

FRACTIONAL POINCARÉ INEQUALITY WITH FINITE TOTAL Q -CURVATURE

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ABSTRACT. In this paper, we prove several Poincaré inequalities of fractional type on conformally flat manifolds with finite total Q -curvature. This shows a new aspect of the Q -curvature on non-compact complete manifolds.

1. INTRODUCTION

The Q -curvature arises naturally as a conformal invariant associated to the Paneitz operator. When $n = 4$, the Paneitz operator is defined as:

$$P_g = \Delta^2 + \delta\left(\frac{2}{3}Rg - 2Ric\right)d,$$

where δ is the divergence, d is the differential, R is the scalar curvature of g , and Ric is the Ricci curvature tensor. The Paneitz Q -curvature is defined as

$$Q_g = \frac{1}{12} \left\{ -\Delta R + \frac{1}{4}R^2 - 3|E|^2, \right\}$$

where E is the traceless part of Ric , and $|\cdot|$ is taken with respect to the metric g . Under the conformal change $g_u = e^{2u}g_0$, the Paneitz operator transforms by $P_{g_u} = e^{-4u}P_{g_0}$, and Q_{g_u} satisfies the fourth order equation

$$(1.1) \quad P_{g_0}u + 2Q_{g_0} = 2Q_{g_u}e^{4u}.$$

This is analogous to the transformation law satisfied by the Laplacian operator $-\Delta_g$ and the Gaussian curvature K_g on surfaces,

$$-\Delta_{g_0}u + K_{g_0} = K_{g_u}e^{2u}.$$

The invariance of Q -curvature in dimension 4 is due to the Chern-Gauss-Bonnet formula for a closed manifold M :

$$(1.2) \quad \chi(M) = \frac{1}{4\pi^2} \int_M \left(\frac{|W|^2}{8} + Q_g \right) dv_g,$$

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where W denotes the Weyl tensor.

A related problem is the classical isoperimetric inequality on a complete simply connected surface M^2 , called Fiala-Huber's [Fia41], [Hub57] isoperimetric inequality

$$(1.3) \quad \text{vol}(\Omega) \leq \frac{1}{2(2\pi - \int_{M^2} K_g^+ dv_g)} \text{Area}(\partial\Omega)^2,$$

where K_g^+ is the positive part of the Gaussian curvature K_g . Also $\int_{M^2} K_g^+ dv_g < 2\pi$ is the sharp bound for the isoperimetric inequality to hold.

In [Wan15], we generalize the Fiala-Huber's isoperimetric inequality to all even dimensions, replacing the role of the Gaussian curvature in dimension two by that of the Q -curvature in higher dimensions:

Let $(M^n, g) = (\mathbb{R}^n, g = e^{2u}|dx|^2)$ be a complete noncompact even dimensional manifold. Let Q^+ and Q^- denote the positive and negative part of Q_g respectively; and dv_g denote the volume form of M . Suppose $g = e^{2u}|dx|^2$ is a "normal" metric, i.e.

$$(1.4) \quad u(x) = \frac{1}{c_n} \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} Q_g(y) dv_g(y) + C;$$

for some constant C . If

$$(1.5) \quad \beta^+ := \int_{M^n} Q^+ dv_g < c_n$$

where $c_n = 2^{n-2} \left(\frac{n-2}{2}\right)! \pi^{\frac{n}{2}}$, and

$$(1.6) \quad \beta^- := \int_{M^n} Q^- dv_g < \infty,$$

then (M^n, g) satisfies the isoperimetric inequality with isoperimetric constant depending only on n, β^+ and β^- . Namely, for any bounded domain $\Omega \subset M^n$ with smooth boundary,

$$(1.7) \quad |\Omega|_g^{\frac{n-1}{n}} \leq C(n, \beta^+, \beta^-) |\partial\Omega|_g.$$

The main purpose of the current paper is to further derive the fractional Poincaré inequality. The constant in this inequality is also controlled by the integral of the Q -curvature.

Theorem 1.1. Let $(M^n, g) = (\mathbb{R}^n, g = e^{2u}|dx|^2)$ be a complete noncompact even dimensional manifold. Let Q^+ and Q^- denote the positive and negative part of Q_g respectively; and dv_g denote the volume form of M . Suppose $g = e^{2u}|dx|^2$ is a "normal" metric, i.e.

$$(1.8) \quad u(x) = \frac{1}{c_n} \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} Q_g(y) dv_g(y) + C;$$

for some constant C . If

$$(1.9) \quad \beta^+ := \int_{M^n} Q^+ dv_g < c_n$$

where $c_n = 2^{n-2}(\frac{n-2}{2})!\pi^{\frac{n}{2}}$, and

$$(1.10) \quad \beta^- := \int_{M^n} Q^- dv_g < \infty,$$

then (M^n, g) satisfies the fractional Poincaré inequality with constant depending only on n, β^+ and β^- . Namely, for $\alpha \in (0, 2)$, there exists $C > 0$ depending only on n, β^+ and β^- , such that for any function f in $C^2(M^n)$ and any Euclidean ball B ,

$$(1.11) \quad \int_B |f(x) - f_{B,\omega}|^2 \omega(x) dx \leq C \int_{2B} \int_{2B} \frac{|f(x) - f(y)|^2}{d_g(x, y)^{n+\alpha}} \omega(x) \omega(y) dx dy.$$

The proof of Theorem 1.1 is based on several steps: the first one consists in deriving a 2-Poincaré inequality relying on the properties of the metric under consideration. This step gives in particular an important geometric meaning of the Q -curvature. The second step consists in using functional calculus to rewrite properly this Poincaré inequality and using spectral theory to estimate powers of a suitable weighted Laplacian. Finally, one derives the desired inequality in Theorem 1.1 by a covering argument and some estimates. This approach has been successfully used in [MRS11, RS11] to derive some types of fractional Poincaré inequalities in some Euclidean or geometric contexts. Notice that in these latter works, the fractional Poincaré inequality is not symmetric with respect to the measure in the right hand side. In our Theorem, this is the case due to a suitable covering and also the fact that we are considering local estimates.

Remark 1.2. We remark that $\liminf_{|x| \rightarrow \infty} R_g(x) \geq 0$ would imply $g = e^{2u}|dx|^2$ is a “normal” metric in dimension four. See [CQY00]. We also remark that the constant c_n in the assumption (1.9) is sharp. In fact, c_n is equal to the integral of the Q -curvature on a half cylinder (a cylinder with a round cap attached to one of its two ends); but obviously a half cylinder fails to satisfy the isoperimetric inequality. We also remark that being a normal metric is a natural and necessary assumption.

Remark 1.3. It is worth noting that normal metric is a necessary assumption in this theorem, because without it, there exist quadratic functions in the kernel of the bi-Laplacian operator Δ^2 (with respect to the flat metric) for which (2.3) fails.

