

# MATRIX WEIGHTED POINCARÉ INEQUALITIES AND APPLICATIONS TO DEGENERATE ELLIPTIC SYSTEMS

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ABSTRACT. We prove Poincaré and Sobolev inequalities in matrix  $A_p$  weighted spaces. We then use these Poincaré inequalities to prove existence and regularity results for degenerate systems of elliptic equations whose degeneracy is governed by a matrix  $A_p$  weight. Such results parallel earlier results by Fabes, Kenig, and Serapioni for a single degenerate equation governed by a scalar  $A_p$  weight. In addition, we prove Cacciopoli and reverse Hölder inequalities for weak solutions of the degenerate systems. As a means to prove the Poincaré inequalities we prove that the Riesz potential and fractional maximal function operators are bounded on matrix weighted  $L^p$  spaces and go on to develop an entire matrix  $A_{p,q}$  theory.

## 1. INTRODUCTION

The classic Poincaré inequality

$$\left( \frac{1}{|Q|} \int_Q |u(x) - u_Q|^q dx \right)^{1/q} \leq C|Q|^{\frac{1}{d}} \left( \frac{1}{|Q|} \int_Q |\nabla u(x)|^p dx \right)^{1/p},$$

holds for all cubes  $Q$  in  $\mathbb{R}^d$  when  $u$  is sufficiently smooth,  $u$  is sufficiently smooth,  $1 \leq p < n$ , and  $q = \frac{dp}{d-p}$ . Such inequalities are vital to the theory of regularity of weak solutions to PDE. Fabes, Kenig, and Serapioni [9] studied the degenerate elliptic equation

$$\operatorname{div}(A(x)\nabla u(x)) = \sum_{\alpha,\beta=1}^n \partial_\alpha(A_\alpha^\beta(x)\partial_\beta u(x)) = -\operatorname{div}\vec{f} \quad (1.1)$$

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where  $A$  is a positive definite matrix that satisfies

$$w(x)|\xi|^2 \simeq \langle A(x)\xi, \xi \rangle, \quad \xi \in \mathbb{R}^n$$

for some  $w \in A_2$  and  $|\vec{f}| \in L^2(\Omega, w^{-1})$ . They proved that weighted Poincaré inequalities of the form

$$\begin{aligned} \left( \frac{1}{w(Q)} \int_Q |u(x) - u_Q|^q w(x) dx \right)^{1/q} \\ \leq C|Q|^{\frac{1}{d}} \left( \frac{1}{w(Q)} \int_Q |\nabla u(x)|^p w(x) dx \right)^{1/p} \end{aligned} \quad (1.2)$$

hold for some  $q > p$  when  $w \in A_p$  and used the inequalities (1.2) to prove that weak solutions to (1.1) (under further assumptions on  $\vec{f}$ ) are Hölder continuous.

In this paper we will consider the system of degenerate elliptic equations of the form

$$\sum_{\alpha, \beta, j=1}^d \partial_\alpha (A_{ij}^{\alpha\beta}(x) \partial_\beta u_j(x)) = -(\operatorname{div} F)_i, \quad i = 1, \dots, d, \quad (1.3)$$

where

$$\sum_{\alpha, \beta, i, j=1}^n A_{ij}^{\alpha\beta}(x) \eta_\alpha^i \overline{\eta_\beta^j} \simeq \|W(x)^{\frac{1}{2}} \eta\|^2, \quad \eta \in \mathcal{M}_n(\mathbb{C}) \quad (1.4)$$

for a matrix weight  $W(x)$  (a positive definite matrix function with locally integrable entries) and  $F \in L^2(\Omega, W^{-1})$  (which will be defined momentarily). To the best of our knowledge, it seems that systems of elliptic equations whose degeneracies are governed by matrix weights have never been considered before.

Given a matrix weight  $W$  and an exponent  $p > 0$  we define  $L^p(\Omega, W)$  to be the collection of all vector valued functions  $\vec{f}$  such that

$$\|\vec{f}\|_{L^p(W, \Omega)}^p = \int_\Omega |W^{\frac{1}{p}}(x) \vec{f}(x)|^p dx < \infty.$$

We will also sometimes let  $L^p(\Omega, W)$  denote the space of all  $n \times n$  matrix functions  $F$  whose norm above is finite. When  $\Omega = \mathbb{R}^n$  we will write  $L^p(W)$ .

A natural solution space for weak solutions of 1.3 is the matrix weight Sobolev space  $H^{1,p}(\Omega, W)$  normed by

$$\|\vec{f}\|_{H^{1,p}(\Omega, W)} = \left( \int_\Omega |W^{\frac{1}{p}}(x) \vec{f}(x)|^p dx \right)^{\frac{1}{p}} + \left( \int_\Omega \|W^{\frac{1}{p}}(x) D \vec{f}(x)\|^p dx \right)^{\frac{1}{p}}$$

where  $\|\cdot\|$  is the matrix norm, which will be defined as the completion of  $C^\infty(\Omega)$  with respect to the above norm.

A matrix weight belongs to  $A_p$  if

$$[W]_{A_p} = \sup_Q \frac{1}{|Q|} \int_Q \left( \frac{1}{|Q|} \int_Q \|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|^{p'} dy \right)^{\frac{p}{p'}} dx < \infty.$$

Treil-Volberg [25] showed that the Hilbert transform, defined component-wise, is bounded on  $L^2(W)$  if and only if the matrix weight  $W$  belongs to  $A_2$ . Nazarov-Treil and Volberg [19, 26] when  $d = 1$  and the first author [12] when  $d > 1$  proved upper and lower matrix weighted Littlewood-Paley  $L^p$  bounds when  $W$  is a matrix  $A_p$  weight. Furthermore, Goldberg [7] characterized the boundedness of singular integral operators and the Hardy-Littlewood maximal operator by the matrix  $A_p$  condition.

We are now ready to state our main results. We begin with Sobolev and Poincaré inequalities in the matrix weighted case.

**Theorem 1.1.** *If  $1 < p < \infty$  and  $W$  is a matrix  $A_p$  weight, then there exists  $\epsilon > 0$  such that*

$$\begin{aligned} \left( \frac{1}{|Q|} \int_Q |W^{\frac{1}{p}}(x)\vec{f}(x)|^{p+\epsilon} dx \right)^{\frac{1}{p+\epsilon}} \\ \lesssim |Q|^{\frac{1}{d}} \left( \frac{1}{|Q|} \int_Q \|W^{\frac{1}{p}}(x)D\vec{f}(x)\|^{p-\epsilon} dx \right)^{\frac{1}{p-\epsilon}} \end{aligned}$$

for each cube  $Q$  and  $\vec{f} \in C_0^1(Q)$ .

**Theorem 1.2.** *If  $1 < p < \infty$ ,  $\vec{f} \in C^1(Q)$ , and  $W$  is a matrix  $A_p$  weight, then there exists  $\epsilon > 0$  such that*

$$\begin{aligned} \left( \frac{1}{|Q|} \int_Q |W^{\frac{1}{p}}(x)(\vec{f}(x) - \vec{f}_Q)|^{p+\epsilon} dx \right)^{\frac{1}{p+\epsilon}} \\ \lesssim |Q|^{\frac{1}{d}} \left( \frac{1}{|Q|} \int_Q \|W^{\frac{1}{p}}(x)D\vec{f}(x)\|^{p-\epsilon} dx \right)^{\frac{1}{p-\epsilon}} \end{aligned}$$

for each cube  $Q$ .

As will be apparent from the prove, note that we can in fact replace the cube  $Q$  in Theorem 1.2 with an open ball  $B$  so long as we have that  $\vec{f} \in C^1(B)$ .

It is well known that Poincaré inequalities follow from bounds on the fractional integral operators

$$I_\alpha f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}} dy, \quad 0 < \alpha < d$$

and their corresponding fractional maximal operators

$$M_\alpha f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha}{d}}} \int_Q |f(y)| dy, \quad 0 \leq \alpha < d.$$

Such operators play a crucial role in the theory of the smoothness of functions. The fractional integral operator acts as an anti-derivative and hence its boundedness implies the Sobolev embedding theorems. While our Sobolev and Poincaré inequalities will not follow from matrix weighted bounds for fractional integral operators, we will nevertheless be interested in proving such bounds for their own sake.

First let us recall the results in the scalar case. Muckenhoupt and Wheeden [18] characterized the weights  $w$  for which  $M_\alpha$  and  $I_\alpha$  are bounded on weighted Lebesgue spaces. In particular, they showed that if  $1 < p < \frac{d}{\alpha}$  and  $q$  is defined by  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$ , then  $I_\alpha$  and  $M_\alpha$  are bounded from  $L^p(w^{\frac{p}{q}})$  to  $L^q(w)$  if and only if  $w \in A_{p,q}$ :

$$[w]_{A_{p,q}} = \sup_Q \left( \frac{1}{|Q|} \int_Q w(x) dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{-\frac{p'}{q}} dx \right)^{\frac{q}{p'}} < \infty.$$

Lacey et. al. [16], found the sharp upper bounds on the operator norms in terms of the constant  $[w]_{A_{p,q}}$  showing that

$$\|M_\alpha\|_{L^p(w^{\frac{p}{q}}) \rightarrow L^q(w)} \lesssim [w]_{A_{p,q}}^{(1-\frac{\alpha}{d})\frac{p'}{q}} \quad (1.5)$$

and

$$\|I_\alpha\|_{L^p(w^{\frac{p}{q}}) \rightarrow L^q(w)} \lesssim [w]_{A_{p,q}}^{(1-\frac{\alpha}{d})\max(1, \frac{p'}{q})}. \quad (1.6)$$

We aim to study the matrix weighted case of these results. Given a matrix weight  $W$  and a pair of exponents  $p$  and  $q$  we define the matrix  $A_{p,q}$  constant as follows

$$[W]_{A_{p,q}} = \sup_Q \frac{1}{|Q|} \int_Q \left( \frac{1}{|Q|} \int_Q \|W^{\frac{1}{q}}(x)W^{-\frac{1}{q}}(y)\|^{p'} dy \right)^{\frac{q}{p'}} dx.$$

where the supremum is over all cubes contained in  $\mathbb{R}^n$ . A matrix weight belongs to  $A_{p,q}$  if  $[W]_{A_{p,q}} < \infty$ . Moreover, we define the weighted

fractional maximal function as follows

$$M_{W,\alpha}\vec{f}(x) = \sup_{Q\ni x} \frac{1}{|Q|^{1-\frac{\alpha}{d}}} \int_Q |W^{\frac{1}{q}}(x)W^{-\frac{1}{q}}(y)\vec{f}(y)| dy$$

where the supremum is over all cubes that contain  $x$ . We will be concerned with  $L^p \rightarrow L^q$  bounds for  $M_{\alpha,W}$ . Our first result is the following.

**Theorem 1.3.** *Suppose  $0 \leq \alpha < d$ ,  $1 < p < \frac{d}{\alpha}$  and  $q$  is defined by  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$ . If  $W \in A_{p,q}$  then*

$$\|M_{\alpha,W}\|_{L^p \rightarrow L^q} \lesssim [W]_{A_{p,q}}^{\frac{p'}{q}(1-\frac{\alpha}{d})} \quad (1.7)$$

and this bound is sharp.

Inequality (1.7) is the matrix valued version of (1.5). In fact, the sharpness of (1.7) follows from the scalar case because a better bound for the matrix case would imply a better bound for the scalar case. We remark that the proof is a modification of the arguments found in [7].

For the fractional integral operator we have the following result.

**Theorem 1.4.** *Suppose  $0 \leq \alpha < d$ ,  $1 < p < \frac{d}{\alpha}$  and  $q$  is defined by  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$ . If  $W \in A_{p,q}$  then  $I_\alpha : L^p(W^{\frac{p}{q}}) \rightarrow L^q(W)$  and*

$$\|I_\alpha\|_{L^p(W^{\frac{p}{q}}) \rightarrow L^q(W)} \lesssim [W]_{A_{p,q}}^{(1-\frac{\alpha}{d})\frac{p'}{q}+1} \quad (1.8)$$

We do not believe the bound (1.8) is sharp and other methods will be needed to find the sharp bound.

Using our Poincaré inequalities we are able to prove regularity results for weak solutions to (1.3). We begin with the following reverse Hölder inequality. In the uniformly elliptic case, this result is due to Meyers.

**Theorem 1.5.** *Let  $W$  be a matrix  $A_2$  weight, let  $\Omega$  be a domain in  $\mathbb{R}^d$ , and let  $F \in L^2(\Omega, W^{-1})$ . If  $A = A_{ij}^{\alpha\beta}$  satisfies (1.4) and  $\vec{u} \in H^{1,2}(\Omega, W)$  is a weak solution to (1.3), then there exists  $q > 2$  such that given  $B_{2r} \subset \Omega$  we have*

$$\begin{aligned} \left( \frac{1}{|B_{r/2}|} \int_{B_{r/2}} \|W^{\frac{1}{2}}(x)D\vec{u}(x)\|^q dx \right)^{\frac{1}{q}} &\lesssim \left( \frac{1}{|B_r|} \int_{B_r} \|W^{\frac{1}{2}}(x)D\vec{u}(x)\|^2 dx \right)^{\frac{q}{2}} \\ &+ \left( \frac{1}{|B_r|} \int_{B_r} \|W^{-\frac{1}{2}}(x)F(x)\|^q dx \right)^{\frac{1}{q}}. \end{aligned}$$

In Section 6 we will also briefly discuss applying some of the simple ideas in the very recent paper [8] to extend this Meyer reverse Hölder inequality to solutions of nonhomogenous degenerate  $p$ -Laplacian systems setting with a matrix  $A_p$  degeneracy (see the beginning of Section 5 for precise definitions.)

Finally, we end with the last of our main result: a regularity theorem for weak solutions in dimension two.

**Theorem 1.6.** *Let  $d = 2$  and  $\vec{u}$  be a weak solution to (1.3) when  $F = 0$ . Then there exists  $\epsilon > 0$  such that for  $x, y \in \Omega$  with  $|x - y| < \frac{1}{2} \text{dist}(\{x, y\}, \Omega^c)$ , we have*

$$|\vec{u}(x) - \vec{u}(y)| \lesssim C_{x,y} |x - y|^\epsilon$$

where

$$C_{x,y} = \left( \sup_{|Q|^{1-\epsilon}} \int_Q \|W^{-\frac{1}{2}}(\xi)\|^2 d\xi \right)^{\frac{1}{2}}$$

where the supremum is over cubes  $Q \subset \Omega$  centered either at  $x$  or  $y$ , and having side length smaller than  $|x - y|$ .

As with Theorem 1.5, we will also extend 1.6 to solutions of homogeneous degenerate  $p$ -Laplacian systems setting in the last section. Note that it would be very interesting to know whether one can use Theorem 1.6 to prove continuity a.e. of weak solutions to (1.3) when  $F = 0$ .

In the special case when  $A_{ij}^{\alpha\beta}(x) = B_{ij}(x)\delta_{\alpha\beta}$  for some matrix  $B$ , the system (1.3) becomes

$$\text{div}(B(x)D\vec{u}(x)) = -(\text{div}F)(x).$$

Such systems were considered by Iwaniec/Martin [13], Huang [11], and Stroffolini [24]. Of particular is when  $B$  itself is a matrix  $A_p$  weight, and Theorems 1.5 and 1.6 are of independent interest themselves in this case.

The plan of the paper will be as follows. In Section 2 we will state some notation that will be used throughout the paper. In Section 3 we will prove Theorem 1.3 and Theorem 1.4. We will prove the Poincaré and Sobolev inequalities in Section 4 and prove the existence results in Section 5. Finally we finish the manuscript with the proof of the local regularity of weak solutions including the proofs of the Meyers' reverse Hölder estimates (Theorem 1.5) and the local regularity in dimension two (Theorem 1.6) in Section 6.

## 2. PRELIMINARIES

We will first need the notion of dyadic grid. Cubes will always be assumed to have sides parallel to the coordinate axes and we will denote the side-length of a cube  $Q$  as  $\ell(Q)$ . A dyadic grid, usually denoted  $\mathcal{D}$  will be a collection of cubes that satisfy the following three properties:

- (1) If  $Q \in \mathcal{D}$  then  $\ell(Q) = 2^k$  for some  $k \in \mathbb{Z}$ .
- (2) If  $\mathcal{D}^k = \{Q \in \mathcal{D} : \ell(Q) = 2^k\}$ , then  $\mathbb{R}^d = \bigcup_{Q \in \mathcal{D}^k} Q$ .
- (3) If  $Q, P \in \mathcal{D}$  then  $Q \cap P$  is either  $\emptyset$ ,  $Q$ , or  $P$ .

