arXiv:1601.00098v1 [math.OC] 1 Jan 2016

Nonlinear Predictor Feedback for Input-Affine Systems with Distributed Input Delays

Anton Ponomarev

Abstract—Prediction-based transformation is applied to control-affine systems with distributed input delays. Transformed system state is calculated as a prediction of the system's future response to the past input with future input set to zero. Stabilization of the new system leads to Lyapunov–Krasovskii proven stabilization of the original one. Conditions on the original system are: smooth linearly bounded open-loop vector field and smooth uniformly bounded input vectors. About the transformed system which turns out to be affine in the undelayed input but with input vectors dependent on the input history and system state, we assume existence of a linearly bounded stabilizing feedback and quadratically bounded control-Lyapunov function. If all assumptions hold globally, then achieved exponential stability is global, otherwise local. Analytical and numerical control design examples are provided.

Index Terms—Nonlinear systems, delayed control, NL predictive control, stability of NL systems.

I. NOTATION

The symbol $PC(T, X)$ stands for the space of piecewise continuous functions mapping $T \subset R$ into a Euclidean space X. The L^2 norm of $\varphi \in PC\big([-h, 0), R^m\big)$ is $\|\varphi\|$, i.e.,

$$\|\varphi\|^2 = \int_{-h}^{0} \|\varphi(\theta)\|^2 \, d\theta. \tag{1}$$

Given $u \in PC\big([t-h, t), R^m\big)$, where $h > 0$, let u_t be a function defined as $u_t(\theta) = u(t + \theta)$ for all $\theta \in [-h, 0)$.

$\mathcal{O}(R)$ is the closed R-ball about the origin in a normed space, specifically,

$$x \in \mathcal{O}(R) \Leftrightarrow \|x\|^2 \leq R^2, \tag{2}$$

$$(x, \varphi) \in \mathcal{O}(R) \Leftrightarrow \|x\|^2 + \|\varphi\|^2 \leq R^2. \tag{3}$$

II. INTRODUCTION

A. Problem statement

Consider the system

$$\dot{x}(t) = f\big(x(t)\big) + B_0\big(x(t)\big)u(t) + B_1\big(x(t)\big)u(t - h)$$
$$+ \int_{-h}^{0} B_{\text{int}}\big(\theta, x(t)\big)u(t + \theta) \, d\theta, \tag{4}$$

where $x \in R^n$, $u \in R^m$, $h > 0$, and the following assumptions hold for some $M_f < \infty$ and $R \in (0, \infty]$:

1) regarding f, for all $x, x_0 \in \mathcal{O}(R)$:

$$\|f(x)\| \leq M_f \|x\|, \tag{5}$$

$$f(x) = f(x_0) + A(x_0)(x - x_0) + o(x - x_0), \tag{6}$$

$$A \in C^0\big(\mathcal{O}(R), R^{n \times n}\big); \tag{7}$$

2) regarding B_1, for all $x, x_0 \in \mathcal{O}(R)$:

$$B_1 \text{ is bounded on } \mathcal{O}(R), \tag{8}$$

$$B_1(x) = B_1(x_0) + \Big(\mathcal{B}_1^1(x_0)(x - x_0), \dots, \mathcal{B}_1^m(x_0)(x - x_0)\Big)$$
$$+ o(x - x_0), \tag{9}$$

$$\mathcal{B}_1^i \in C^0\big(\mathcal{O}(R), R^{n \times n}\big) \quad \forall i \in \overline{1, m}; \tag{10}$$

3) regarding B_{int}, for all $x, x_0 \in \mathcal{O}(R)$ and $\theta \in [-h, 0]$:

$$B_{\text{int}} \text{ is bounded on } [-h, 0] \times \mathcal{O}(R), \tag{11}$$

$$B_{\text{int}}(\cdot, x) \in PC\big([-h, 0], R^{n \times m}\big), \tag{12}$$

$$B_{\text{int}}(\theta, x) = B_{\text{int}}(\theta, x_0)$$
$$+ \Big(\mathcal{B}_{\text{int}}^1(\theta, x_0)(x - x_0), \dots, \mathcal{B}_{\text{int}}^m(\theta, x_0)(x - x_0)\Big)$$
$$+ o(x - x_0), \tag{13}$$

$$\mathcal{B}_{\text{int}}^i(\theta, \cdot) \in C^0\big(\mathcal{O}(R), R^{n \times n}\big) \quad \forall i \in \overline{1, m}. \tag{14}$$

The linear bound (5) is assumed primarily to establish simple bounds on the system's solutions.

This paper is concerned with stabilization of the origin of (4), however, the approach presented here is applicable to time-variant systems with more point-wise and integral delays.

As for the practical importance of (4), such systems are used, e.g., for modeling of networked control systems where control delay value is unknown and varies rapidly [1].

