points of the critical strip ([Ma1], [Ma2] and many other works); they can be used in order to produce "local zeta–factors" in the mythical algebraic geometry of characteristic 1 ([Ma6]); they describe relations between certain inner derivations of a free Lie algebra ([Po], [Hai], [HaiMo], [BaSch]), essentially because iterated period polynomials define representations of unipotent completion of basic fundamental modular groupoids.

Iterated period polynomials of real weights can be compared to various other constructions where interpolation from integer values to real values occurs, e. g. Deligne's theory of " symmetric groups S_w, $w \in \mathbf{R}$" using a categorification. It would be very interesting to find similar categorifying constructions also in the case of modular forms of real weights. One can expect perhaps appearance of "modular spaces $\overline{M}_{1,w}, w \in \mathbf{R}$." Notice that certain p–adic interpolations appeared already long time ago in the theory of p–adic L–functions.

2. Modular forms of real weight and their period integrals

In this section I fix notation and give a brief survey of relevant definitions and results from [Kn], [KnMa], and [BrCh]. I adopt conventions of [BrCh], where modular forms of real weights are holomorphic functions on the upper complex half-plane, whereas their period integrals, analogs of (1.3), are holomorphic functions on the lower half–plane.

2.1. Growth conditions for holomorphic functions in upper/lower complex half–planes. Let $\mathbf{P}^1(\mathbf{C})$ be the set of $\mathbf{C}$–points of the projective line endowed with a fixed projective coordinate z. This coordinate identifies the complex plane $\mathbf{C}$ with the maximal subset of $\mathbf{P}^1(\mathbf{C})$ where z is holomorphic.

We put

$$H^+ := \{z \in \mathbf{C} \,|\, \operatorname{Im} z > 0\}, \quad H^- := \{t \in \mathbf{C} \,|\, \operatorname{Im} t < 0\}.$$

As in [BrCh], we identify holomorphic functions on H^+ and H^- using *antiholomorphic* involution: $F(z) \mapsto \overline{F(\overline{z})}$. In the future holomorphic functions on H^- will often be written using coordinate $t = \overline{z}$. On the other hand, the standard hyperbolic metric of curvature -1 on $H^+ \cup H^-$, $ds^2 = |dz|^2/(\operatorname{Im} z)^2$ looks identically in both coordinates.

Cusps $\mathbf{P}^1(\mathbf{Q})$ are rational points on the common boundary of H^+ and H^- (including infinite point). Denote by $\mathcal{P}^+$, resp. $\mathcal{P}^-$ the space of functions $F(z)$ holomorphic in H^+, resp. H^- satisfying for some constants $K, A > 0$ and all $z \in H^\pm$ inequality

$$|F(z)| < K(|z|^A + |\operatorname{Im}z|^{-A}). \tag{2.1}$$

This is called the *polynomial growth* condition.

Cusp forms and their iterated period functions with which we will be working actually satisfy stronger growth conditions near the boundary: see 2.4 below.

2.2. Actions of modular group. The standard left action of $PSL(2, \mathbf{Z})$ upon $\mathbf{P}^1(\mathbf{C})$ by linear fractional transformations of z

$$z \mapsto \gamma(z) = \frac{az + b}{cz + d}$$

defines the right action upon holomorphic functions in $H^\pm$. In the theory of automorphic forms of even integral weight this natural right action is considered first

upon tensor powers of 1–forms $F(z)(dz)^{k/2}$ and then transported back to holomorphic functions by dividing the result by $(dz)^{k/2}$. Equivalently, the last action on functions can be defined using integral powers of $j(\gamma, z) := cz + d$:

$$(F|_k^1 \gamma)(z) = j(\gamma, z)^{-k} F(\gamma z) \tag{2.2}$$

and similarly in t–coordinate.

In the theory of automorphic forms of general real weight k, the relevant generalisation requires two additional conventions. First, we define $(cz + d)^k$ using the following choice of arguments:

$$\arg(cz + d) \in (-\pi, \pi] \text{ for } z \in H^+,$$

$$\arg(ct + d) \in [-\pi, \pi) \text{ for } t \in H^-.$$

Second, multiplication by $(cz + d)^k$ is completed by an additional complex factor depending on γ.