We will use the following well known fact about dyadic grids whose proof can be found in a recent manuscript by Lerner and Nazarov [17].

**Proposition 2.1.** *Let  $\mathcal{D}^t = \{2^{-k}([0, 1]^d + m + (-1)^k t) : k \in \mathbb{Z}, m \in \mathbb{Z}^d\}$ , then given any cube  $Q$ , there exists  $1 \leq t \leq 2^d$  and  $Q_t \in \mathcal{D}^t$  such that  $Q \subset Q_t$  and  $\ell(Q_t) \leq 6\ell(Q)$ .*

We now establish the machinery of the matrix weights needed for the paper. Given a cube  $Q$ , let  $\tilde{V}_Q, \tilde{V}'_Q$  be a reducing operator (i.e. a positive definite  $n \times n$  matrix) where

$$|\tilde{V}_Q \vec{e}| \approx \left( \frac{1}{|Q|} \int_Q |W^{-\frac{1}{q}}(x) \vec{e}|^{p'} dx \right)^{\frac{1}{p'}}, \quad |\tilde{V}'_Q \vec{e}| \approx \left( \frac{1}{|Q|} \int_Q |W^{\frac{1}{q}}(x) \vec{e}|^q dx \right)^{\frac{1}{q}}.$$

In fact we can pick the reducing operators in such a way that

$$\left( \frac{1}{|Q|} \int_Q |W^{-\frac{1}{q}}(x) \vec{e}|^{p'} dx \right)^{\frac{1}{p'}} \leq |\tilde{V}_Q \vec{e}| \leq \sqrt{n} \left( \frac{1}{|Q|} \int_Q |W^{-\frac{1}{q}}(x) \vec{e}|^{p'} dx \right)^{\frac{1}{p'}}$$

and a similar statement holds for  $|\tilde{V}'_Q \vec{e}|$  (see Proposition 1.2 in [7]). Using the reducing operators we see that

$$\begin{aligned} \sup_Q \|\tilde{V}_Q \tilde{V}'_Q\|^q &\approx [W]_{A_{p,q}} \\ &= \sup_Q \frac{1}{|Q|} \int_Q \left( \frac{1}{|Q|} \int_Q \|W^{\frac{1}{q}}(x) W^{-\frac{1}{q}}(y)\|^{p'} dy \right)^{\frac{q}{p'}} dx. \end{aligned} \quad (2.1)$$

Let  $\rho$  be a norm on  $\mathbb{C}^n$  and let  $\rho^*$  be the dual norm defined by

$$\rho^*(\vec{e}) = \sup_{\vec{f} \in \mathbb{C}^n} \frac{|\langle \vec{e}, \vec{f} \rangle_{\mathbb{C}^n}|}{\rho(\vec{f})}.$$

By elementary arguments we have that  $(\rho^*)^* = \rho$  for any norm  $\rho$ . Also let

$$\rho_{q,Q}(\vec{e}) = \left( \frac{1}{|Q|} \int_Q |W^{\frac{1}{q}}(x) \vec{e}|^q dx \right)^{\frac{1}{q}}$$

so that  $\rho_{q,Q}(\vec{e}) \approx |\tilde{V}'_Q \vec{e}|$  and by trivial arguments  $\rho_{q,Q}^*(\vec{e}) \approx |(\tilde{V}'_Q)^{-1} \vec{e}|$ .

### 3. BOUNDS FOR FRACTIONAL OPERATORS

In this section we will prove Theorems 1.3 and 1.4. We begin with some facts about the matrix  $A_{p,q}$  condition. Throughout this section we will assume that  $0 \leq \alpha < d$  and  $p$  and  $q$  and satisfy the Sobolev relationship

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}.$$

**Proposition 3.1.**  *$W$  is an  $A_{p,q}$  weight if and only if the averaging operators*

$$\vec{f} \mapsto \frac{1_Q}{|Q|^{1-\frac{\alpha}{d}}} \int_Q \vec{f}(x) dx$$

are uniformly bounded from  $L^p(W^{\frac{p}{q}})$  to  $L^q(W)$ .

*Proof.* The proof is similar to Proposition 2.1 in [7]. In particular, since  $L^p(W^{\frac{p}{q}})$  is the dual space of  $L^{p'}(W^{-\frac{p'}{q}})$  under the usual unweighted pairing

$$L_{\vec{g}}(\vec{f}) = \left\langle \vec{f}, \vec{g} \right\rangle_{L^2(\mathbb{R}^d; \mathbb{C}^n)}$$

for  $\vec{g} \in L^p(W^{\frac{p}{q}})$ , we have that

$$\begin{aligned} \sup_{\|\vec{f}\|_{L^p(W^{\frac{p}{q}})}=1} \left\| \frac{1_Q}{|Q|^{1-\frac{\alpha}{d}}} \int_Q \vec{f}(x) dx \right\|_{L^q(W)} &= \sup_{\|\vec{f}\|_{L^p(W^{\frac{p}{q}})}=1} |Q|^{-\frac{1}{p'}} \rho_{q,Q} \left( \int_Q \vec{f} dx \right) \\ &= \sup_{\|\vec{f}\|_{L^p(W^{\frac{p}{q}})}=1} \sup_{\vec{e} \in \mathbb{C}^n} |Q|^{-\frac{1}{p'}} \frac{\left| \int_Q \langle \vec{f}(x), \vec{e} \rangle_{\mathbb{C}^n} dx \right|}{(\rho_{q,Q})^*(\vec{e})} \\ &= \sup_{\vec{e} \in \mathbb{C}^n} |Q|^{-\frac{1}{p'}} \frac{\|1_Q \vec{e}\|_{L^{p'}(W^{-\frac{p'}{q}})}}{(\rho_{q,Q})^*(\vec{e})} \end{aligned}$$

and the last term here being uniformly finite (with respect to all cubes  $Q$ ) is easily seen to be equivalent to  $W$  being an  $A_{p,q}$  weight.  $\square$



**3.1. The fractional maximal operator.** Recall that the natural definition of the maximal operator on matrix weighted spaces is given by

$$M_{W,\alpha}\vec{f}(x) = \sup_{Q\ni x} \frac{1}{|Q|^{1-\frac{\alpha}{d}}} \int_Q |W^{\frac{1}{q}}(x)W^{-\frac{1}{q}}(y)\vec{f}(y)| dy.$$

We will also need the following auxiliary fractional maximal operator:

$$M'_{W,\alpha}\vec{f}(x) = \sup_{Q\ni x} \frac{1}{|Q|^{1-\frac{\alpha}{d}}} \int_Q |\tilde{V}_Q^{-1}W^{-\frac{1}{q}}(y)\vec{f}(y)| dy.$$

**Corollary 3.2.** *If  $M_{W,\alpha} : L^p \rightarrow L^q$  boundedly then  $W$  is a matrix  $A_{p,q}$  weight.*

*Proof.* For each cube  $Q$  containing  $x$  we have

$$\begin{aligned} \left| \frac{1_Q(x)}{|Q|^{1-\frac{\alpha}{d}}} \int_Q W^{\frac{1}{q}}(x)\vec{f}(y) dy \right| &\leq \frac{1_Q(x)}{|Q|^{1-\frac{\alpha}{d}}} \int_Q |W^{\frac{1}{q}}(x)\vec{f}(y)| dy \\ &\leq M_{W,\alpha}(W^{\frac{1}{q}}\vec{f}) \end{aligned}$$

so that

$$\begin{aligned} \sup_Q \left\| \frac{1_Q}{|Q|^{1-\frac{\alpha}{d}}} \int_Q \vec{f}(y) dy \right\|_{L^q(W)} &\leq \left\| M_{W,\alpha}(W^{\frac{1}{q}}\vec{f}) \right\|_{L^q} \\ &\lesssim \|\vec{f}\|_{L^p(W^{\frac{p}{q}})}. \end{aligned}$$

□

**Corollary 3.3.** *If  $W$  is a matrix  $A_{p,q}$  weight then for any unit vector  $\vec{e}$  we have that  $|W^{\frac{1}{q}}\vec{e}|^q$  is a scalar  $A_{p,q}$  weight with  $A_{p,q}$  characteristic  $\lesssim [W]_{A_{p,q}}$ .*

*Proof.* Let  $\phi$  be any scalar function and let  $\vec{f} = \phi\vec{e}$ . By Proposition 3.1, we have that

$$\phi \mapsto \frac{1_Q}{|Q|^{1-\frac{\alpha}{d}}} \int_Q \phi(x) dx$$

are uniformly bounded from the scalar weighted space  $L^p(|W^{\frac{1}{q}}\vec{e}|^p)$  to the scalar weighted space  $L^q(|W^{\frac{1}{q}}\vec{e}|^q)$ . But Proposition 3.1 again in scalar setting then gives us that  $|W^{\frac{1}{q}}\vec{e}|^q$  is a scalar  $A_{p,q}$  weight with  $A_{p,q}$  characteristic  $\lesssim [W]_{A_{p,q}}$ . □

**Corollary 3.4.**  *$W$  is an  $A_{p,q}$  weight if and only if  $W^{-\frac{p'}{q}}$  is an  $A_{q',p'}$  weight.*

*Proof.* By Proposition 3.1 and duality, we have that  $W$  is an  $A_{p,q}$  weight if and only if the averaging operators

$$\vec{f} \mapsto \frac{1_Q}{|Q|^{1-\frac{\alpha}{d}}} \int_Q \vec{f}(x) dx$$

are uniformly bounded from  $L^{q'}(W^{-\frac{d'}{q}})$  to  $L^{p'}(W^{-\frac{d'}{q}})$ . However, trivially we have

$$\frac{1}{p'} = \frac{1}{q'} - \frac{\alpha}{d}$$

so another application of Proposition 3.1 tells us that  $W^{-\frac{d'}{q}}$  is an  $A_{q',p'}$  weight if and only if  $W$  is an  $A_{p,q}$  weight.  $\square$

**Remark.** Let  $r = 1 + \frac{q}{p'}$ . Clearly these two corollaries also means that the  $A_r$  characteristic of each  $|W^{-\frac{1}{q}} \vec{e}|^{p'}$  is bounded by  $[W]_{A_{p,q}}^{r'-1}$ .

Furthermore, it is easy to see that  $w$  is a scalar  $A_{p,q}$  weight if and only if  $w$  is a scalar  $A_r$  weight. In the matrix case, however, there is no reason to believe that this is true. In particular,  $W$  is a matrix  $A_r$  weight precisely when

$$\sup_Q \frac{1}{|Q|} \int_Q \left( \frac{1}{|Q|} \int_Q \|W^{\frac{p'}{p'+q}}(x) W^{-\frac{p'}{p'+q}}(y)\|_{\frac{p'+q}{q}} dy \right)^{\frac{q}{p'}} dx < \infty$$

which is unlikely to imply, or be implied by (2.1).

**Lemma 3.5.** *If  $W$  is an  $A_{p,q}$  weight then  $\|M'_{W,\alpha}\|_{L^p \rightarrow L^q}^q \lesssim [W]_{A_{p,q}}^{r'-1}$*

*Proof.* By the scalar reverse Hölder inequality for  $A_\infty$  weights and the above remark, we can pick  $\epsilon \approx [W]_{A_{p,q}}^{1-r'}$  where

$$\left( \frac{1}{|Q|} \int_Q |W^{-\frac{1}{q}}(x) \vec{e}|^{\frac{p-\epsilon}{p-\epsilon-1}} dx \right)^{\frac{p-\epsilon-1}{p-\epsilon}} \lesssim \left( \frac{1}{|Q|} \int_Q |W^{-\frac{1}{q}}(x) \vec{e}|^{p'} dx \right)^{\frac{1}{p'}}.$$

Let  $\{\vec{e}_i\}_{i=1}^n$  be any orthonormal basis of  $\mathbb{C}^n$  and for any fixed  $y \in \mathbb{R}^d$  let  $Q$  be a cube that contains  $y$ . Then by Hölder's inequality we have that

$$\begin{aligned} & \frac{1}{|Q|^{1-\frac{\alpha}{d}}} \int_Q |\tilde{V}_Q^{-1} W^{-\frac{1}{q}}(x) \vec{f}(x)| dx \\ & \leq |Q|^{\frac{\alpha}{d}} \left( \frac{1}{|Q|} \int_Q \|W^{-\frac{1}{q}}(x) \tilde{V}_Q^{-1}\|_{\frac{p-\epsilon}{p-\epsilon-1}} dx \right)^{\frac{p-\epsilon-1}{p-\epsilon}} \left( \frac{1}{|Q|} \int_Q |f(x)|^{p-\epsilon} dx \right)^{\frac{1}{p-\epsilon}} \end{aligned}$$

However, by the reverse Hölder inequality, we have

$$\begin{aligned}
& \left( \frac{1}{|Q|} \int_Q \|W^{-\frac{1}{q}}(x)\tilde{V}_Q^{-1}\|_{\frac{p-\epsilon}{p-\epsilon-1}} dx \right)^{\frac{p-\epsilon-1}{p-\epsilon}} \\
& \approx \sum_{i=1}^n \left( \frac{1}{|Q|} \int_Q |W^{-\frac{1}{q}}(x)\tilde{V}_Q^{-1}\vec{e}_i|_{\frac{p-\epsilon}{p-\epsilon-1}} dx \right)^{\frac{p-\epsilon-1}{p-\epsilon}} \\
& \lesssim \sum_{i=1}^n \left( \frac{1}{|Q|} \int_Q |W^{-\frac{1}{q}}(x)\tilde{V}_Q^{-1}\vec{e}_i|_{p'} dx \right)^{\frac{1}{p'}} \\
& \approx \sum_{i=1}^n \|\tilde{V}_Q\tilde{V}_Q^{-1}\| \lesssim 1.
\end{aligned}$$

Thus, if  $M$  is the ordinary maximal function then an application of Hölder's inequality gives us that

$$\begin{aligned}
& \left( \frac{1}{|Q|^{1-\frac{\alpha}{d}}} \int_Q |\tilde{V}_Q^{-1}W^{-\frac{1}{q}}(x)\vec{f}(x)| dx \right)^q \lesssim |Q|^{\frac{q\alpha}{d}} \left( \frac{1}{|Q|} \int_Q |\vec{f}(x)|^{p-\epsilon} dx \right)^{\frac{q}{p-\epsilon}} \\
& = |Q|^{\frac{q\alpha}{d}} \left( \frac{1}{|Q|} \int_Q |\vec{f}(x)|^{p-\epsilon} dx \right)^{\frac{q-p}{p-\epsilon}} \left( \frac{1}{|Q|} \int_Q |\vec{f}(x)|^{p-\epsilon} dx \right)^{\frac{p}{p-\epsilon}} \\
& \leq |Q|^{\frac{q\alpha}{d}} \left( \frac{1}{|Q|} \int_Q |\vec{f}(x)|^p dx \right)^{\frac{q-p}{p}} \left( M(|\vec{f}|^{p-\epsilon})(y) \right)^{\frac{p}{p-\epsilon}} \\
& = \left( \int_{\mathbb{R}^d} |\vec{f}(x)|^p dx \right)^{\frac{q-p}{p}} \left( M(|\vec{f}|^{p-\epsilon})(y) \right)^{\frac{p}{p-\epsilon}}
\end{aligned}$$

since  $\frac{q\alpha}{d} - \frac{q}{p} + 1 = 0$ . Thus, the standard  $L^t$  bound for  $M$  with  $t > 1$  gives us

$$\begin{aligned}
\int_{\mathbb{R}^d} (M'_{W,\alpha}\vec{f}(y))^q dx & \lesssim \|\vec{f}\|_{L^p}^{q-p} \|M(|\vec{f}|^{p-\epsilon})\|_{L^{\frac{p}{p-\epsilon}}}^{\frac{p}{p-\epsilon}} \\
& \lesssim \epsilon^{-1} \|\vec{f}\|_{L^p}^q.
\end{aligned}$$

□

**Lemma 3.6.** *Let  $Q$  be a dyadic cube (in some fixed dyadic lattice) and*

$$N_Q(x) = \sup_{Q \supseteq R \ni x} \|W^{\frac{1}{q}}(x)\tilde{V}_R\|$$

where the supremum is taken over all dyadic cubes  $R \subseteq Q$  containing  $x$ . If  $W$  is an  $A_{p,q}$  weight then we can pick  $\delta \approx [W]_{A_{p,q}}^{-1}$  where

$$\int_Q (N_Q(x))^{q+\epsilon} dx \lesssim |Q|[W]_{A_{p,q}}$$

for all  $0 \leq \epsilon < \delta$ .