There may be quite a few ways to stabilize (4). See, e.g., [2] where even more general time-variant systems with matrices B_0, B_1, and B_{int} dependent on x_t are tackled, the result being that sometimes a nonlinear feedback of $x(t)$ is stabilizing. However, we are concerned here with the so-called *predictor feedback*. It got the name because, in a sense, it predicts the system's future behavior and how the control currently being chosen affects this behavior. Formally speaking, it is based on a state transformation which puts the system into a delay-free form. If the transformed system is stabilized with a feedback, it is possible to prove that the feedback will also stabilize the original system.

The predictor feedback theory is currently well-developed for linear systems with distributed input delays [3], [4], [5] and nonlinear single-delay systems [6]. This paper extends the methodology to nonlinear systems with distributed input delays (4) for the first time.

B. Outline of the paper

In Section III, already known predictor feedbacks are summarized.

In Section IV, nonlinear predictor-based state transformation is defined for (4). The transformed system is derived in Theorem 1.

In Section V, an approach to stabilization of (4) is proposed (see Theorem 2). It is based on assumed existence of a quadratically bounded control-Lyapunov function and linearly bounded stabilizing feedback for the predictor-transformed system. Stability of the original system's closed loop is proved by Lyapunov–Krasovskii method.

In Section VI, some illustrative problems are discussed.

III. PREDICTOR FEEDBACK OVERVIEW

Let us recall a control methodology first designed for linear systems in [3], [4], [5] and named there "finite spectrum assignment", "receding horizon approach", and "model reduction". We describe here a certain rehash of the same method based on the "prediction" concept and called "predictor feedback".

A. Linear predictor feedback: one delay

To begin with, consider the linear system with one discrete delay:

$$\dot{x}(t) = Ax(t) + B_1 u(t - h). \tag{15}$$

The principles of predictor feedback in this case are:

- given the current state $x(t)$ and past input u_t, *predictor* $Y\big(x(t), u_t\big)$ of the future state $x(t + h)$ is calculated using the variation of constants as

$$Y\big(x(t), u_t\big) = e^{Ah}x(t) + \int_{-h}^{0} e^{-A\theta}B_1 u(t + \theta)\, d\theta; \quad (16)$$

- *state transformation* is applied to the system:

$$y(t) = Y\big(x(t), u_t\big); \quad (17)$$

- *predictor-transformed system* is delay-free:

$$\dot{y}(t) = Ay(t) + B_1 u(t); \quad (18)$$

- $y(t)$ is stabilized, if possible, with a feedback $u(t) = Ky(t)$;
- the feedback for the original system is $u(t) = KY\big(x(t), u_t\big)$.

B. Linear predictor feedback: distributed delay

Suppose now that distributed delays are present:

$$\dot{x}(t) = Ax(t) + B_0 u(t) + B_1 u(t-h) + \int_{-h}^{0} B_{\text{int}}(\theta)u(t+\theta)\, d\theta. \quad (19)$$

In this case *literal* prediction of $x(t + h)$ at the current time t would be challenging because it requires either *knowledge* of the future input or *assumptions* about the future input. The former leads to a non-causal and thus non-implementable feedback, the latter – to an implicitly defined controller.

In [3], [4], [5], however, "prediction with *zero future input*" is used. It is not the *literal* prediction, i.e., actual future input is not required to be zero. The feedback is constructed as follows:

- *predictor* $Y\big(x(t), u_t\big)$ is defined as a prediction of $x(t + h)$ with past input u_t and future input set to zero, i.e., assuming $u(t + \theta) = 0$ for all $\theta \in [0, h]$; formally,

$$Y\big(x(t), u_t\big) = \xi(h), \quad (20)$$
$$\xi'(s) = A\xi(s) + B_1 u(t + s - h)$$
$$+ \int_{-h}^{-s} B_{\text{int}}(\theta)u(t + s + \theta)\, d\theta, \quad (21)$$
$$\xi(0) = x(t) \quad (22)$$

or, equivalently,

$$Y\big(x(t), u_t\big) = e^{Ah}x(t) + \int_{-h}^{0} Q(\theta)u(t + \theta)\, d\theta, \quad (23)$$

where

$$Q(\theta) = e^{-A\theta}B_1 + \int_{-h}^{\theta} e^{A(h-\theta+\tau)}B_{\text{int}}(\tau)\, d\tau; \quad (24)$$

- predictor-based *state transformation* is applied:

$$y(t) = Y\big(x(t), u_t\big); \quad (25)$$

- *predictor-transformed system* is again delay-free:

$$\dot{y}(t) = Ay(t) + \big(e^{Ah}B_0 + Q(0)\big)u(t); \quad (26)$$

- $y(t)$ is stabilized with $u(t) = Ky(t)$;
- the final feedback is

$$u(t) = KY\big(x(t), u_t\big). \quad (27)$$

Observe that the y-system (26) is clearly exponentially stable under the feedback $u = Ky$ if suitable K exists. Stability of (19) under (27) is less obvious as the closed loop is a time-delay system. Lyapunov–Krasovskii analysis is highly desirable. The central problem in this analysis is to find a functional $v(x, \varphi)$ defined for all $x \in R^n$, $\varphi \in PC\big([-h, 0), R^m\big)$ which is positive definite in terms of the norm $\|x\|^2 + \|\varphi\|^2$ and decreasing along the closed-loop solutions,

i.e., when $x = x(t)$, $\varphi = u_t$. It is convenient to construct the functional starting with a Lyapunov function $y^T Vy$ for the closed-loop y-system. The functional may be then designed as

$$v(x, \varphi) = Y^T(x, \varphi)VY(x, \varphi) + \int_{-h}^{0} e^{\sigma\theta}\|\varphi(\theta)\|^2\, d\theta, \quad (28)$$

where $\sigma > 0$ (see [7] for proofs).

