Long-Time Predictability in Disordered Spin Systems Following a Deep Quench

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We study the problem of predictability, or "nature vs. nurture", in several disordered Ising spin systems evolving at zero temperature from a random initial state: how much does the final state depend on the information contained in the initial state, and how much depends on the detailed history of the system? Our numerical studies of the "dynamical order parameter" in Edwards-Anderson Ising spin glasses and random ferromagnets indicate that the influence of the initial state decays as dimension increases. Similarly, this same order parameter for the Sherrington-Kirkpatrick infinite-range spin glass indicates that this information decays as the number of spins increases. Based on these results, we conjecture that the influence of the initial state on the final state decays to zero in finite-dimensional random-bond spin systems as dimension goes to infinity, regardless of the presence of frustration. We also study the rate at which spins "freeze out" to a final state as a function of dimensionality and number of spins; here the results indicate that the number of "active" spins at long times increases with dimension (for short-range systems) or number of spins (for infinite-range systems). We provide heuristic arguments to support these conjectures, and also analyze theoretically several mean-field models: the random energy model, the uniform Curie-Weiss ferromagnet, and the disordered Curie-Weiss ferromagnet. We find that for these models, the information contained in the initial state does *not* decay in the thermodynamic limit— in fact, it *fully determines* the final state. Unlike in short-range models, the presence of frustration in mean-field models dramatically alters the dynamical behavior with respect to the issue of predictability.

I. INTRODUCTION

The dynamical properties of Ising spin systems far from equilibrium, and in particular those following a deep quench, continue to be a major focus of research. There are multiple directions along which this research has been pursued, including the general areas of phase-ordering kinetics [1–5], persistence [6–10], and damage spreading [11–14], among many others. In [15] another line of investigation was proposed: the problem of *predictability* and retention of information in discrete spin systems evolving far from equilibrium.

Following a deep quench an Ising spin system will be in a random configuration. One can then ask, how rapidly

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does this information in the initial state decay over time, as a function of dimension, lattice type, model (uniform ferromagnet, random ferromagnet, spin glass, and so on), and other characteristics of the system under study? The nature of information retention or decay depends on two sources of randomness: that contained in the initial configuration and that generated through the dynamical realization governing an individual history. This dynamical randomness is present even at zero temperature, in the order that spins are chosen to attempt to flip (and in the tie-breaking rule for homogeneous systems when a spin flip would cost zero energy). Disordered Ising models such as random ferromagnets and spin glasses bring in a third source of randomness, namely in the couplings determining the local environment of an individual spin. We are interested in the question of how much information contained in the configuration at time t depends on the initial configuration and how much depends on the specific dynamical path the system has followed. We will formalize these remarks in Sect. II below. We have colloquially referred to the above as the "nature vs. nurture" problem, where nature represents the information contained in the initial configuration and (quenched) random couplings, while nurture refers to the history (i.e., dynamical realization) of the subsequent evolution of the system.

It was shown in early work that this problem can be solved exactly for $1D$ random ferromagnets and spin glasses [16]. Preliminary numerical studies on the $2D$ *homogeneous* ferromagnet on the square lattice were reported in [17]. More recently, extensive numerical studies [18] have largely solved the nature vs. nurture problem for the uniform $2D$ ferromagnet, so that the rate of decay of initial information for this model is now understood quantitatively. These results will be briefly reviewed in the next section.

In this paper we turn our attention to *disordered* Ising models in dimensions greater than one. Our particular focus will be on the behavior of the random ferromagnet and the Edwards-Anderson (EA) spin glass [19] as a function of dimension. We will also consider the infinite-range Sherrington-Kirkpatrick (SK) spin glass [20] as a function of system size N. Our conclusions will be based on numerical studies, but in Sect. V we also present an analytical discussion of these models, as well as the random energy model (REM) [21] and the uniform and disordered Curie-Weiss ferromagnets.

II. PRELIMINARIES

Consider a set of Ising spins $S_i = \pm 1$ on the sites i of the Euclidean lattice $\mathbb{Z}^d$ with periodic boundary conditions. The system Hamiltonian is

$$\mathcal{H} = -\sum_{\langle i,j\rangle} J_{ij} S_i S_j \, , \tag{1}$$

where $\langle i, j\rangle$ indicates a sum over all nearest neighbor pairs and the couplings are independent, identically distributed random variables chosen from a common distribution (which depends on the exact model under consideration). We study numerically three types of models: the first is the Edwards-Anderson (EA) Ising spin glass [19] in d dimensions, in which the common distribution of the couplings is a Gaussian with mean zero and variance one. The second is the random ferromagnet, where the couplings are all positive; here the common distribution is taken to be a one-sided Gaussian, in which each bond is chosen as the absolute value of a standard Gaussian random variable (again with mean zero and variance one). Finally, we consider the infinite-range Sherrington-Kirkpatrick (SK) spin glass. Here the spins sit on the N sites of a complete graph, with a modified Hamiltonian

$$\mathcal{H} = -\frac{1}{\sqrt{N}} \sum_{i<j} J_{ij} S_i S_j \, . \tag{2}$$

The couplings are again chosen from a Gaussian distribution with mean zero and variance one, and the rescaling factor $N^{-1/2}$ ensures a sensible thermodynamic limit of the energy and free energy per spin. (Note, however, that the rescaling factor plays no role in the dynamics described below.)

