Covariant renormalizable gravity model as a mimetic Horndeski model: cosmological solutions and perturbations

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We consider the Nojiri-Odintsov covariant Hořava-like gravitational model, where diffeomorphism invariance is broken dynamically via a non-standard coupling to a perfect fluid. The theory allows to address some of the potential instability problems present in Hořava-Lifshitz gravity due to explicit diffeomorphism invariance breaking. The fluid is instead constructed from a scalar field constrained by a Lagrange multiplier. This construction allows to identify the scalar field with the mimetic field of the recently proposed mimetic gravity. Subsequently, we thoroughly explore the consequences of this identification. By adding a potential for the scalar field, we show how one can reproduce a number of interesting cosmological scenarios. We then turn to the study of perturbations around a flat FLRW background, showing that the fluid in question behaves as an irrotational fluid, with zero sound speed. To address this problem, we consider a modified version of the theory, adding higher derivative terms in a way which brings us beyond the Horndeski model. We compute the sound speed in this modified higher order mimetic model and show that it is non-zero, which means that perturbations therein can be sensibly defined. In conclusion, we construct a renormalizable theory of gravity which preserves diffeomorphism invariance at the level of the action but breaks it dynamically in the UV, reduces to General Relativity in the IR, allows the realization of a number of interesting cosmological scenarios and is well defined when considering perturbations around a flat FLRW background.

I. INTRODUCTION

Of the four fundamental interactions, gravity retains a particularly problematic status in that the corresponding quantum theory has thus far proved elusive, due to the non-renormalizability of General Relativity (GR). It is clear that GR should be viewed as an effective field theory (EFT) bound to break down at some high energy scale (presumably the Planck scale), and which contains only the leading term in a curvature expansion. One possibility to render gravity renormalizable is to modify the UV behaviour of the graviton propagator, for instance through the addition of higher order curvature terms. While it is known that actions constructed from invariants quadratic in curvature are renormalizable [1], the addition of higher curvature in fact correspond to the addition of higher order time derivatives, which lead to the appearance of ghost degrees of freedom and hence a loss of unitarity (see e.g. [2] for a general review). Clearly, then, a solution is to improve the UV behaviour of the graviton propagator by adding higher order spatial but not time derivatives, which in turn requires treating space and time on a different footing, leading to Lorentz violation. Using Lorentz violation in the UV as a field theory regulator is a possibility which has been already considered in the past, although the amount of Lorentz violation should be appropriately suppressed in the IR. In particular, the formulation of non-relativistic theories of gravity has been driven by the endeavours to describe non-relativistic field theories via through AdS/CFT.

Hořava-Lifshitz gravity (HLG) [3], proposed by Petr Hořava in 2009, is an attempt to embed these heuristic ideas into a rigorous framework. The idea behind the proposal is then very simple: to render gravity power-counting

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renormalizable by abandoning Lorentz symmetry in favour of a Lifshitz-type anisotropic scaling in the UV. The theory is compatible with anisotropic scaling on the space and time coordinates ($\mathbf{x}$ and t respectively) with dynamical critical exponent z, that is:

$$t \to b^z t\,, \quad \mathbf{x} \to b\mathbf{x}\,. \tag{1}$$

The above should be understood in the sense that the theory possesses a solution which describes an UV fixed point whose scaling properties are described by Eq.(1). If anisotropic scaling with $z \geq 3$ is realized in the UV, the theory is power-counting renormalizable in 3+1 dimensions, which leads to its conjectured renormalizability. In the IR, the theory naturally flows to $z = 1$. It is clear that for $z \neq 1$, Lorentz invariance is lost.[1] The theory is instead invariant under space-independent time reparametrizations and time-dependent spatial diffeomorphisms, that is, transformations of the form:

$$t \to t'(t)\,, \quad \mathbf{x} \to \tilde{\mathbf{x}}(t, \mathbf{x})\,. \tag{2}$$

The map in Eq.(2) is known as a foliation preserving diffeomorphism. In HLG, unlike GR, the foliation of spacetime by constant hypersurfaces is therefore more than just a choice of coordinates. Hořava-Lifshitz gravity has attracted much interest lately and lead to a spree of publications. For recent comprehensive reviews, see for instance [4], with [5] describing $F(R)$ HLG. Refer instead to [6–11] for discussions on cosmology within HLG.

Despite its success, HLG has been criticized by several authors (see e.g. [12–14] for some of the early criticisms). It has been argued that the theory possesses additional unphysical modes, associated to the explicit breaking of diffeomorphism invariance, which do not decouple and actually become strongly coupled in the IR, hence preventing the recovery of the perturbative GR limit at low energies. To circumvent this problem, in 2009 Nojiri and Odintsov put forward a Hořava-like model which retains full diffeomorphism invariance at the level of the action [15], which we will refer to as covariant renormalizable gravity (CRG henceforth). The model features a non-standard coupling of curvature ($R_{\mu\nu}$ and R) to the energy-momentum tensor of a perfect exotic fluid. When considering perturbations around the background, diffeomorphism invariance is broken dynamically by this non-standard coupling. Due to the full diffeomorphism invariance, only physical transverse modes are present in the model. Much as in the case of HLG, the graviton propagator behaves as $1/\mathbf{k}^{2z}$ in the UV, where $\mathbf{k}$ denotes spatial momenta and $z \geq 3$ in order for the theory to be power-counting (super-)renormalizable, and a consistent theory was constructed for any integer value of z. The model was reformulated and generalized in [16], where an effective fluid with arbitrary equation of state (EoS) parameter w was constructed by means of a Lagrange multiplier constrained scalar field, following ideas first presented in [17–19] to unify dark matter and dark energy.