A lot of problems regarding linear predictor feedback have been addressed since its discovery including stability [8], [9], [10], robustness [7], [11], [12], [13], delay-adaptive versions [14], [15], and practical implementation issues [16], [17], [18], [19]. The approach is available for systems with state delays and an input delay [20], [21].

C. Nonlinear predictor feedback

Predictor feedback technique has been already expanded on nonlinear systems with one discrete delay in the input:

$$\dot{x}(t) = f\big(x(t), u(t - h)\big). \quad (29)$$

The feedback was constructed and studied in [6] using the same idea as in the linear case: construct a predictor $Y\big(x(t), u_t\big)$ of $x(t + h)$; introduce new variable $y(t) = Y\big(x(t), u_t\big)$ which transforms the system to

$$\dot{y}(t) = f\big(y(t), u(t)\big) \quad (30)$$

and stabilize it with a feedback $u = \kappa(y)$. An analysis is then conducted which shows that stability of the original system is achieved by $u(t) = \kappa\big(Y\big(x(t), u_t\big)\big)$.

As some examples of further development, let us mention the delay-adaptive variation of the feedback [22] and feedback for state-dependent delay [23].

IV. PREDICTOR-BASED STATE TRANSFORMATION

Following the linear theory described in Subsection III-B, we define predictor $Y\big(x(t), u_t\big)$ for (4) via prediction of $x(t + h)$ under *zero future input*. The nonlinear version of (25) is

$$y(t) = Y\big(x(t), u_t\big) \quad (31)$$

with mapping $Y(x, \varphi)$ defined for $x \in R^n$, $\varphi \in PC\big([-h, 0], R^m\big)$ by the system

$$Y(x, \varphi) = \xi(h), \quad (32)$$
$$\xi'(s) = f\big(\xi(s)\big) + B_1\big(\xi(s)\big)\varphi(s - h)$$
$$+ \int_{-h}^{-s} B_{\text{int}}\big(\theta, \xi(s)\big)\varphi(s + \theta)\, d\theta, \quad (33)$$
$$\xi(0) = x. \quad (34)$$

The definition is algorithmic: to find $Y(x, \varphi)$, solve (33) with initial condition (34) for $s \in [0, h]$.

There may be problems with this definition if $\xi(s)$ goes to infinity between $s = 0$ and $s = h$. However, the following lemma shows that the mapping is defined at least in a zero neighborhood.

Lemma 1: $Y(x, \varphi)$ is defined globally if (5) is satisfied for $R = \infty$ or, otherwise, locally for

$$(x, \varphi) \in \mathcal{O}\left(\frac{R}{\rho}\right), \quad (35)$$

where $\rho = \sqrt{2}e^{M_f h}\max\big\{1, h\max_{\theta \in [-h, 0], x \in \mathcal{O}(R)}\|B_{\text{int}}(\theta, x)\| + \max_{x \in \mathcal{O}(R)}\|B_1(x)\|\big\}$. Furthermore, for these x and φ

$$\|x\|^2 \le \rho^2\big(\|Y(x, \varphi)\|^2 + \|\varphi\|^2\big), \quad (36)$$
$$\|Y(x, \varphi)\|^2 \le \rho^2\big(\|x\|^2 + \|\varphi\|^2\big). \quad (37)$$

Proof: Suppose that (5) holds true along the solution of the problem (33), (34). Applying the Gronwall–Bellman, Cauchy–Schwarz, and Young inequalities, one deduces

$$\|\xi(s)\|^2 \leq \rho^2 \left(\|x\|^2 + \|\varphi\|^2 \right) \tag{38}$$

for all $s \in [0, h]$. It leads to the conclusion that if (35) is satisfied, then $\|\xi(s)\| \leq R$, so (5) does hold along $\xi(s)$, and $\xi(s)$ does not go to infinity. Let $s = h$ to get (37). To arrive at (36), use the same estimations when s goes backwards from h to 0. ∎

Let us ignore the possibility of predictor nonexistence for a while (until Theorem 2) and derive the transformed system.