2.2.1. Definition. *A unitary multiplier system v of weight $k \in \mathbf{R}$ (for the group $SL(2, \mathbf{Z})$) is a map $v : SL(2, \mathbf{Z}) \to \mathbf{C}$, $|v(\gamma)| = 1$, satisfying the following conditions. Put*

$$j_{v,k}(\gamma, z) := v(\gamma)(cz + d)^k.$$

Then we have

$$j_{v,k}(\gamma\delta, z) = j_{v,k}(\gamma, \delta z) \cdot j_{v,k}(\delta, z) \tag{2.3}$$

and

$$j_{v,k}(-\gamma, z) = j_{v,k}(\gamma, z). \tag{2.4}$$

Identities (2.3) and (2.4) imply that the formula

$$F(z) \mapsto v(\gamma) j(\gamma, z)^k F(\gamma z) = j_{v,k}(\gamma, z) F(\gamma z) \tag{2.5}$$

defines a right action of $SL(2, \mathbf{Z})/(\pm \mathrm{id}) = PSL(2, \mathbf{Z})$ upon functions holomorphic in H^+. Functions invariant with respect to this action and having exponential decay at cusps (in terms of geodesic distance, cf. [KnMa] and BrChDi]) are called *cusp forms* for the full modular group of weight k with multiplier system v.

For such a form $F(z)$, one can define its *period function* $P_F(t)$ by the formula similar to (1.3). Generally, it is defined only on H^- and satisfies the polynomial growth condition near the boundary.

Moreover, behaviour of period functions of modular forms with respect to modular transformations involves the action

$$(P|_{-k}^{v}\gamma)(t) := v(\gamma)^{-1}j(\gamma,t)^k P(\gamma t) = j_{v,k}(\gamma,t)P(\gamma t) \tag{2.6}$$

which is the right action of $PSL(2,\mathbf{Z})$ upon functions holomorphic in H^-.

2.3. Modular forms. Let $F \in \mathcal{P}^+$ be a holomorphic function of polynomial growth in H^+ (see 2.1) satisfying the $SL(2,\mathbf{Z})$–invariance condition:

$$(F|_{k+2}^{v}\gamma)(z) = F(z) \text{ for all } \gamma \in SL(2,\mathbf{Z}). \tag{2.7}$$

It is called *a modular form of weight $k+2$ and multiplier system v*. Such a modular form is called *a cusp form* if in addition its Fourier series at all cusps contain only positive powers of the relevant exponential function (cf. [KnMa]).

The space of all such forms is denoted $C^0(\Gamma, k+2, v)$. It can be non–trivial only if $k > 0$.

2.4. Period integrals. For a cusp form $F \in C^0(\Gamma, k, v)$ and points a, b in $H^+ \cup \{cusps\}$ we put $\omega_F(z;t) := F(z)(z-t)^k dz$ and define its integral as a function of $t \in H^-$:

$$I_a^b(\omega_F; t) := \int_a^b \omega_F(z; t).$$

If a and/or b is a cusp, then the integration path near it must follow a segment of geodesic connecting a and b. We may and will assume that in our (iterated) integrals the integration path is always the segment of geodesic connecting limits of integration.

More generally, for a finite sequence of cusp forms $F_j \in C^0(\Gamma, k_j + 2, v_j)$ and $\omega_j(z) = \omega_j(z;t) := F_j(z)(z-t)^{k_j} dz$, $j = 1, \ldots, n$, where t is considered as parameter, we put

$$I_a^b(\omega_1, \ldots, \omega_n; t) := \int_a^b \omega_1(z_1) \int_a^{z_1} \omega_2(z_2) \cdots \int_a^{z_{n-1}} \omega_n(z_n). \tag{2.8}$$

These functions of t are holomorphic on H^- and extend holomorphically to a neighbourhood of $\mathbf{P}^1(\mathbf{R}) \setminus \{a, b\}$ (we assume here that a, b are cusps). More precisely, they belong to the $PSL(2, \mathbf{R})$–module $\mathcal{D}_{\mathbf{v}, -\mathbf{k}}^{\omega^0, \infty}$ defined in Sec. 1.6 of [BrChDi], where $\mathbf{v}, \mathbf{k}$ are defined by

$$\mathbf{v}(\gamma) := v_1(\gamma) v_2(\gamma) \ldots v_n(\gamma), \quad \mathbf{k} = k_1 + k_2 + \cdots + k_n.$$