*Proof.* By an approximation argument in [2] p. 12, we may assume that the weights are truncated, i.e.,

$$W, W^{-1} \leq k \text{Id}_{n \times n}.$$

Our bounds will not depend on  $k$  and hence we will be to obtain the desired result by a limiting argument. As in [7], we will show that  $B \lesssim [W]_{A_{p,q}}$  if we assume that

$$\int_Q (N_Q(x))^{q+\epsilon} dx \lesssim B|Q|.$$

Let  $\{R_j\}$  be maximal dyadic subcubes of  $Q$  satisfying

$$\|\tilde{V}_Q^{-1}\tilde{V}_{R_j}\| > C$$

for some large  $C$  independent of  $W$  to be determined. Note that if  $x \in Q \setminus \cup_j R_j$  then for any dyadic cube  $R \subseteq Q$  containing  $x$  we have

$$\begin{aligned} \|W^{\frac{1}{q}}(x)\tilde{V}_R\| &\leq \|W^{\frac{1}{q}}(x)\tilde{V}_Q\| \|\tilde{V}_Q^{-1}\tilde{V}_R\| \\ &\leq C \|W^{\frac{1}{q}}(x)\tilde{V}_Q\| \end{aligned}$$

so that

$$\begin{aligned} \int_{Q \setminus \cup_j R_j} (N_Q(x))^{q+\epsilon} dx &\leq C^{q+\epsilon} \int_Q \|W^{\frac{1}{q}}(x)\tilde{V}_Q\|^{q+\epsilon} dx \\ &\leq C^q \sum_{i=1}^n \int_Q |W^{\frac{1}{q}}(x)\tilde{V}_Q \vec{e}_i|^{q+\epsilon} dx \\ &\leq C^q [W]_{A_{p,q}}^{\frac{q+\epsilon}{q}} |Q| \\ &\lesssim C^q [W]_{A_{p,q}} |Q| \end{aligned}$$

since  $0 \leq \epsilon \lesssim [W]_{A_{p,q}}^{-1}$ . On the other hand,

$$\begin{aligned} C^{p'} \sum_j |R_j| &\leq \sum_j |R_j| \|\tilde{V}_Q^{-1}\tilde{V}_{R_j}\|^{p'} \\ &\lesssim \sum_{i=1}^n \sum_j \int_{R_j} |W^{-\frac{1}{q}}(x)\tilde{V}_Q^{-1}\vec{e}_i|^{p'} dx \lesssim |Q|. \end{aligned}$$

Thus for  $C$  large enough independent of  $W$  we have  $\sum_j |R_j| \leq \frac{1}{2}|Q|$ .

By the definition of the  $R_j$ 's and their maximality we can assume for each  $x \in R_j$  that  $N_Q(x) = N_{R_j}(x)$  since otherwise  $N_Q(x) \leq C\|W^{\frac{1}{q}}(x)\tilde{V}_Q\|$ . Then we have

$$\int_{\cup_j R_j} (N_Q(x))^{q+\epsilon} dx = \sum_j \int_{R_j} (N_{R_j}(x))^{q+\epsilon} dx \leq B \sum_j |R_j| \leq \frac{1}{2}B|Q|.$$

Finally this implies that there exists  $C$  independent of  $W$  where  $B \leq \frac{1}{2}B + C[W]_{A_{p,q}}$  which completes the proof.  $\square$

We are now ready to prove Theorem 1.3

*Proof of Theorem 1.3.* By Proposition 2.1 we may assume that the supremum defining  $M_{W,\alpha}$  is over all cubes from a fixed dyadic grid  $\mathcal{D}$ . For each  $x \in \mathbb{R}^d$  let  $R_x$  be dyadic cube containing  $x$  such that

$$\begin{aligned} \frac{1}{2}(M_{W,\alpha}\vec{f})(x) &\leq \frac{1}{|R_x|^{1-\frac{\alpha}{d}}} \int_{R_x} |W^{\frac{1}{q}}(x)W^{-\frac{1}{q}}(y)\vec{f}(y)| dy \\ &\leq \|W^{\frac{1}{q}}(x)\tilde{V}_{R_x}\| \left( \frac{1}{|R_x|^{1-\frac{\alpha}{d}}} \int_{R_x} |\tilde{V}_{R_x}^{-1}W^{-\frac{1}{q}}(y)\vec{f}(y)| dy \right). \end{aligned} \quad (3.1)$$

For  $x \in \mathbb{R}^d$  pick  $j \in \mathbb{Z}$  where

$$2^j \leq \frac{1}{|R_x|^{1-\frac{\alpha}{d}}} \int_{R_x} |\tilde{V}_{R_x}^{-1}W^{-\frac{1}{q}}(y)\vec{f}(y)| dy < 2^{j+1} \quad (3.2)$$

and let  $\mathcal{S}_j$  be the collection of all cubes  $R = R_x$  for all  $x \in \mathbb{R}^d$  that are maximal with respect to (3.2) (note that Hölder's inequality implies that such a maximal cube exists). Then, for every  $x \in \mathbb{R}^d$  we have that  $R_x \subseteq S \in \mathcal{S}_j$  for some  $j = j_x \in \mathbb{Z}$  and  $S \in \mathcal{S}_j$ . Then for such  $S \in \mathcal{S}_j$  we have

$$\begin{aligned} (M_{W,\alpha}\vec{f})(x) &\leq 2\|W^{\frac{1}{q}}(x)\tilde{V}_{R_x}\| \left( \frac{1}{|R_x|^{1-\frac{\alpha}{d}}} \int_{R_x} |\tilde{V}_{R_x}^{-1}W^{-\frac{1}{q}}(y)\vec{f}(y)| dy \right) \\ &\leq 2(2^{j+1})N_S(x) \end{aligned}$$

so that finally the previous two lemmas give us that

$$\begin{aligned} \int_{\mathbb{R}^d} |M_{W,\alpha}\vec{f}(x)|^q dx &\lesssim \sum_{j \in \mathbb{Z}, S \in \mathcal{S}_j} 2^{qj} \int_S (N_S(x))^q dx \\ &\lesssim [W]_{A_{p,q}} \sum_{j \in \mathbb{Z}} 2^{qj} |\bigsqcup \mathcal{S}_j| \end{aligned}$$

$$\begin{aligned}
&\leq [W]_{A_{p,q}} \sum_{j \in \mathbb{Z}} 2^{qj} |\{x : M'_{W,\alpha} \vec{f}(x) \geq 2^j\}| \\
&\approx [W]_{A_{p,q}} \|M'_{W,\alpha} \vec{f}\|_{L^q}^q \\
&\lesssim [W]_{A_{p,q}}^{r'} \|\vec{f}\|_{L^p}^p
\end{aligned}$$

which completes the proof.  $\square$

We end our discussion of the fractional maximal function on matrix weighted spaces with an observation that operator  $M'_{W,\alpha}$  defined over dyadic cubes is weak type  $(p, q)$  for any matrix weight  $W$ .

**Proposition 3.7.** *If  $W$  is any matrix weight then  $M'_{W,\alpha}$  is weak  $(p, q)$*

*Proof.* Let  $\lambda > 0$  and pick maximal dyadic cubes  $Q_j$  such that

$$\frac{1}{|Q_j|^{1-\frac{\alpha}{d}}} \int_{Q_j} |\tilde{V}_{Q_j}^{-1} W^{-\frac{1}{q}}(y) \vec{f}(y)| dy > \lambda$$

so that

$$\{x : M'_{W,\alpha} \vec{f}(x) > \lambda\} = \bigsqcup_j Q_j.$$

However, by Hölder's inequality we have that

$$\begin{aligned}
\sum_j |Q_j| &= \sum_j \frac{|Q_j|^{q-\frac{q\alpha}{d}}}{|Q_j|^{q-1-\frac{q\alpha}{d}}} \\
&\leq \frac{1}{\lambda^q} \sum_j \left( \frac{1}{|Q_j|^{1-\frac{1}{q}-\frac{\alpha}{d}}} \int_{Q_j} |\tilde{V}_{Q_j}^{-1} W^{-\frac{1}{q}}(y) \vec{f}(y)| dy \right)^q \\
&\leq \frac{1}{\lambda^q} \sum_j \left( \frac{1}{|Q_j|} \int_{Q_j} \|\tilde{V}_{Q_j}^{-1} W^{-\frac{1}{q}}(y)\|^{p'} dy \right)^{\frac{q}{p'}} \left( \int_{Q_j} |\vec{f}(y)|^p dy \right)^{\frac{q}{p}} \\
&\lesssim \frac{1}{\lambda^q} \left( \sum_j \int_{Q_j} |\vec{f}(y)|^p dy \right)^{\frac{q}{p}} \\
&\leq \frac{1}{\lambda^q} \left( \int_{\mathbb{R}^d} |\vec{f}(y)|^p dy \right)^{\frac{q}{p}}
\end{aligned}$$

since  $\frac{q}{p} > 1$ .  $\square$

Unfortunately, it is not clear whether this result can be used to sharpen any of the results in this paper with respect to the  $A_{p,q}$  characteristic.

**3.2. Fractional integral operators.** Let  $I_\alpha$  be the Riesz potential defined by

$$I_\alpha \vec{f}(x) = \int_{\mathbb{R}^d} \frac{\vec{f}(y)}{|x-y|^{d-\alpha}} dy.$$

We begin by approximating  $I_\alpha$  by a dyadic operator. Recall

**Lemma 3.8.** *Let  $\mathcal{D}^t$  be collection of dyadic grids from Proposition 2.1 then*

$$\begin{aligned} & \left| \left\langle W^{\frac{1}{q}} I_\alpha W^{-\frac{1}{q}} \vec{f}, \vec{g} \right\rangle \right| \\ & \lesssim \sum_{t \in \{0, \frac{1}{3}\}^d} \sum_{Q \in \mathcal{D}^t} \frac{1}{|Q|^{1-\frac{\alpha}{d}}} \int_Q \int_Q \left| \left\langle W^{-\frac{1}{q}}(y) \vec{f}(y), W^{\frac{1}{q}}(x) \vec{g}(x) \right\rangle_{\mathbb{C}^n} \right| dx dy \end{aligned}$$

*Proof.* The proof requires nothing new in the matrix setting: let  $|\cdot|_\infty$  be the standard  $L^\infty$  norm on  $\mathbb{R}^d$  and let  $Q(x, r)$  be the ball with center  $x \in \mathbb{R}^d$  in this norm. Then for each  $k \in \mathbb{Z}$  there exists  $t \in \{0, \frac{1}{3}\}^d$  and  $Q_t \in \mathcal{D}^t$  such that  $Q(x, 2^k) \subset Q_t$  and

$$2^{k+1} = \ell(Q) \leq \ell(Q_t) \leq 6\ell(Q(x, 2^k)) = 12 \cdot 2^k.$$

Thus, we have

$$\begin{aligned} & \left| \left\langle W^{\frac{1}{q}} I_\alpha W^{-\frac{1}{q}} \vec{f}, \vec{g} \right\rangle \right| \\ & \leq \int_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}} \int_{2^{k-1} \leq |x-y|_\infty < 2^k} \frac{\left| \left\langle W^{-\frac{1}{q}}(y) \vec{f}(y), W^{\frac{1}{q}}(x) \vec{g}(x) \right\rangle_{\mathbb{C}^n} \right|}{|x-y|^{n-\alpha}} dy dx \\ & \lesssim \int_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}} \sum_{t \in \{0, \frac{1}{3}\}^d} \sum_{\substack{Q \in \mathcal{D}^t \\ 2^{k-1} \leq \ell(Q) < 2^k}} \frac{\chi_Q(x)}{|Q|^{1-\frac{\alpha}{d}}} \int_Q \left| \left\langle W^{-\frac{1}{q}}(y) \vec{f}(y), W^{\frac{1}{q}}(x) \vec{g}(x) \right\rangle_{\mathbb{C}^n} \right| dy \\ & \lesssim \sum_{t \in \{0, \frac{1}{3}\}^d} \sum_{Q \in \mathcal{D}^t} \frac{1}{|Q|^{1-\frac{\alpha}{d}}} \int_Q \int_Q \left| \left\langle W^{-\frac{1}{q}}(y) \vec{f}(y), W^{\frac{1}{q}}(x) \vec{g}(x) \right\rangle_{\mathbb{C}^n} \right| dx dy. \end{aligned}$$

□

We are now ready to prove Theorem 1.4.

*Proof of Theorem 1.4.* We will show that

$$W^{\frac{1}{q}} I_\alpha W^{-\frac{1}{q}} : L^p \rightarrow L^q$$

which is equivalent to the boundedness

$$I_\alpha : L^p(W^{\frac{2}{q}}) \rightarrow L^q(W).$$

By Lemma 3.8 is enough to estimate

$$\begin{aligned} & \left| \left\langle W^{\frac{1}{q}} I_\alpha W^{-\frac{1}{q}} \vec{f}, \vec{g} \right\rangle \right| \\ & \lesssim \sum_{t \in \{0, \frac{1}{3}\}^d} \sum_{Q \in \mathcal{D}^t} \frac{1}{|Q|^{1-\frac{\alpha}{d}}} \int_Q \int_Q \left| \left\langle W^{-\frac{1}{q}}(y) \vec{f}(y), W^{\frac{1}{q}}(x) \vec{g}(x) \right\rangle_{\mathbb{C}^n} \right| dx dy. \end{aligned}$$

By a standard approximation argument we will assume that  $\vec{f}$  and  $\vec{g}$  are bounded functions with compact support. Let  $\mathcal{D}$  be a fixed dyadic grid and pick  $a > 2^{n+1}$  to determined later in the argument and let  $\mathcal{Q}^k$  denote the collection

$$\mathcal{Q}^k = \{Q \in \mathcal{D} : a^k < \frac{1}{|Q|} \int_Q |\tilde{V}_Q^{-1} W^{-\frac{1}{q}}(y) \vec{f}(y)| dy \leq a^{k+1}\}$$

and let  $\mathcal{S}^k$  the collection of  $Q \in \mathcal{D}$  that are maximal with respect to the inequality

$$\frac{1}{|Q|} \int_Q |\tilde{V}_Q^{-1} W^{-\frac{1}{q}}(y) \vec{f}(y)| dy > a^k.$$

Finally, set  $\mathcal{S} = \bigcup_k \mathcal{S}^k$ . Since

$$\begin{aligned} & \sum_{Q \in \mathcal{D}} \frac{1}{|Q|^{1-\frac{\alpha}{d}}} \int_Q \int_Q \left| \left\langle W^{-\frac{1}{q}}(y) \vec{f}(y), W^{\frac{1}{q}}(x) \vec{g}(x) \right\rangle_{\mathbb{C}^n} \right| dx dy \\ & \leq \sum_{Q \in \mathcal{D}} |Q|^{\frac{\alpha}{d}} \left( \int_Q |\tilde{V}_Q^{-1} W^{-\frac{1}{q}}(y) \vec{f}(y)| dy \right) \left( \frac{1}{|Q|} \int_Q |\tilde{V}_Q W^{\frac{1}{q}}(x) \vec{g}(x)| dx \right) \end{aligned}$$

we can estimate

$$\begin{aligned} & \sum_{Q \in \mathcal{D}} \frac{1}{|Q|^{1-\frac{\alpha}{d}}} \int_Q \int_Q \left| \left\langle W^{-\frac{1}{q}}(y) \vec{f}(y), W^{\frac{1}{q}}(x) \vec{g}(x) \right\rangle_{\mathbb{C}^n} \right| dx dy \\ & \leq \sum_k \sum_{Q \in \mathcal{Q}^k} |Q|^{\frac{\alpha}{d}} \left( \int_Q |\tilde{V}_Q^{-1} W^{-\frac{1}{q}}(y) \vec{f}(y)| dy \right) \left( \frac{1}{|Q|} \int_Q |\tilde{V}_Q W^{\frac{1}{q}}(x) \vec{g}(x)| dx \right) \\ & \leq \sum_k a^{k+1} \sum_{Q \in \mathcal{Q}^k} |Q|^{\frac{\alpha}{d}} \int_Q |\tilde{V}_Q W^{\frac{1}{q}}(x) \vec{g}(x)| dx \\ & = \sum_k a^{k+1} \sum_{P \in \mathcal{S}^k} \sum_{\substack{Q \in \mathcal{Q}^k \\ Q \subset P}} |Q|^{\frac{\alpha}{d}} \int_Q |\tilde{V}_Q W^{\frac{1}{q}}(x) \vec{g}(x)| dx. \end{aligned}$$