Theorem 1: System (4) after transformation (31) assumes the form

$$\dot{y}(t) = f\big(y(t)\big) + B\big(y(t), u_t\big)u(t) \tag{39}$$

with $B(y, \varphi)$ defined for every y and φ implicitly by the equations

$$B(y, \varphi) = B_1\big(\xi(h)\big) + \beta(h), \tag{40}$$

$$\beta'(s) = \tilde{A}\big(s, \xi(s), \varphi\big)\beta(s) + B_{\text{int}}\big(-s, \xi(s)\big), \tag{41}$$

$$\beta(0) = B_0\big(\xi(0)\big), \tag{42}$$

$$\xi'(s) = f\big(\xi(s)\big) + B_1\big(\xi(s)\big)\varphi(s - h)$$
$$+ \int_{-h}^{-s} B_{\text{int}}\big(\theta, \xi(s)\big)\varphi(s + \theta)\, d\theta, \tag{43}$$

$$\xi(h) = y, \tag{44}$$

$$\tilde{A}(s, \xi, \varphi) = A(\xi) + \sum_{i=1}^{m} \mathcal{B}_1^i(\xi)\varphi_i(s - h)$$
$$+ \sum_{i=1}^{m} \int_{-h}^{-s} \mathcal{B}_{\text{int}}^i(\theta, \xi)\varphi_i(s + \theta)\, d\theta \tag{45}$$

granted that $\xi(s)$ exists on the interval $[0, h]$.

Proof: We are going to calculate the difference between $y(t)$ and $y(t + \Delta)$, where Δ is a small number, and find $\dot{y}(t)$ from there.

According to (31), $y(t) = \xi(h)$, where $\xi(s)$ satisfies

$$\xi'(s) = f\big(\xi(s)\big) + B_1\big(\xi(s)\big)u(t + s - h)$$
$$+ \int_{-h}^{-s} B_{\text{int}}\big(\theta, \xi(s)\big)u(t + s + \theta)\, d\theta, \tag{46}$$

$$\xi(0) = x(t). \tag{47}$$

Notice that this function ξ is the same as in the statement of the theorem, just defined from the other end of the interval $[0, h]$. Likewise, $y(t + \Delta) = \tilde{\xi}(h)$, where $\tilde{\xi}(s)$ satisfies

$$\tilde{\xi}'(s) = f\big(\tilde{\xi}(s)\big) + B_1\big(\tilde{\xi}(s)\big)u(t + \Delta + s - h)$$
$$+ \int_{-h}^{-s} B_{\text{int}}\big(\theta, \tilde{\xi}(s)\big)u(t + \Delta + s + \theta)\, d\theta, \tag{48}$$

$$\tilde{\xi}(0) = x(t + \Delta). \tag{49}$$

Consider now $\zeta(s) = \tilde{\xi}(s) - \xi(s + \Delta)$ for $s \in [0, h - \Delta]$. It follows from the definition of ξ and $\tilde{\xi}$ that

$$\zeta(0) = x(t + \Delta) - \xi(\Delta) = B_0\big(\xi(0)\big)u(t)\Delta + o(\Delta), \tag{50}$$

and

$$\zeta'(s) = \tilde{\xi}'(s) - \xi'(s + \Delta)$$
$$= f\big(\tilde{\xi}(s)\big) - f\big(\xi(s + \Delta)\big)$$
$$+ \Big(B_1\big(\tilde{\xi}(s)\big) - B_1\big(\xi(s + \Delta)\big)\Big)u(t + \Delta + s - h)$$
$$+ \int_{-h}^{-s} B_{\text{int}}\big(\theta, \tilde{\xi}(s)\big)u(t + \Delta + s + \theta)\, d\theta$$
$$- \int_{-h}^{-s - \Delta} B_{\text{int}}\big(\theta, \xi(s + \Delta)\big)u(t + \Delta + s + \theta)\, d\theta$$

$$= A\big(\xi(s)\big)\zeta(s) + \sum_{i=1}^{m} \mathcal{B}_1^i\big(\xi(s)\big)\zeta(s)u_i(t + s - h)$$
$$+ \sum_{i=1}^{m} \int_{-h}^{-s} \mathcal{B}_{\text{int}}^i\big(\theta, \xi(s)\big)\zeta(s)u_i(t + s + \theta)\, d\theta$$
$$+ B_{\text{int}}\big(-s, \xi(s)\big)u(t)\Delta + o(\Delta)$$
$$= \tilde{A}\big(s, \xi(s), u_t\big)\zeta(s) + B_{\text{int}}\big(-s, \xi(s)\big)u(t)\Delta + o(\Delta). \tag{51}$$

Equation (51) holds only if $\xi(s) \in \mathcal{O}(R)$. Observe that, up to $o(\Delta)$, the solution of the problem (51), (50) coincides with that of the problem (41), (42) multiplied by $u(t)\Delta$:

$$\zeta(h - \Delta) = \beta(h)u(t)\Delta + o(\Delta). \tag{52}$$

Therefore,

$$\frac{y(t + \Delta) - y(t)}{\Delta} = \frac{\tilde{\xi}(h) - \xi(h)}{\Delta}$$
$$= \frac{\tilde{\xi}(h) - \tilde{\xi}(h - \Delta) + \zeta(h - \Delta)}{\Delta}$$
$$= f\big(y(t)\big) + B\big(y(t), u_t\big)u(t) + \frac{o(\Delta)}{\Delta}, \tag{53}$$

which proves the theorem. ∎

V. Feedback Design

The following result is inspired by the control-Lyapunov function concept.