The key role in our constructions is played by the following result ([BrCh], Lemma 3.2):

2.4.1. Lemma. *The iterated period integral (2.8) as a function of $t \in H$ satisfies for all $\gamma \in SL(2, \mathbf{Z})$ the following functional equations:*

$$(I_a^b(\omega_1, \ldots, \omega_n; \cdot)|_{-\mathbf{k}}^{\mathbf{v}} \gamma)(t) = I_{\gamma^{-1}a}^{\gamma^{-1}b}(\omega_1, \ldots, \omega_n; t) \tag{2.9}$$

(see (2.6)).

3. Generalized reciprocity functions from iterated period integrals

3.1. Non–commutative generating series. Fix a finite family of cusp forms as in 2.4 and the respective family of 1–forms $\omega := (\omega_j(z; t))$. Let $(A_j), j = 1, \ldots, l$, be independent associative but non–commuting formal variables.

As in [Ma3], we produce the generating series of all integrals of the type (2.8):

$$J_a^b(\Omega; t) := 1 + \sum_{n \geq 1} \sum_{1 \leq m_1, \ldots, m_n \leq l} I_a^b(\omega_{m_1}, \ldots, \omega_{m_n}; t) A_{m_1} \ldots A_{m_n}.$$

Consider the multiplicative group $\mathbf{G}$ of the formal series in (A_j) with coefficients in functions of t and lower term 1 (A_j commute with coefficients). The right action of $SL(2, \mathbf{Z})$ upon this group is defined coefficientwise. In particular, the action upon $J_a^b(\Omega; t)$ is given by:

$$(J_a^b(\Omega; \cdot)|_{-\mathbf{k}}^{\mathbf{v}} \gamma)(t) =:= 1 + \sum_{n \geq 1} \sum_{1 \leq m_1, \ldots, m_n \leq l} (I_a^b(\omega_{m_1}, \ldots, \omega_{m_n}; \cdot)|_{-\mathbf{k}}^{\mathbf{v}} \gamma)(t) A_{m_1} \ldots A_{m_n}.$$

Moreover, the linear action of the group $GL(l, \mathbf{C})$ upon $(A_1, \ldots, A_l)$ extends to its action upon the group of formal series $\mathbf{G}$. We will need here only the action of the subgroup of diagonal matrices.

Let $\gamma \in PSL(2, \mathbf{Z})$ and $(p, q) \in W$. Then we will denote by $\mathbf{v}(\gamma)*$ the automorphism of such a ring sending A_m to $v_m(\gamma)^{-1} A_m$.

3.1.1. Definition. *The generalised reciprocity function associated to the family $\omega := (\omega_j)$ is the map of the set of coprime pairs of positive integers $(p, q) \mapsto f_\omega(p, q) \in \mathbf{G}$ defined by*

$$f_\omega(p, q) = f(p, q) := 1 + \sum_{n \geq 1} \sum_{1 \leq m_1, \ldots, m_n \leq l} p^{k_{m_1} + \cdots + k_{m_n}} I_0^\infty(\omega_{m_1}, \ldots, \omega_{m_n}; qp^{-1}) A_{m_1} \ldots A_{m_n}.$$

Put now

$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \theta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

3.2. Theorem. *The function f satisfies the following functional equation generalising (1.8):*

$$f(p, q) = (\mathbf{v}(\theta) * f(p, q + p)) \cdot (\mathbf{v}(\theta\sigma\theta) * f(q + p, q)), \tag{3.1}$$

It can be extended to the function defined on coprime pairs $(p, q), p > 0, q < 0$, using the functional equation generalising (1.7)

$$f(p, q) := (\mathbf{v}(\sigma) * f(-q, p))^{-1} \tag{3.2}$$

and defining the right hand side via 3.1.1.

Remark. According to [BrChDi], (2.12), we have

$$v_m(\sigma) = e^{-\pi i(k_m + 2)/2}, \quad v_m(\theta) = e^{\pi i(k_m + 2)/6}.$$

Hence $v_m(\theta\sigma\theta) = e^{-\pi i(k_m + 2)/6}$.