The price to pay in CRG is the presence of this exotic fluid, which does not correspond to the usual perfect fluids found in cosmology, whose origin is unknown or possibly string-inspired. However, it was recently realized in [20] that CRG is intimately connected to the framework of mimetic gravity, which also features fluids constructed from Lagrange multiplier constrained scalar fields. In mimetic gravity, one isolates the conformal degree of freedom in a covariant way, by parametrizing the physical metric in terms of an auxiliary metric and a scalar field (the mimetic field). The Lagrange multiplier term in the action constrains the 4-gradient of the mimetic field, which in turn induces an effective fluid which can be associated to collisionless cold dark matter. Hence, the exotic fluid in CRG is in a sense related to the conformal mode of gravity. Mimetic gravity is also a special case of the most general second-order scalar-tensor theory of gravity, known as Horndeski gravity [21], and is related to GR by singular disformal transformations. In fact, recently a general scalar-tensor mimetic Horndeski theory has been considered, and it has been argued that the two approaches to mimetic gravity (i.e. disformal transformations and Lagrange multiplier constraints) are indeed equivalent. Motivated by the recent spur of interest towards Hořava-Lifshitz gravity, mimetic gravity and Horndeski gravity, our aim in this work is to rigorously explore the connections between these different theories, and in particular to covariant renormalizable gravity. In particular, it is our objective to inspect the consequences of identifying the exotic fluid in CRG with the mimetic fluid.

This paper is organized as follows. Having already reviewed the basic ideas of Hořava-Lifshitz gravity in Section I, we will proceed to briefly review CRG and mimetic gravity in Section II, outlining the connection between the two, and determining under what conditions the two are equivalent. Section III will be devoted to generalizing CRG with

[1] In fact, in the context of a more fundamental theory where spacetime itself is emergent, it is difficult to conceive how Lorentz invariance could be preserved at a fundamental level.

the addition of a potential for the mimetic scalar field. In Section IV we will explore some cosmological solutions of generalized CRG. In the same section we will consider perturbations around a flat FLRW background, and note that the speed of sound therein is identically zero, thus preventing us from defining perturbations in the mimetic field in the usual way. To overcome this drawback, in Section V we consider a modified generalized version of our original mimetic CRG model, by adding higher derivative terms. We show that in the modified version, the speed of sound is nonvanishing. We draw some concluding remarks in Section VI. Throughout the paper, we will set $8\pi M_{Pl}^2 = 1$, where M_{Pl} is the Planck mass.

II. COVARIANT RENORMALIZABLE GRAVITY AND MIMETIC GRAVITY

A. Nojiri-Odintsov model

Let us first review Nojiri-Odintsov covariant renormalizable gravity. The basic action of the theory (to be later generalized) takes the form:

$$S = \int d^4x \, \sqrt{-g} \left[\frac{R}{2} - \alpha \left(T^{\mu\nu} R_{\mu\nu} + \beta T R \right)^2 \right] , \quad \beta = \frac{w-1}{2(1-3w)} . \tag{3}$$

In the above, $T_{\mu\nu}$ is the energy-momentum tensor of the perfect fluid, whose EoS parameter is w. When perturbing around a flat background as, the term in the action [Eq.(3)] proportional to α will only spatial and not time derivatives, and hence breaks full diffeomorphism invariance dynamically. It can be shown that the graviton propagator will now behave like $1/\mathbf{k}^4$ in the UV: that is, we have recovered a dynamical $z = 2$ Hořava-like theory. The argument holds if $w \neq -1, 1/3$ for in the former case the coupling to the fluid vanishes and hence the graviton propagator behaviour is not modified, whereas in the latter the coupling itself diverges. As was shown in [15], due to the absence of explicit symmetry breaking terms, only transverse physical modes propagate in the theory, unlike the case of HLG where unphysical longitudinal modes which are strongly coupled in the IR might appear. The above arguments hold in a curved background as well [15].

The action in Eq.(3) gave us the $z = 2$ Hořava Lisfhitz-like theory, but can be easily generalized to accommodate any $z \geq 3$. The general action given by:

$$S = \int d^4x \, \sqrt{-g} \left\{ \frac{R}{2} - \alpha \left[\left(T^{\mu\nu} \nabla_\mu \nabla_\nu + \gamma T \nabla^\rho \nabla_\rho \right)^n \left(T^{\mu\nu} R_{\mu\nu} + \beta T R \right) \right]^2 \right\} , \quad \beta = \frac{w-1}{2(1-3w)} , \quad \gamma = \frac{1}{3w-1} . \tag{4}$$

which corresponds to the $z = 2n + 2$ Hořava-like theory. Clearly when $n = 0$ we recover the basic action in Eq.(3). The terms inducing dynamical Lorentz symmetry breaking contain higher derivatives and hence are relevant only in the UV region. In the IR limit GR is recovered. Whereas n is usually an integer, it is possible to consider pseudo-local differential operators with non-integer n.

The model was reformulated in [16] following ideas presented in [17–19]. The idea is to construct the fluid from a scalar field with arbitrary EoS parameter w from a scalar field satifsying a constraint enforced through a Lagrange multiplier term. Because of the constraint, the scalar field can be non-dynamical and even in the high energy region one can obtain a non-relativistic fluid, as required in CRG. The Nojiri-Odintsov action for $z = 2n+2$ CRG formulated with a Lagrange multiplier (which we will refer to as LCRG) reads:

$$S = \int d^4x \, \sqrt{-g} \left\{ \frac{R}{2} - \alpha \left[\left(\partial^\mu \phi \partial^\nu \phi \nabla_\mu \nabla_\nu + 2U_0 \nabla^\rho \nabla_\rho \right)^n \left(\partial^\mu \phi \partial^\nu \phi R_{\mu\nu} + U_0 R \right) \right]^2 - \lambda \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + U_0 \right) \right\} . \tag{5}$$

As expected, the Nojiri-Odintsov action, which is diffeomorphism invariant, is now expressed entirely in terms of local fields. Again, given a certain n, the graviton propagator behaves as $1/\mathbf{k}^{2n+2}$ in the UV, thus leading to (super-)renormalizability in 3+1 dimensions if $z = (\geq)3$. In [22–26], the $F(R)$ gravity version of Hořava gravity has been formulated, whereas the $F(R)$ version of CRG and LCRG have been constructed in [27] and [16] respectively, and are straightforward generalizations of Eqs.(4,5). See also [28] for further generalizations of CRG. In the IR, all these theories reduce to GR and have been shown to recover Newton's law.