Theorem 2: If for all $(y, \varphi) \in \mathcal{O}(R)$ it is possible to define smooth functions $v_0(y)$ and $w_0(y)$ and a mapping $\kappa(y, \varphi)$ which satisfy the inequalities

$$m_{v_0}\|y\|^2 \leq v_0(y) \leq M_{v_0}\|y\|^2, \tag{54}$$

$$w_0(y) \geq m_{w_0}\|y\|^2, \tag{55}$$

$$\|\kappa(y, \varphi)\| \leq M_\kappa\|y\|, \tag{56}$$

$$\big(f(y) + B(y, \varphi)\kappa(y, \varphi)\big)^T \nabla v_0(y) \leq -w_0(y), \tag{57}$$

where $m_{v_0} > 0$ and $m_{w_0} > 0$, then:

1) the origin is exponentially stable in the loop of (4) closed by the feedback

$$u(t) = \kappa\big(Y(x, u_t), u_t\big); \tag{58}$$

2) the stability is global if $R = \infty$ or local if $R < \infty$ with the region of attraction containing at least the ball

$$\big(x(t), u_t\big) \in \mathcal{O}\left(\sqrt{\frac{m_v}{M_v}}\frac{R}{\rho}\right), \tag{59}$$

where ρ is given by Lemma 1, and

$$m_v = \frac{1}{2\rho^2} \min\big\{m_{v_0}, \gamma e^{-\sigma h}, \rho^2\gamma e^{-\sigma h}\big\}, \tag{60}$$

$$M_v = \gamma + \rho^2 M_{v_0}, \tag{61}$$

$$\gamma = \frac{m_{w_0}}{2M_\kappa^2}, \tag{62}$$

$$\sigma = \frac{m_{w_0}}{2M_{v_0}}; \tag{63}$$

3) solutions of the closed loop (4), (58) exhibit the exponential decay property

$$\|x(t)\|^2 + \|u_t\|^2 \leq \frac{M_v}{m_v}e^{-\sigma t}\big(\|x(0)\|^2 + \|u_0\|^2\big). \tag{64}$$

Proof: The proof is by the Lyapunov–Krasovskii method. Consider the functional for the closed loop (4), (58):

$$v(x, \varphi) = v_0\big(Y(x, \varphi)\big) + \gamma \int_{-h}^{0} e^{\sigma\theta}\|\varphi(\theta)\|^2\, d\theta. \tag{65}$$

Using (36), (37), and (54), one can estimate

$$m_v\big(\|x\|^2 + \|\varphi\|^2\big) \le v(x,\varphi) \le M_v\big(\|x\|^2 + \|\varphi\|^2\big) \quad (66)$$

with m_v and M_v given in the statement of the theorem.

Let $(x(t), u_t)$ be an arbitrary solution of the closed loop and suppose that (31) is defined along the whole trajectory $(x(t), u_t)$. Denote the value of $v(x,\varphi)$ along this solution as $v(t)$, i.e.,

$$v(t) = v\big(x(t), u_t\big). \quad (67)$$

It follows from the choice of γ and σ and from (55), (56), (57) that

$$\dot{v}(t) \le \big(-m_{w_0} + \gamma M_\kappa^2\big)\big\|Y\big(x(t), u_t\big)\big\|^2$$
$$-\sigma\gamma\int_{-h}^0 e^{\sigma\theta}\|u(t+\theta)\|^2\,d\theta$$
$$\le -\sigma v(t). \quad (68)$$

Combining (66) and (68) leads to (64). Finally, (64) together with (59) taken at $t = 0$ ensures that at no time $t \ge 0$ does the system leave the region $(x(t), u_t) \in \mathcal{O}(R/\rho)$ which, according to Lemma 1, is sufficient for (31) to be defined for all $t \ge 0$. Thus, the proof just given is correct inside (59). ∎

The next statement is supplemental to Theorem 2. It presents a way to construct the feedback $\kappa(y,\varphi)$, given $v_0(y)$ and $w_0(y)$. Its application is demonstrated in Section VI-C.

Corollary 1: Define $\mathbb{B} \subset R^{n \times m}$ as the set of all possible values of $B(y,\varphi)$ when $(y,\varphi) \in \mathcal{O}(R)$. Suppose $\mathbb{B}$ is bounded.

If there are smooth functions $v_0(y)$ and $w_0(y)$ meeting the conditions (54), (55), and

$$\|\nabla v_0(y)\| \le M_{\nabla v_0}\|y\|, \quad (69)$$
$$f^T(y)\nabla v_0(y) - k\|B^T\nabla v_0(y)\|^2 \le -w_0(y) \quad (70)$$

for some constants $M_{\nabla v_0} > 0$, k, all $B \in \mathbb{B}$, and all $y \in \mathcal{O}(R)$, then the feedback

$$\kappa(y,\varphi) = -kB^T(y,\varphi)\nabla v_0(y) \quad (71)$$

satisfies (56) and (57). In other words, the implications of Theorem 2 hold true for these v_0, w_0, and κ.