Initially the spins are in a random initial configuration, in which each spin is chosen to be $+1$ or -1 with probability $1/2$, independently of all the others. This corresponds to an infinite-temperature spin configuration. The subsequent evolution is governed by zero-temperature Glauber dynamics, which in the simulations to be described below is implemented as follows. At each step, a site i is selected uniformly at random and the energy change ΔE_i associated with flipping the associated spin is computed: $\Delta E_i = \Delta\mathcal{H}$ when $S_i \to -S_i$ and all other spins remain fixed. If the energy decreases ($\Delta E_i < 0$) as a result of the flip, the flip is carried out. If the energy increases ($\Delta E_i > 0$), the flip is not accepted. If the energy remains the same ($\Delta E = 0$), a flip is carried out with probability $1/2$. This last "tie-breaking" rule is relevant only for models (such as the homogenous ferromagnet, or the $\pm J$ spin glass) in which a zero-energy flip can occur. For disordered models with continuous coupling distributions, such as those considered here, this possibility never arises. These dynamics are run until the system reaches an absorbing state that is stable against all single spin flips.

In [18] we defined a "heritability exponent" by preparing two Ising systems (regarded as identical twins) on a square of side L with the same initial configuration and then allowing them to evolve independently using different dynamical realizations of zero-temperature Glauber dynamics [22]. The spin overlap between the resulting copies is $q_t(L) = \frac{1}{L^2}\sum_{i=1}^{L^2} S_i^1(t)S_i^2(t)$, where $S_i^k(t)$ denotes the state of the i^{th} spin at time t in twin k, where $k = 1, 2$. The influence of initial conditions is quantified by $q_t(L)$, where $q_0(L) = 1$ for any L. We examined the average $\overline{q_t(L)}$ over both initial conditions and dynamics to investigate two relevant quantities: $\overline{q_\infty(L)} = \lim_{t\to\infty} \overline{q_t(L)}$, the size dependence of the overlap at large times, and $\overline{q_t} = \lim_{L\to\infty} \overline{q_t(L)}$, the time dependence of the overlap in large volumes.

It is important to note that heritability is not the same as persistence, although the two ask related questions. Persistence asks which spins have not flipped up to a time t, while heritability asks to what extent the *information* contained in the initial state persists up to time t. A spin may have flipped multiple times during this interval but its final state might still be predictable knowing the initial condition.

Our key finding for the $2D$ uniform ferromagnet [18] was that heritability decays as a power law at long times: $\overline{q_t} \sim t^{-\theta_h}$. The power-law exponent θ_h is the "heritability exponent" referred to above. The size dependence of the final overlap between twins on a finite lattice similarly decays as a power law: $\overline{q_\infty(L)} \sim L^{-b}$, The exponents θ_h and b were shown, through a finite size scaling ansatz, to be related by $b = 2\theta_h$, consistent with our numerically determined values and with exact $1D$ values.

Before turning to a study of nature vs. nurture in disordered Ising systems, we point out an important difference between the uniform ferromagnet on $\mathbb{Z}^d$ and the models considered here: as noted above, there can be no "ties", or zero-energy flips in disordered models whose couplings are random variables arising from continuous distributions. (This is equally true for the homogeneous ferromagnet with an odd number of neighbors, such as the $d = 2$ honeycomb lattice, but we do not consider those models here.) It was proved in [16] that, in the uniform ferromagnet on the square lattice under the dynamics described here, every spin (on the infinite lattice) flips infinitely often. It was also proved in [16] that, in any discrete spin model such as those considered here, every spin makes only a finite number of energy-lowering flips in any dynamical run; consequently, $\overline{q_t}$ does not decay to zero as $t \to \infty$ in any finite-dimensional EA spin glass or random ferromagnet with a continuous coupling distribution [23]. In [16], we defined a dynamical order parameter q_D (D for dynamical) which effectively corresponds to

$$q_D = \lim_{t\to\infty} \overline{q_t}\,. \tag{3}$$

Because we are now considering heritability as a function of dimension d, to avoid confusion we will hereafter refer to the dynamical order parameter as q_∞ rather than q_D. The interesting question is how does q_∞ behave as a function of dimension d for these models, and in particular does it tend to zero as $d \to \infty$? For the SK model, in contrast, one is forced to consider q_∞ to be the long-time limit of $\overline{q_t(N)}$ for finite number of spins N, which of course will be nonzero for any finite N. The corresponding question is then whether $\overline{q_\infty(N)} \to 0$ as $N \to \infty$.

III. MODELS AND METHODS

In order to distinguish the influence of nature and nurture in the above three models, we use the twin method described above. As a function of time t, we look at the overlap q between the twins for a system of size $N = L^d$:

$$q_t(N) = \frac{1}{N}\sum_{i=1}^{N} S_i^1(t)S_i^2(t) \tag{4}$$

where $S_i^1(t)$ denotes the state of i^{th} spin at time t in twin 1 and $S_i^2(t)$ is the spin state for twin 2. Here and throughout this work, time t is measured in sweeps, where one sweep corresponds to N spin-flip attempts. The overlap is initially unity: $q_0(N) = 1$ and, on average, decays in time. For each finite realization, it reaches a final value when both twins are in absorbing states; we are therefore interested, in addition to q_∞ defined above, in the N (equivalently, L) dependence of $q_t(N)$ for the three models. We also study the time to reach the absorbing state and the fraction of active or flippable spins as a function of time.

We use the algorithm introduced in [18]. For the first t_0 sweeps we implement Glauber dynamics directly: A spin S_i is randomly selected, the energy change ΔE_i is computed, and the spin is flipped ($S_i \to -S_i$) if $\Delta E_i < 0$ and not flipped otherwise. After each attempted spin flip, time is incremented by $1/N$ sweeps. After t_0 sweeps, only a few active spins (such that $\Delta E_i < 0$) remain and the dynamics is significantly accelerated using kinetic Monte Carlo methods. To implement kinetic Monte Carlo, a list of active spins is maintained. A single step of kinetic Monte Carlo consists of selecting a spin at random from the active list and then carrying out the flip. The time is incremented by $1/f_t(N)$, where $f_t(N)$ is the number of active spins before the spin flip. After the spin flip the active list is updated: the spin at i is removed from the list and its neighbors are all checked to see if they must be added to or removed from the active list. Kinetic Monte Carlo dramatically improves the run time for the d-dimensional models. For the finite-dimensional models, t_0 is set to 10 while for the SK model, $t_0 = 20$.