The paper will be organized as follows. In section 2, we present preliminaries on the Q -curvature in conformal geometry. In section 3, we discuss A_p weights, their properties and relations to various inequalities. We then devote section 4 to the volume growth estimate of geodesic balls, which will be used in later sections, and discuss the p -Poincaré inequality in section 5. Finally, in section 6, we finish the proof of Theorem 1.1.

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2. PRELIMINARIES IN CONFORMAL GEOMETRY

In the past decades, there have been many works focusing on the study of the Q -curvature equation and the associated conformal covariant operators, both from PDE point of view and from the geometry point of view. We now discuss some background of it in conformal geometry. Consider a 4-manifold (M^4, g) , the Branson's Q -curvature of g is defined as

$$Q_g := \frac{1}{12} \left\{ -\Delta R_g + \frac{1}{4} R_g^2 - 3|E|^2, \right\}$$

where R_g is the scalar curvature, E_g is the traceless part of Ric_g , and $|\cdot|$ is taken with respect to the metric g . It is well known that the Q -curvature is an integral conformal invariant associated to the fourth order Paneitz operator P_g

$$P_g := \Delta^2 + \delta \left(\frac{2}{3} R_g g - 2 Ric_g \right) d.$$

Under the conformal change Q_{g_u} satisfies the fourth order differential equation. In the particular situation when the background metric $g_0 = |dx|^2$, the equation (1.1) reduces to

$$(-\Delta)^2 u = 2Q_{g_u} e^{4u},$$

where Δ is the Laplacian operator of the flat metric g_0 .

Another analogy between the Q -curvature and the Gaussian curvature is the invariance of the integral of the Q -curvature, due to the Chern-Gauss-Bonnet formula for closed manifold M^4 :

$$(2.1) \quad \chi(M^4) = \frac{1}{4\pi^2} \int_{M^4} \left(\frac{|W_g|^2}{8} + Q_g \right) dv_M,$$

where W_g denotes the Weyl tensor. There has been great progress in understanding the Q -curvature. For example see the work of Fefferman-Graham [FG12] on the study of the Q -curvature and ambient metrics, that of Chang-Qing-Yang [CQY00] on the Q -curvature and Cohn-Vossen inequality; and that of Malchiodi [Mal07], Chang-Gursky-Yang [CGY02] on the existence and regularity of constant Q -curvature metrics, etc.

For higher dimensions, the Q -curvature is defined via the analytic continuation in the dimension and the formula is not explicit in general. However when the background metric is flat, it satisfies, under the conformal change of metric $g_u = e^{2u}|dx|^2$, the n -th order differential equation

$$(-\Delta)^{\frac{n}{2}}u = 2Q_{g_u}e^{nu},$$

where Δ is the Laplacian operator of $|dx|^2$.

For complete conformally flat manifolds, the Gauss-Bonnet formula (2.1) is no longer valid. Chang-Qing-Yang [CQY00] proved instead the following result: let $(M^4, g) = (\mathbb{R}^4, e^{2u}|dx|^2)$ be a noncompact complete conformally flat manifold with finite total Q -curvature, i.e. $\int_{M^4} |Q_g| dv_g < \infty$. If the metric is normal, i.e.

$$(2.2) \quad u(x) = \frac{1}{4\pi^2} \int_{\mathbb{R}^4} \log \frac{|y|}{|x-y|} Q_g(y) e^{4u(y)} dy + C,$$

or if the scalar curvature R_g is nonnegative at infinity, then

$$(2.3) \quad \frac{1}{4\pi^2} \int_{M^4} Q_g dv_g \leq \chi(\mathbb{R}^4) = 1,$$

and

$$(2.4) \quad \chi(\mathbb{R}^4) - \frac{1}{4\pi^2} \int_{\mathbb{R}^4} Q_g dv_g = \sum_{j=1}^k \lim_{r \rightarrow \infty} \frac{\text{vol}_g(\partial B_j(r))^{4/3}}{4(2\pi^2)^{1/3} \text{vol}_g(B_j(r))},$$

where $B_j(r)$ denotes the Euclidean ball with radius r at the j -th end.

Chang, Qing and Yang's theorem asserts that for 4-manifolds (in fact, their theorem is valid for all even dimensions) which is conformal to the Euclidean space, the integral of the Q -curvature controls the asymptotic isoperimetric ratio at the end of this complete manifold. This is analogous to the two-dimensional result by Cohn-Vossen [CV35], who studied the Gauss-Bonnet integral for a noncompact complete surface M^2 with analytic metric. He showed that if the Gaussian curvature K_g is absolutely integrable (in which case we say the manifold has finite

total curvature), then

$$(2.5) \quad \frac{1}{2\pi} \int_M K_g dv_g \leq \chi(M),$$

where $\chi(M)$ is the Euler characteristic of M . Later, Huber [Hub57] extended this inequality to metrics with much weaker regularity. More importantly, he proved that such a surface M^2 is conformally equivalent to a closed surface with finitely many points removed. The difference of the two sides in inequality (2.5) encodes the asymptotic behavior of the manifold at its ends. The precise geometric interpretation has been given by Finn [Fin65] as follows. Suppose a noncompact complete surface has absolutely integrable Gaussian curvature. Then one may represent each end conformally as $\mathbb{R}^2 \setminus K$ for some compact set K . Define the asymptotic isoperimetric constant of the j -th end to be

$$\nu_i = \lim_{r \rightarrow \infty} \frac{L_g^2(\partial B(0, r) \setminus K)}{4\pi A_g(B(0, r) \setminus K)},$$

where $B(0, r)$ is the Euclidean ball centered at origin with radius r , L is the length of the boundary, and A is the area of the domain. Then

$$(2.6) \quad \chi(M) - \frac{1}{2\pi} \int_M K_g dv_g = \sum_{j=1}^N \nu_j,$$

where N is the number of ends on M . This result tells us that the condition of finite total Gaussian curvature has rigid geometric and analytical consequences.

Chang, Qing and Yang's results (2.3), (2.4) are higher dimensional counterparts of (2.5), (2.6).

3. A_p WEIGHTS AND STRONG A_∞ WEIGHTS

In this section, we are going to present the definitions and the properties of A_p weights and strong A_∞ weights.

In harmonic analysis, A_p weights ($p \geq 1$) are introduced to characterize when a function ω could be a weight such that the associated measure $\omega(x)dx$ has the property that the maximal function M of an L^1 function is weakly L^1 , and that the maximal function of an L^p function is L^p if $p > 1$.