We now examine the inner most sum:

$$\begin{aligned}
& \sum_{\substack{Q \in \mathcal{Q}^k \\ Q \subset P}} |Q|^{\frac{\alpha}{d}} \int_Q |\tilde{V}_Q W^{\frac{1}{q}}(x) \vec{g}(x)| dx \\
& \leq \sum_{\substack{Q \in \mathcal{Q} \\ Q \subset P}} |Q|^{\frac{\alpha}{d}} \int_Q |\tilde{V}_Q W^{\frac{1}{q}}(x) \vec{g}(x)| dx \\
& = \sum_{j=0}^{\infty} \sum_{\substack{Q \subset P \\ \ell(Q)=2^{-j}\ell(P)}} |Q|^{\frac{\alpha}{d}} \int_Q |\tilde{V}_Q W^{\frac{1}{q}}(x) \vec{g}(x)| dx \\
& = |P|^{\frac{\alpha}{d}} \sum_{j=0}^{\infty} 2^{-j\alpha} \sum_{\substack{Q \subset P \\ \ell(Q)=2^{-j}\ell(P)}} \int_Q |\tilde{V}_Q W^{\frac{1}{q}}(x) \vec{g}(x)| dx \\
& \lesssim |P|^{\frac{\alpha}{d}} \int_P N_P(x) |\vec{g}(x)| dx
\end{aligned}$$

where as before

$$N_P(x) = \sup_{P \supseteq Q \ni x} \|W^{\frac{1}{q}}(x) \tilde{V}_Q\|.$$

Plugging this back into the original sum gives us

$$\begin{aligned}
& \sum_{Q \in \mathcal{Q}} \frac{1}{|Q|^{1-\frac{\alpha}{d}}} \int_Q \int_Q \left| \left\langle W^{-\frac{1}{q}}(y) \vec{f}(y), W^{\frac{1}{q}}(x) \vec{g}(x) \right\rangle_{\mathbb{C}^n} \right| dx dy \\
& \lesssim \sum_k a^{k+1} \sum_{P \in \mathcal{S}^k} |P|^{\frac{\alpha}{d}} \int_P N_P(x) |\vec{g}(x)| dx \\
& \leq a \sum_k \sum_{P \in \mathcal{S}^k} |P| \left( \frac{1}{|P|^{1-\frac{\alpha}{d}}} \int_P |\tilde{V}_P^{-1} W^{-\frac{1}{q}}(y) \vec{f}(y)| dy \right) \\
& \quad \times \left( \frac{1}{|P|} \int_P N_P(x) |\vec{g}(x)| dx \right). \quad (3.3)
\end{aligned}$$

However, for any  $u \in P$ ,

$$\begin{aligned}
& \frac{1}{|P|} \int_P N_P(x) |\vec{g}(x)| dx \\
& \lesssim \left( \frac{1}{|P|} \int_P (N_P(x))^{\frac{q'-\epsilon}{q'-\epsilon-1}} dx \right)^{\frac{q'-\epsilon-1}{q'-\epsilon}} \left( \frac{1}{|P|} \int_P |\vec{g}(x)|^{q'-\epsilon} dy \right)^{\frac{1}{q'-\epsilon}} \\
& \lesssim \left( M(|\vec{g}|^{q'-\epsilon})(u) \right)^{\frac{1}{q'-\epsilon}}
\end{aligned}$$

for  $\epsilon > 0$  small by Lemma 3.6. We now show that if  $Q \in \mathcal{S}$  and if  $E_Q$  is defined by

$$E_Q = Q \setminus \bigcup_{\substack{Q' \in \mathcal{S} \\ Q' \subsetneq Q}} Q',$$

then  $|E_Q| \geq \frac{1}{2}|Q|$ . Pick  $k$  such that  $Q \in \mathcal{S}_k$ . By maximality we have that

$$E_Q = Q \setminus \left( \bigcup_{\substack{Q' \in \mathcal{S}^{k+1} \\ Q' \subsetneq Q}} Q' \right).$$

Note that if  $Q \in \mathcal{S}^k$  and  $\tilde{Q}$  is the parent of  $Q$  then for any  $\vec{e} \in \mathbb{C}^n$  we have

$$|\tilde{V}_Q^{-1}\vec{e}| \leq |\tilde{V}'_Q\vec{e}| \lesssim |\tilde{V}'_{\tilde{Q}}\vec{e}| \leq [W]_{\tilde{A}_{p,q}}^{\frac{1}{q}} |\tilde{V}_{\tilde{Q}}^{-1}\vec{e}|.$$

Thus, there exists  $C > 0$  depending on  $W$  where (by maximality)

$$\frac{1}{|Q|} \int_Q |\tilde{V}_Q^{-1}W^{-\frac{1}{q}}(y) \vec{f}(y)| dy \leq C \frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |\tilde{V}_{\tilde{Q}}^{-1}W^{-\frac{1}{q}}(y) \vec{f}(y)| dy \leq Ca^k. \quad (3.4)$$

We now break up the disjoint collection  $\{Q' \in \mathcal{S}^{k+1} : Q' \subsetneq Q\}$  into two disjoint collections via the “stopping time” from Lemma 3.6. In particular let  $A_{Q,k}$  be those cubes  $Q'$  in  $\{Q' \in \mathcal{S}^{k+1} \text{ s.t. } Q' \subsetneq Q\}$  that are maximal with respect to the property  $\|\tilde{V}_{Q'}\tilde{V}_Q^{-1}\| > a'$  so that by the proof of Lemma 3.6 we have that  $|\cup A_{Q,k}| \leq \frac{1}{4}|Q|$  for  $a' > 0$  large enough (independent of  $W$ ). Therefore, (3.4) implies that

$$\begin{aligned} \left| \bigcup_{\substack{Q' \in \mathcal{S}^{k+1} \\ Q' \subsetneq Q}} Q' \right| &\leq |\cup A_{Q,k}| + \frac{1}{a^{k+1}} \sum_{\substack{Q' \in \mathcal{S}^{k+1} \\ Q' \subsetneq Q, Q' \not\subset \cup A_{Q,k}}} \int_{Q'} |\tilde{V}_{Q'}W^{\frac{1}{q}}(x) \vec{f}(x)| dx \\ &\leq \frac{1}{4}|Q| + \frac{a'}{a^{k+1}} \sum_{\substack{Q' \in \mathcal{S}^{k+1} \\ Q' \subsetneq Q, Q' \not\subset \cup A_{Q,k}}} \int_{Q'} |\tilde{V}_{Q'}W^{\frac{1}{q}}(x) \vec{f}(x)| dx \\ &\leq \frac{1}{4}|Q| + \frac{a'}{a^{k+1}} \int_Q |\tilde{V}_QW^{\frac{1}{q}}(x) \vec{f}(x)| dx \\ &\leq \frac{1}{4}|Q| + \frac{Ca^ka'}{a^{k+1}}|Q| \\ &\leq \frac{1}{2}|Q|. \end{aligned}$$

if  $a = 4a'C \approx [W]_{A_{p,q}}^{\frac{1}{q}}$ .

Finally we return to our estimate for the sum (3.3). Notice that the sets  $\{E_P\}_{P \in \mathcal{S}}$  are disjoint and hence

$$\begin{aligned}
(3.3) &\lesssim a \sum_{P \in \mathcal{S}} \int_{E_P} M'_{W,\alpha}(\vec{f})(u) (M(|\vec{g}|^{q'-\epsilon})(u))^{\frac{1}{q'-\epsilon}} du \\
&\lesssim a \int_{\mathbb{R}^d} M'_{W,\alpha}(\vec{f})(u) (M(|\vec{g}|^{q'-\epsilon})(u))^{\frac{1}{q'-\epsilon}} du \\
&\leq a \left( \int_{\mathbb{R}^d} (M'_{W,\alpha}(\vec{f})(u))^q du \right)^{\frac{1}{q}} \left( \int_{\mathbb{R}^d} (M(|\vec{g}|^{q'-\epsilon})(u))^{\frac{q'}{q'-\epsilon}} du \right)^{\frac{1}{q'}} \\
&\lesssim a \|\vec{f}\|_{L^p} \|\vec{g}\|_{L^{q'}}
\end{aligned}$$

by Lemma 3.5. □

**Remark.** Combining everything and tracing back the  $A_{p,q}$  dependence, we obtain that

$$\|W^{\frac{1}{q}} I_\alpha W^{-\frac{1}{q}}\|_{L^p \rightarrow L^q} \lesssim [W]_{A_{p,q}}^{(1-\frac{\alpha}{d})\frac{p'}{q}+1}$$

which we do not believe to be sharp.

#### 4. MATRIX WEIGHTED POINCARÉ AND SOBOLEV INEQUALITIES

We now prove our matrix weighted Poincaré and Sobolev inequalities. Recall, that in the scalar case the following representation formulas hold:

$$|f(x) - f_Q| \lesssim I_1(|\nabla f| \chi_Q)(x), \quad x \in Q, f \in C^1(\mathbb{R}^d)$$

and

$$|f(x)| \lesssim I_1(|\nabla f|)(x), \quad f \in C_0^1(\mathbb{R}^d).$$

**Lemma 4.1.** *For  $\vec{f}, \vec{g} \in C_0^1(\mathbb{R}^d)$ , we have that*

$$\left| \langle W^{\frac{1}{q}} \vec{f}, \vec{g} \rangle_{L^2} \right| \lesssim \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\left| \langle (W^{\frac{1}{q}}(x) D\vec{f}(y))(x-y), \vec{g}(x) \rangle_{\mathbb{C}^n} \right|}{|x-y|^n} dx dy$$

where  $D\vec{f}(x)$  is the standard Jacobian matrix of  $\vec{f}$  at  $x$ .

*Proof.* Let  $\vec{f} = (f_1, \dots, f_n)$  so by standard arguments

$$f_i(x) = -\frac{1}{d\omega_d} \int_{\mathbb{R}^d} \frac{\langle \nabla f_i(y), (x-y) \rangle_{\mathbb{R}^d}}{|x-y|^d} dy$$

where  $\omega_d$  is the volume of the unit ball in  $\mathbb{R}^d$ . Thus, by elementary matrix manipulations and the definition of  $D\vec{f}$  we have that

$$W^{\frac{1}{q}}(x)\vec{f}(x) = -\frac{1}{d\omega_d} \int_{\mathbb{R}^d} \frac{(W^{\frac{1}{q}}(x)D\vec{f}(y))(x-y)}{|x-y|^d} dy$$

which implies the lemma.  $\square$

With the help of Lemma 4.1, the proof of the following is very similar to the proof Theorem 1.4, and therefore we will only sketch the details.

**Theorem 4.2.** *If  $W$  is a matrix  $A_{p,q}$  weight where*

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{d}$$

then

$$\|W^{\frac{1}{q}}\vec{f}\|_{L^q} \lesssim \|W^{\frac{1}{q}}D\vec{f}\|_{L^p}$$

for Schwartz functions  $\vec{f}$  and  $\vec{g}$ .

*Proof.* The arguments in Lemma 3.8 and Lemma 4.1 give us that

$$\begin{aligned} & \left| \langle W^{\frac{1}{q}}\vec{f}, \vec{g} \rangle_{L^2} \right| \\ & \lesssim \int_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}} \sum_t \sum_{\substack{Q \in \mathcal{Q}^t \\ 2^{k-1} \leq \ell(Q) < 2^k}} \chi_Q(x) \int_Q \frac{\left| \langle (W^{\frac{1}{q}}(x)D\vec{f}(y))(x-y), \vec{g}(x) \rangle_{\mathbb{C}^n} \right|}{|x-y|^d} dy dx \\ & \lesssim \sum_t \sum_{Q \in \mathcal{Q}^t} \frac{1}{|Q|} \int_Q \int_Q \left| \langle (W^{\frac{1}{q}}(x)D\vec{f}(y))(x-y), \vec{g}(x) \rangle_{\mathbb{C}^n} \right| dy dx \\ & \lesssim \sum_t \sum_{Q \in \mathcal{Q}^t} \frac{1}{|Q|^{1-\frac{1}{d}}} \int_Q \int_Q \|\tilde{V}_Q^{-1}W^{-\frac{1}{q}}(y)(W^{\frac{1}{q}}(y)D\vec{f}(y))\| \|\tilde{V}_Q W^{\frac{1}{q}}(x)\vec{g}(x)\| dy dx. \end{aligned}$$

Repeating the stopping time arguments from the proof of Theorem 1.4 to estimate the last term, we get that

$$\left| \langle W^{\frac{1}{q}}\vec{f}, \vec{g} \rangle_{L^2} \right| \lesssim \|W^{\frac{1}{q}}D\vec{f}\|_{L^p} \|\vec{g}\|_{L^{q'}}.$$

$\square$

As for local Poincaré/Sobolev inequalities with gains, let  $V_Q, V'_Q$  be the  $A_p$  reducing operators:

$$|V_Q \vec{e}| \approx \left( \frac{1}{|Q|} \int_Q |W^{-\frac{1}{p}}(x)\vec{e}|^{p'} dx \right)^{\frac{1}{p'}}, \quad |V'_Q \vec{e}| \approx \left( \frac{1}{|Q|} \int_Q |W^{\frac{1}{p}}(x)\vec{e}|^p dx \right)^{\frac{1}{p}}.$$

Furthermore, for this section, we let

$$M'_{W,1}\vec{f}(x) = \sup_{\substack{Q \ni x \\ Q \in \mathcal{Q}}} \frac{1}{|Q|^{1-\frac{1}{d}}} \int_Q |(V_Q)^{-1}W^{-\frac{1}{p}}(y)\vec{f}(y)| dy.$$

**Lemma 4.3.** *For any  $P \subseteq Q$  and  $\vec{e} \in \mathbb{C}^n$  we have that*

$$|(V_Q)^{-1}\vec{e}| \lesssim \left(\frac{|Q|}{|P|}\right)^{\frac{1}{p'}} |(V_P)^{-1}\vec{e}|$$

*Proof.* We have

$$\begin{aligned} |V_Q\vec{e}| &\approx \frac{1}{|Q|^{\frac{1}{p'}}} \left( \int_Q |W^{-\frac{1}{p}}(y)\vec{e}|^{p'} dy \right)^{\frac{1}{p}} \\ &\geq \left(\frac{1}{|Q|}\right)^{\frac{1}{p'}} \left( \int_P |W^{-\frac{1}{p}}(y)\vec{e}|^{p'} dy \right)^{\frac{1}{p}} \\ &\approx \left(\frac{|P|}{|Q|}\right)^{\frac{1}{p'}} |V_P\vec{e}| \end{aligned}$$

which implies that

$$\|(V_Q)^{-1}V_P\| = \|V_P(V_Q)^{-1}\| \lesssim \left(\frac{|Q|}{|P|}\right)^{\frac{1}{p'}}.$$

□

Our next result is a of matrix version of Lemma 1.1 in [9], for the fractional matrix weighted maximal function  $M'_{W,\alpha}$ .

**Lemma 4.4.** *Let  $W$  be a matrix  $A_p$  weight, let  $p \leq d$ , and let  $1 \leq k < \frac{d}{d-p}$  (where  $k \geq 1$  can be chosen arbitrarily if  $p = d$ ). Then there exists  $q^* < p$  where*

$$\left( \frac{1}{|Q|} \int_Q (M'_{W,1}\vec{f}(x))^{kq^*} dx \right)^{\frac{1}{kq^*}} \lesssim |Q|^{\frac{1}{d}} \left( \frac{1}{|Q|} \int_Q |\vec{f}(x)|^{q^*} dx \right)^{\frac{1}{q^*}}$$

for all  $\vec{f}$  supported on  $Q$  and all  $q^* > q$ .