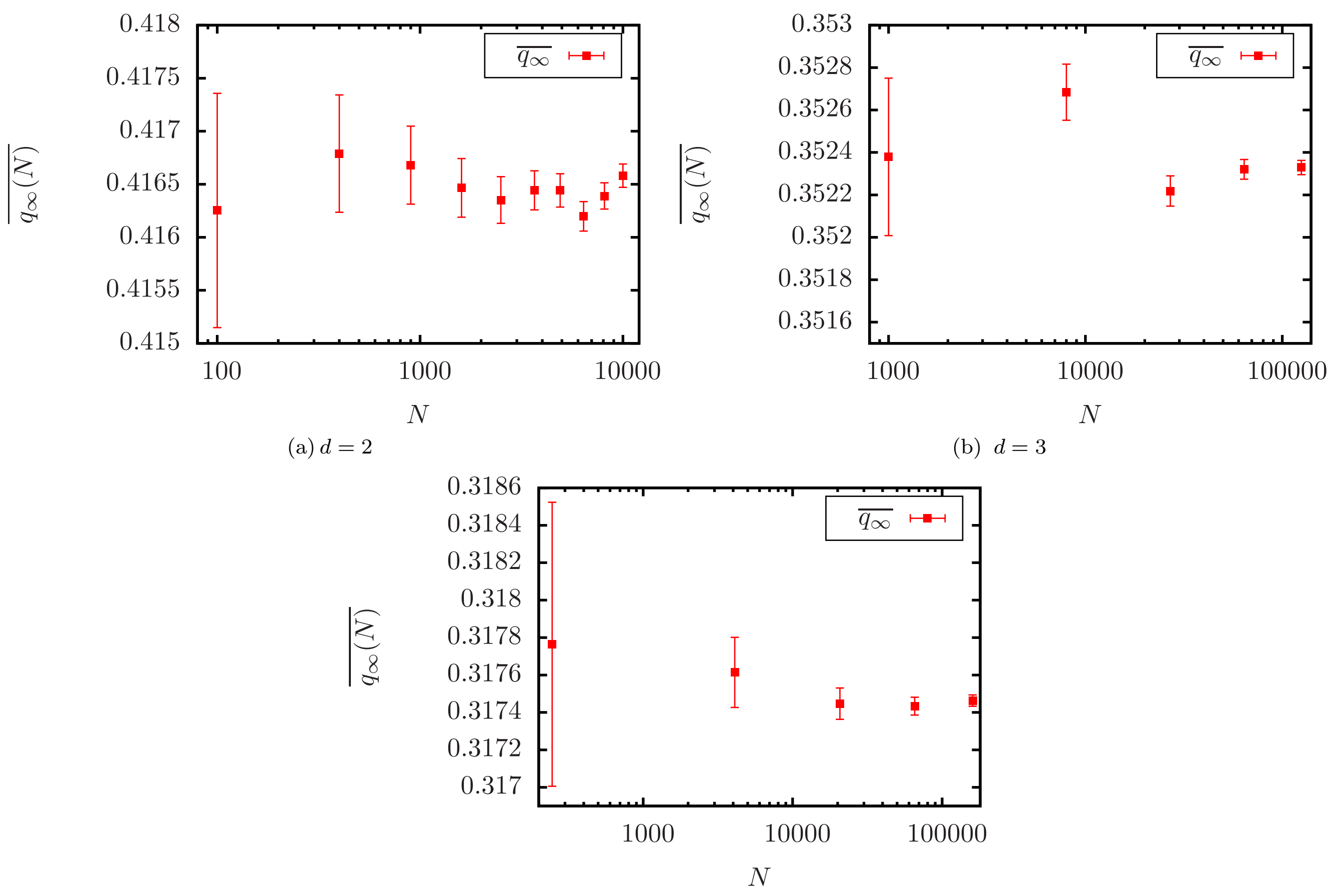
For each system, we study 30,000 independent pairs of twins (i. e. 60,000 systems). From $q_t(N)$, we compute the mean $\overline{q_t}$, the standard deviation σ_t and the standard error of the mean. We are mostly interested in the final value $\overline{q_\infty}$ when both twins have reached the absorbing states.

IV. RESULTS

In this section we present numerical results for the Edwards-Anderson (EA) spin glass in dimensions $d = 2$, 3 and 4, the random ferromagnet in $d = 2$, and then the Sherrington-Kirkpatrick (SK) model.

A. Edwards-Anderson model in d dimensions

Figure 1 is a plot of $\overline{q_\infty(N)}$ as a function of number of spins N for the EA spin glass, with subfigures (a), (b), and (c) representing $d = 2$, 3, and 4 dimensions, respectively. It is clear that $\overline{q_\infty(N)}$ is rapidly converging to a non-zero constant as $N \to \infty$.



(a) $d = 2$

(b) $d = 3$

(c) $d = 4$

FIG. 1: The overlap $\overline{q_\infty(N)}$ vs. system size N. Panels (a), (b) and (c) represent Edwards-Anderson spin glasses in $d = 2$, 3 and 4 dimensions, respectively.

The fast convergence of $\overline{q_\infty(N)}$ to a constant value, motivates using $\overline{q_\infty(N_{\max})}$ as an estimator for $\overline{q_\infty(N \to \infty)}$

where $N_{\rm max}$ is the largest size simulated for each system. For $d = 1$, it is known that $\overline{q_\infty(N \to \infty)} = 1/2$ [16]. The results for $\overline{q_\infty(N_{\rm max})}$ and the values of $N_{\rm max}$ for each dimension are presented in Table I. The errors listed in the Table include only statistical errors. It is possible that systematic errors due to estimating $\overline{q_\infty(N \to \infty)}$ at a finite $N_{\rm max}$ are larger. Figure 2 shows $\overline{q_\infty(N_{\rm max})}$ vs. d and indicates that $\overline{q_\infty(N \to \infty)}$ decreases with increasing dimension though with only four data points it is unclear whether in the limit of high dimension, $d \to \infty$, $\overline{q_\infty(N \to \infty)} = 0$ or $\overline{q_\infty(N \to \infty)} > 0$.