B. Mimetic gravity

Fluids constructed from scalar fields constrained by a Lagrange multiplier abound in the recent literature. For instance, this construction is at the heart of the mimetic gravity framework. In mimetic gravity one isolates the conformal degree of freedom of gravity in a covariant way, by parametrizing the physical metric g in terms of an auxiliary metric $\tilde{g}$ and the mimetic field ϕ as follows [30]:

$$g_{\mu\nu} = -\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi \,. \tag{6}$$

The corresponding equations for the gravitational field are equivalent to the Einstein field equations, with the addition of a source term, which describes a pressureless fluid with 4-velocity $\partial_\mu\phi$. This additional degree of freedom can mimic collisionless cold dark matter. Of course, the following equality has to hold for consistency:

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = -1 \,. \tag{7}$$

In fact, this suggests that the constraint given by Eq.(7) can be implemented in the action by means of a Lagrange multiplier [31]. In other words, the action for mimetic gravity can be written as:

$$S = \int d^4x \, \sqrt{-g} \left[\frac{R}{2} - \lambda \left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + 1 \right) - V(\phi) \right] \,, \tag{8}$$

where $V(\phi)$ represents a potential for the mimetic field.

It became soon understood that mimetic gravity is related to GR via a singular disformal transformation. Recall that, by virtue of diffeomorphism invariance of GR, any metric can be parametrized in terms of a fiducial metric and a scalar field [32]. If the transformation is invertible, the number of degrees of freedom is unchanged by the transfomation and one recovers GR. If, as is the case in mimetic gravity, the transformation is singular, then the number of degrees of freedom can change and one has, in general, equations of motion which differ from those of GR [33–35]. More recently still, [36] has shown how the two approaches to mimetic gravity, that is, singular disformal transformations and Lagrange multiplier, are in fact equivalent.[2] Mimetic gravity has received a tremendous amount of interest over the past years, with several extensions and solutions being formulated and derived, and various studies conducted on the stability against negative energy states. It is beyond the scope of this section to review in detail findings concerning mimetic gravity, for which instead we refer the reader to the rapidly developing literature on the subject [40–85].

Early on it was realized that the original mimetic gravity proposal suffers a serious problem, namely that the perturbations in the mimetic field behave as dust with a speed of sound which is identically zero irrespective of wavelength [31]. That is, there is no dependence on the Laplacian of the perturbation in the perturbation equation itself. This is unacceptable, as it implies that it is not possible to define quantum perturbations in mimetic matter as one would usually do, else these would fail to seed the observed large-scale structure in the Universe. This is of course not unexpected, given identical results obtained in [17]. Parallel work has shown this to be the case in the most generic mimetic Horndeski theory of gravity [86]. A possible solution is the addition of higher derivative (HD) terms, which modify the sound speed and even have the potential to address some of the outstanding problems of the standard collisionless cold dark matter picture on small scales [49, 50, 65].

By comparing the actions given by Eqs.(5,8), it is clear that up to the potential for the scalar field ϕ, LCRG and mimetic gravity are equivalent in the limit where $\alpha \to 0$ and when $U_0 = 1/2$. In other words, we can identify the exotic fluid in LCRG with the mimetic fluid. It is therefore our goal to explore the consequences of this very interesting identification. Our purpose in this work is twofold. First, having identified the connection between LCRG and mimetic gravity, we generalize the former by adding a potential for the mimetic field (and obviously keeping the non-standard coupling between curvature and the mimetic fluid), and examine whether this theory exhibits interesting cosmological solutions. Second, we exploit this identification to address the aforementioned problem of vanishing sound speed in the original mimetic gravity proposal. Given that the sound speed problem can be addressed by the addition of HD terms, we explore whether the HD terms which in LCRG serve the purpose of breaking Lorentz invariance dynamically can cure the problem of perturbations in mimetic gravity. The end product of our study is a renormalizable theory of

[2] See also [37–39] for recent works on the role of disformal transformations in cosmology

gravity which preserves diffeomorphism invariance at the level of the action (thus circumventing the possible strong-couling IR instability issues of HLG) but breaks it dynamically in the UV, reduces to GR in the IR, allows the realization of a number of interesting cosmological scenarios (including the presence of cosmological dark matter) and is well defined when considering perturbations which will seed the observed large-scale structure.

III. GENERALIZED NOJIRI-ODINTSOV COVARIANT RENORMALIZABLE GRAVITY

In the light of the previous discussion concerning the identification of the exotic LCRG fluid with the mimetic fluid, we generalize the action of LCRG by adding a potential for the mimetic field, which induces the exotic fluid coupling to curvature. In Eq.(5), as anticipated, we set $U_0 = 1/2$, and for definiteness we also set the dimensionless parameter $\beta = 1$. This choice will not affect our discussions. The action of the theory is therefore given by:

$$I = \frac{1}{2}\int d^4x\sqrt{-g}\left\{R - 2\Lambda - \alpha\left[\left(R^{\mu\nu} - \frac{R}{2}g^{\mu\nu}\right)\nabla_\mu\phi\nabla_\nu\phi\right]^n - \frac{\lambda}{2}\left(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + 1\right) - V(\phi)\right\} \, , \tag{9}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$, $R_{\mu\nu}$ and R are the Ricci tensor and scalar respectively, n is a positive integer, ϕ is the mimetic field with potential $V(\phi)$, Λ is the cosmological constant which can be incorporated in $V(\phi)$, and ∇_μ denotes the covariant derivative (see also [28]).[3] A theory with the action given by Eq.(9) may also be seen as an example of $F(R, T, R_{\mu\nu}T^{\mu\nu})$ gravity, see [87]. Variation with respect to the Lagrange multiplier λ yields the constraint equation on the scalar field:

$$g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi = -1 \, . \tag{10}$$

Variation with respect to the scalar field leads to the following equation of motion:

$$\begin{aligned} V'(\phi) &= \nabla_\mu\left[\left(2n\alpha F^{n-1}G^{\mu\nu} + \lambda g^{\mu\nu}\right)\partial_\nu\phi\right] \\ &= \frac{1}{\sqrt{-g}}\partial_\mu\left\{\sqrt{-g}\left[\left(2n\alpha F^{n-1}G^{\mu\nu} + \lambda g^{\mu\nu}\right)\partial_\nu\phi\right]\right\} \, , \end{aligned} \tag{11}$$

where $V'(\phi)$ indicates the derivative of the potential respect to the field, $G^{\mu\nu} \equiv R^{\mu\nu} - g^{\mu\nu}R/2$, and we have defined:

$$F \equiv T_{\mu\nu}R^{\mu\nu} - \frac{RT}{2} \, , \qquad T_{\mu\nu} \equiv \nabla_\mu\phi\nabla_\nu\phi \, , \qquad T \equiv g^{\mu\nu}T_{\mu\nu} = -1 \, . \tag{12}$$

Finally, by varying with respect to the metric we obtain the equations for the gravitational field (see also [29]):

$$\begin{aligned} G_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{\alpha}{2}F^n g_{\mu\nu} &= n\alpha F^{n-1}\left[R^\rho_\mu T_{\rho\nu} + R^\rho_\nu T_{\rho\mu} - \frac{1}{2}\left(TR_{\mu\nu} + RT_{\mu\nu}\right)\right] + \frac{\lambda}{2}T_{\mu\nu} \\ &+ n\alpha\left[D_{\alpha\beta\mu\nu}(T^{\alpha\beta}F^{n-1}) - \frac{1}{2}D_{\mu\nu}\left(TF^{n-1}\right)\right] + \Omega^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}} - g_{\mu\nu}\frac{V(\phi)}{2} \, , \end{aligned} \tag{13}$$

where we have defined the following differential operators:

$$\begin{aligned} D_{\alpha\beta\mu\nu} &\equiv \frac{1}{4}\left[(g_{\mu\alpha}g_{\nu\beta} + g_{\nu\alpha}g_{\mu\beta})\Box + g_{\mu\nu}\left(\nabla_\alpha\nabla_\beta + \nabla_\beta\nabla_\alpha\right) - \left(g_{\mu\alpha}\nabla_\beta\nabla_\nu + g_{\nu\alpha}\nabla_\beta\nabla_\mu + g_{\mu\beta}\nabla_\alpha\nabla_\nu + g_{\nu\beta}\nabla_\alpha\nabla_\mu\right)\right] \, , \\ D_{\mu\nu} &\equiv g_{\mu\nu}\Box - \frac{1}{2}\left(\nabla_\mu\nabla_\nu + \nabla_\nu\nabla_\mu\right) \, . \end{aligned} \tag{14}$$

Here, $\Box \equiv \nabla^i\nabla_i$ is the d'Alambertian operator. In Eq.(13), $\Omega_{\mu\nu}$ is a tensor that will not play any role if $T_{\mu\nu}$ does not have any metric dependence, as in the case we are considering: for the purpose of the ensuing discussions, we will omit it. The trace of Eq.(13) reads:

$$-R + 4\Lambda - \frac{\lambda}{2}T = 2\alpha F^n(n-1) + \frac{n\alpha}{2}\left(g_{\mu\nu}\Box + \nabla_\mu\nabla_\nu + \nabla_\nu\nabla_\mu\right)\left(T^{\mu\nu}F^{n-1}\right) - \frac{3n\alpha}{2}\Box\left(TF^{n-1}\right) - 2V(\phi) \, . \tag{15}$$

As we will demonstrate further on, the case where $n = 1$ is particularly interesting given that, as anticipated, it is equivalent to a specific case of a mimetic Horndeski model.

[3] Note that $\nabla_\mu\phi \equiv \partial_\mu\phi$, being ϕ a scalar field.

IV. COSMOLOGICAL SOLUTIONS

In what follows, we shall only consider the flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric, whose line element is given by:

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j\,, \tag{16}$$

where $a \equiv a(t)$ is the scale factor, and $i, j = 1, 2, 3$.

Taking the hypersurfaces of constant time to be equal to those of constant ϕ, and making use of the constraint on the gradient of the scalar field given by Eq.(10), we see that the field can be identified (up to an integration constant) with time:

$$\phi = t\,, \tag{17}$$

and thus $\nabla_i\phi = (\dot{\phi}, 0, 0, 0)$. In this case, the only non-vanishing component of the tensor $T_{\mu\nu}$ is the $(0, 0)$ one, i.e.:

$$T_{00} = \dot{\phi}^2 = 1\,, \quad T_{0i} = T_{i0} = T_{ij} = 0\,, \quad i, j = 1, 2, 3\,. \tag{18}$$

We can now choose two independent equations to study the system, and our choice will fall upon Eqs.(11,15). We evaluate the trace equation [Eq.(15)], which considerably simplifies given that all quantities now depend exclusively on time. The calculation is laborious but relatively straightforward and will be sketched in Appendix A. The final equation we obtain is:

$$\frac{\lambda}{2} = 6\dot{H} + 12H^2 - 4\Lambda + \alpha(5n-2)(3H^2)^n + 3^n n\alpha(2n-1)H^{2n-2}\dot{H} - 2V(\phi)\,. \tag{19}$$

From Eq.(11) instead we obtain:

$$\frac{1}{a^3}\partial_0\left[a^3\left(2n\alpha(3H^2)^n - \lambda\right)\right] = V'(\phi)\,. \tag{20}$$

The two above equations are of course equivalent to the two equations of motions derived from (13) on an FLRW metric, namely:

$$\begin{aligned} 0 &= \Lambda - 3H^2 + \frac{\alpha}{2}(1-4n)(3H^2)^n + \frac{\lambda}{2} + \frac{V(\phi)}{2}\,, && (21)\\ 0 &= \Lambda - 3H^2 - 2\dot{H} + \frac{\alpha}{2}(1-2n)(3H^2)^n + 3^{n-1}\alpha n(1-2n)\dot{H}H^{2n-2} + \frac{V(\phi)}{2}\,. && (22)\end{aligned}$$

A final remark is in order. For $V(\phi) = 0$, from Eq.(20) one obtains:

$$\left[2n\alpha(3H^2)^n - \lambda\right] = \frac{C_0}{a^3}\,, \tag{23}$$

where C_0 is an integration constant. One can interpret Eq.(23) as a generalized Friedmann equation, with C_0 setting the amount of mimetic dark matter in the Universe (in fact, the corresponding energy density scales as a^{-3}, as is expected for cold dark matter. It is also clear that for $\alpha = 0$ one recovers the mimetic gravity model presented in [30, 31].