Proof. The direct computation shows that

$$\theta^{-1}(0) = -1, \; \theta^{-1}(\infty) = \infty, \quad (\theta\sigma\theta)^{-1}(0) = 0, \; (\theta\sigma\theta)^{-1}(\infty) = -1.$$

Therefore

$$J_0^\infty(\Omega; t) = J_{-1}^\infty(\Omega; t) J_0^{-1}(\Omega; t) = J_{\theta^{-1}(0)}^{\theta^{-1}(\infty)}(\Omega; t) \cdot J_{(\theta\sigma\theta)^{-1}(0)}^{(\theta\sigma\theta)^{-1}(\infty)}(\Omega; t). \tag{3.3}$$

We now apply (2.9) to both factors in the right hand side of (3.3) term wise. To avoid cumbersome notations, we will restrict ourselves to simple integrals, that is, to the coefficients of terms linear in (A_j). The general case easily follows from this one.

We have then from (2.6):

$$I_{\theta^{-1}(0)}^{\theta^{-1}(\infty)}(\omega_j(z;t)) = v(\theta)^{-1}j(\theta,t)^{k_j} \cdot I_0^\infty(\omega_j(z;\theta(t))) =$$

$$= v(\theta)^{-1} \cdot I_0^\infty(\omega_j(z;t+1)) \tag{3.4}$$

and similarly

$$I_{(\theta\sigma\theta)^{-1}(0)}^{(\theta\sigma\theta)^{-1}(\infty)}(\omega_j(z;t)) = v(\theta\sigma\theta)^{-1}j(\theta\sigma\theta,t)^{k_j} \cdot I_0^\infty(\omega_j(z;\theta(t))) =$$

$$v(\theta\sigma\theta)^{-1} \cdot (t+1)^{k_j} \cdot I_0^\infty(\omega_j(z;\frac{t}{t+1})). \tag{3.5}$$

Thus, from (3.3) we get the following identity for A_j–coefficients:

$$I_0^\infty(\omega_j(z;t)) = v(\theta)^{-1}I_0^\infty(\omega_j(z;t+1))+v(\theta\sigma\theta)^{-1}\cdot(t+1)^{k_j}\cdot I_0^\infty(\omega_j(z;\frac{t}{t+1})). \tag{3.6}$$

In view of growth estimates of period functions in [BrCh] and [BrChDi], we may put here $t = q/p$, where q, p are coprime integers, $p > 0$. We get

$$I_0^\infty(\omega_j(z;qp^{-1})) =$$

$$v(\theta)^{-1}I_0^\infty(\omega_j(z;(q+p)p^{-1}))+v(\theta\sigma\theta)^{-1}\cdot((q+p)p^{-1})^{k_j}\cdot I_0^\infty(\omega_j(z;q(q+p)^{-1})) \tag{3.7}$$

that is,

$$p^{k_j}I_0^\infty(\omega_j(z;qp^{-1})) =$$

$$v(\theta)^{-1}p^{k_j}I_0^\infty(\omega_j(z;(q+p)p^{-1}))+v(\theta\sigma\theta)^{-1}\cdot(q+p)^{k_j}\cdot I_0^\infty(\omega_j(z;q(q+p)^{-1})). \tag{3.8}$$

Thus, if we put, interpolating formula (1.4) (established for even integer weights)

$$f_{0,j}(p,q) := p^{k_j}I_0^\infty(\omega_j(z;qp^{-1})), \tag{3.9}$$

then we get from (3.7) functional equations

$$f_{0,j}(p,q) = v(\theta)^{-1} f_{0,j}(p, q+p) + v(\theta\sigma\theta)^{-1} f_{0,j}(q+p, q). \tag{3.10}$$

This is the A_j–linear part of (3.1). The general case is obtained in the same way.

Similarly, looking at the A_j–coefficients in the identity $J_0^\infty(\Omega; t) J_{\sigma(0)}^{\sigma(\infty)}(\Omega; t) = 1$ and applying (2.6) for $t = -pq^{-1}$ we get

$$f_{0,j}(p,q) + v(\sigma) f_{0,j}(-q, p) = 0.$$

The reasoning as above completes the proof.