For a nonnegative locally integrable function ω , we call it an A_p weight $p > 1$, if

$$(3.1) \quad \frac{1}{|B|} \int_B \omega(x) dx \cdot \left(\frac{1}{|B|} \int_B \omega(x)^{-p'/p} dx \right)^{p/p'} \leq C < \infty,$$

for all balls B in \mathbb{R}^n . Here p' is conjugate to p : $\frac{1}{p'} + \frac{1}{p} = 1$. The constant C is uniform for all B and we call the smallest such constant C the A_p bound of ω . The definition of A_1 weight is given by taking limit of $p \rightarrow 1$ in (3.1), which gives

$$\frac{1}{|B|} \int_B \omega \leq C\omega(x),$$

for almost all $x \in B$. Thus it is equivalent to say the maximal function of the weight is bounded by the weight itself:

$$M\omega(x) \leq C'\omega(x),$$

for a uniform constant C' . Another extreme case is the A_∞ weight. ω is called an A_∞ weight if it is an A_p weight for some $p > 1$. It is not difficult to see $A_1 \subseteq A_p \subseteq A_{p'} \subseteq A_\infty$ when $1 \leq p \leq p' \leq \infty$.

One of the most fundamental property of A_p weight is the reverse Hölder inequality: if ω is A_p weight for some $p \geq 1$, then there exists an $r > 1$ and a $C > 0$, such that

$$(3.2) \quad \left(\frac{1}{|B|} \int_B \omega^r dx \right)^{1/r} \leq \frac{C}{|B|} \int_B \omega dx,$$

for all balls B . This would imply that any A_p weight ω satisfies the doubling property: there is a $C > 0$ (it might be different from the constant C in (3.2)), such that

$$\int_{B(x_0, 2r)} \omega(x) dx \leq C \int_{B(x_0, r)} \omega(x) dx$$

for all balls $B(x_0, r) \subset \mathbb{R}^n$.

Suppose ω_1 and ω_2 are A_1 weights, and let t be any positive real number. Then it is not hard to show that $\omega_1\omega_2^{-t}$ is an A_∞ weight. Conversely, the factorization theorem of A_∞ weight proved by Peter Jones [Jon80] asserts: if ω is an A_∞ weight, then there exist ω_1 and ω_2 which are both A_1 weights, and $t > 1$ such that $\omega = \omega_1\omega_2^{-t}$. Later, in the proof of the main theorem, we will decompose the volume form e^{nu} into two pieces. The idea to decompose e^{nu} is inspired by Peter Jones' factorization theorem. In our case, we give an explicit decomposition of the weight e^{nu} , and by analyzing each part in the decomposition we finally prove that e^{nu} is a strong A_∞ weight, a class of weights much stronger than A_∞ that we will introduce in the following.

The notion of strong A_∞ weight was first proposed by David and Semmes in [DS90]. Given a positive continuous weight ω , we define

$\delta_\omega(x, y)$ to be:

$$(3.3) \quad \delta_\omega(x, y) := \left(\int_{B_{xy}} \omega(z) dz \right)^{1/n},$$

where B_{xy} is the ball with diameter $|x - y|$ that contains x and y . One can prove that δ_ω is only a quasi-distance in the sense that it satisfies the quasi-triangle inequality

$$\delta_\omega(x, y) \leq C(\delta_\omega(x, z) + \delta_\omega(z, y)).$$

On the other hand, for a continuous function ω , by taking infimum over all rectifiable arc $\gamma \subset B_{xy}$ connecting x and y , one can define the ω -distance to be

$$(3.4) \quad d_\omega(x, y) := \inf_\gamma \int_\gamma \omega^{\frac{1}{n}}(s) |ds|.$$

If ω is an A_∞ weight, then it is easy to prove (see for example Proposition 3.12 in [Sem93])

$$(3.5) \quad d_\omega(x, y) \leq C\delta_\omega(x, y)$$

for all $x, y \in \mathbb{R}^n$. If in addition to the above inequality, ω also satisfies the reverse inequality, i.e.

$$(3.6) \quad \delta_\omega(x, y) \leq Cd_\omega(x, y),$$

for all $x, y \in \mathbb{R}^n$, then we say ω is a strong A_∞ weight, and C is the bound of this strong A_∞ weight.

Every A_1 weight is a strong A_∞ weight, but for any $p > 1$ there is an A_p weight which is not strong A_∞ . Conversely, for any $p > 1$ there is a strong A_∞ weight which is not A_p . It is easy to verify by definition the function $|x|^\alpha$ is A_1 thus strong A_∞ if $-n < \alpha \leq 0$; it is not A_1 but still strong A_∞ if $\alpha > 0$. And $|x_1|^\alpha$ is not strong A_∞ for any $\alpha > 0$ as one can choose a curve γ contained in the x_2 -axis.

The notion of strong A_∞ weight was initially introduced in order to study weights that are comparable to the Jacobian of quasi-conformal maps. It was proved by Gehring that the Jacobian of a quasiconformal map on \mathbb{R}^n is always a strong A_∞ weight, and it was conjectured that the converse was assertive: every strong A_∞ weight is comparable to the Jacobian of a quasi-conformal map. Later, however, counter-examples were found by Semmes [Sem96] in dimension $n \geq 3$, and by Laakso [Laa02] in dimension 2. Nevertheless, it was proved by David and Semme that a strong A_∞ weight satisfies the Sobolev inequality:

Theorem 3.1. [DS90] Let ω be a strong A_∞ weight. Then for $f \in C_0^\infty(\mathbb{R}^n)$,

$$(3.7) \quad \left(\int_{\mathbb{R}^n} |f(x)|^{p^*} \omega(x) dx \right)^{1/p^*} \leq C \left(\int_{\mathbb{R}^n} (\omega^{-\frac{1}{n}}(x) |\nabla f(x)|)^p \omega(x) dx \right)^{1/p},$$

where $1 \leq p < n$, $p^* = \frac{np}{n-p}$. Take $p = 1$, it is the standard isoperimetric inequality. The constant C in the inequality only depends on the strong A_∞ bound of ω and n .

By taking f to be a smooth approximation of the indicator function of domain Ω , this implies the validity of the isoperimetric inequality with respect to the weight ω . In this paper, we will take $\omega = e^{nu}$, the volume form of $(\mathbb{R}^n, e^{2u}|dx|^2)$. We aim to show e^{nu} is a strong A_∞ weight. By Theorem 3.1, this implies the isoperimetric inequality on $(\mathbb{R}^n, e^{2u}|dx|^2)$:

$$\left(\int_{\Omega} e^{nu(x)} dx \right)^{\frac{n-1}{n}} \leq C \int_{\partial\Omega} e^{(n-1)u(x)} d\sigma_x,$$

or equivalently, for $g = e^{2u}|dx|^2$,

$$|\Omega|_g^{\frac{n-1}{n}} \leq C |\partial\Omega|_g.$$

A good reference for A_p weights is Chapter 5 in [Ste93]. For more details on strong A_∞ weight, we refer the readers to [DS90], where the concept was initially proposed.