*Proof.* The proof is similar to the proof of Lemma 1.1 in [9]. If  $P \subseteq Q$  then the previous lemma gives us that there exists  $C > 0$  independent of  $\vec{f}$  (and in fact independent of  $W$ ) where

$$\frac{1}{|Q|^{1-\frac{1}{d}}} \int_Q |(V_Q)^{-1}W^{-\frac{1}{p}}(y)\vec{f}(y)| dy$$

$$\begin{aligned}
&\leq C \left( \frac{|Q|}{|P|} \right)^{\frac{1}{p'}} \frac{1}{|Q|^{1-\frac{1}{d}}} \int_P |(V_P)^{-1} W^{-\frac{1}{p}}(y) \vec{f}(y)| dy \\
&= C \left( \frac{|Q|}{|P|} \right)^{\frac{1}{p'} - 1 + \frac{1}{d}} \frac{1}{|I|^{1-\frac{1}{d}}} \int_P |(V_P)^{-1} W^{-\frac{1}{p}}(y) \vec{f}(y)| dy \\
&\leq C \frac{1}{|P|^{1-\frac{1}{d}}} \int_P |(V_P)^{-1} W^{-\frac{1}{p}}(y) \vec{f}(y)| dy
\end{aligned}$$

since  $\vec{f}$  is supported on  $P$  and  $\frac{1}{d} - 1 + \frac{1}{p'} \leq 0$  by assumption.

Now for  $P \in \mathcal{D}$ , let  $\lambda > 0$ , and let

$$E_\lambda = \{x \in P : M'_{W,1} \vec{f}(x) > \lambda\}.$$

Let  $\{P_j\}$  be the maximal dyadic subintervals of  $P$  such that

$$\frac{1}{|P_j|^{1-\frac{1}{d}}} \int_{P_j} |(V_{P_j})^{-1} W^{-\frac{1}{p}}(y) \vec{f}(y)| dy > \frac{\lambda}{C}$$

where  $C$  is above. Then by maximality we have  $E_\lambda \subset \bigsqcup_j P_j$ .

Pick  $q < p$  such that

$$\sup_{J \in \mathcal{D}} \frac{1}{|J|} \int_J \|(V_J)^{-1} W^{-\frac{1}{p}}(y)\|^{q'} dy < \infty$$

and  $1 \leq k \leq \frac{d}{d-q}$ . We then have by Hölder's inequality that

$$\begin{aligned}
|E_\lambda| &\leq \sum_j |P_j| \\
&\lesssim \frac{1}{\lambda^{kq}} \sum_j \frac{1}{|P_j|^{kq - \frac{kq}{d} - 1}} \left( \int_{P_j} |(V_{P_j})^{-1} W^{-\frac{1}{p}}(y) \vec{f}(y)| dy \right)^{kq} \\
&\lesssim \frac{1}{\lambda^{kq}} \sum_j |P_j|^{1 + \frac{kq}{d} - k} \left( \int_{P_j} |\vec{f}(y)|^q dy \right)^k
\end{aligned}$$

However,  $1 + \frac{kq}{d} - k \geq 0$  iff  $k \leq \frac{d}{d-q}$ . Thus, since  $\bigsqcup_j P_j \subset P$  and  $k \geq 1$  we have

$$|E_\lambda|^{\frac{1}{kq}} \leq \frac{C'}{\lambda} |P|^{\frac{1}{kq} + \frac{1}{d} - \frac{1}{q}} \|\vec{f}\|_{L^q(P)}$$

which means that  $M'_{W,1}$  is bounded from  $L^q(P, \frac{dx}{|P|})$  into  $L^{qk, \infty}(P, \frac{dx}{|P|})$  with

$$\|M'_{W,1} \vec{f}\|_{L^{qk, \infty}(P, \frac{dx}{|P|})} \lesssim |P|^{\frac{1}{d}} \|\vec{f}\|_{L^q(P, \frac{dx}{|P|})}.$$

A similar argument shows that  $M'_{W,1}$  is bounded from  $L^{\tilde{p}}(P, \frac{dx}{|P|})$  into weak- $L^{\tilde{p}k}(P, \frac{dx}{|P|})$  with

$$\|M'_{W,1}\vec{f}\|_{L^{\tilde{p}k,\infty}(P, \frac{dx}{|P|})} \lesssim |P|^{\frac{1}{d}} \|\vec{f}\|_{L^{\tilde{p}}(P, \frac{dx}{|P|})}.$$

for all  $\tilde{p} > p$ . A straight forward application of the Marcinkiewicz interpolation theorem now completes the proof.  $\square$

Before we state and prove our local matrix weighted Poincare and Sobolev inequalities, we need to two remarks.

**Remark.** First, note that the proof of Theorem 4.4 is exactly the same if  $\vec{f}$  is replaced by an  $\mathcal{M}_n(\mathbb{C})$  valued function  $F$ , where here

$$M'_{W,1}F(x) = \sup_{\substack{Q \ni x \\ Q \in \mathcal{D}}} \frac{1}{|Q|^{1-\frac{1}{d}}} \int_Q \|(V_Q)^{-1}W^{-\frac{1}{p}}(y)F(y)\| dy.$$

**Remark.** Second, note that  $W$  is a matrix  $A_p$  weight if and only if  $\widetilde{W} := W^{-\frac{p'}{p}}$  is a matrix  $A_{p'}$  weight. Furthermore, it is easy to see that we can take  $V_Q(\widetilde{W}) = V'_Q(W)$  and  $V'_Q(W) = V_Q(\widetilde{W})$ . Thus, if

$$M''_{W,1}\vec{f}(x) = \sup_{\substack{Q \ni x \\ Q \in \mathcal{D}}} \frac{1}{|Q|^{1-\frac{1}{d}}} \int_Q |V_Q W^{\frac{1}{p}}(y)\vec{f}(y)| dy$$

then since the matrix  $A_p$  condition gives us that

$$\begin{aligned} & \sup_{\substack{Q \ni x \\ Q \in \mathcal{D}}} \frac{1}{|Q|^{1-\frac{1}{d}}} \int_Q |V_Q W^{\frac{1}{p}}(y)\vec{f}(y)| dy \\ & \lesssim \sup_{\substack{Q \ni x \\ Q \in \mathcal{D}}} \frac{1}{|Q|^{1-\frac{1}{d}}} \int_Q |(V'_Q)^{-1}W^{\frac{1}{p}}(y)\vec{f}(y)| dy, \end{aligned}$$

an application of Lemma 4.4 immediately gives us the following.

**Lemma 4.5.** *Let  $W$  be a matrix  $A_p$  weight, let  $p' \leq d$ , and let  $1 \leq k < \frac{d}{d-p'}$  (where  $k \geq 1$  can be chosen arbitrarily if  $p = d$ ). Then there exists  $q < p'$  where*

$$\left( \frac{1}{|Q|} \int_Q (M''_{W,1}\vec{f}(x))^{kq^*} dx \right)^{\frac{1}{kq^*}} \lesssim |Q|^{\frac{1}{d}} \left( \frac{1}{|Q|} \int_Q |\vec{f}(x)|^{q^*} dx \right)^{\frac{1}{q^*}}$$

for all  $\vec{f}$  supported on  $Q$  and all  $q^* > q$ .

Moreover, if

$$N'_Q(x) = \sup_{Q \supseteq R \ni x} \|W^{-\frac{1}{p}}(x)V_R^{-1}\|$$

then since

$$\sup_{Q \supseteq R \ni x} \|W^{-\frac{1}{p}}(x)V_R^{-1}\| \lesssim \sup_{Q \supseteq R \ni x} \|W^{-\frac{1}{p}}(x)V'_R\|,$$

Lemma 3.6 immediately says that there exists  $\epsilon > 0$  where

$$\sup_{Q \in \mathcal{D}} \frac{1}{|Q|} \int_Q (N'_Q(x))^{p'+\epsilon} dx < \infty. \quad (4.1)$$

We are now ready to prove our matrix weighted Poincaré and Sobolev inequalities with gains. Recall the statement of Theorem 1.1 that if  $W \in \mathbb{A}_p$  then there exists  $\epsilon > 0$  such that

$$\left( \frac{1}{|Q|} \int_Q |W^{\frac{1}{p}}(x)\vec{f}(x)|^{p+\epsilon} dx \right)^{\frac{1}{p+\epsilon}} \lesssim |Q|^{\frac{1}{d}} \left( \frac{1}{|Q|} \int_Q |W^{\frac{1}{p}}(x)D\vec{f}(x)|^{p-\epsilon} dx \right)^{\frac{1}{p-\epsilon}}$$

*Proof of Theorem 1.1.* Pick  $\tilde{q} > p$  (to be determined momentarily). Let  $\vec{g} \in L^{\tilde{q}}$ , and assume  $\vec{f}$  and  $\vec{g}$  are supported on  $Q$ . By the arguments in the previous theorem we have that

$$\begin{aligned} & \left| \langle W^{\frac{1}{p}}\vec{f}, \vec{g} \rangle_{L^2} \right| \\ & \lesssim \sum_{t \in \{0,1/3\}^d} \sum_{I \in \mathcal{D}^t} \frac{1}{|I|} \int_I \int_I \left| \langle (W^{\frac{1}{p}}(x)D\vec{f}(y))(x-y), \vec{g}(x) \rangle_{\mathbb{C}^n} \right| dy dx \\ & \lesssim \sum_{t \in \{0,1/3\}^d} \sum_{I \in \mathcal{D}^t} \frac{1}{|I|^{1-\frac{1}{d}}} \int_I \int_I \|V_I^{-1}W^{-\frac{1}{p}}(y)(W^{\frac{1}{p}}(y)D\vec{f}(y))\| \|V_I W^{\frac{1}{p}}(x)\vec{g}(x)\| dx dy. \end{aligned}$$

Now fix a dyadic grid  $\mathcal{D}$ . Assume that  $Q \in \mathcal{D}$ , so in this case we can assume that  $I \in \mathcal{D}(Q)$  since

$$\begin{aligned} & \sum_{\substack{I \in \mathcal{D} \\ I \supseteq Q}} \frac{1}{|I|} \int_I \int_I \left| \langle (W^{\frac{1}{p}}(x)D\vec{f}(y))(x-y), \vec{g}(x) \rangle_{\mathbb{C}^n} \right| dy dx \\ & \lesssim \frac{1}{|Q|} \int_Q \int_Q \left| \langle (W^{\frac{1}{p}}(x)D\vec{f}(y))(x-y), \vec{g}(x) \rangle_{\mathbb{C}^n} \right| dy dx \\ & \lesssim \frac{1}{|Q|^{1-\frac{1}{d}}} \int_Q \|V_Q^{-1}W^{-\frac{1}{p}}(y)(W^{\frac{1}{p}}(y)D\vec{f}(y))\| \|V_Q W^{\frac{1}{p}}(x)\vec{g}(x)\| dx dy \end{aligned}$$

which will be easily estimated later in the proof.



We now have to modify the stopping time argument in the proof of Theorem 1.4. Let  $F(y) = W^{\frac{1}{p}}(y)D\vec{f}(y)$ . As before let  $a > 0$  be large, let  $\mathcal{Q}^k$  denote the collection

$$\mathcal{Q}^k = \left\{ P \in \mathcal{D} : a^k < \frac{1}{|P|} \int_P \|V_P^{-1}W^{-\frac{1}{p}}(y)F(y)\| dy \leq a^{k+1} \right\},$$

and let  $\mathcal{S}^k$  be the collection of  $P \in \mathcal{D}$  that are maximal with respect to the inequality

$$\frac{1}{|P|} \int_P \|V_P^{-1}W^{-\frac{1}{p}}(y)F(y)\| dy > a^k.$$

Set  $\mathcal{S} = \bigcup_k \mathcal{S}^k$  and note that  $\mathcal{S}^k \cap \mathcal{S}^{k'} = \emptyset$  if  $k \neq k'$  by maximality if  $a$  is large enough (see (3.4) in the proof of Theorem 1.4).

Thus, we need to estimate

$$\begin{aligned} & \sum_{I \in \mathcal{D}(Q)} \frac{1}{|I|^{1-\frac{1}{d}}} \int_I \int_I \|V_I^{-1}W^{-\frac{1}{p}}(y)F(y)\| \|V_I W^{\frac{1}{p}}(x)\vec{g}(x)\| dx dy \\ &= \sum_k \sum_{I \in \mathcal{Q}^k \cap \mathcal{D}(Q)} \frac{1}{|I|^{1-\frac{1}{d}}} \int_I \int_I \|V_I^{-1}W^{-\frac{1}{p}}(y)F(y)\| \|V_I W^{\frac{1}{p}}(x)\vec{g}(x)\| dx dy \\ &\leq \sum_k a^{k+1} \sum_{I \in \mathcal{Q}^k \cap \mathcal{D}(Q)} |I|^{\frac{1}{d}} \int_I |V_I W^{\frac{1}{p}}(x)\vec{g}(x)| dx \\ &\leq \sum_k a^{k+1} \sum_{P \in \mathcal{S}^k} \sum_{I \in \mathcal{D}(P) \cap \mathcal{D}(Q)} |I|^{\frac{1}{d}} \int_I |V_I W^{\frac{1}{p}}(x)\vec{g}(x)| dx. \end{aligned} \quad (4.2)$$

We now break up (4.2) into two sums corresponding to  $P \subseteq Q$  and  $P \supset Q$ . In the later case, note that

$$\begin{aligned} & \sum_{I \in \mathcal{D}(Q) \cap \mathcal{D}(P)} |I|^{\frac{1}{d}} \int_I |V_I W^{\frac{1}{p}}(x)\vec{g}(x)| dx \\ &= \sum_{I \in \mathcal{D}(Q)} |I|^{\frac{1}{d}} \int_I |V_I W^{\frac{1}{p}}(x)\vec{g}(x)| dx \\ &= \sum_{j=0}^{\infty} \sum_{\substack{I \subset Q \\ \ell(I)=2^{-j}\ell(Q)}} |I|^{\frac{1}{d}} \int_I |V_I W^{\frac{1}{p}}(x)\vec{g}(x)| dx \end{aligned}$$

$$\begin{aligned}
&= |Q|^{\frac{1}{d}} \sum_{j=0}^{\infty} 2^{-j} \sum_{\substack{I \subset Q \\ \ell(I)=2^{-j}\ell(Q)}} \int_I |V_I W^{\frac{1}{p}}(x) \vec{g}(x)| dx \\
&\lesssim |Q|^{\frac{1}{d}} \int_Q N_Q(x) |\vec{g}(x)| dx
\end{aligned}$$

where as before

$$N_Q(x) = \sup_{Q \supseteq I \ni x} \|W^{\frac{1}{p}}(x) V_I\|.$$

Thus,

$$\begin{aligned}
&\sum_k a^{k+1} \sum_{\substack{P \in \mathcal{S}^k \\ P \supset Q}} \sum_{I \in \mathcal{D}(P) \cap \mathcal{D}(Q)} |I|^{\frac{1}{d}} \int_I |V_I W^{\frac{1}{p}}(x) \vec{g}(x)| dx \\
&\lesssim \sum_k a^{k+1} \sum_{\substack{P \in \mathcal{S}^k \\ P \supset Q}} |Q|^{\frac{1}{d}} \int_Q N_Q(x) |\vec{g}(x)| dx \\
&\leq \sum_k \sum_{\substack{P \in \mathcal{S}^k \\ P \supset Q}} |Q|^{\frac{1}{d}} \left( \frac{1}{|P|} \int_P \|V_P^{-1} W^{-\frac{1}{p}}(y) F(y)\| dy \right) \left( \int_Q N_Q(x) |\vec{g}(x)| dx \right) \\
&\lesssim \sum_k \sum_{\substack{P \in \mathcal{S}^k \\ P \supset Q}} |Q|^{\frac{1}{d}} \left( \frac{|P|}{|Q|} \right)^{\frac{1}{p'}} \left( \frac{1}{|P|} \int_Q \|V_Q^{-1} W^{-\frac{1}{p}}(y) F(y)\| dy \right) \\
&\quad \times \left( \int_Q N_Q(x) |\vec{g}(x)| dx \right) \\
&\lesssim \sum_k \sum_{\substack{P \in \mathcal{S}^k \\ P \supset Q}} |Q|^{\frac{1}{d}} \left( \frac{|P|}{|Q|} \right)^{\frac{1}{p'}-1} \left( \int_Q \|V_Q^{-1} W^{-\frac{1}{p}}(y) F(y)\| dy \right) \\
&\quad \times \left( \frac{1}{|Q|} \int_Q N_Q(x) |\vec{g}(x)| dx \right) \\
&\lesssim |Q| \left( \frac{1}{|Q|^{1-\frac{1}{d}}} \int_Q \|V_Q^{-1} W^{-\frac{1}{p}}(y) F(y)\| dy \right) \left( \frac{1}{|Q|} \int_Q N_Q(x) |\vec{g}(x)| dx \right)
\end{aligned} \tag{4.3}$$

by Lemma 4.3 and using the fact that  $\mathcal{S}^k \cap \mathcal{S}^{k'} = \emptyset$  if  $k \neq k'$ .