Remark. In this statement 3.2, we avoided the simultaneous direct treatment of all cusps qp^{-1} because in our context we have to use the identities of the type $p^k \cdot ((p+q)p^{-1})^k = (p+q)^k$ which require a separate treatment depending on the signs of integers involved, and the cusps 0 and ∞ should also be treated separately already in the definition of $f(p,q)$. We leave this as an exercise for the reader.

3.3. Dedekind cocycles. In the last section of [Ma5], equations for reciprocity functions of even integer weights (cf. Definition 1.3.1 above) were interpreted as defining a special class of 1–cocycles of $\Gamma := PSL(2, \mathbf{Z})$.

More precisely, let $\mathbf{G}$ be a possibly noncommutative left Γ–module. It is known that $PSL(2, \mathbf{Z})$ is the free product of its two subgroups $\mathbf{Z}_2$ and $\mathbf{Z}_3$ generated respectively by

$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Restriction to (σ, τ) of any cocycle in $Z^1(PSL(2, \mathbf{Z}), \mathbf{G})$ belongs to the set

$$\{\, (X, Y) \in \mathbf{G} \times \mathbf{G} \mid X \cdot \sigma X = 1,\ Y \cdot \tau Y \cdot \tau^2 Y = 1 \,\}. \tag{3.11}$$

In fact, this defines a bijection between cocycles and pairs (3.11).

An element (X, Y) of (3.11) is called (the representative of) *the Dedekind cocycle*, iff it satisfies the relation

$$Y = \tau X. \tag{3.12}$$

Let now $\mathbf{G}_0$ be a group. Denote by $\mathbf{G}$ the group of functions $f : \mathbf{P}^1(\mathbf{Q}) \to \mathbf{G}_0$ with pointwise multiplication. Define the left action of Γ upon $\mathbf{G}$ by

$$(\gamma f)(x) = f(\gamma^{-1}x); \quad f \in \mathbf{G}, \ x \in \mathbf{P}^1(\mathbf{Q}), \ \gamma \in \Gamma. \tag{3.13}$$

Let $f : W \to \mathbf{G}_0$ be a $\mathbf{G}_0$–valued reciprocity function, as in Definition 1.3.1. Define elements $X_f, Y_f \in \mathbf{G}$ as the following functions $\mathbf{P}^1(\mathbf{Q}) \to \mathbf{G}_0$:

$$X_f(qp^{-1}) := f(p, q), \tag{3.14}$$

$$Y_f(qp^{-1}) := (\tau X_f)(qp^{-1}) = X_f(\tau^{-1}(qp^{-1})) = f(q, q - p). \tag{3.15}$$

Then the map $f \mapsto (X_f, Y_f)$ establishes a bijection between the set of $\mathbf{G}_0$–valued reciprocity functions and the set of (representatives of) Dedekind cocycles from $Z^1(\Gamma, \mathbf{G})$ ([Ma5], Theorem 3.6).

We will now show that generalised Dedekind cocycles also can be constructed from iterated integrals of cusp forms of real weights, although the respective Γ–module of coefficients will not be of the form (3.13).

3.4. A digression: left vs. right. Treating cocycles with non commutative coefficients, we may prefer to work with left or right modules of coefficients, depending on the concrete environment. We will give a description of Dedekind cocycles with coefficients in a right module.

The sets of left/right 1–cocycles with coefficients in a left/right module $\mathbf{G}$ are defined by

$$Z_l^1(\Gamma, \mathbf{G}) := \{\lambda : \Gamma \to \mathbf{G} \,|\, \lambda(\gamma_1\gamma_2) = \lambda(\gamma_1) \cdot \gamma_1\lambda(\gamma_2)\,\}. \tag{3.16$_l$}$$

$$Z_r^1(\Gamma, \mathbf{G}) := \{\,\rho : \Gamma \to \mathbf{G} \,|\, \rho(\gamma_1\gamma_2) = (\rho(\gamma_1)|\gamma_2) \cdot \rho(\gamma_2)\,\}. \tag{3.16$_r$}$$

where $*|\gamma$ denotes the right action of γ.