Proof: (56) is satisfied with

$$M_\kappa = |k|\max_{B \in \mathbb{B}}\|B\|M_{\nabla v_0}. \quad (72)$$

(57) turns into (70) after substitution of (71). ∎

Assuming infinitely fast computations, the control algorithm suggested by Corollary 1 is this:

1) At time t, given $x(t)$ and u_t, solve (46), (47) for $\xi(s)$.
2) Assign $y(t) := \xi(h)$.
3) Solve (41), (42) for $\beta(h)$.
4) Assign $B\big(y(t), u_t\big) := B_1 + \beta(h)$.
5) Apply $u(t) = -kB^T\big(y(t), u_t\big)\nabla v_0\big(y(t)\big)$ and repeat.

VI. EXAMPLES

A. Scalar case

The most trivial examples are equations of the form

$$\dot{x}(t) = f\big(x(t)\big) + b_0 u(t) + b_1 u(t-h) + \int_{-h}^0 b_{\text{int}}(\theta)u(t+\theta)\,d\theta \quad (73)$$

with same-sign input coefficients, e.g., $b_0 > 0$, $b_1 > 0$, and $b_{\text{int}} > 0$. In this case $B(y,\varphi)$ is a positive scalar separated from zero, so a possible feedback is (58) with

$$\kappa(y,\varphi) = \frac{-f(y) - y}{B(y,\varphi)}. \quad (74)$$

When input coefficients are of different signs, $B(y,\varphi)$ may or may not be zero. If it is zero for some y and φ, then the origin of (73) may or may not be stabilizable. The exact case is unclear because, unlike in the linear systems, here $B(y,\varphi)$ is not constant.

B. Explicit prediction

As explained in [6], for some nonlinear systems with one input delay the predictor can be obtained by quadrature. The same holds true for distributed input delays. Let us give an example.

The system

$$\begin{cases} \dot{x}_1(t) = x_2^2(t) + u(t-h), \\ \dot{x}_2(t) = x_2(t) + u(t) \end{cases} \quad (75)$$

allows explicit predictor transformation

$$y_1 = x_1 + \frac{e^{2h}-1}{2}x_2^2 + \int_{-h}^0 u(t+\theta)\,d\theta, \quad (76)$$
$$y_2 = e^h x_2 \quad (77)$$

which results in the system

$$\begin{cases} \dot{y}_1 = y_2^2 + \big(1 + \big(e^h - e^{-h}\big)y_2\big)u, \\ \dot{y}_2 = y_2 + e^h u. \end{cases} \quad (78)$$

The origin of (78) is globally asymptotically stabilizable. Indeed, another state and input transformation

$$z_1 = y_1 - e^{-h}y_2 + \frac{e^{-2h}-1}{2}y_2^2, \quad (79)$$
$$z_2 = e^{-h}y_2, \quad (80)$$
$$u = -2z_2 + \tilde{u} \quad (81)$$

puts (78) into the cascade form

$$\begin{cases} \dot{z}_1 = z_2^2 - z_2, \\ \dot{z}_2 = -z_2 + \tilde{u}. \end{cases} \quad (82)$$

Consider the positive definite function (the design is from [24])

$$V(z) = \big(z_1 + \tfrac{1}{2}z_2(z_2 - 2)\big)^2 + z_2^2. \quad (83)$$

Its time-derivative along the solutions of (82)

$$\dot{V} = -2z_2^2 + \frac{\partial V}{\partial z_2}\tilde{u} \quad (84)$$

shows that the feedback

$$\tilde{u} = -\frac{\partial V}{\partial z_2} \quad (85)$$

is globally asymptotically stabilizing.

On a side note, the overall transformation from (75) to (82) happens to be linear:

$$z_1 = x_1 - x_2 + \int_{-h}^0 u(t+\theta)\,d\theta, \quad (86)$$
$$z_2 = x_2, \quad (87)$$
$$u = -2x_2 + \tilde{u}. \quad (88)$$

Remark: Returning from z back through y to x leads to a rather complex feedback for the original system (75). Does it achieve global asymptotic stability (GAS)? *On the one hand*, transformation $x \mapsto y$ is well defined globally and transformed system is closed-loop GAS. *On other*, assumption (5) and conditions (54)–(56) hold only locally if anywhere. Therefore, we are bound to conclude that our proofs do not warrant GAS of the original systems. This observation is valid not only in this example. It represents an inherent limitation of our proofs of stability. See Conclusions for further remarks.
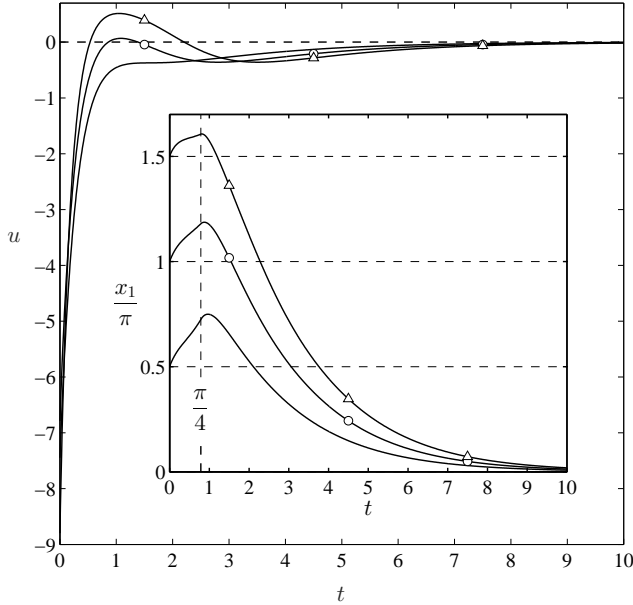
Fig. 1: Simulation of (89) under the feedback (58), (104) with initial conditions $x_1(0) \in \{\pi/2, \pi, 3\pi/2\}$, $x_2(0) = \pi/2$, $u_0 \equiv 1$.