A. The $n = 1$ case

Let us consider a more specific model of generalized LCRG [Eq.(5)] where we set $n = 1$ (and thus α is dimensionless). We take $\Lambda = 0$, given that the cosmological constant can be incorporated in the potential. In this case Eqs.(19,20) lead to:

$$\begin{aligned} \lambda &= 6(2+\alpha)\dot{H} + 6(4+3\alpha)H^2 - 4V(\phi)\,, && (24)\\ V'(\phi) &= \frac{1}{a^3}\partial_0\left[a^3\left(6\alpha H^2 - \lambda\right)\right]\,. && (25)\end{aligned}$$

Further, note that the derivative with respect to ϕ is equal to the time derivative (provided the hypersurfaces of ϕ have been appropriately chosen), therefore Eq.(25) becomes:

$$\frac{1}{a^3}\partial_t\left[a^3\left(6\alpha H^2-\lambda\right)\right]=\frac{\partial V}{\partial t}\,. \tag{26}$$

Note also that, for $n=1$ and $\Lambda=0$, one obtains from Eq.(22):

$$2\dot{H}+3H^2=\frac{V}{(2+\alpha)}\,. \tag{27}$$

This once more demonstrates that the model we are considering on the FLRW metric is essentially equivalent to the model proposed in [31] (see also [73]) in the limit where $\alpha\to 0$.

Furthermore, for consistency, it it easy to show that Eq.(25) is a consequence of Eqs.(25,27). Thus, one may choose to deal only with (27). This equation is a non-linear Riccati type equation. It is a well known fact that it may be transformed in a linear second order differential equation by means of the Sturm-Liouville canonical substitution:

$$H=\frac{2}{3}\frac{\dot{u}}{u}\,. \tag{28}$$

After performing this substitution, we arrive at:

$$a(t)=u^{2/3}\,, \tag{29}$$

from which we easily derive:

$$\ddot{u}-\frac{3}{4(2+\alpha)}Vu=0\,. \tag{30}$$

We can use Eq.(30) as a reconstruction equation and as starting point for discussing a number of examples. We begin by noting that, if V is a constant, one recovers the de Sitter solution with $u\sim\exp[H_0 t]$, H_0 being a constant Hubble parameter. Another quite natural choice for the potential is a quadratic one, i.e.:

$$V(\phi)=3(2+\alpha)\left[H_0^2+\beta^2\left(2\phi-\phi_0\right)\left(-H_0+\frac{\beta^2}{4}(2\phi-\phi_0)\right)-\frac{2}{3}\beta^2\right]\,, \tag{31}$$

where H_0 is a fixed Hubble parameter and $\beta\,,\phi_0$ are dimensional constants, with dimensions $[\beta]=[\phi_0^{-1}]=[H]$. Given that $\phi=t$, we find that the exact solution for u reads:

$$u(t)=u_0 e^{\frac{3}{2}H_0 t-\frac{3\beta^2}{4}t(t-2t_0)}\,,\quad t_0\equiv\frac{\phi_0}{2}\,, \tag{32}$$

where u_0 is a constant and t_0 is a fixed time. This solution can be interpreted as describing a Starobinski-like inflationary epoch [88] in the Jordan frame. The Hubble parameter can be derived as:

$$H\equiv\frac{2}{3}\frac{\dot{u}}{u}=H_0-\beta^2\left(t-t_0\right)\,. \tag{33}$$

From Eq.(33) one sees that, for t close to t_0, one has a quasi-de Sitter expansion. On the other hand, for large $t_0\ll t$, H approaches zero. We provide another example starting from the following choice:

$$V(\phi)=\frac{4A^2(2+\alpha)}{3}\frac{\sinh A\phi}{1+\sinh A\phi}\,, \tag{34}$$

where A is a constant with mass-dimension 1. The exact solution is readily found, and is given by:

$$u(t)=1+\sinh At\,. \tag{35}$$

In terms of $a(t)$, this solution represents a cosmological bounce (see e.g. [89] for a recent review on bounce cosmologies). This example shows that the mimetic fluid may act as a phantom fluid.

B. Perturbations around the FLRW metric

In order to investigate and ascertain the cosmological viability of the $n = 1$ Nojiri-Odintsov model, one has to consider perturbations around the flat FLRW metric, Eq.(16). We consider only scalar perturbations, given that vector perturbations are not produced in the most common models of inflation and quickly decay with the expansion of the Universe, and we are not interested in primordial tensor modes. Thus, it is simplest for us to work in conformal Newtonian gauge, where the line element reads:

$$ds^2 = -(1 + 2\Phi(t, \mathbf{x}))dt^2 + a^2(t)(1 - 2\Psi(t, \mathbf{x}))\delta_{ij}dx^i dx^j \,, \quad i, j = 1, 2, 3 \,, \tag{36}$$

where $\Phi(t, \mathbf{x})$ and $\Psi(t, \mathbf{x})$ are functions of the space-time coordinates and $|\Phi(t, x) \,, \Psi(t, x)| \ll 1$. Thus, to lowest order, $g^{00}(t, x) \simeq -1 + 2\Phi(t, x)$ and $g^{11}(t, x) \simeq a(t)^{-2}(1 + 2\Psi(t, x))$. We perturb the mimetic field as:

$$\phi = t + \delta\phi(t, x) \,, \tag{37}$$

$\phi(t, x)$ being a function of the space-time coordinates. The mimetic constraint, Eq.(10), implies the following relation:

$$\delta\dot{\phi}(t, x) = \Phi(t, x) \,. \tag{38}$$

The following relations hold true as well:

$$T_{00} = 1 + 2\delta\dot{\phi} \,, \quad T_{0i} = \partial_i \delta\phi \,, \quad T = -1 + \mathcal{O}(\Phi(t, x)^2) \,. \tag{39}$$