TABLE I: The largest simulated system size, $N_{\rm max}$ and $\overline{q_\infty(N_{\rm max})}$ for dimension d.

d	$N_{\rm max}$	$\overline{q_\infty(N_{\rm max})}$
2	100^2	0.4166(1)
3	50^3	0.35233(3)
4	20^4	0.31746(3)

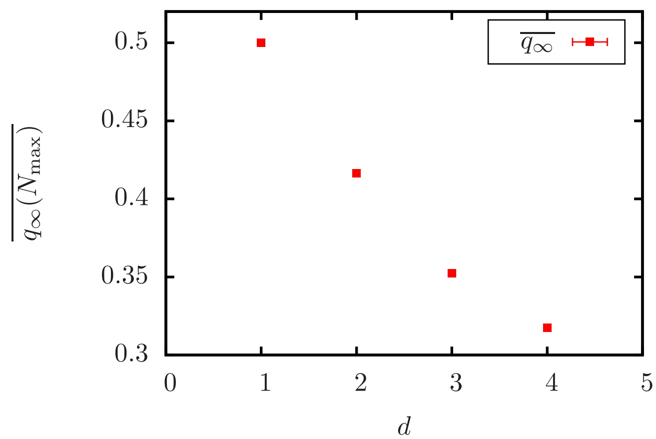


FIG. 2: The average overlap in the absorbing state for the largest systems size $\overline{q_\infty(N_{\rm max})}$ vs. dimension d for the EA spin glass models.

Next we consider the mean survival time $\tau(N)$ as a function of number of spins N. The survival time for each system is the (integer) number of sweeps immediately prior to reaching the absorbing state. Figure 3 shows $\tau(N)$ vs. N for each dimension d.

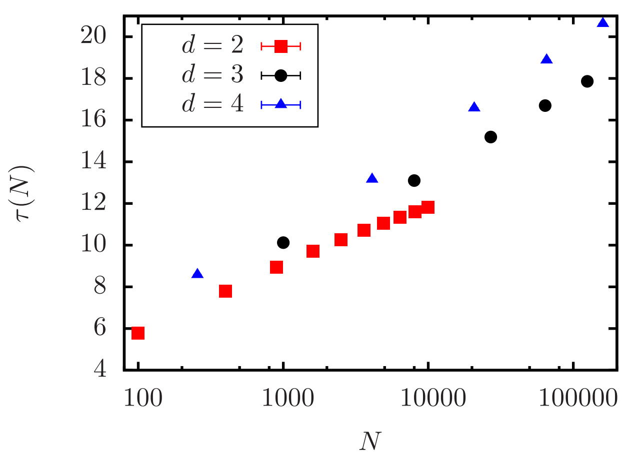


FIG. 3: (color online) Mean survival time $\tau(N)$ vs. number of spins N for the EA spin glass models $d = 2$ (red squares), $d = 3$ (black circles), and $d = 4$ (blue triangles).

A slight downward curvature of $\tau(N)$ can be seen in two dimensions, but not in three and four dimensions, where

$\tau(N)$ shows no sign of saturating up to the largest sizes studied. At the same time, we know from [16] that in all dimensions the number of spin flips is finite. It is also curious that in all cases the overlap saturates to a constant very quickly with N, but the mean survival time is still increasing, even for the largest sizes. How can we reconcile these disparities? In fact, these observations are not in contradiction. The most likely scenario is that $\tau(N)$ approaches a finite dimension-dependent constant for all finite d, and our simulations simply have not gone to sufficiently large N. But other possibilities are also consistent with observations. The fact that each spin flips finitely often of course does *not* necessarily imply that the survival time, or (roughly) equivalently, the mean number of spin flips per site, is finite: it could be the case that a small number of spins continue to flip long after most of the others have reached their final state. This would also be consistent with the overlap saturating while the survival time is still increasing with N. A more detailed numerical study is needed to determine whether this is indeed happening.

However, our main interest here is the dependence on d of the mean number of spin flips per site at fixed N. This is not the same quantity as $\tau(N)$, for the reasons discussed in the preceding paragraph; nevertheless, the two are related, and the clear increase in $\tau(N)$ with dimension d that can be seen in Fig. 3 provides reasonable supporting (more accurately, necessary but not sufficient) evidence that the typical number of spin flips per site increases with d, and we will take this as one of our conjectures in Sect. V.

B. Random Ferromagnet in 2D

Figure 4 is a plot $\overline{q_\infty(N)}$ vs. N for the $d = 2$ random ferromagnet. Similarly to the EA spin glass, $\overline{q_\infty(N)}$ quickly saturates to a constant and it is reasonable to assume that at the largest system size, $\overline{q_\infty(N_{\max})} \approx \overline{q_\infty(N \to \infty)}$. We note that $\overline{q_\infty(N_{\max})} = 0.4198(1)$ for $N_{\max} = 100^2$, and it is interesting to compare this result to that obtained for the $d = 2$ EA spin glass, where $\overline{q_\infty(N_{\max})} = 0.4166(1)$ for the same largest size $N_{\max} = 100^2$. The two results are very close, within 1% of each other, though they differ by many standard errors. Since the error bar accounts only for statistical errors, and not finite-size systematic errors, it is possible that $\overline{q_\infty(N \to \infty)}$ is identical for spin glasses and random ferromagnets, though we believe it is more likely that they differ. In either case, the closeness of the two values of $\overline{q_\infty(N \to \infty)}$ suggest that frustration, which is present in the spin glass but not the random ferromagnet, plays little or no role in the nature vs. nurture problem for finite dimensions.