C. Numerical prediction

Consider the nonlinear inverted pendulum described by the system

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = \sin x_1(t) + u(t) + u(t-h), \quad h = \pi/4. \end{cases} \quad (89)$$

The design presented below is also possible for all $h \in [0, \pi/2]$.

In the above notation,

$$f(x) = \begin{pmatrix} x_2 \\ \sin x_1 \end{pmatrix}, \quad B_0 = B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (90)$$

Assumptions (5) and (6) hold true with

$$A(x) = \begin{pmatrix} 0 & 1 \\ \cos x_1 & 0 \end{pmatrix}, \quad M_f = 1, \quad R = \infty. \quad (91)$$

We are going to use Corollary 1 to design the feedback function κ and then simulate the closed loop (89), (58).

Step 1: estimate $\mathbb{B}$ in order to apply Corollary 1. The problem (41), (42) is

$$\beta'(s) = A\big(\xi(s)\big)\beta(s), \quad (92)$$

$$\beta(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (93)$$

From the inclusions

$$\frac{d}{ds}\big(\beta_1^2 + \beta_2^2\big) \in \big[0, 2\big(\beta_1^2 + \beta_2^2\big)\big], \quad (94)$$

$$\frac{d}{ds}\left(\frac{\beta_1}{\beta_2}\right) \in \left[1 - \left(\frac{\beta_1}{\beta_2}\right)^2, 1 + \left(\frac{\beta_1}{\beta_2}\right)^2\right] \quad (95)$$

valid for $s \in [0, \pi/2)$ we find

$$\beta_1^2(h) + \beta_2^2(h) \in \big[1, e^{2h}\big], \quad (96)$$

$$\frac{\beta_1(h)}{\beta_2(h)} \in \big[\tanh h, \tan h\big], \quad (97)$$

which means that $\beta(h)$ resides in a sector of a circular ring centered at the origin. It implies a rough estimation of $\mathbb{B}$ as the sector moved by B_1.

Step 2: find v_0, w_0 and k that satisfy Corollary 1. Let

$$v_0(y) = y^T V y \quad (98)$$

with some positive definite V. The requirement of Corollary 1 is then

$$2f^T(y)Vy - 4k(Vy)^T BB^T Vy \leq -w_0(y). \quad (99)$$

It is convenient to decompose $f(y)$ as

$$f(y) = F(y)y, \quad (100)$$

where

$$F(y) = \begin{pmatrix} 0 & 1 \\ \alpha(y) & 0 \end{pmatrix}, \quad (101)$$

$$\alpha(y) = \frac{\sin y_1}{y_1} \in [-0.22, 1], \quad (102)$$

then (99) will follow from negative definiteness of the matrix

$$V^{-1}F^T(y) + F(y)V^{-1} - 4kBB^T \quad (103)$$

for all $y \in R^n$, all $B \in \mathbb{B}$, and some k. We found computationally that for $h = \pi/4$ it is valid to take $V = I$ and $k = 1$, so the feedback function suggested by Corollary 1 is

$$\kappa(y, \varphi) = -B^T(y, \varphi)y. \quad (104)$$

Fig. 1 shows the simulation results of (89) under the feedback (58), (104) for different initial conditions. Euler's approximation with time step of 0.01 was used for calculating $x(t)$, $\beta(s)$, and $\xi(s)$.

VII. Conclusions

Let us pose some problems for a future discussion.

Firstly, concerning practical implementation of the proposed feedback, it should be mentioned that solving nonlinear equations may be costly, and any approximation, strictly speaking, requires robustness analysis.

Secondly, it is demonstrated in Section VI-B that our linear-quadratic bounds (5) and (54)–(56) may prohibit one from declaring global stabilization even when it is likely to be achieved. In fact, such restrictive bounds are not necessary. Their purpose is to simplify stability analysis by making the case "almost linear". The simplification is evinced in the exponential rate of decay (64) which is generally not expected from nonlinear systems. A possible way to avoid assumptions (5) and (54)–(56) and to achieve class $\mathcal{KL}$ rate of decay in (64) is to follow the methodology of [6].

References

[1] G. Goebel, U. Münz, and F. Allgöwer, "Stabilization of linear systems with distributed input delay," in *2010 American Control Conference*, Marriott Waterfront, Baltimore, MD, USA, 2010, pp. 5800–5805.