Let $\rho \in Z_r^1(PSL(2, \mathbf{Z}), \mathbf{G})$ and set $U := \rho(\sigma), V := \rho(\tau)$. From $\sigma^2 = \tau^3 = 1$ and from (3.16)$_r$ it follows that

$$(U|\sigma) \cdot U = 1, \ (V|\tau^2) \cdot (V|\tau) \cdot V = 1. \tag{3.17}$$

Moreover, any pair $(U, V) \in \mathbf{G} \times \mathbf{G}$ satisfying (3.17) comes from a unique cocycle $\rho \in Z_r^1(PSL(2, \mathbf{Z}), \mathbf{G})$.

3.4.1. Lemma. *(a) If the map* $\mathbf{G} \times \Gamma \to \mathbf{G}$*:* $(g, \gamma) \mapsto g|\gamma$ *defines on* $\mathbf{G}$ *the structure of right* Γ*–module, then the map* $(\gamma, g) \mapsto \gamma g := g|\gamma^{-1}$ *defines on* $\mathbf{G}$ *the structure of left* Γ*–module.*

This construction establishes a bijection between the sets of structures of left/right Γ*–modules on* $\mathbf{G}$*.*

(b) For such a pair of left/right structures and an element $\lambda \in Z_l^1(PSL(2, \mathbf{Z}), \mathbf{G})$*, define*

$$\rho : \Gamma \to \mathbf{G} : \quad \rho(\gamma) = (\lambda(\gamma^{-1}))^{-1}. \tag{3.18}$$

This establishes a bijection of the respective set of 1–cocycles $Z_l^1(\Gamma, \mathbf{G})$ *and* $Z_r^1(\Gamma, \mathbf{G})$*.*

(c) Assume now that $\Gamma = PSL(2, \mathbf{Z})$*. If* (λ, ρ) *is a pair of cocycles connected by (3.18), put as above* $U := \rho(\sigma), V := \rho(\tau)$*, and as in [Ma5], sec. 3,* $X := \lambda(\sigma), Y := \lambda(\tau)$*.*

In this way, Dedekind right cocycles defined by the additional condition $V = U|\sigma$ *bijectively correspond to the Dedekind left cocycles defined in [Ma5] by the condition* $Y = \tau X$*.*

This can be checked by straightforward computations.

3.5. Dedekind cocycles from cusp forms of real weights. In this subsection, we take for $\mathbf{G}$ the subgroup of non–commutative series in A_i (with coefficients in functions on H^-) generated by all series of the form $J_a^b(\Omega; t)$, $a, b \in \mathbf{P}^1(\mathbf{Q})$ and fixed Ω (cf. 3.1).

The left action of $\Gamma = PSL(2, \mathbf{Z})$ upon $\mathbf{G}$ is defined by

$$\gamma J_a^b(\Omega; t) := J_{\gamma a}^{\gamma b}(\Omega; t).$$

Every cusp a defines the left cocycle $\lambda_a : \Gamma \to \mathbf{G}$:

$$\lambda_a(\gamma) := J_{\gamma a}^a(\Omega; t)$$

because

$$J_{\gamma_1 \gamma_2 a}^a = J_{\gamma_1 a}^a \cdot J_{\gamma_1 \gamma_2 a}^{\gamma_1 a} = J_{\gamma_1 a}^a \cdot \gamma_1 J_{\gamma_2 a}^a.$$

In order to construct our Dedekind cocycle, we combine the σ–component of λ_∞ with τ–component of λ_0:

3.5.1. Theorem. *The pair*

$$X := J_0^\infty(\Omega; t), \quad Y := J_1^0(\Omega; t) \tag{3.19}$$

is (the representative of) a left Dedekind cocycle.

Proof. We have

$$X \cdot \sigma X = J_0^\infty(\Omega; t) \cdot J_\infty^0(\Omega; t) = 1,$$

$$Y \cdot \tau Y \cdot \tau^2 Y = J_1^0(\Omega; t) \cdot J_\infty^1(\Omega; t) \cdot J_0^\infty(\Omega; t) = 1,$$

and finally

$$\tau X = J_{\tau 0}^{\tau \infty}(\Omega; t) = J_1^0(\Omega; t) = Y.$$

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