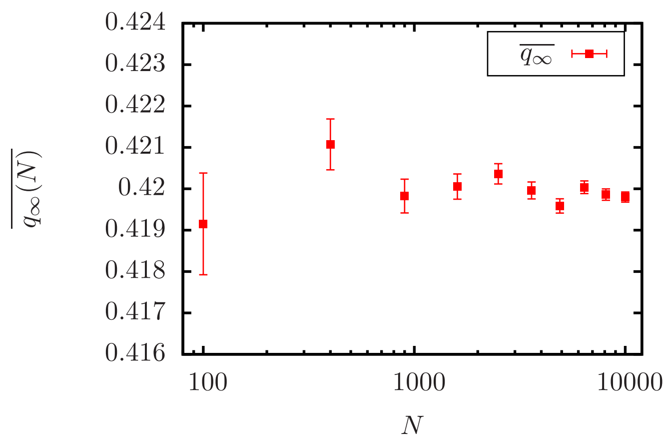


FIG. 4: $\overline{q_\infty(N)}$ vs. N for the $d = 2$ random ferromagnet.

C. Sherrington-Kirkpatrick model

This section presents results for Sherrington-Kirkpatrick (SK) model, the Ising spin glass on the complete graph. Figure 5 is a plot of $\overline{q_\infty(N)}$ as a function of N. While it is clear that $\overline{q_\infty(N)}$ is decreasing with N it is not obvious whether $\overline{q_\infty(N \to \infty)}$ is zero or greater than zero.

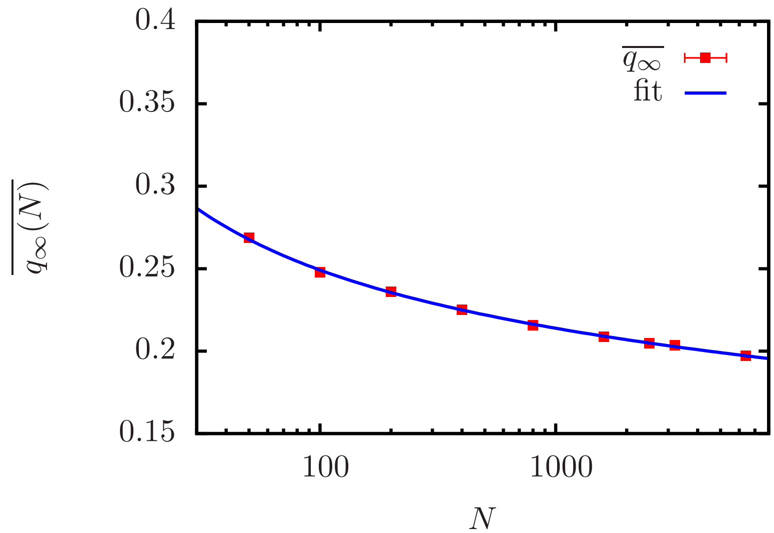


FIG. 5: The points are simulation results for $\overline{q_\infty(N)}$ vs. N for the SK model. The curve is the highest quality fit, fit 1 with $N_{\text{min}} = 50$.

To attempt to determine which possibility holds we fit $\overline{q_\infty(N)}$ to three functional forms:

1. $\overline{q_\infty(N)} = \dfrac{a}{(\log N)^{1/3}} + \dfrac{b}{N}$

2. $\overline{q_\infty(N)} = aN^c + b$

3. $\overline{q_\infty(N)} = \dfrac{a}{(\log N)^b}$

For each of these forms, we carried out fits for two values of the minimum size used in the fit $N_{\text{min}} = 30$ and 50. The fitting coefficients and quality of the fits are summarized in Table II and Fig. 5 shows the highest quality fit, which is fit 1 with $N_{\text{min}} = 50$.

TABLE II: Parameters and fit quality for the three fits of $\overline{q_\infty(N)}$ vs. N

	$\frac{a}{(\log N)^{1/3}} + \frac{b}{N}$		$aN^c + b$		$\frac{a}{(\log N)^b}$	
N_{min}	30	50	30	50	30	50
a	0.4063(3)	0.4064(4)	0.29(2)	0.24(2)	0.439(8)	0.421(6)
b	0.49(5)	0.41(13)	0.175(3)	0.166(6)	0.369(9)	0.349(7)
c			$-0.29(2)$	$-0.23(3)$		
Reduced χ^2	0.99484	1.12424	2.44157	1.65737	4.43254	1.5121
Quality of Fit	0.4416	0.345	0.0123	0.1411	7.9458×10^{-6}	0.1695

The first fitting function is the best fit to the data for both values of N_{min} and implies that $\overline{q_\infty(N \to \infty)} = 0$. Nonetheless, the second functional form also provides a reasonable fit and implies that $\overline{q_\infty(N \to \infty)} > 0$. It is also noteworthy that the leading coefficient, a of the first functional form is stable with respect to changing the fitting range while this is not true of the second functional form. Although the numerics are not definitive, we believe that it is most likely that $\overline{q_\infty(N)}$ converges very slowly, i.e. as $1/\log(N)^{1/3}$, to $\overline{q_\infty(N \to \infty)} = 0$. Since, in some sense, the SK model is believed to correspond to the EA model at $d = \infty$, these results provide further support to the conjecture that $q_\infty \to 0$ in the EA model as dimensionality goes to infinity.

Next we consider the survival time as a function of N. For the SK model, we define $\tau(N)$ as the median survival time for a system of N spins. We note that distribution of survival times is well described by a lognormal and obtain the median by fitting the data to a lognormal. Figure 6 is a log-log plot of $\tau(N)$ vs. N. A power law fit with a $1/N$ correction to scaling, $\tau(N) = aN^b(1 + c/N)$ does a reasonable job of fitting the data for $N \geq 800$. The fitted exponent is $b = 0.69$. A pure power law is a worse fit and yields a smaller value of the exponent, $b = 0.64$. The pure power law with $b = 0.64$ is shown as the solid curve in Fig. 6. Although we have low confidence in the value of the exponent describing τ, in contrast to the finite-dimensional models, it seems clear that the survival time diverges in N, most likely as a power near 2/3.