[2] F. Mazenc, S.-I. Niculescu, and M. Bekaik, "Stabilization of time-varying nonlinear systems with distributed input delay by feedback of plant's state," *IEEE Trans. Autom. Control*, vol. 58, no. 1, pp. 264–269, Jan. 2013.

[3] A. Manitius and A. W. Olbrot, "Finite spectrum assignment problem for systems with delays," *IEEE Trans. Autom. Control*, vol. AC-24, no. 4, pp. 541–552, Aug. 1979.

[4] W. H. Kwon and A. E. Pearson, "Feedback stabilization of linear systems with delayed control," *IEEE Trans. Autom. Control*, vol. AC-25, no. 2, pp. 266–269, Apr. 1980.

[5] Z. Artstein, "Linear systems with delayed control: a reduction," *IEEE Trans. Autom. Control*, vol. AC-27, no. 4, pp. 869–879, Aug. 1982.

[6] M. Krstic, "Input delay compensation for forward complete and strict-feedforward nonlinear systems," *IEEE Trans. Autom. Control*, vol. 5, no. 2, pp. 287–303, Feb. 2010.

[7] A. Ponomarev, "Reduction-based robustness analysis of linear predictor feedback for distributed input delays," *IEEE Trans. Autom. Control*, to be published, doi:10.1109/TAC.2015.2437520, available online: http://arxiv.org/abs/1505.02402.

[8] M. Krstic, "Lyapunov tools for predictor feedbacks for delay systems: inverse optimality and robustness to delay mismatch," *Automatica*, vol. 4, no. 11, pp. 2930–2935, Nov. 2008.

[9] N. Bekiaris-Liberis and M. Krstic, "Lyapunov stability of linear predictor feedback for distributed input delays," *IEEE Trans. Autom. Control*, vol. 56, no. 3, pp. 655–660, Mar. 2011.

[10] F. Mazenc, S.-I. Niculescu, and M. Krstic, "Lyapunov–Krasovskii functionals and application to input delay compensation for linear time-invariant systems," *Automatica*, vol. 48, no. 7, pp. 1317–1323, Jul. 2012.

[11] W. Michiels and S.-I. Niculescu, "On the delay sensitivity of Smith Predictors," *Int. J. Syst. Sci.*, vol. 34, no. 8–9, pp. 543–551, Jul. 2003.

[12] M. Jankovic, "Forwarding, backstepping, and finite spectrum assignment for time delay systems," *Automatica*, vol. 45, no. 1, pp. 2–9, Jan. 2009.

[13] I. Karafyllis and M. Krstic, "Delay-robustness of linear predictor feedback without restriction on delay rate," *Automatica*, vol. 49, no. 6, pp. 1761–1767, Jun. 2013.

[14] D. Bresch-Pietri and M. Krstic, "Delay-adaptive predictor feedback for systems with unknown long actuator delay," *IEEE Trans. Autom. Control*, vol. 55, no. 9, pp. 2106–2112, Sep. 2010.

[15] D. Bresch-Pietri, J. Chauvin, and N. Petit, "Adaptive control scheme for uncertain time-delay systems," *Automatica*, vol. 48, no. 8, pp. 1536–1552, Aug. 2012.

[16] Z. Palmor, "Stability properties of Smith dead-time compensator controller," *Int. J. Control*, vol. 32, no. 6, pp. 937–949, 1980.

[17] S. Mondié, M. Dambrine, and O. Santos, "Approximation of control laws with distributed delays: a necessary condition for stability," *Kybernetika*, vol. 38, no. 5, pp. 541–551, 2002.

[18] S. Mondié and W. Michiels, "A safe implementation for finite spectrum assignment: robustness analysis," in *Proc. 42nd IEEE Conf. Decision and Control*, vol. 5, Dec. 2003, pp. 4569–4574.

[19] S. Evesque, A. M. Annaswamy, S. Niculescu, and A. P. Dowling, "Adaptive control of a class of time-delay systems," *J. Dyn. Sys., Meas., Control*, vol. 125, no. 2, pp. 186–193, Jun. 2003.

[20] V. L. Kharitonov, "An extension of the prediction scheme to the case of systems with both input and state delay," *Automatica*, vol. 50, no. 1, pp. 211–217, Jan. 2014.

[21] ——, "Predictor based stabilization of neutral type systems with input delay," *Automatica*, vol. 52, no. 2, pp. 125–134, Feb. 2015.

[22] D. Bresch-Pietri and M. Krstic, "Delay-adaptive control for nonlinear systems," *IEEE Trans. Autom. Control*, vol. 59, no. 5, pp. 1203–1218, May 2014.

[23] N. Bekiaris-Liberis and M. Krstic, "Compensation of state-dependent input delay for nonlinear systems," *IEEE Trans. Autom. Control*, vol. 58, no. 2, pp. 275–289, Feb. 2013.

[24] M. Jankovic, R. Sepulchre, and P. V. Kokotovic, "Constructive Lyapunov stabilization of nonlinear cascade systems," *IEEE Trans. Autom. Control*, vol. 41, no. 12, pp. 1723–1735, Dec. 1996.