From the (i, j) components of (13), when $i, j = 1, 2, 3 \,, i \neq j$, we obtain to first order in $\delta\phi(t, x)$ [we can use the (1-2)-component]:

$$G_{12}\left(1 - \frac{\alpha}{2}\right) = \alpha D_{\alpha\beta 12}T^{\alpha\beta} \,, \tag{40}$$

where:

$$G_{12} = -\partial_x\partial_y(\Phi - \Psi) \,, \quad D_{\alpha\beta 12}T^{\alpha\beta} = H\partial_x\partial_y\delta\phi + \partial_x\partial_y\delta\dot{\phi} \,. \tag{41}$$

It then follows that:

$$\Psi = \Phi + \left(\frac{2\alpha}{2 - \alpha}\right)(H\delta\phi + \delta\dot{\phi}) \,. \tag{42}$$

The (0,1)-component of Eq.(9) reads:

$$G_{01}\left(1 + \frac{\alpha}{2}\right) = \alpha\dot{H}\partial_x\delta\phi + \frac{\lambda}{2}\partial_x\delta\phi + \alpha D_{\alpha\beta 01}T^{\alpha\beta} \,, \tag{43}$$

where:

$$G_{01} = 2\partial_x\left(\dot{\Psi} + H\Phi\right) \,, \quad D_{\alpha\beta 01}T^{\alpha\beta} = -(H^2 + \dot{H})\partial_x\delta\phi \,, \quad \lambda = 6\alpha H^2 - 4\dot{H} - 2\alpha\dot{H} \,. \tag{44}$$

Thus, one arrives at:

$$\delta\ddot{\phi} + H\delta\dot{\phi} + \dot{H}\delta\phi = 0 \,. \tag{45}$$

Some remarks are in order at this point. First, the perturbation equation is equal to the one obtained in [31]. Furthermore, as noticed there, the sound speed is vanishing. As a consequence, it is not possible to define the quantum fluctuations of the mimetic field as in the standard inflation models. In order to overcome this drawback, one has to modify the model. In our case, one could investigate the $n = 2$ case, which will be done elsewhere. Alternatively, one can try to modify the model on the lines of [31, 49].

V. MODIFIED HIGHER ORDER MIMETIC MODEL

In order to modified our original $n = 1$ CRG mimetic model, we first recall the identity [90]:

$$-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi R + (\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2 = G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + \text{total derivative} \,. \tag{46}$$

and rewrite the $n=1$ action [Eq.(9)] in the form:

$$S=\frac{1}{2}\int d^4x\sqrt{-g}\left[R(1+\frac{\alpha}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi)-\alpha(\Box\phi)^2+\alpha(\nabla_\mu\nabla_\nu\phi)^2-\frac{\lambda}{2}\left(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi+1\right)-V(\phi)\right]. \tag{47}$$

In this form, the above action is still in the mimetic Horndeski form [36, 73, 80]. We now modify the model as follows:

$$S=\frac{1}{2}\int d^4x\sqrt{-g}\left[R(1+ag^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi)-\frac{c}{2}(\Box\phi)^2+\frac{b}{2}(\nabla_\mu\nabla_\nu\phi)^2-\frac{\lambda}{2}\left(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi+1\right)-V(\phi)\right]. \tag{48}$$

In this form, and without the mimetic constraint, the action describes a higher order derivative model in the scalar sector, namely with equations of motion which are of fourth order. The presence of the mimetic constraint renders the potential instability problem milder. Our original $n=1$ model is recovered when $a=\frac{\alpha}{2}$ and $b=c=2\alpha$. Thus we can interpret $(b-c)$ as a "Horndeski breaking parameter".

The equations of motion for the gravitational field, implemented by the mimetic constraint, read:

$$\begin{aligned}(1-a)G_{\mu\nu} &= \frac{1}{2}g_{\mu\nu}\left[\frac{b}{2}\phi^{\alpha\beta}\phi_{\alpha\beta}-\frac{c}{2}(\Box\phi)^2-V(\phi)\right]+\lambda\nabla_\mu\phi\nabla_\nu\phi \\ &- b\phi_{\mu\rho}\phi^\rho_\nu+\frac{b}{2}g^{\alpha\beta}\left[\nabla_\alpha(\phi_{\mu\nu}\nabla_\beta\phi)-\nabla_\alpha(\phi_{\mu\beta}\nabla_\nu\phi)-\nabla_\alpha(\phi_{\nu\beta}\nabla_\mu\phi)\right] \\ &+ c\left[\phi_{\mu\nu}+g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha(\Box\phi\nabla_\beta\phi)-\nabla_\mu\Box\phi\nabla_\nu\phi-\nabla_\nu\Box\phi\nabla_\mu\phi\right],\end{aligned} \tag{49}$$

where $\phi_{\alpha\beta}\equiv\nabla_\alpha\nabla_\beta\phi$. On a flat FLRW spacetime, the relevant bulk equation reads:

$$2\dot{H}+c_hH^2=c_vV(\phi), \tag{50}$$

where we have defined c_h and c_v as follows:

$$c_h\equiv\frac{12a+3b-9c-12}{4a+b-3c-4}, \quad c_v\equiv\frac{2}{4-4a-b+3c}. \tag{51}$$