Note that for  $\tilde{q} > p$  close enough to  $p$  and  $\epsilon > 0$  small enough, we have

$$\sup_{P \in \mathcal{D}} \frac{1}{|P|} \int_P (N_P(x))^{\tilde{q}+\epsilon} dx < \infty$$

(which is possible by Lemma 3.6). Then we have by Hölder's inequality

$$\begin{aligned} \frac{1}{|Q|} \int_Q N_Q(x) |\vec{g}(x)| dx &\leq \left( \frac{1}{|Q|} \int_Q (N_Q(x))^{\tilde{q}+\epsilon} dx \right)^{\frac{1}{\tilde{q}+\epsilon}} \left( \frac{1}{|Q|} \int_Q |\vec{g}(x)|^{\frac{\tilde{q}+\epsilon}{\tilde{q}+\epsilon-1}} dx \right)^{\frac{\tilde{q}+\epsilon-1}{\tilde{q}+\epsilon}} \\ &\lesssim \inf_{u \in Q} \left( M(|\vec{g}|^{\frac{\tilde{q}+\epsilon}{\tilde{q}+\epsilon-1}})(u) \right)^{\frac{\tilde{q}+\epsilon-1}{\tilde{q}+\epsilon}}. \end{aligned}$$

Thus,

$$(4.3) \lesssim |Q|^{\frac{1}{d}} \int_Q (M'_{W,1}F)(u) \left( M(|\vec{g}|^{\frac{\tilde{q}+\epsilon}{\tilde{q}+\epsilon-1}})(u) \right)^{\frac{\tilde{q}+\epsilon-1}{\tilde{q}+\epsilon}} du.$$

On the other hand, if  $P \subseteq Q$  in (4.2) then we estimate

$$\begin{aligned} &\sum_k a^{k+1} \sum_{\substack{P \in \mathcal{F}^k \\ P \subseteq Q}} \sum_{I \in \mathcal{D}(P)} |I|^{\frac{1}{d}} \int_I |V_I W^{\frac{1}{p}}(x) \vec{g}(x)| dx \\ &\lesssim \sum_k a^{k+1} \sum_{\substack{P \in \mathcal{F}^k \\ P \subseteq Q}} |P|^{\frac{1}{d}} \int_P N_P(x) |\vec{g}(x)| dx \\ &\lesssim \sum_k \sum_{\substack{P \in \mathcal{F}^k \\ P \subseteq Q}} |P| \left( \frac{1}{|P|^{1-\frac{1}{d}}} \int_P \|V_P^{-1} W^{-\frac{1}{p}}(y) F(y)\| dy \right) \left( \frac{1}{|P|} \int_P N_P(x) |\vec{g}(x)| dx \right) \\ &\lesssim \sum_k \sum_{\substack{P \in \mathcal{F}^k \\ P \subseteq Q}} |E_P| \left( \frac{1}{|P|^{1-\frac{1}{d}}} \int_P \|V_P^{-1} W^{-\frac{1}{p}}(y) F(y)\| dy \right) \left( \frac{1}{|P|} \int_P N_P(x) |\vec{g}(x)| dx \right) \\ &\lesssim \int_Q (M'_{W,1}F)(u) \left( M(|\vec{g}|^{\frac{\tilde{q}+\epsilon}{\tilde{q}+\epsilon-1}})(u) \right)^{\frac{\tilde{q}+\epsilon-1}{\tilde{q}+\epsilon}} du \end{aligned}$$

by the sparseness of the family  $\{E_P\}$ . Thus, we have (plugging back in for  $F$ )

$$\left| \langle W^{\frac{1}{p}} \vec{f}, \vec{g} \rangle_{L^2} \right| \lesssim \int_Q \left( M'_{W,1}(W^{\frac{1}{p}} D \vec{f})(u) \right) \left( M(|\vec{g}|^{\frac{\tilde{q}+\epsilon}{\tilde{q}+\epsilon-1}})(u) \right)^{\frac{\tilde{q}+\epsilon-1}{\tilde{q}+\epsilon}} du.$$

But then another application of Hölder's inequality gives us that

$$\begin{aligned} \left| \langle W^{\frac{1}{p}} \vec{f}, \vec{g} \rangle_{L^2} \right| &\lesssim \|M'_{W,1}(W^{\frac{1}{p}} D\vec{f})\|_{L^{\tilde{q}}(Q)} \left\| \left( M(|\vec{g}|^{\frac{\tilde{q}+\epsilon}{\tilde{q}+\epsilon-1}}) \right)^{\frac{\tilde{q}+\epsilon-1}{\tilde{q}+\epsilon}} \right\|_{L^{\tilde{q}'}(Q)} \\ &\lesssim \|M'_{W,1}(W^{\frac{1}{p}} D\vec{f})\|_{L^{\tilde{q}}(Q)} \|\vec{g}\|_{L^{\tilde{q}'}(Q)}. \end{aligned}$$

Finally, by Lemma 4.4, we can pick  $1 < k < \frac{d}{d-p}$  and  $q^* < p$  such that  $\tilde{q} := kq^* > p$  is enough close to  $p$  and

$$\left( \frac{1}{|Q|} \int_Q (M'_{W,1}(W^{\frac{1}{p}} D\vec{f})(x))^{\tilde{q}} dx \right)^{\frac{1}{\tilde{q}}} \lesssim |Q|^{\frac{1}{d}} \left( \frac{1}{|Q|} \int_Q \|(W^{\frac{1}{p}} D\vec{f})(x)\|^{q^*} dx \right)^{\frac{1}{q^*}}$$

so we have

$$\begin{aligned} \left| \langle W^{\frac{1}{p}} \vec{f}, \vec{g} \rangle_{L^2} \right| &\lesssim \|M'_{W,1}(W^{\frac{1}{p}} D\vec{f})\|_{L^{\tilde{q}}(Q)} \|\vec{g}\|_{L^{\tilde{q}'}(Q)} \\ &= |Q|^{\frac{1}{\tilde{q}}} \|M'_{W,1}(W^{\frac{1}{p}} D\vec{f})\|_{L^{\tilde{q}}(Q, \frac{dx}{|Q|})} \|\vec{g}\|_{L^{\tilde{q}'}(Q)} \\ &\lesssim |Q|^{\frac{1}{d} + \frac{1}{\tilde{q}} - \frac{1}{q^*}} \|W^{\frac{1}{p}} D\vec{f}\|_{L^{q^*}(Q)} \|\vec{g}\|_{L^{\tilde{q}'}(Q)}. \end{aligned}$$

which completes the proof when  $p \leq d$  if  $Q \in \mathcal{D}$ .

If  $Q \notin \mathcal{D}$ , then we can obviously pick disjoint cubes  $Q_j \in \mathcal{D}$  for  $j = 1, \dots, 2^d$  with  $\ell(Q) \leq \ell(Q_j) < 2\ell(Q)$  and  $Q \subseteq \sqcup_j Q_j$ . Writing  $\vec{f}_j = \chi_{Q_j} \vec{f}$  and defining  $\vec{g}_j$  similarly, we then obviously have

$$\begin{aligned} &\sum_{I \in \mathcal{D}} \frac{1}{|I|} \int_I \int_I \left| \left\langle (W^{\frac{1}{p}}(x) D\vec{f}(y))(x-y), \vec{g}(x) \right\rangle_{\mathbb{C}^n} \right| dy dx \\ &\leq \sum_{i,j=1}^{2^d} \sum_{I \in \mathcal{D}} \frac{1}{|I|} \int_I \int_I \left| \left\langle (W^{\frac{1}{p}}(x) D\vec{f}_i(y))(x-y), \vec{g}_j(x) \right\rangle_{\mathbb{C}^n} \right| dy dx. \end{aligned}$$

If  $i \neq j$  then obviously

$$\begin{aligned} &\sum_{I \in \mathcal{D}} \frac{1}{|I|} \int_I \int_I \left| \left\langle (W^{\frac{1}{p}}(x) D\vec{f}_i(y))(x-y), \vec{g}_j(x) \right\rangle_{\mathbb{C}^n} \right| dy dx \\ &= \sum_{\substack{I \in \mathcal{D} \\ I \supseteq Q_i \cup Q_j}} \frac{1}{|I|} \int_{Q_i} \int_{Q_j} \left| \left\langle (W^{\frac{1}{p}}(x) D\vec{f}_i(y))(x-y), \vec{g}_j(x) \right\rangle_{\mathbb{C}^n} \right| dy dx \\ &\lesssim \frac{1}{|Q|} \int_Q \int_Q \left| \left\langle (W^{\frac{1}{p}}(x) D\vec{f}(y))(x-y), \vec{g}(x) \right\rangle_{\mathbb{C}^n} \right| dy dx \end{aligned}$$

which can easily be estimated as above. Finally if  $i = j$  then this reduces to the proof above when  $\vec{f}$  and  $\vec{g}$  are supported on the same dyadic cube in  $\mathcal{D}$ , which completes the proof when  $p \leq d$ .

Now if  $p > d$  then clearly  $p' < \frac{d}{d-1} \leq d$  since  $d \geq 2$ . Letting again  $F(y) = W^{\frac{1}{p}}(y)D\vec{f}(y)$ , then as before, we have

$$\left| \langle W^{\frac{1}{p}}\vec{f}, \vec{g} \rangle_{L^2} \right| \lesssim \sum_{t \in \{0,1/3\}^d} \sum_{I \in \mathcal{D}^t} \frac{1}{|I|} \int_I \int_I \left| \langle (W^{\frac{1}{p}}(x)D\vec{f}(y))(x-y), \vec{g}(x) \rangle_{\mathbb{C}^n} \right| dy dx.$$

Arguing as before, we fix  $t \in \{0,1/3\}^d$  and set  $\mathcal{D} = \mathcal{D}^t$ , assume that  $Q \in \mathcal{D}$ , and assume that  $I \in \mathcal{D}(Q)$ . Thus we need to estimate

$$\sum_{I \in \mathcal{D}(Q)} \frac{1}{|I|^{1-\frac{1}{d}}} \int_I \int_I \|V_I^{-1}W^{-\frac{1}{p}}(y)F(y)\| |V_I W^{\frac{1}{p}}(x)\vec{g}(x)| dy dx.$$

Now for  $a > 0$  large enough, let  $\mathcal{S}^k$  be the collection of dyadic cubes that are maximal with respect to the inequality

$$\frac{1}{|P|} \int_P |V_P W^{\frac{1}{p}}(x)\vec{g}(x)| dx > a^k$$

and set  $\mathcal{S} = \bigcup_k \mathcal{S}^k$ .

Pick some  $q^* < p < \tilde{q}$  close to  $p$  and assume that  $\vec{g} \in L^{\tilde{q}'}(Q)$ . Arguing as before (or just replacing the roles of  $F$  and  $\vec{g}$ , and using the fact that  $\widetilde{W} = W^{-\frac{p'}{p}}$  is a matrix  $A_{p'}$  weight with  $V_I(\widetilde{W}) = V_Q'(W)$  and  $V_Q'(W) = V_Q(\widetilde{W})$ ), we have by (4.1)

$$\left| \langle W^{\frac{1}{p}}\vec{f}, \vec{g} \rangle_{L^2} \right| \lesssim \int_Q M''_{\widetilde{W},1}\vec{g}(u) \left( M(\|F\|_{\frac{(q^*)'+\epsilon}{(q^*)'+\epsilon-1}})(u) \right)^{\frac{(q^*)'+\epsilon-1}{(q^*)'+\epsilon}} du$$

for  $q^*$  close enough to  $p$ .

Another application of Hölder's inequality again gives us that

$$\begin{aligned} \left| \langle W^{\frac{1}{p}}\vec{f}, \vec{g} \rangle_{L^2} \right| &\lesssim \|M''_{\widetilde{W},1}\vec{g}\|_{L^{(q^*)}'(Q)} \left\| \left( M(\|F\|_{\frac{(q^*)'+\epsilon}{(q^*)'+\epsilon-1}}) \right)^{\frac{(q^*)'+\epsilon-1}{(q^*)'+\epsilon}} \right\|_{L^{q^*}(Q)} \\ &\lesssim \|M''_{\widetilde{W},1}\vec{g}\|_{L^{(q^*)}'(Q)} \|F\|_{L^{q^*}(Q)} \\ &= |Q|^{\frac{1}{(q^*)'}} \|M''_{\widetilde{W},1}\vec{g}\|_{L^{(q^*)}'(Q, \frac{dx}{|Q|})} \|F\|_{L^{q^*}(Q)} \\ &\lesssim |Q|^{\frac{1}{d} + \frac{1}{(q^*)'} - \frac{1}{\tilde{q}'}} \|\vec{g}\|_{L^{\tilde{q}'}(Q)} \|W^{\frac{1}{p}}D\vec{f}\|_{L^{q^*}(Q)} \\ &= |Q|^{\frac{1}{d} - \frac{1}{q^*} + \frac{1}{\tilde{q}}} \|\vec{g}\|_{L^{\tilde{q}'}(Q)} \|W^{\frac{1}{p}}D\vec{f}\|_{L^{q^*}(Q)} \end{aligned}$$

by Lemma 4.5 (for  $q^*$  and  $\tilde{q}$  close enough to  $p$ )

□

Finally we end this section with the proof of Theorem 1.2.

*Proof of Theorem 1.2.* Without loss of generality, assume  $\vec{f}$  and  $\vec{g}$  are supported on  $Q$ . Then again by standard arguments, we have for  $x \in Q$  that

$$f_i(x) - (f_i)_Q = -\frac{1}{|Q|} \int_Q \int_0^{|x-y|} \frac{\left\langle \nabla f_i \left( x + \frac{r(y-x)}{|y-x|} \right), (x-y) \right\rangle_{\mathbb{R}^d}}{|x-y|} dr dy \quad (4.4)$$

so that

$$\begin{aligned} & |\langle W^{\frac{1}{p}}(x)(\vec{f}(x) - \vec{f}_Q), \vec{g}(x) \rangle_{\mathbb{C}^n}| \\ & \leq \frac{1}{|Q|} \int_Q \int_0^{|x-y|} \frac{|\langle W^{\frac{1}{p}}(x)(D\vec{f}(x + \frac{r(y-x)}{|x-y|})) (x-y), \vec{g}(x) \rangle_{\mathbb{C}^n}|}{|y-x|^d} dr dy \end{aligned}$$

By again standard arguments (see [14] p. 226 for example), we therefore have

$$|W^{\frac{1}{p}}(x)(\vec{f}(x) - \vec{f}_Q)|_{\chi_Q(x)} \lesssim \int_Q \frac{|\langle W^{\frac{1}{p}}(x)(D\vec{f}(y))(x-y), \vec{g}(y) \rangle_{\mathbb{C}^n}|}{|x-y|^d} dy.$$

The proof is now the same as the proof of the previous theorem.  $\square$

Note that if  $\vec{f} \in C^1(B)$  for an open ball  $B$  then (4.4) holds with  $Q$  replaced by  $B$ , so in this case if  $Q$  is a cube containing  $B$  with comparable side length, then

$$|W^{\frac{1}{p}}(x)(\vec{f}(x) - \vec{f}_B)|_{\chi_B(x)} \lesssim \int_Q \frac{|\langle W^{\frac{1}{p}}(x)(\chi_B(y)D\vec{f}(y))(x-y), \vec{g}(y) \rangle_{\mathbb{C}^n}|}{|x-y|^d} dy$$

so arguing as we did before immediately proves Theorem 1.2 for open balls.