4. VOLUME GROWTH OF GEODESIC BALLS

We will study in this section that the volume growth of geodesic balls is Euclidean. There are two different cases. In Case 1, we suppose $\int_M Q_g^+ dv_g < 4\pi^2$, and $\int_M Q_g^- dv_g < \infty$. Then the volume growth is Euclidean

$$(4.1) \quad C_2 r^n \leq Vol_g(B^g(0, r)) \leq C_1 r^n,$$

and the constants are uniformly controlled by n and the integral of Q -curvature. More precisely, C_i , $i = 1, 2$ depend only on n , $4\pi^2 - \int_M Q_g^+ dv_g > 0$, $\int_M Q_g^- dv_g$. This uniform result is derived from strong A_∞ property of the conformal factor e^{nu} , which was proved in [Wan15]. In the other case, which we call Case 2, we assume weaker assumptions on the integrals of the Q -curvature, namely $\int_M Q_g dv_g < 4\pi^2$, and

$\int_M |Q_g| dv_g < \infty$. As a consequence, we show that the volume growth is still Euclidean. However, the constants are not uniformly controlled by the integrals of the Q -curvature. In our proof of Theorem 1.1, since we need to localize arguments on each end of the locally conformally flat manifold, we will apply the consequence of Case 2. Case 1 is a stronger result, and is of independent interest. Therefore, we also provide proof in this section.

This result is going to be used in the proof of Theorem 1.1.

Proposition 4.1. Let $B^g(x_0, r)$ be a geodesic ball centered at x_0 , with radius r measured by the metric g . Then

$$C_2 r^n \leq \text{Vol}_g(B^g(x_0, r)) \leq C_1 r^n,$$

where C_i , $i = 1, 2$ depend only on n and $\int_M |Q_g| dv_g$.

Proof. By the main theorem of [Wan15], on a conformally flat manifold M with totally finite Q -curvature and normal metric $g = e^{2u}|dx|^2$, the isoperimetric inequality is valid. Moreover, the isoperimetric constant is uniformly controlled by n and $\int_M |Q_g| dv_g$. This gives directly the lower bound of volume growth of geodesic balls. Namely, there exists C_2 , depending only on the isoperimetric constant, thus only on n and $\int_M |Q_g| dv_g$, such that

$$C_2 r^n \leq \text{Vol}_g(B^g(0, r)).$$

On the other hand, [Wan15] also proves that the volume form e^{nu} is a strong A_∞ weight. We recall the definition of strong A_∞ weights.

Definition 4.2. Given a positive continuous weight ω , we define $\delta_\omega(x, y)$ to be:

$$(4.2) \quad \delta_\omega(x, y) := \left(\int_{B_{xy}} \omega(z) dz \right)^{1/n},$$

where B_{xy} is the ball with diameter $|x - y|$ that contains x and y . On the other hand, for a continuous function ω , by taking infimum over all rectifiable arc $\gamma \subset B_{xy}$ connecting x and y , one can define the ω -distance to be

$$(4.3) \quad d_\omega(x, y) := \inf_\gamma \int_\gamma \omega^{\frac{1}{n}}(s) |ds|.$$

If ω satisfies

$$(4.4) \quad d_\omega(x, y) \leq C\delta_\omega(x, y);$$

and the reverse inequality

$$(4.5) \quad \delta_\omega(x, y) \leq Cd_\omega(x, y),$$

for all $x, y \in \mathbb{R}^n$, then we say ω is a strong A_∞ weight, and C is the bound of this strong A_∞ weight.

Being a strong A_∞ weight, e^{nu} relates the volume of a geodesic sphere and the distance function. Given a geodesic ball $B^g(0, r)$, the Euclidean diameter of this ball is realized by two points x, y on the boundary $\partial B^g(0, r)$. Thus $B^g(x_0, r) \subset B^0(x, R)$, where R is the Euclidean distance of x, y , and $B^0(x, R)$ denotes the Euclidean ball centered at x with radius R .

Let us denote by p the middle point of x and y . Then by the doubling property of strong A_∞ weight e^{nu} ,

$$(4.6) \quad Vol_g(B^g(0, r)) \leq Vol_g(B^0(x, R)) \leq C_3 Vol_g(B^0(p, R/2)).$$

Notice x, y lie on $\partial B^0(p, R/2)$, and the line segment between them is the diameter of this Euclidean ball. By the definition of strong A_∞ weight,

$$(4.7) \quad Vol_g(B^0(p, R/2)) \leq C_4 d_g(x, y)^n.$$

Using the triangle inequality, it is obvious that

$$(4.8) \quad d_g(x, y) \leq 2r.$$

Thus we obtain

$$(4.9) \quad Vol_g(B^g(x_0, r)) \leq C_1 r^n.$$

C_3 depends only on the strong A_∞ bound of e^{nu} , and C_4 depends on n . Thus C_1 that is determined by C_3 and C_4 depends only on n and

$$\int_M |Q_g| dv_g. \quad \square$$

5. p -POINCARÉ INEQUALITY

By [Wan15], $\omega = e^{nu}$ is strong A_∞ . Thus by [DS90], we have point-wise Poincaré inequality: for any $x, y \in B$,

$$(5.1) \quad |f(x) - f(y)| \leq C \int_{B_{xy}} (\omega(B_{xu})^{-\frac{n-1}{n}} + \omega(B_{yu})^{-\frac{n-1}{n}}) |\nabla f(u)| \omega(u)^{\frac{n-1}{n}} du.$$

Here we denote by B_{xy} the smallest Euclidean ball that covers x and y .

Lemma 5.1. For a strong A_∞ weight ω , the above pointwise Poincaré inequality implies the p -Poincaré inequality, $p > 1$: For any Euclidean ball B , let $2B$ denote the concentric ball with double radius. Then

$$(5.2) \quad \begin{aligned} & \int_B |f(x) - f_{B,\omega}|^p \omega(x) dx \\ & \leq C \omega(B)^{\frac{p}{n}} \int_{2B} |\nabla f(u)|^p \omega(u)^{1-\frac{p}{n}} du, \end{aligned}$$

where $f_{B,\omega} = \frac{1}{\omega(B)} \int_B f(x) \omega(x) dx$.