Again, as a consistency check, we verify that by setting $b=c=2\alpha$ and $a=\frac{\alpha}{2}$, we recover the previous results. Concerning the bulk equation [Eq.(50)], we once again introduce the auxiliary function u as:

$$H=\frac{2}{c_h}\frac{\dot{u}}{u}, \tag{52}$$

from which one has:

$$a(t)=u^{\frac{2}{c_h}}, \tag{53}$$

and:

$$\ddot{u}-\frac{c_hc_vV(\phi)}{4}u=0. \tag{54}$$

One can use Eq.(54) to obtain exact solutions as in Section IV. For instance, making the following choice:

$$V(\phi)=\frac{4A^2}{c_vc_h}\frac{\sinh A\phi}{1+\sinh A\phi}, \tag{55}$$

one recovers the bounce solution :

$$u(t)=1+\sinh At. \tag{56}$$

Alternatively, along the lines of [73], one may introduce the the e-fold time $N=\ln a$. As a result, the equation of motion for H becomes:

$$\frac{dH^2}{dN}+c_hH^2=c_vV(N). \tag{57}$$

The general solution reads:

$$H^2(N) = \mathrm{e}^{-c_h N}\left(C + \int dN \mathrm{e}^{c_h N} c_v V(N)\right) . \tag{58}$$

The term depending on the constant C plays the role of cosmological dark matter. Other exact solutions can be obtained with a suitable choice of the potential. For example, by making the choice:

$$V(N) = V_0 N , \tag{59}$$

V_0 being a positive constant, one finds:

$$H^2(N) = \mathrm{e}^{-c_h N} C + \frac{V_0 c_v}{c_h}\left(N - \frac{1}{c_h}\right) . \tag{60}$$

Thus, if the conditions $c_h\,, c_v < 0$ are met, one can describe inflation when $1 \ll N$ (see e.g. [80]).

A. Perturbations of the modified higher order mimetic model

We have seen that in the FLRW space-time bulk, the exact cosmological solutions for the modified higher order mimetic model are similar to the ones of the Horndeski mimetic model we are interested in. We now turn to the study of cosmological perturbations. We work once more in the comoving Newtonian gauge, with line element defined by Eq.(36). Thus, as before $\delta\dot{\phi} = \Phi$. The spatial components of the perturbed equations give:

$$\Psi = \Phi + \frac{b}{2 - 2\alpha}\left(\delta\dot{\phi} + H\delta\phi\right) . \tag{61}$$

Finally, the $_{0i}$ and $_{i0}$ components of the perturbed equations give:

$$\delta\ddot{\phi} + H\delta\dot{\phi} - \frac{c_s^2}{a^2}\nabla^2\delta\phi + \dot{H} + \delta\phi = 0 , \tag{62}$$

where c_2 is defined as:

$$c_2 \equiv \frac{(2 + b - 2a)(4 + 3c - 4a - b)}{4(a - 1)} , \tag{63}$$

and the non-vanishing squared sound speed, c_s^2, reads:

$$c_s^2 \equiv \frac{b - c}{2c_2} . \tag{64}$$

The result is completely analogous to that obtained in [31, 49], albeit with a modified sound speed. As expected, we recover the result presented in [31] when we set $a = 0$, $b = 0$. Also as expected, the sound speed vanishes in the Horndeski case, when $b = c$, being proportional to the Horndeski breaking parameter $b - c$.

VI. CONCLUDING REMARKS

In this paper, we have explored connections between two modified gravity frameworks. The first is the Nojiri-Odintsov covariant renormalizable gravity, a fully diffeomorphism invariant Hořava-like theory of gravity which breaks diffeomorphism invariance dynamically, by means of a non-standard coupling to a perfect fluid. The second is mimetic gravity, which by means of a scalar field (the mimetic field) constrained by a Lagrange multiplier, provides the fluid in question. The identification of the fluid in CRG and the mimetic fluid is very interesting and we have provided a first inspection of the consequences of this association.

We have thus considered a generalization of the mimetic covariant renormalizable gravity model. For the case $n = 1$, we have shown that there subsists an equivalence with a particularly simple class of Horndeski models. Much as in the case of the original mimetic gravity proposal of [30, 31], we have shown that with the addition of a suitable potential it is possible to reconstruct several viable cosmological scenarios, by using Eq.(30). As an example, we have shown how to realize a Starobinsky-like inflationary epoch and a bounce solution starting from well motivated potentials.

By studying perturbations of the $n = 1$ model around a flat FLRW background, we have shown that the correspondence with the original mimetic gravity proposal persists. In fact, the fluid in question behaves as an irrotational fluid with vanishing sound speed, which renders the usual definition of quantum perturbations problematic. Parallel work has shown this to be the case in the most general mimetic Horndeski model [86]. We address this problem by going beyond the Horndeski form, modifying the action as in Eq.(48). When we consider perturbations in this model around a flat FLRW background, the resulting sound speed is no longer zero, but proportional to the Horndeski breaking parameter. Thus perturbations which will then grow under gravitational instability to seed the large-scale structure in the Universe can be defined sensibly in this model.

In conclusion, we have modified the Nojiri-Odintsov covariant Hořava-like mimetic model. The resulting model is thus a renormalizable theory of gravity which preserves diffeomorphism invariance at the level of the action, reduces to General Relativity in the infrared, allows the realization of a number of interesting cosmological scenarios (including the presence of cosmological dark matter) and is well defined when considering perturbations around a flat FLRW background. Depending on the size of the Horndeski breaking parameter, and hence of the sound speed, one can even hope to use this model to address some of the outstanding problems of collisionless cold dark matter on small scales, by appropriately suppressing small-scale power. In the future, it would be interesting to consider the $n \geq 2$ models, together with further interesting cosmological solutions. In particular it would be certainly important to study the evolution of the gravitational potential and the form of the resulting late-time matter power spectrum. We reserve the study of these issues for future work.