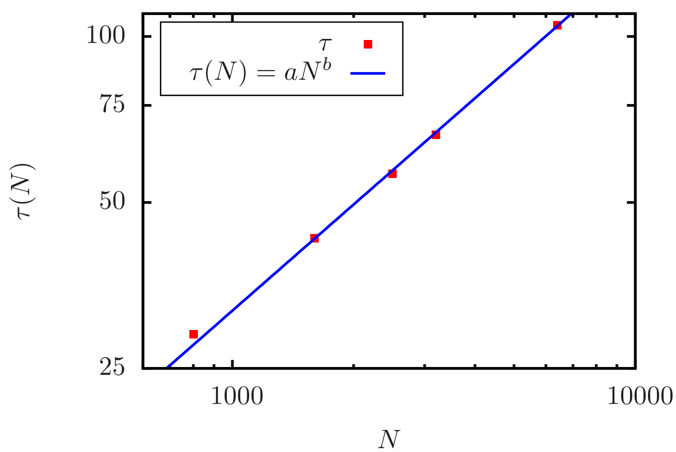


FIG. 6: The median time for reaching the absorbing state, $\tau(N)$ vs. number of spins, N for the SK model.

Finally, we study the fraction of active (flippable) spins $f_t(N)$ as a function of time t. Figure 7 is a log-log plot of $f_t(N)$ vs. t. We find that for the first 10 sweeps, $f_t(N)$ has a power law decay. At longer times the curves fall much more steeply. However, for intermediate times and the larger system sizes there appears to be a flattening before the steep fall-off. Thus the asymptotic behavior in time for $f_t(N)$ in the limit $N \to \infty$ is not clear. A power law $f_t(N) = (1/2)(t+1)^{-1.52}$ in the range $0 \le t \le 10$ is a good fit to the data for all values of N.

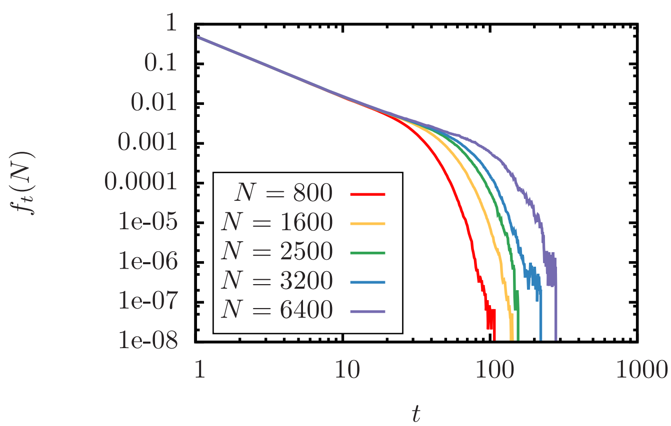


FIG. 7: Log-log plot of the fraction of active spins $f_t(N)$ vs. t for the SK model for different sizes N with sizes increasing from left to right.

V. DISCUSSION

In this section we summarize the numerical results presented above, and discuss them within the context of the theory of the nature vs. nurture problem. We also examine some other mean-field models for which rigorous conclusions can be stated, and whose dynamical behavior is surprising given our knowledge of the behavior of finite-dimensional systems and the SK model.

The numerical results discussed above focused on three disordered systems: the EA spin glass and random ferromagnet as a function of dimension d, and the SK spin glass as a function of the number of spins N. While no firm conclusions can yet be drawn for these models, the results suggest several broad conclusions, which for now remain as conjectures:

1. In the numerical simulations of the EA model and random ferromagnet on $\mathbb{Z}^d$ reported in Sect. IV, the survival time is seen to increase with dimension d; based on this (see the discussion at the end of Sect. IV A), we conjecture that the number of spin flips per site diverges as $d \to \infty$.

2. The dynamical order parameter q_∞ in the EA model and random ferromagnet on $\mathbb{Z}^d$ is nonzero for any fixed $d < \infty$ but tends to zero as $d \to \infty$, and similarly in the SK model as $N \to \infty$.

3. In short-range models, the dynamical behavior studied here is relatively insensitive to frustration, in that the spin glass and random ferromagnet behave similarly. However, frustration appears to play an important role in mean-field models, as discussed below.

While we do not as yet have analytical results on these conjectures, we can provide heuristic arguments in their support. We first do so for short range models, then present some analytical results for mean field models and a corresponding heuristic for the SK model.

A. Short-range models

We begin with some relevant rigorous results obtained in an earlier paper [15]. These are the following: first, for both the EA spin glass and the random ferromagnet on the infinite Euclidean lattice $\mathbb{Z}^d$ for any finite d, there is an *uncountable* infinity of k-spin-flip stable states, for any finite $k \geq 1$ (states with $k = 1$ comprise the set of final possible states for the dynamics described in this paper). The overlap distribution for the set of such states is a δ-function at 0.

Related to the above result is that if one begins with two independently chosen random initial configurations for either the spin glass or random ferromagnet in any finite dimension, and lets their respective (zero-temperature Glauber) dynamics proceed independently, then the two dynamical runs will almost surely "land" in separate 1-spin-flip stable states with overlap zero. However, the nature vs. nurture question focuses on a *single* randomly chosen initial configuration and asks for the spin overlap between the final states obtained through different runs under independent dynamical realizations. In [15], it was proved that with probability one the spin overlap between final states obtained under two independent dynamical runs starting from the *same* initial configuration is none other than $q_\infty > 0$.