Proof. By pointwise Poincaré inequality (5.1)

$$(5.3) \quad |f(x) - f(y)|^p \leq C \left(\int_{B_{xy}} (\omega(B_{xu})^{-\frac{n-1}{n}} + \omega(B_{yu})^{-\frac{n-1}{n}}) |\nabla f(u)| \omega(u)^{\frac{n-1}{n}} du \right)^p.$$

Applying Hölder's inequality, we obtain

$$(5.4) \quad \begin{aligned} |f(x) - f(y)|^p & \leq C \left(\int_{B_{xy}} |\nabla f(u)|^p \omega(u)^{-\frac{p}{n}} \left(\omega(B_{xu})^{-\frac{n-1}{n}} + \omega(B_{yu})^{-\frac{n-1}{n}} \right) \omega(u) du \right) \\ & \cdot \left(\int_{B_{xy}} \left(\omega(B_{xu})^{-\frac{n-1}{n}} + \omega(B_{yu})^{-\frac{n-1}{n}} \right) \omega(u) du \right)^{p-1}. \end{aligned}$$

$$(5.5) \quad \begin{aligned} \int_{2B} \omega(B_{xu})^{-\frac{n-1}{n}} \omega(u) du & \leq C \sum_{k \geq 0} \int_{u \in 2B, \omega(B_{xu}) \approx 2^{-k} \omega(2B)} 2^{\frac{k(n-1)}{n}} \omega(2B)^{-\frac{n-1}{n}} \omega(u) du \\ & \leq C \omega(B)^{-\frac{n-1}{n}} \sum_{k \geq 0} 2^{\frac{k(n-1)}{n}} 2^{-k} \omega(B) \\ & = C \omega(B)^{\frac{1}{n}}. \end{aligned}$$

Therefore,

$$(5.6) \quad \begin{aligned} & \int_{B_{xy}} \left(\omega(B_{xu})^{-\frac{n-1}{n}} + \omega(B_{yu})^{-\frac{n-1}{n}} \right) \omega(u) du \\ & \leq 2C \int_{2B} \omega(B_{xu})^{-\frac{n-1}{n}} \omega(u) du \\ & \leq C \omega(B)^{\frac{1}{n}}. \end{aligned}$$

Using it in (5.7), we have

$$(5.7) \quad \begin{aligned} & |f(x) - f(y)|^p \\ & \leq C\omega(B)^{\frac{p-1}{n}} \int_{B_{xy}} |\nabla f(u)|^p \omega(u)^{\frac{-p}{n}} \left(\omega(B_{xu})^{-\frac{n-1}{n}} + \omega(B_{yu})^{-\frac{n-1}{n}} \right) \omega(u) du. \end{aligned}$$

Hence

$$(5.8) \quad \begin{aligned} & \int_B \int_B |f(x) - f(y)|^p \omega(x) \omega(y) dx dy \\ & \leq C\omega(B)^{\frac{p-1}{n}} \int_{B_{xy}} \int_B \int_B |\nabla f(u)|^p \omega(u)^{\frac{-p}{n}} \left(\omega(B_{xu})^{-\frac{n-1}{n}} + \omega(B_{yu})^{-\frac{n-1}{n}} \right) \\ & \quad \omega(x) \omega(y) dx dy \omega(u) du. \end{aligned}$$

We use inequality (5.6) again to see that

$$(5.9) \quad \int_B \int_B \left(\omega(B_{xu})^{-\frac{n-1}{n}} + \omega(B_{yu})^{-\frac{n-1}{n}} \right) \omega(x) \omega(y) dx dy \leq C\omega(B)^{1+\frac{1}{n}}.$$

Therefore (5.8) becomes

$$(5.10) \quad \begin{aligned} & \int_B \int_B |f(x) - f(y)|^p \omega(x) \omega(y) dx dy \\ & \leq C\omega(B)^{1+\frac{p}{n}} \int_{B_{xy}} |\nabla f(u)|^p \omega(u)^{\frac{-p}{n}} \omega(u) du. \end{aligned}$$

Therefore,

$$(5.11) \quad \begin{aligned} & \int_B |f(x) - f_{B,\omega}|^p \omega(x) dx \\ & \leq C\omega(B)^{\frac{p}{n}} \int_{2B} |\nabla f(u)|^p \omega(u)^{1-\frac{p}{n}} du, \end{aligned}$$

where $f_{B,\omega} = \frac{1}{\omega(B)} \int_B f(x) \omega(x) dx$.

Since $g = e^{2u}|dx|^2$, $dv_g = e^{nu} dx$, this is equivalent to

$$(5.12) \quad \int_B |f(x) - f_{B,\omega}|^p dv_g \leq C \text{vol}_g(B)^{\frac{p}{n}} \int_{2B} |\nabla_g f(u)|^p dv_g.$$

□

In particular, let $p = 2$, we have

$$(5.13) \quad \int_B |f(x) - f_{B,\omega}|^2 dv_g \leq C \text{vol}_g(B)^{\frac{2}{n}} \int_{2B} |\nabla_g f(u)|^2 dv_g.$$

We define the operator m by multiplication with χ_B

$$m(f) := f\chi_B;$$

the measure μ_2 by

$$\mu_2 := \omega(x)\chi_{2B}dx;$$

and the weighted divergence operator L_{μ_2} by

$$L_{\mu_2}f := \Delta_g f = \sum_{i=1}^n \frac{1}{\omega(x)} \partial_i (\omega(x)^{1-\frac{2}{n}} \partial_i f).$$

Thus (5.13) is equivalent to the following statement: for $f_{B,\omega} = 0$

$$(5.14) \quad \int_{\mathbb{R}^n} m(f)^2 d\mu_2 \leq C \text{vol}_g(B)^{\frac{2}{n}} \int_{\mathbb{R}^n} |L_{\mu_2}^{1/2} f|^2 d\mu_2.$$

Thus

$$\|m(f)\|_{L^2(\mathbb{R}^n, \mu_2)} \leq C \text{vol}_g(B)^{\frac{1}{n}} \|L_{\mu_2}^{1/2} f\|_{L^2(\mathbb{R}^n, \mu_2)}.$$

By spectral theory, $m(f) \leq CL_{\mu_2}^{1/2} f$ in $L^2(\mathbb{R}^n, \mu_2)$ implies $m^\alpha(f) \leq CL_{\mu_2}^{\alpha/2} f$ in $L^2(\mathbb{R}^n, \mu_2)$. Thus for f satisfying $f_{B,\omega} = 0$,

$$\int_{\mathbb{R}^n} m^\alpha(f) \cdot f d\mu_2 \leq C \int_{\mathbb{R}^n} L_{\mu_2}^{\alpha/2} f \cdot f d\mu_2.$$

In other words, for any f

$$\left(\int_B |f(x) - f_{B,\omega}|^2 \omega(x) dx \right)^{1/2} = \left(\int_{\mathbb{R}^n} m^\alpha(f) \cdot f d\mu_2 \right)^{1/2} \leq C \|L_{\mu_2}^{\alpha/4} f\|_{L^2(\mathbb{R}^n, \mu_2)}.$$

6. CONTROL OF $\|L_{\mu_2}^{\alpha/4} f\|_{L^2}$

We start with several lemmas. The following lemma provides off-diagonal estimates. This is simply on an energy inequality and we refer the reader to [RS11] for a proof.