## 5. EXISTENCE OF DEGENERATE ELLIPTIC SYSTEMS

For our existence results we will consider general nonlinear elliptic equations whose degeneracy is governed by a matrix  $A_p$  weight. Consider a mapping  $\mathcal{A} : \mathbb{R}^n \times \mathcal{M}_n(\mathbb{C}) \rightarrow \mathcal{M}_n(\mathbb{C})$  such that  $x \mapsto \mathcal{A}(x, \eta)$  is measurable for all  $\eta \in \mathcal{M}_n(\mathbb{C})$  and  $\eta \mapsto \mathcal{A}(x, \eta)$  is continuous for almost all  $x \in \mathbb{R}^n$ . Note that this makes the mapping  $x \mapsto \mathcal{A}(x, \eta(x))$  a measurable mapping whenever  $\eta(x)$  is a measurable matrix valued function. We will assume that  $1 < p < \infty$  and  $\mathcal{A}$  satisfies

- (i)  $\langle \mathcal{A}(x, \eta), \eta \rangle_{\text{tr}} \geq c \|W^{1/p}(x)\eta\|^p, \quad \eta \in \mathcal{M}_n(\mathbb{C})$
  - (ii)  $|\langle \mathcal{A}(x, \eta), \nu \rangle_{\text{tr}}| \leq C \|W^{1/p}(x)\eta\|^{p-1} \|W^{1/p}\nu\|, \quad \eta, \nu \in \mathcal{M}_n(\mathbb{C})$
  - (iii)  $\langle \mathcal{A}(x, \eta) - \mathcal{A}(x, \nu), \eta - \nu \rangle_{\text{tr}} \geq 0, \quad \eta, \nu \in \mathcal{M}_n(\mathbb{C}).$
- where  $\langle \cdot, \cdot \rangle_{\text{tr}}$  is the Frobenius inner product defined by

$$\langle A, B \rangle_{\text{tr}} = \text{tr}(B^* A) = \sum_{j=1}^n \sum_{k=1}^n A_{jk} \overline{B_{kj}}.$$

A typical example of a non-linear operator  $\mathcal{A}$  (and one that will be discussed more in the last section) is given by the degenerate system of  $p$ -Laplace operators

$$\mathcal{A}(x, \eta) = \langle \eta G(x), \eta \rangle_{\text{tr}}^{\frac{p-2}{2}} \eta G(x).$$

Such degenerate systems arise from minimizing the energy functional

$$\mathcal{E}(\vec{g}) = \int_{\Omega} \langle D\vec{g}(x)G(x), D\vec{g}(x) \rangle_{\text{tr}}^{\frac{p}{2}} dx.$$

Such systems also arise naturally in the theory mappings of finite distortion [13, Chapter 15].

In the rest of the paper we will be concerned with the following system of equations in a domain  $\Omega$ :

$$\text{Div } \mathcal{A}(x, D\vec{u}(x)) = -(\text{Div } F)(x), \quad (5.1)$$

where  $\vec{u} : \mathbb{R}^n \rightarrow \mathbb{C}^n$  and  $\text{Div } F(x) = (\text{div } F^1(x), \dots, \text{div } F^n(x))$  and  $F^i$  are the row vectors of  $F(x)$ . We will focus on weak solutions to (5.1):

$$\int_{\Omega} \langle \mathcal{A}(x, D\vec{u}(x)), D\vec{\varphi}(x) \rangle_{\text{tr}} dx = - \int_{\Omega} \langle F(x), D\vec{\varphi}(x) \rangle_{\text{tr}} dx \quad (5.2)$$

for any  $\vec{\varphi} \in C_c^\infty(\Omega)$ . As mentioned in the introduction, a natural domain for these types of systems of equations is given by the matrix weighted Sobolev space  $H^{1,p}(\Omega, W)$  defined as the completion of  $\{\vec{u} \in C^\infty(\Omega) : \vec{u}, D\vec{u} \in L^p(\Omega, W)\}$  with respect to the norm:

$$\|\vec{u}\|_{H^{1,p}(\Omega, W)} = \left( \int_{\Omega} |W^{\frac{1}{p}}(x)\vec{u}(x)|^p dx + \int_{\Omega} \|W^{\frac{1}{p}}(x)D\vec{u}(x)\|^p dx \right)^{\frac{1}{p}}.$$

Moreover, the space  $H_0^{1,p}(\Omega, W)$  is the completion of  $C_0^\infty(\Omega)$  in the norm  $\|\cdot\|_{H^{1,p}(\Omega, W)}$ . While we do not need it, it should be noted that the exact same arguments that are in [6], Sections 1-5 prove that

$$H^{1,p}(\Omega, W) = \{\vec{u} \in \mathcal{W}_{loc}^{1,1} : \vec{u}, D\vec{u} \in L^p(\Omega, W)\}.$$

As is customary we will only prove the existence of weak solutions when  $F = 0$ .

**Theorem 5.1.** *Suppose that  $W \in A_p$  and  $\mathcal{A}$  satisfies (i), (ii), and (iii) above. If  $\vec{h} \in H^{1,p}(\Omega, W)$ , then the system*

$$\operatorname{Div} \mathcal{A}(x, D\vec{u}) = 0$$

*has a weak solution such that  $\vec{u} - \vec{h} \in H_0^{1,p}(\Omega, W)$ .*

We will follow Kinderlehrer and Stampacchia [15]. Let  $X$  be a reflexive Banach space with dual space  $X^*$ . If  $K$  is a convex subset of  $X$  then a mapping  $\mathfrak{A} : K \rightarrow X'$  is said to be *monotone* if

$$\langle \mathfrak{A}u - \mathfrak{A}v, u - v \rangle \geq 0, \quad u, v \in K$$

and is *coercive* on  $K$  if there exists  $\varphi \in K$  such that

$$\frac{\langle \mathfrak{A}u_j - \varphi, u_j - \varphi \rangle}{\|u_j - \varphi\|} \rightarrow \infty \quad (5.3)$$

whenever  $\{u_j\}$  is a sequence in  $K$  with  $\|u_j\| \rightarrow \infty$ . The following proposition is in Kinderlehrer and Stampacchia [15, p. 87].

**Proposition 5.2.** *Let  $K \neq \emptyset$  be a closed convex subset of a reflexive Banach space  $X$  and  $\mathfrak{A} : K \rightarrow X^*$  be monotone, coercive, and weakly continuous on  $K$ . Then there exists  $u \in K$  such that*

$$\langle \mathfrak{A}u, g - u \rangle \geq 0, \quad \forall g \in K.$$

*Proof of Theorem 5.1.* Let  $X = L^p(\Omega, \mathbb{R}^{n \times n}, W)$  be the space of  $n \times n$  matrix valued functions  $F : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  such that

$$\|F\|_{L^p(W)}^p = \int_{\Omega} \|W^{1/p}(x)F(x)\|^p dx < \infty,$$

with dual space  $X^* = L^{p'}(\Omega, \mathbb{R}^{n \times n}, W^{-p'/p})$  under the usual pairing

$$\langle F, G \rangle = \int_{\Omega} \langle F, G \rangle_{\operatorname{tr}} dx.$$

For  $\vec{\psi} \in H^{1,p}(\Omega, W)$  define

$$U_{\vec{\psi}} = \{\vec{u} \in H^{1,p}(\Omega, W) : \vec{u} - \vec{\psi} \in H_0^{1,p}(\Omega, W)\}$$

and

$$K = \{D\vec{u} : \vec{u} \in U_{\vec{\psi}}\}.$$



Then  $K$  is a nonempty convex subset of  $X$ . To see that  $K$  is closed suppose that  $D\vec{v}_k \rightarrow V$  in  $X$ . Since  $W \in A_p$ , by the Sobolev inequality we have that

$$\int_{\Omega} |W^{\frac{1}{p}}(x)(\vec{v}_k - \vec{\psi})|^p dx \lesssim \int_{\Omega} \|W^{\frac{1}{p}}(x)(D\vec{v}_k - D\vec{\psi})\|^p dx \leq C.$$

Since  $U_{\psi}$  is a closed convex subset of  $H^{1,p}(\Omega, W)$  there exists a subsequence  $\{\vec{v}_{k_j}\}$  and function  $\vec{v} \in U_{\psi}$  such that  $\vec{v}_{k_j} \rightarrow \vec{v}$  in  $H^{1,p}(\Omega, W)$ . In particular  $V = D\vec{v} \in K$  and hence  $K$  is closed.

For  $F, G \in X$  define

$$\langle \mathfrak{A}F, G \rangle = \int_{\Omega} \langle \mathcal{A}(x, F(x)), G(x) \rangle_{\text{tr}} dx.$$

Notice by assumption (ii) on  $\mathcal{A}$  we have that

$$|\langle \mathfrak{A}F, G \rangle| \leq \|F\|_{L^p(W)}^{p-1} \|G\|_{L^p(W)}$$

so that  $\mathfrak{A} : X \rightarrow X^*$ . From assumption (iii) on  $\mathcal{A}$  we have that  $\mathfrak{A}$  is monotone. Thus we need to check that  $\mathfrak{A}$  is coercive, i.e. satisfies condition (5.3). Suppose  $U_k = D\vec{u}_k \in K$  satisfies  $\|U_k\|_{L^p(W)} \rightarrow \infty$ . Then, given  $V = D\vec{v} \in K$  we have  $\|U_k - V\|_{L^p(W)} \rightarrow \infty$  as well. Fix  $V = D\vec{v} \in K$  and use assumption (i) on  $\mathcal{A}$  to get

$$\begin{aligned} \langle \mathfrak{A}U_k - \mathfrak{A}V, U_k - V \rangle &= \int_{\Omega} \langle \mathcal{A}(x, U_k) - \mathcal{A}(x, V), U_k - V \rangle_{\text{tr}} dx \\ &= \int_{\Omega} \langle \mathcal{A}(x, U_k), U_k \rangle_{\text{tr}} dx + \int_{\Omega} \langle \mathcal{A}(x, V), V \rangle_{\text{tr}} dx \\ &\quad - \int_{\Omega} \langle \mathcal{A}(x, U_k), V \rangle_{\text{tr}} dx - \int_{\Omega} \langle \mathcal{A}(x, V), U_k \rangle_{\text{tr}} dx \\ &\geq c(\|U_k\|_{L^p(W)}^p + \|V\|_{L^p(W)}^p) \\ &\quad - C(\|U_k\|_{L^p(W)}^{p-1} \|V\|_{L^p(W)} + \|U_k\|_{L^p(W)} \|V\|_{L^p(W)}^{p-1}) \\ &\geq c2^{-p} \|U_k - V\|_{L^p(W)}^p \\ &\quad - C2^{1-p} [\|V\|_{L^p(W)} (\|V\|_{L^p(W)}^{p-1} + \|U_k - V\|_{L^p(W)}^{p-1}) \\ &\quad - C(\|V\|_{L^p(W)}^{p-1} (\|V\|_{L^p(W)} + \|U_k - V\|_{L^p(W)})] \end{aligned}$$

which obviously means that

$$\frac{\langle \mathfrak{A}U_k - \mathfrak{A}V, U_k - V \rangle}{\|U_k - V\|_{L^p(W)}} \rightarrow \infty$$

as  $\|U_k\|_{L^p(W)} \rightarrow \infty$  so that  $\mathfrak{A}$  is coercive. Finally, the weak continuity follows from the continuity of  $\eta \mapsto \mathcal{A}(x, \eta)$ . By Proposition 5.2 there exists  $U = D\vec{u} \in K$  such that

$$\langle \mathfrak{A}U, G - U \rangle \geq 0, \quad \forall G \in K.$$

If  $\vec{\varphi} \in C_c(\Omega, \mathbb{R}^n)$ , then  $\vec{u} - \vec{\varphi}$  and  $\vec{u} + \vec{\varphi}$  belong to  $U_{\vec{\psi}}$  and hence

$$\int_{\Omega} \langle \mathcal{A}(x, D\vec{u}), D\vec{\varphi} \rangle_{\text{tr}} = 0.$$

□

We now consider the case when  $\mathcal{A}$  is linear, that is,

$$\mathcal{A}(x, \eta) = \sum_{i, \beta=1}^d A_{ij}^{\alpha\beta}(x) \eta_{j\beta}.$$

In this case we will consider the nonhomogeneous system

$$\text{Div } \mathcal{A}(x, D\vec{u}(x)) = -\text{Div}(F(x)),$$

which can be written as the system

$$\sum_{\alpha, \beta, j=1}^d \partial_{\alpha}(A_{ij}^{\alpha\beta} \partial_{\beta} u_j) = -(\text{Div } F)_i \quad i = 1, \dots, d. \quad (5.4)$$

Moreover, when  $p = 2$  conditions (i) and (ii) on  $\mathcal{A}$  become

$$\sum_{i, j, \beta, \alpha}^n A_{ij}^{\alpha\beta}(x) \eta_{\beta}^j \overline{\eta_{\alpha}^i} \gtrsim \|W^{\frac{1}{2}}(x) \eta\|^2, \quad (5.5)$$

and

$$\left| \sum_{i, j, \beta, \alpha}^n A_{ij}^{\alpha\beta}(x) \nu_{\beta}^j \overline{\eta_{\alpha}^i} \right| \lesssim \|W^{\frac{1}{2}}(x) \eta\| \|W^{\frac{1}{2}}(x) \nu\| \quad (5.6)$$

respectively. Moreover, condition (iii) on  $\mathcal{A}$  is automatically satisfied by the linearity of  $\mathcal{A}$ . In this case we have an existence and uniqueness result, which follows from a standard use of the Lax-Milgram theorem.

**Theorem 5.3.** *Let  $\mathcal{A}$  satisfy (5.5) and (5.6),  $\vec{h} \in H^{1,2}(\Omega, W)$ , and  $F \in L^2(\Omega, W^{-1})$ . Then the system (5.4) has a unique weak solution  $\vec{u} \in H^{1,2}(\Omega, W)$  such that  $\vec{u} - \vec{h} \in H_0^{1,2}(\Omega, W)$*

## 6. BASIC REGULARITY RESULTS

We now discuss some deeper results (that still are fairly elementary from the elliptic PDE point of view.) The first is a degenerate Caccioppoli inequality.

**Lemma 6.1.** *Assume that  $\Omega$  is some open set and  $B_r$  is any ball of radius  $r$  whose closure is contained in  $\Omega$ . If  $A = A_{ij}^{\alpha\beta}$  satisfies (5.5) and (5.6), and if  $\vec{u} \in H^{1,2}(\Omega, W)$  is a solution to (5.4) for  $F \in L^2(\Omega, W^{-1})$ , then*

$$\begin{aligned} \int_{B_{r/2}} \|W^{\frac{1}{2}}(x)D\vec{u}(x)\|^2 dx &\lesssim \frac{1}{r^2} \int_{B_r \setminus B_{r/2}} |W^{\frac{1}{2}}(x)\vec{u}(x)|^2 dx \\ &+ \int_{B_r} \|W^{-\frac{1}{2}}(x)F(x)\|^2 dx \end{aligned} \quad (6.1)$$

**Remark.** We do not need to assume any conditions on our matrix weight  $W$  other than positive definiteness a.e. In particular, the constants in our Caccioppoli inequality do not depend on the  $A_2$  characteristic.