The question then becomes how, given this perspective on the problem, we would expect this overlap to change with dimension. To address this, we note another relevant result from [15]. This asserts that the union of the basins of attraction of all k-spin flip stable states in any of these models has measure zero in the space of all randomly chosen initial conditions. Moreover, any initial condition (except for a set of measure zero) is on the boundary of an uncountably infinite number of basins of attraction of 1-spin-flip stable states. It follows that two dynamical realizations starting from the same initial condition have probability zero of landing in the same 1-spin-flip state for any fixed d.

So far all of our arguments are rigorously supported, but at this point we need to turn to heuristic arguments. It seems reasonable, and is (weakly) supported by the numerical results in Sect. IV that the number of spin flips per site in a typical dynamical run diverges as $d \to \infty$. As a consequence, it is reasonable to expect that the system can access an increasingly large subset of the set of all 1-spin-flip stable states. One would then expect the overlap of the 1-spin flip stable states within reach of a typical initial configuration to decay to zero as the dimension increases. This would then lead to $q_\infty \to 0$ as $d \to \infty$. Such an argument, of course, is only suggestive, and a fully rigorous approach will be pursued elsewhere.

B. Mean-field models

It is also of interest to consider some mean-field models which have not previously been studied from the nature vs. nurture perspective. The previous discussion may lead one to suppose that for a mean field model, $q_\infty \to 0$ as $N \to \infty$ should be normally expected. However, as we will see below, this is not the case.

We consider the random energy model (REM) [21], the Curie-Weiss ferromagnet, and (briefly) the random ferromagnet. While the results described below can be proved, we will present them here informally. We begin with the REM. Although the REM is not typically thought of as a dynamical model, there is a natural dynamics associated with it. For any N-spin system there are 2^N corresponding spin configurations, which are the corners of the hypercube. The REM assigns a random energy independently to each corner (or site) of the hypercube. The distribution of the energies could be Gaussian, as in the original formulation of the REM, or flat, or some similar distribution; the results are independent of the specific form, as long as the variance of the distribution is finite. A local minimum then corresponds to a site whose N neighboring sites on the hypercube all have larger energy. (This corresponds to a 1-spin flip stable state, because the spin configuration corresponding to each neighbor on the hypercube differs from that corresponding to the original site by a single spin flip.)

Consider now a random walk starting at an arbitrary site, corresponding to a uniformly chosen point on the hypercube. If the starting site is a local minimum the walk goes no farther. Otherwise, if the site has k (out of N) neighbors with lower energy, the walk chooses one uniformly at random among the k. This is equivalent to the usual zero-temperature Glauber or Metropolis dynamics on the Ising spin system formulation of the REM. One then continues the process until it ends at a local minimum.

We can then ask a number of well-posed questions; in particular, how long is the length of a typical walk? If such a walk does not proceed, on average, macroscopically far in the hypercube (i.e., has $o(N)$ steps), then q_∞ should go to 1. But this is precisely the case for the REM. In fact, this problem was solved in a different context by Kauffman and Levin [24], who found that, on average, such a walk takes only $O(\log N)$ steps. Consequently, $q_\infty \geq 1 - O(\log N/N)$. Therefore, as $N \to \infty$, $q_\infty \to 1$: nature always wins! This is an unusual result, which we have not seen for any nontrivial short-range model. (The largest q_∞ found for other models is 1/2, for a random Ising chain in one dimension [16].) The REM is of great interest because, despite its simplicity, it mimics much (though certainly not all) of the thermodynamics of the far more complex SK model. These arguments demonstrate, though, that the dynamics of the two models, at least from the viewpoint espoused in this paper, are very different.

We turn now to the mean-field ferromagnetic models. In the case of the homogeneous Curie-Weiss model with N spins, a typical initial condition will have an excess of spins (of order $\sqrt{N}$) in one state (say the plus state) over the other. Given the usual Glauber dynamics, it's clear that the final state will then be all plus, so the typical initial condition completely determines the final configuration. The only initial condition in which this will not be true is when exactly half the spins (assuming N is even) in the initial condition are plus and half minus. It is easy to see that the contribution to q_∞ from such configurations goes to zero as $N \to \infty$.

For disordered Curie-Weiss models, in which the couplings are i.i.d. nonnegative random variables, it can be shown that q_∞ still goes to 1 as $N \to \infty$, but more extensive arguments are required. A brief sketch of such an argument is as follows. A typical initial spin configuration will have on order $\sqrt{N}$ excess of plus or minus spins. Without loss of generality suppose the initial spin configuration has a positive magnetization of order $\sqrt{N}$. It is not hard to show, after taking into account the effect of fluctuations in the magnitudes of the couplings, that the difference between the number of sites with positive and negative effective field at time 0 is thus of order $\sqrt{N}$ with probability going to one as $N \to \infty$. Now letting the time increase from 0 to a time $t = o(\sqrt{N})$ later, one can use standard arguments based on the law of large numbers and central limit theorem to show that the magnetization increases by order t during this time interval. This continues for all times of order $N^{\frac{1}{2}-\epsilon}$ for any $\epsilon > 0$ during which, with probability approaching 1, no site's effective field changes sign. But once $O(\sqrt{N})$ flips have taken place, the magnetization has shifted on order $O(\sqrt{N})$ causing a corresponding shift, of the same order as the fluctuations in the effective field, in the effective field at every site. Thus every site's effective field after a time of order $\sqrt{N}$ is shifted in the positive direction by a quantity converging to a point mass at $c > 0$; consequently, some sites that had previously had negative effective fields now have positive ones.

Repeating this until a time $t = O(N^{\frac{1}{2}+\epsilon})$ (for any $\epsilon > 0$) leads, with high probability, to every site having positive effective field. Since the dynamics of the disordered Curie-Weiss model are monotone, then from that time on, spins flip only from minus to plus and the system steadily drifts to the all plus state. The conclusion is that as $N \to \infty$, with probability approaching 1, the system (thought of as a random walker on the 2^N hypercube) avoids all local minima and rapidly descends down the energy landscape to the all plus 1-spin flip stable state.