Lemma 6.1. There exists C with the following property: for all closed disjoint subsets $E, F \subset G$ with $d(E, F) =: d > 0$, all function $f \in L^2(G, d\mu_M)$ supported in E and all $t > 0$,

$$\begin{aligned} & \left\| (I + t L_{\mu_2})^{-1} f \right\|_{L^2(F, \mu_2)} + \left\| t L_{\mu_2} (I + t L_{\mu_2})^{-1} f \right\|_{L^2(F, \mu_2)} \leq \\ & 8 e^{-C \frac{d}{\sqrt{t}}} \|f\|_{L^2(E, \mu_2)}. \end{aligned}$$

The next lemma is just spectral theory (see [RS11] for a proof).

Lemma 6.2. Let $\alpha \in (0, 2)$. Let $\mathcal{D}(L_{\mu_2})$ be the domain of functions on which L_{μ_2} is well defined. There exists $C > 0$ such that for all $f \in \mathcal{D}(L_{\mu_2})$,

$$(6.1) \quad \|L_{\mu_2}^{\alpha/4} f\|_{L^2(\mathbb{R}^n, \mu_2)}^2 \leq C \int_0^\infty t^{-1-\alpha/2} \|tL_{\mu_2}(I + tL_{\mu_2})^{-1} f\|_{L^2(\mathbb{R}^n, \mu_2)}^2 dt.$$

Lemma 6.3. Fix $t > 0$. $B^g(x_j^t, \sqrt{t})$, $j = 1, \dots$ is a Vitali's covering of $(M^n, g) = (\mathbb{R}^n, e^{2u}|dx|^2)$, such that they are mutually disjoint, and

$$M^n = \bigcup_{j \in \mathbb{N}} B^g(x_j^t, 2\sqrt{t}).$$

Then there exists \tilde{C} such that for all $\theta > 1$ and $x \in M^n$, there are at most $\tilde{C}\theta^{2k}$ indices j satisfying $d_g(x, y) \leq \theta\sqrt{t}$.

Proof. We follow the same proof as that in [Kan85], as well as in [RS11]. Let $x \in M^n$ and

$$I(x) := \left\{ j \in \mathbb{N}; d_g(x - x_j^t) \leq \theta\sqrt{t}. \right\}$$

For all $j \in I(x)$

$$B^g(x_j^t, \sqrt{t}) \subset B^g(x, (1 + \theta)\sqrt{t}),$$

and

$$B^g(x, \sqrt{t}) \subset B^g(x_j^t, (1 + \theta)\sqrt{t}).$$

Also, by Proposition 4.1, the measure $\omega(x)dx$ has doubling property. Therefore,

$$(6.2) \quad \begin{aligned} I(x)\omega(B^g(x, \sqrt{t})) &\leq \sum_{j \in I(x)} \omega(B^g(x_j^t, (1 + \theta)\sqrt{t})) \\ &\leq C(1 + \theta)^k \sum_{j \in I(x)} \omega(x_j^t, \sqrt{t}) \\ &\leq C(1 + \theta)^k \omega(x, (1 + \theta)\sqrt{t}) \\ &\leq C(1 + \theta)^{2k} \omega(x, \sqrt{t}). \end{aligned}$$

□

Lemma 6.4. There exists $C > 0$ such that for all $t > 0$ and all $j \in \mathbb{N}$

$$(6.3) \quad \|g_0^{j,t}\|_{L^2(C_k^{j,t}, \mu_2)}^2 \leq \frac{C}{\omega(B^g(\sqrt{t}))} \int_{B^g(x_j^t, 4\sqrt{t})} \int_{B^g(x_j^t, 4\sqrt{t})} |f(x) - f(y)|^2 d\mu_2(x) d\mu_2(y).$$

B. For all $k \geq 1$,

$$(6.4) \quad \begin{aligned} & \|g_k^{j,t}\|_{L^2(C_k^{j,t}, d\mu_2)}^2 \\ & \leq \frac{C}{\omega(B^g(2^k\sqrt{t}))} \int_{B^g(x_j^t, 2^{k+2}\sqrt{t})} \int_{B^g(x_j^t, 2^{k+2}\sqrt{t})} |f(x) - f(y)|^2 d\mu_2(x) d\mu_2(y). \end{aligned}$$

Proof.

$$(6.5) \quad \begin{aligned} g_0^{j,t}(x) &= f(x) - \frac{1}{\omega(B^g(2\sqrt{t}))} \int_{B^g(x_j^t, 2\sqrt{t})} f(y) d\mu_2(y) \\ &= \frac{1}{\omega(B^g(2\sqrt{t}))} \int_{B^g(x_j^t, 2\sqrt{t})} (f(x) - f(y)) d\mu_2(y). \end{aligned}$$

By Cauchy-Schwarz inequality,

$$(6.6) \quad |g_0^{j,t}(x)|^2 \leq \frac{C}{\omega(B^g(4\sqrt{t}))} \int_{B^g(x_j^t, 2\sqrt{t})} |f(x) - f(y)|^2 d\mu_2(y).$$

Hence

$$(6.7) \quad \begin{aligned} & \|g_0^{j,t}(x)\|_{L^2(C_0^{j,t}, \mu_2)}^2 \\ & \leq \frac{C}{\omega(B^g(2\sqrt{t}))} \int_{B^g(x_j^t, 4\sqrt{t})} \int_{B^g(x_j^t, 4\sqrt{t})} |f(x) - f(y)|^2 d\mu_2(x) d\mu_2(y). \end{aligned}$$

By a similar argument, we prove Part B of the lemma as well:

$$(6.8) \quad \begin{aligned} & \|g_k^{j,t}(x)\|_{L^2(C_k^{j,t}, \mu_2)}^2 \\ & \leq \frac{C}{\omega(B^g(2^k\sqrt{t}))} \int_{B^g(x_j^t, 2^{k+2}\sqrt{t})} \int_{B^g(x_j^t, 2^{k+2}\sqrt{t})} |f(x) - f(y)|^2 d\mu_2(x) d\mu_2(y). \end{aligned}$$

□

Lemma 6.5. Let $\alpha \in (0, 2)$. Let $\mathcal{D}(L_{\mu_2})$ be the domain of functions on which L_{μ_2} is well defined. There exists $C > 0$ such that for all $f \in \mathcal{D}(L_{\mu_2})$,

$$(6.9) \quad \begin{aligned} & \int_0^\infty t^{-1-\alpha/2} \|tL_{\mu_2}(I + tL_{\mu_2})^{-1}f\|_{L^2(\mathbb{R}^n, \mu_2)}^2 dt \\ & \leq C \int_{2B} \int_{2B} \frac{|f(x) - f(y)|^2}{d_g(x, y)^{n+\alpha}} \omega(x)\omega(y) dx dy. \end{aligned}$$