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Appendix A

Here we sketch the derivation of Eq.(19). To begin we evaluate F:

$$F = T_{\mu\nu}R^{\mu\nu} - \frac{RT}{2} = T^{00}R_{00} - \frac{RT}{2} = 3H^2 \ , \tag{65}$$

where we have used the fact that the only non-zero component of T_{ij} is T_{00}, and in FRW $R_{00} = -3\ddot{a}/a$, $R = 6[\ddot{a}/a + (\dot{a}/a)^2]$. Next, we evaluate $\Box(TF^{n-1})$, where $T = g^{\mu\nu}T_{\mu\nu} = -1$:

$$\begin{aligned} \Box(TF^{n-1}) &= 3^{n-1}\left(\frac{\partial^2}{\partial t^2} + 3H\frac{\partial}{\partial t}\right)H^{2n-2} \\ &= 3^{n-1}(2n-2)(2n-3)\dot{H}^2H^{2n-4} + 3^{n-1}(2n-2)\ddot{H}H^{2n-3} + 3^n(2n-2)\dot{H}H^{2n-2} \end{aligned} \tag{66}$$

Next, we evaluate $g_{\mu\nu}\Box(T^{\mu\nu}F^{n-1})$. Metric compatibility can be used to argue that this term is actually equal to the one we just calculated:

$$\begin{aligned} g_{\mu\nu}\Box(T^{\mu\nu}F^{n-1}) &= \Box(TF^{n-1}) \\ &= 3^{n-1}(2n-2)(2n-3)\dot{H}^2H^{2n-4} + 3^{n-1}(2n-2)\ddot{H}H^{2n-3} + 3^n(2n-2)\dot{H}H^{2n-2} \end{aligned} \tag{67}$$

Obviously, $(\nabla_\mu\nabla_\nu + \nabla_\nu\nabla_\mu)(T^{\mu\nu}F^{n-1}) = 2\nabla_\mu\nabla_\nu(T^{\mu\nu}F^{n-1})$, so we only need to evaluate $\nabla_\mu\nabla_\nu(T^{\mu\nu}F^{n-1})$. We first evaluate $\nabla_\nu(T^{\mu\nu}F^{n-1})$:

$$\nabla_\nu(T^{\mu\nu}F^{n-1}) = F^{n-1}(\partial_\nu T^{\mu\nu} + \Gamma^\mu_{\beta\sigma}T^{\sigma\beta} + \Gamma^\nu_{\nu\sigma}T^{\mu\sigma}) + T^{\mu\nu}\partial_\nu(F^{n-1}) \equiv T^\mu \tag{68}$$

Therefore:

$$\nabla_\mu\nabla_\nu(T^{\mu\nu}F^{n-1}) = \nabla_\mu T^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}T^\mu) = \partial_\mu T^\mu + \frac{3}{a}T^\mu\partial_\mu a \tag{69}$$

By making use of Eq.(68) and expanding, we get:

$$\nabla_\mu\nabla_\nu(T^{\mu\nu}F^{n-1}) = \partial_\mu\left[\nabla_\nu(T^{\mu\nu}F^{n-1})\right] + \frac{3}{a}\left[\nabla_\nu(T^{\mu\nu}F^{n-1})\right]\partial_\mu a = \tag{70}$$

$$(\partial_\nu T^{\mu\nu} + \Gamma^\mu_{\beta\sigma}T^{\sigma\beta} + \Gamma^\nu_{\nu\sigma}T^{\mu\sigma})\partial_\mu(F^{n-1}) + F^{n-1}(\partial_\mu\partial_\nu T^{\mu\nu} + T^{\sigma\beta}\partial_\mu\Gamma^\mu_{\beta\sigma} + \Gamma^\mu_{\beta\sigma}\partial_\mu T^{\sigma\beta} + T^{\mu\sigma}\partial_\mu\Gamma^\nu_{\nu\sigma} + \Gamma^\nu_{\nu\sigma}\partial_\mu T^{\mu\sigma})$$

$$+\partial_\mu\partial_\nu(F^{n-1})T^{\mu\nu} + \partial_\mu(T^{\mu\nu})\partial_\nu(F^{n-1}) + \frac{3}{a}F^{n-1}(\partial_\nu T^{\mu\nu} + \Gamma^\mu_{\beta\sigma}T^{\sigma\beta} + \Gamma^\nu_{\nu\sigma}T^{\mu\sigma})\partial_\mu a + \frac{3}{a}T^{\mu\nu}\partial_\nu(F^{n-1})\partial_\mu a$$

Of the fourteen terms in Eq.(71), only five are nonzero. This can easily be shown by repeatedly using the fact that $T^{\mu\nu} = 0$ if $\mu,\nu \neq 0$, $\partial_\mu T^{\alpha\beta} = 0$ and $\Gamma^0_{00} = \Gamma^i_{00} = 0$. The only five nonzero terms are:

$$\Gamma^\nu_{\nu\sigma}T^{\mu\sigma}\partial_\mu F^{n-1} = 3^n(2n-2)\dot{H}H^{2n-2} \tag{71}$$

$$\partial_\mu\partial_\nu(F^{n-1})T^{\mu\nu} = 3^{n-1}\left[(2n-2)(2n-3)\dot{H}^2H^{2n-4} + (2n-2)\ddot{H}H^{2n-3}\right] \tag{72}$$

$$F^{n-1}\partial_\mu(\Gamma^\nu_{\nu\sigma})T^{\mu\sigma} = 3^{n+1}H^{2n} + 3^n(4n-3)\dot{H}H^{2n-2} + 3^{n-1}\left[(2n-2)(2n-3)\dot{H}^2H^{2n-4} + (2n-2)\ddot{H}H^{2n-3}\right] \tag{73}$$

$$\frac{3}{a}F^{n-1}\Gamma^\nu_{\nu\sigma}T^{\mu\sigma}\partial_i a = 3(3H^2)^n \tag{74}$$

$$\frac{3}{a}T^{\mu\nu}\partial_\nu(F^{n-1})\partial_\mu a = 3^n(2n-2)\dot{H}H^{2n-2} \tag{75}$$

It is then easy to see that:

$$(g_{ij}\Box + \nabla_i\nabla_j + \nabla_j\nabla_i)(T^{rs}F^{n-1}) - 3\Box(TF^{n-1}) = 6(3H^2)^n + 3^n(4n-2)\dot{H}H^{2n-2}\ . \tag{76}$$

From this one then easily obtains Eq.(19).

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