We conclude with a heuristic argument suggesting that the behavior of the SK model is very different: for this case we conjecture that $q_\infty(N) \to 0$ as $N \to \infty$. The SK model has exponentially many local minima as $N \to \infty$, and they are believed to be uniformly distributed on the hypercube (i.e., have typical overlap 0). Unlike the random ferromagnet, however, there is no natural "downward drift" toward a global energy minimum in the SK model owing to the randomness in the signs of the couplings. Moreover, unlike in the REM model, the energies assigned to vertices of the hypercube are highly correlated: the difference in energy of neighbors on the hypercube should be distributed as $\sim N(0, 1)$ (where $N(0, V)$ is the normal, or Gaussian, distribution of mean zero and variance V), while the energy of a typical site is distributed as $\sim N(0, N)$.

The numerics (Fig. 6) also suggest that the SK spin glass dynamics take more than $O(N)$ steps on the hypercube before reaching a local minimum, implying that a typical dynamical realization can traverse most of the hypercube as $N \to \infty$. Thus it should be the case that as $N \to \infty$, the set of accessible local minima approaches the set of all local minima, suggesting that $q_\infty(N) \to 0$ as $N \to \infty$.

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[1] A. J. Bray. Theory of phase-ordering kinetics. *Adv. Phys.*, 43: 357-459, 1994.
[2] D. Stauffer. Ising Spinodal Decomposition at T=0 in one to five dimensions. *J. Phys. A: Math. Gen.*, 27: 5029–5032, 1994.
[3] V. Spirin, P. L. Krapivsky and S. Redner. Fate of zero-temperature Ising ferromagnets. *Phys. Rev. E*, 63: 036118, 2000.
[4] V. Spirin, P. L. Krapivsky and S. Redner. Freezing in Ising ferromagnet. *Phys. Rev. E*, 65:016119, 2001.
[5] K. Barros, P. L. Krapivsky and S. Redner. Freezing into stripe states in two-dimensional ferromagnets and crossing probabilities in critical percolation. *Phys. Rev. E*, 80:040101, 2009.
[6] B. Derrida, A. J. Bray and C. Godreche. Non-trivial exponents in the zero temperature dynamics of the 1D Ising and Potts models. *J. Phys. A*, Math. Gen. 27: L357-L361, 1994.
[7] S. Jain. Zero-temperature dynamics of the weakly disordered Ising model. *Phys. Rev. E* 59: R2493-R2495, 1999.
[8] S. N. Majumdar and C. Sire. Survival probability of a Gaussian non-Markovian process: Application to the T=0 dynamics of the Ising model. *Phys. Rev. Lett*, 77: 1420–1423, 1996.
[9] G. Manoj and P. Ray. Persistence in higher dimension: A finite size scaling study. *Phys. Rev. E*, 62: 7755–7758, 2000.
[10] B. Derrida, V. Hakim and V. Pasquier. Exact exponent for the number of persistent spins in the zero-temperature dynamics of the one-dimensional Potts model. *J. Stat. Phys.*, 85: 763–797, 1996.
[11] S.A. Kauffman. Metabolic stability and epigenesis in randomly constructed genetic nets. *J. Theor. Biol.*, 22: 437-467, 1969.
[12] M. Creutz. Deterministic Ising dynamics. *Ann. Phys.*, 167:62-72, 1986.
[13] H.E. Stanley, D. Stauffer, J. Kertesz and H. Hermann. Dynamics of spreading phenomena in two-dimensional Ising models. *Phys. Rev. Lett.*, 59:2326-2328, 1987.
[14] P. Grassberger. Damage spreading and critical exponents for model A Ising dynamics. *Physica A*, 214:547-559, 1995.
[15] C.M. Newman and D.L. Stein. Equilibrium pure states and nonequilibrium chaos. *J. Stat. Phys.*, 94: 709-722, 1999.
[16] S. Nanda, C.M. Newman, and D.L. Stein. Dynamics of Ising spin systems at zero temperature. in *On Dobrushin's Way (from Probability Theory to Statistical Physics)*, R. Minlos, S. Shlosman and Y. Suhov, eds., *Amer. Math. Soc. Transl.* 2: 183-194, 2000.
[17] P.M.C. de Oliveira, C.M. Newman, V. Sidoravicious, and D.L. Stein. Ising ferromagnet: Zero-temperature dynamical evolution. *J. Phys. A*, 39: 6841-6849, 2006.
[18] J. Ye , J. Machta, C.M. Newman and D.L. Stein, Nature versus nurture: Predictability in low-temperature Ising dynamics. *Phys. Rev. E*, 88:040101, 2013.
[19] S. Edwards and P.W. Anderson, Theory of spin glasses. *J. Phys. F*, 5:965-974, 1975.
[20] D. Sherrington and S. Kirkpatrick, Solvable model of a spin glass. *Phys. Rev. Lett.*, 35:1792-1796, 1975.
[21] B. Derrida, Random-energy model: Limit of a family of disordered models. *Phys. Rev. Lett.*, 45:79-82, 1980.
[22] Although quantities introduced in this section, such as the heritability exponent and the spin overlap, are discussed within the context of the two-dimensional uniform ferromagnet, their definitions are valid (suitably generalized) in any dimension.
[23] It is possible in principle for $q_\infty = 0$ for a system where every spin flips only finitely many times. However, in all of the nontrivial physical models studied so far in which $q_\infty = 0$, q_t approaches zero asymptotically as $t \to \infty$, and we expect this to be the case more generally.
[24] S. Kauffman and S.A. Levin, Towards a general theory of adaptive walks on rugged landscapes. *J. Theor. Biol.*, 1281:11-45 (1987).