Proof. Given $t \in (0, +\infty)$, we will give an upper bound containing the first order difference of f . By Proposition 4.1, volume of geodesic balls on (M^n, g) satisfies the doubling property. Thus we can find a Vitali's

covering of $(M^n, g) = (\mathbb{R}^n, e^{2u}|dx|^2)$: there exists a countable family of balls $B^g(x_j^t, \sqrt{t})$, such that they are mutually disjoint, and

$$M^n = \bigcup_{j \in \mathbb{N}} B^g(x_j^t, 2\sqrt{t}).$$

By Lemma 6.3, there exists a constant C , such that for all $\theta > 1$, and $x \in \mathbb{R}^n$, there are at most $C\theta^{2\kappa}$ indexes j such that $d_g(x, x_j^t) \leq \theta\sqrt{t}$ where κ is given by

For a fixed j , one has

$$tL_{\mu_2}(I + tL_{\mu_2})^{-1}f = tL_{\mu_2}(I + tL_{\mu_2})^{-1}g^{j,t}$$

where $g^{j,t}$ is defined to be

$$g^{j,t}(x) := f(x) - m^{j,t},$$

$$m^{j,t} := \frac{1}{\omega(B^g(x_j^t, 2\sqrt{t}))} \int_{B^g(x_j^t, 2\sqrt{t})} f(y) d\mu_2(y).$$

Then

$$(6.10) \quad \begin{aligned} & \|tL_{\mu_2}(I + tL_{\mu_2})^{-1}f\|_{L^2(\mathbb{R}^n, \mu_2)}^2 \\ & \leq \sum_{j \in \mathbb{N}} \|tL_{\mu_2}(I + tL_{\mu_2})^{-1}f\|_{L^2(B^g(x_j^t, 2\sqrt{t}), \mu_2)}^2 \\ & \leq \sum_{j \in \mathbb{N}} \|tL_{\mu_2}(I + tL_{\mu_2})^{-1}g^{j,t}\|_{L^2(B^g(x_j^t, 2\sqrt{t}), \mu_2)}^2. \end{aligned}$$

To estimate

$$\sum_{j \in \mathbb{N}} \|tL_{\mu_2}(I + tL_{\mu_2})^{-1}g^{j,t}\|_{L^2(B^g(x_j^t, 2\sqrt{t}), \mu_2)}^2,$$

we set

$$C_0^{j,t} = B^g(x_j^t, 4\sqrt{t}),$$

and

$$C_k^{j,t} = B^g(x_j^t, 2^{k+2}\sqrt{t}) \setminus B^g(x_j^t, 2^{k+1}\sqrt{t}),$$

for $k \geq 1$. We also define

$$g_k^{j,t}(x) := g^{j,t}(x)\chi_{C_k^{j,t}}(x),$$

where $\chi_{C_k^{j,t}}(x)$ is the characteristic function of a set $C_k^{j,t}$. Since $g^{j,t} = \sum_{k \geq 0} g_k^{j,t}$, we have

$$(6.11) \quad \begin{aligned} & \|tL_{\mu_2}(I + tL_{\mu_2})^{-1}g^{j,t}\|_{L^2(B^g(x_j^t, 2\sqrt{t}), \mu_2)} \\ & \leq \sum_{k \geq 0} \|tL_{\mu_2}(I + tL_{\mu_2})^{-1}g_k^{j,t}\|_{L^2(B^g(x_j^t, 2\sqrt{t}), \mu_2)}. \end{aligned}$$

Using Lemma 6.1 we obtain that

$$(6.12) \quad \begin{aligned} & \|tL_{\mu_2}(I + tL_{\mu_2})^{-1}g^{j,t}\|_{L^2(B^g(x_j^t, 2\sqrt{t}), \mu_2)} \\ & \leq C \left(\|g_0^{j,t}\|_{L^2(C_0^{j,t}, \mu_2)} + \sum_{k \geq 1} e^{-c2^k} \|g_k^{j,t}\|_{L^2(C_k^{j,t}, \mu_2)} \right). \end{aligned}$$

By Cauchy-Schwarz's inequality, we deduce

$$(6.13) \quad \begin{aligned} & \|tL_{\mu_2}(I + tL_{\mu_2})^{-1}g^{j,t}\|_{L^2(B^g(x_j^t, 2\sqrt{t}), \mu_2)}^2 \\ & \leq C' \left(\|g_0^{j,t}\|_{L^2(C_0^{j,t}, \mu_2)}^2 + \sum_{k \geq 1} e^{-c2^k} \|g_k^{j,t}\|_{L^2(C_k^{j,t}, \mu_2)}^2 \right). \end{aligned}$$

Therefore,

$$(6.14) \quad \begin{aligned} & \int_0^\infty t^{-1-\alpha/2} \|tL_{\mu_2}(I + tL_{\mu_2})^{-1}f\|_{L^2(\mathbb{R}^n, \mu_2)}^2 dt \\ & \leq C' \int_0^\infty t^{-1-\alpha/2} \sum_{j \geq 0} \|g_0^{j,t}\|_{L^2(C_0^{j,t}, \mu_2)}^2 dt + \\ & \quad C' \int_0^\infty t^{-1-\alpha/2} \sum_{k \geq 1} e^{-c2^k} \sum_{j \geq 0} \|g_k^{j,t}\|_{L^2(C_k^{j,t}, \mu_2)}^2 dt. \end{aligned}$$

We now apply Lemma 6.4 to finish the proof. Using part A, and integrating t over $(0, \infty)$, we get

$$(6.15) \quad \begin{aligned} & \int_0^\infty t^{-1-\alpha/2} \sum_{j \geq 0} \|g_0^{j,t}\|_{L^2(C_0^{j,t}, \mu_2)}^2 dt \\ & = \sum_{j \geq 0} \int_0^\infty t^{-1-\alpha/2} \|g_0^{j,t}\|_{L^2(C_0^{j,t}, \mu_2)}^2 dt \\ & \leq C \sum_{j \geq 0} \int_0^\infty \frac{t^{-1-\alpha/2}}{\omega(B^g(\sqrt{t}))} \left(\int_{B^g(x_j^t, 4\sqrt{t})} \int_{B^g(x_j^t, 4\sqrt{t})} |f(x) - f(y)|^2 d\mu_2(x) d\mu_2(y) \right) dt \\ & \leq C \sum_{j \geq 0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x) - f(y)|^2 \\ & \quad \cdot \left(\int_{t \geq \max\left\{\frac{d_g(x, x_j^t)^2}{16}, \frac{d_g(y, x_j^t)^2}{16}\right\}} \frac{t^{-1-\alpha/2}}{\omega(B^g(2^k \sqrt{t}))} dt \right) d\mu_2(x) d\mu_2(y). \end{aligned}$$

By Lemma $\omega(B^g(x, r))$ satisfies

$$C_2 r^n \leq \omega(B^g(x, r)) \leq C_1 r^n,$$

where C_1, C_2 are independent of x . Therefore, in the above formula, we denote volume of a ball with radius \sqrt{t} by $\omega(B^g(\sqrt{t}))$, disregarding where the center is. Notice that

$$(6.16) \quad \begin{aligned} & \sum_{j \geq 0} \int_{t \geq \max\left\{\frac{d_g(x, x_j^t)^2}{16}, \frac{d_g(y, x_j^t)^2}{16}\right\}} \frac{t^{-1-\alpha/2}}{\omega(B^g(2^k \sqrt{t}))} dt = \\ & \int_0^\infty \frac{t^{-1-\alpha/2}}{\omega(B^g(\sqrt{t}))} \sum_{j \geq 0} \chi\left(\max\left\{\frac{d_g(x, x_j^t)^2}{16}, \frac{d_g(y, x_j^t)^2}{16}\right\}, \infty\right)(t) dt. \end{aligned}$$

By Lemma 6.3, there exists a constant $N \in \mathbb{N}$, such that for all $t > 0$, there are at most N indexes j such that $d_g(x, x_j^t) < 16t$, and $d_g(y, x_j^t) < 16t$, and for these indexes j , $d_g(x, y) < 8t$. Thus

$$(6.17) \quad \sum_{j \geq 0} \chi\left(\max\left\{\frac{d_g(x, x_j^t)^2}{16}, \frac{d_g(y, x_j^t)^2}{16}\right\}, \infty\right)(t) dt \leq N \chi\left(\frac{d_g(x, y)^2}{64}, \infty\right).$$

We therefore obtain

$$(6.18) \quad \begin{aligned} & \int_0^\infty t^{-1-\alpha/2} \sum_{j \geq 0} \|g_0^{j,t}\|_{L^2(C_0^{j,t}, \mu_2)}^2 dt \\ & \leq CN \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x) - f(y)|^2 \\ & \quad \cdot \left(\int_{t \geq d_g(x, y)^2/64} \frac{t^{-1-\alpha/2}}{\omega(B^g(2^k \sqrt{t}))} dt \right) d\mu_2(x) d\mu_2(y) \\ & \leq CN \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{\omega(B_{xy}^g) \cdot d_g(x, y)^\alpha} d\mu_2(x) d\mu_2(y). \end{aligned}$$

By part B in Lemma 6.4, we have for all $j \geq 0$ and $k \geq 1$,

$$(6.19) \quad \begin{aligned} & \int_0^\infty t^{-1-\alpha/2} \sum_{j \geq 0} \|g_k^{j,t}\|_{L^2(C_0^{j,t}, \mu_2)}^2 dt \\ & \leq C \sum_{j \geq 0} \int_0^\infty \frac{t^{-1-\alpha/2}}{\omega(B^g(2^k \sqrt{t}))} \int_{B^g(x_j^t, 2^{k+2}\sqrt{t})} \int_{B^g(x_j^t, 2^{k+2}\sqrt{t})} |f(x) - f(y)|^2 d\mu_2(x) d\mu_2(y) dt \\ & \leq C \sum_{j \geq 0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x) - f(y)|^2 \\ & \quad \cdot \left(\int_0^\infty \frac{t^{-1-\alpha/2}}{\omega(B^g(2^k \sqrt{t}))} \chi\left(\max\left\{\frac{d_g(x, x_j^t)^2}{4k+2}, \frac{d_g(y, x_j^t)^2}{4k+2}\right\}, \infty\right)(t) dt \right) d\mu_2(x) d\mu_2(y). \end{aligned}$$

With a given $t > 0$, $x, y \in \mathbb{R}^n$, by Lemma 6.4, there exist at most $C2^{2k\kappa}$ indexes j such that

$$d_g(x, x_j^t) \leq 2^{k+2}\sqrt{t},$$

and

$$d_g(y, x_j^t) \leq 2^{k+2}\sqrt{t},$$

For these j 's,

$$d_g(x, y) \leq 2^{k+3}\sqrt{t}.$$

Therefore,

$$\begin{aligned} & \int_0^\infty \frac{t^{-1-\alpha/2}}{\omega(B^g(2^k\sqrt{t}))} \chi_{\left(\max\left\{\frac{d_g(x, x_j^t)^2}{4^{k+2}} - \frac{d_g(y, x_j^t)^2}{4^{k+2}}, \infty\right\}, \infty\right)}(t) dt \\ (6.20) \quad & \leq C2^{2k\kappa} \int_{t \geq \frac{d_g(x, y)^2}{4^{k+3}}} \frac{t^{-1-\alpha/2}}{\omega(B^g(2^k\sqrt{t}))} dt \\ & \leq C \frac{2^{k(2\kappa+\alpha)}}{\omega(B_{xy}^g) \cdot d_g(x, y)^\alpha}. \end{aligned}$$

We therefore get

$$\begin{aligned} & \int_0^\infty \frac{t^{-1-\alpha/2}}{\omega(B^g(2^k\sqrt{t}))} \sum_{j \geq 0} \|g_k^{j,t}\|_{L^2(C_0^{j,t}, \mu_2)}^2 dt \\ (6.21) \quad & \leq C2^{k(2\kappa+\alpha)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{\omega(B_{xy}^g) \cdot d_g(x, y)^\alpha} d\mu_2(x) d\mu_2(y). \end{aligned}$$

In conclusion, we have

$$\begin{aligned} & \int_0^\infty t^{-1-\alpha/2} \|tL_{\mu_2}(I + tL_{\mu_2})^{-1}f\|_{L^2(\mathbb{R}^n, \mu_2)}^2 dt \\ & \leq CN \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{\omega(B_{xy}^g) \cdot d_g(x, y)^\alpha} d\mu_2(x) d\mu_2(y) \\ (6.22) \quad & + \sum_{k \geq 1} C2^{k(2\kappa+\alpha)} e^{-c2^k} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{\omega(B_{xy}^g) \cdot d_g(x, y)^\alpha} d\mu_2(x) d\mu_2(y) \\ & \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{\omega(B_{xy}^g) \cdot d_g(x, y)^\alpha} d\mu_2(x) d\mu_2(y) \\ & = C \int_{2B} \int_{2B} \frac{|f(x) - f(y)|^2}{\omega(B_{xy}^g) \cdot d_g(x, y)^\alpha} \omega(x)\omega(y) dx dy. \end{aligned}$$

Finally, by the fact that ω is a strong A_∞ weight,

$$\omega(B_{xy}^g) \approx d_g(x, y)^n.$$

Plugging this into (6.22), we completes the proof of the lemma. \square

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