*Proof.* The proof is classical, and the only nontrivial thing to check is that our system and degeneracy is “decoupled” enough. Pick some  $\eta \in C_c^\infty(B_r)$  such that  $\eta \equiv 1$  in  $B_{r/2}$ ,  $0 \leq \eta \leq 1$  in  $B_r$ , and  $|\nabla\eta| \leq \frac{4}{r}\chi_{B_r \setminus B_{r/2}}$  and let  $\vec{\varphi} := \eta^2\vec{u} \in H_0^{1,2}(\Omega, W)$ . By definition, we have that

$$\sum_{\alpha,\beta,i,j} \int_{B_r} A_{ij}^{\alpha\beta} (\partial_\beta u_j) \overline{(\partial_\alpha (u_i \eta^2))} dx = \int_{B_r} \langle F, D(\eta^2\vec{u}) \rangle_{\text{tr}} dx. \quad (6.2)$$

However,

$$\partial_\alpha (u_i \eta^2) = u_i (2\eta \partial_\alpha \eta) + \eta^2 \partial_\alpha u_i$$

so that

$$D(\eta^2\vec{u}) = 2(\eta\vec{u}) \otimes \nabla\eta + \eta^2 D\vec{u} \quad (6.3)$$

so combining this with (5.5), (5.6), and (6.2) gives

$$\begin{aligned} &\int_{B_r} |\eta|^2 \|W^{\frac{1}{2}}D\vec{u}\|^2 dx \\ &\lesssim \sum_{\alpha,\beta,i,j} \int_{B_r} A_{ij}^{\alpha\beta} (\partial_\beta u_j) \overline{(\eta^2 \partial_\alpha u_i)} dx \\ &\leq \left| \sum_{\alpha,\beta,i,j} \int_{B_r} A_{ij}^{\alpha\beta} \partial_\beta u_j \overline{(2\eta \partial_\alpha \eta)} \right| dx + \int_{B_r} \|W^{-\frac{1}{2}}F\| \|W^{\frac{1}{2}}D(\eta^2\vec{u})\| dx \end{aligned}$$

$$\begin{aligned}
&= 2 \left| \sum_{\alpha, \beta, i, j} \int_{B_r} A_{ij}^{\alpha\beta} (\partial_\beta u_j) (\overline{(\eta \vec{u} \otimes \nabla \eta)_{i\alpha}}) \right| dx + \int_{B_r} \|W^{-\frac{1}{2}} F\| \|W^{\frac{1}{2}} D(\eta^2 \vec{u})\| dx \\
&\lesssim \int_{B_r} |\eta| \|W^{\frac{1}{2}} D\vec{u}\| \|W^{\frac{1}{2}} (\vec{u} \otimes \nabla \eta)\| dx + \int_{B_r} \|W^{-\frac{1}{2}} F\| \|W^{\frac{1}{2}} D(\eta^2 \vec{u})\| dx \\
&\leq \int_{B_r} |\eta| \|\nabla \eta\| \|W^{\frac{1}{2}} D\vec{u}\| \|W^{\frac{1}{2}} \vec{u}\| dx + \int_{B_r} \|W^{-\frac{1}{2}} F\| \|W^{\frac{1}{2}} D(\eta^2 \vec{u})\| dx
\end{aligned}$$

Thus, by the ‘‘Cauchy-Schwarz inequality with  $\epsilon$ ’’ we have

$$\begin{aligned}
\int_{B_r} |\eta|^2 \|W^{\frac{1}{2}} D\vec{u}\|^2 dx &\lesssim \epsilon \int_{B_r} |\eta|^2 \|W^{\frac{1}{2}} D\vec{u}\|^2 dx + \frac{C(\epsilon)}{r^2} \int_{B_r \setminus B_{r/2}} |W^{\frac{1}{2}} \vec{u}|^2 dx \\
&\quad + \epsilon \int_{B_r} \|W^{\frac{1}{2}} D(\eta^2 \vec{u})\|^2 dx + C(\epsilon) \int_{B_r} \|W^{-\frac{1}{2}} F\|^2 dx
\end{aligned}$$

for some  $C(\epsilon) > 0$ .

However, (6.3) gives us that

$$\epsilon \int_{B_r} \|W^{\frac{1}{2}} D(\eta^2 \vec{u})\|^2 dx \lesssim \frac{\epsilon}{r^2} \int_{B_r \setminus B_{r/2}} |W^{\frac{1}{2}} \vec{u}|^2 dx + \epsilon \int_{B_r} |\eta|^2 \|W^{\frac{1}{2}} D\vec{u}\|^2 dx$$

so finally

$$\begin{aligned}
&\int_{B_r} |\eta|^2 \|W^{\frac{1}{2}} D\vec{u}\|^2 dx \\
&\lesssim \epsilon \int_{B_r} |\eta|^2 \|W^{\frac{1}{2}} D\vec{u}\|^2 dx + \frac{C(\epsilon)}{r^2} \int_{B_r \setminus B_{r/2}} |W^{\frac{1}{2}} \vec{u}|^2 dx + C(\epsilon) \int_{B_r} \|W^{-\frac{1}{2}} F\|^2 dx
\end{aligned}$$

Setting  $\epsilon > 0$  small enough and remembering that  $\eta \equiv 1$  on  $B_{r/2}$  finishes the proof.  $\square$

We now prove Theorem 1.5 as a Corollary of our Caccioppoli inequality.

*Proof of Theorem 1.5.* Let  $\epsilon > 0$  be chosen where Theorem 1.2 is true, so by (6.1)

$$\begin{aligned}
&\left( \frac{1}{|B_{r/2}|} \int_{B_{r/2}} \|W^{\frac{1}{2}} D\vec{u}\|^2 dx \right)^{\frac{1}{2}} \\
&\lesssim \left( \frac{1}{|B_r|} \int_{B_r} \|W^{-\frac{1}{2}} F\|^2 dx \right)^{\frac{1}{2}} + \frac{1}{r} \left( \frac{1}{|B_r|} \int_{B_r} |W^{\frac{1}{2}} (\vec{u} - \vec{u}_{B_r})|^2 dx \right)^{\frac{1}{2}}
\end{aligned}$$

$$\lesssim \left( \frac{1}{|B_r|} \int_{B_r} \|W^{-\frac{1}{2}}F\|^2 dx \right)^{\frac{1}{2}} + \left( \frac{1}{|B_r|} \int_{B_r} \|W^{\frac{1}{2}}D\vec{u}\|^{2-\epsilon} dx \right)^{\frac{1}{2-\epsilon}}$$

However, setting

$$U(x) = \|W^{\frac{1}{2}}(x)D\vec{u}(x)\|^{2-\epsilon}, \quad G(x) = \|W^{-\frac{1}{2}}(x)F(x)\|^{2-\epsilon}, \quad \text{and } s = \frac{2}{2-\epsilon}$$

we have that

$$\frac{1}{|B_{r/2}|} \int_{B_{r/2}} (U(x))^s dx \lesssim \left( \frac{1}{|B_r|} \int_{B_r} U(x) dx \right)^s + \frac{1}{|B_r|} \int_{B_r} (G(x))^s dx.$$

A classical result of Giaquinta and Modica (see Lemma 2.2 in [8]) now says that there exists  $t > s = \frac{2}{2-\epsilon}$  where

$$\begin{aligned} & \left( \frac{1}{|B_{r/2}|} \int_{B_{r/2}} \|W^{\frac{1}{2}}D\vec{u}\|^{t(2-\epsilon)} dx \right)^{\frac{1}{t}} \\ & \lesssim \left( \frac{1}{|B_r|} \int_{B_r} \|W^{\frac{1}{2}}D\vec{u}\|^2 dx \right)^{\frac{2-\epsilon}{2}} + \left( \frac{1}{|B_r|} \int_{B_r} \|W^{-\frac{1}{2}}F\|^{t(2-\epsilon)} dx \right)^{\frac{1}{t}}. \end{aligned}$$

Setting  $q = t(2-\epsilon) > 2$  clearly completes the proof.  $\square$ .

We can now prove a decay of solutions type theorem, where here we do assume that  $W$  is a matrix  $A_2$  weight. For simplicity we will assume  $F = 0$  in our linear elliptic system. First however we need the following two lemmas, which will prove a sort of “weak” Poincaré inequality for annuli.

**Lemma 6.2.** *For any  $\vec{a} \in \mathbb{C}^n$ , a matrix  $A_p$  weight  $W$ , and  $B = B_{r/2} \setminus B_{r/4}$  where  $B_{r/2}$  and  $B_{r/4}$  are concentric balls of radius  $r/2$  and  $r/4$ , respectively, we have*

$$\begin{aligned} & \left( \frac{1}{|B|} \int_B |W^{\frac{1}{p}}(x)(\vec{u}(x) - \vec{u}_B)|^p dx \right)^{\frac{1}{p}} \\ & \lesssim [W]_{A_p} \left( \frac{1}{|B|} \int_B |W^{\frac{1}{p}}(x)(\vec{u}(x) - \vec{a})|^p dx \right)^{\frac{1}{p}} \end{aligned}$$

**Remark.** As will be apparent from the proof, one can state and prove a similar result for sets  $B$  that aren't necessarily annuli as above. We will leave this for the interested reader to do this.

*Proof.* By the triangle inequality,

$$\begin{aligned} \left( \frac{1}{|B|} \int_B |W^{\frac{1}{p}}(x)(\vec{u}(x) - \vec{u}_B)|^p dx \right)^{\frac{1}{p}} &\leq \left( \frac{1}{|B|} \int_B |W^{\frac{1}{p}}(x)(\vec{u}(x) - \vec{a})|^p dx \right)^{\frac{1}{p}} \\ &\quad + \left( \frac{1}{|B|} \int_B |W^{\frac{1}{p}}(x)(\vec{a} - \vec{u}_B)|^p dx \right)^{\frac{1}{p}} \end{aligned}$$

However,

$$\begin{aligned} |W^{\frac{1}{p}}(x)(\vec{a} - \vec{u}_B)|^p &= \left| \frac{1}{|B|} \int_B W^{\frac{1}{p}}(x)(\vec{u}(y) - \vec{a}) dy \right|^p \\ &= \left| \frac{1}{|B|} \int_B (W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y))W^{\frac{1}{p}}(y)(\vec{u}(y) - \vec{a}) dy \right|^p \\ &\leq \left( \frac{1}{|B|} \int_B \|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|^{p'} dy \right)^{\frac{p}{p'}} \\ &\quad \times \left( \frac{1}{|B|} \int_B |W^{\frac{1}{p}}(y)(\vec{u}(y) - \vec{a})|^p dy \right) \end{aligned}$$

Plugging this in and using the  $A_p$  definition immediately finishes the proof.  $\square$

**Lemma 6.3.** *Let  $W$  be a matrix  $A_p$  weight and assume that  $\vec{u} \in H^{1,2}(\Omega, W)$  for some open set  $\Omega$ . If  $B_r \subseteq \Omega$ , then*

$$\int_{B_{r/2} \setminus B_{r/4}} |W^{\frac{1}{p}}(x)(\vec{u}(x) - \vec{u}_{B_{r/2} \setminus B_{r/4}})|^p dx \lesssim r^p \int_{B_r \setminus B_{r/8}} |W^{\frac{1}{p}}(x)D\vec{u}(x)|^p dx$$

where  $B_r, B_{r/2}, B_{r/4}$  and  $B_{r/8}$  are concentric balls.

*Proof.* The proof utilizes standard geometric ideas. Let  $\{x_j\}_{j=1}^N \subseteq B_{r/2} \setminus B_{r/4}$  be a maximal set satisfying

$$\min_{i \neq j} |x_i - x_j| \geq r/16.$$

Obviously the balls  $\{B_{r/16}(x_j)\}_{j=1}^N$  cover  $B_{r/2} \setminus B_{r/4}$  and a trivial volume-count gives us that we can find an upper bound for  $N$  depending only on  $d$ . Finally, by introducing repeats if necessary, we can without loss of generality assume that

$$B_{r/16}(x_j) \cap B_{r/16}(x_{j+1}) \neq \emptyset \text{ for all } j = 1, \dots, N-1$$

so that for each  $j = 1, \dots, N-1$  there exists  $v_j$  where

$$B_{r/16}(v_j) \subseteq B_{r/8}(x_j) \cap B_{r/8}(x_{j+1}).$$

For notational ease, let  $\tilde{B}_j = B_{r/8}(x_j) \cap B_{r/8}(x_{j+1})$ . Clearly by the previous lemma it is enough to prove that

$$\int_{B_{r/2} \setminus B_{r/4}} |W^{\frac{1}{p}}(x)(\vec{u}(x) - \vec{u}_{B_{r/8}(x_1)})|^p dx \lesssim r^p \int_{B_r \setminus B_{r/8}} |W^{\frac{1}{p}}(x)D\vec{u}(x)|^p dx.$$

To that end, we have

$$\begin{aligned} & \int_{B_{r/2} \setminus B_{r/4}} |W^{\frac{1}{p}}(x)(\vec{u}(x) - \vec{u}_{B_{r/8}(x_1)})|^p dx \\ & \leq \sum_{j=1}^N \int_{B_{r/16}(x_j)} |W^{\frac{1}{p}}(x)(\vec{u}(x) - \vec{u}_{B_{r/8}(x_1)})|^p dx \\ & \leq \sum_{j=1}^N \int_{B_{r/8}(x_j)} |W^{\frac{1}{p}}(x)(\vec{u}(x) - \vec{u}_{B_{r/8}(x_j)})|^p dx \\ & \quad + \sum_{j=1}^N \int_{B_{r/8}(x_j)} |W^{\frac{1}{p}}(x)(\vec{u}_{B_{r/8}(x_j)} - \vec{u}_{B_{r/8}(x_1)})|^p dx \\ & \lesssim r^p \int_{B_r \setminus B_{r/8}} |W^{\frac{1}{p}}(x)D\vec{u}(x)|^p dx \\ & \quad + \sum_{j=1}^N \int_{B_{r/8}(x_j)} |W^{\frac{1}{p}}(x)(\vec{u}_{B_{r/8}(x_j)} - \vec{u}_{B_{r/8}(x_1)})|^p dx \end{aligned}$$

However,

$$\begin{aligned} & |W^{\frac{1}{p}}(x)(\vec{u}_{B_{r/8}(x_j)} - \vec{u}_{B_{r/8}(x_1)})|^p \\ & \lesssim \sum_{i=1}^{j-1} |W^{\frac{1}{p}}(x)(\vec{u}_{B_{r/8}(x_{i+1})} - \vec{u}_{B_{r/8}(x_i)})|^p \\ & \leq \sum_{i=1}^{j-1} \left( |W^{\frac{1}{p}}(x)(\vec{u}_{B_{r/8}(x_{i+1})} - \vec{u}_{\tilde{B}_i})|^p + |W^{\frac{1}{p}}(x)(\vec{u}_{\tilde{B}_i} - \vec{u}_{B_{r/8}(x_i)})|^p \right) \end{aligned}$$

Moreover,

$$\begin{aligned} & |W^{\frac{1}{p}}(x)(\vec{u}_{B_{r/8}(x_{i+1})} - \vec{u}_{\tilde{B}_i})|^p \\ & \leq \left( \frac{1}{|\tilde{B}_i|} \int_{\tilde{B}_i} |W^{\frac{1}{p}}(x)(\vec{u}(y) - \vec{u}_{B_{r/8}(x_{i+1})})| dy \right)^p \\ & \lesssim \left( \frac{1}{|B_{r/8}(x_{i+1})|} \int_{B_{r/8}(x_{i+1})} \|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|^{p'} dy \right)^{\frac{p}{p'}} \end{aligned}$$

$$\times \frac{1}{|B_{r/8}(x_{i+1})|} \int_{B_{r/8}(x_{i+1})} |W^{\frac{1}{p}}(y)(\vec{u}(y) - \vec{u}_{B_{r/8}(x_{i+1})})|^p dy$$

and a similar estimate holds for  $|W^{\frac{1}{p}}(x)(\vec{u}_{B_{r/8}(x_i)} - \vec{u}_{\tilde{B}_i})|^p$ .

Thus, by the matrix  $A_p$  property

$$\sum_{j=1}^N \int_{B_{r/8}(x_j)} |W^{\frac{1}{p}}(x)(\vec{u}_{B_{r/8}(x_j)} - \vec{u}_{B_{r/4}})|^p dx \lesssim r^p \int_{B_r \setminus B_{r/8}} |W^{\frac{1}{p}}(x)D\vec{u}(x)|^p dx$$

which completes the proof.  $\square$

**Theorem 6.4.** *Assume  $A = A_{ij}^{\alpha\beta}$  satisfies (5.5) and (5.6) for some  $W \in A_2$  and that  $\vec{u} \in H^{1,2}(\Omega, W)$  is a weak solution to (5.4) for  $F = 0$ . Then there exists  $C > 0$  and  $0 < \delta < 1$  independent of  $\vec{u}$  and  $R$  where*

$$\int_{B_r} \|W^{\frac{1}{2}}(x)D\vec{u}(x)\|^2 dx \lesssim \left(\frac{r}{R}\right)^\delta \int_{B_R} \|W^{\frac{1}{2}}(x)D\vec{u}(x)\|^2 dx.$$

for every concentric ball  $B_r \subset B_R$  with the closure of  $B_R$  contained in  $\Omega$ .

*Proof.* The proof involves a ‘‘Widman hole filling technique’’ argument. Note that if  $\vec{u}$  is a weak solution then obviously  $\vec{u} - \vec{u}_{B_{r/2} \setminus B_{r/4}}$  is also a weak solution. Thus, by the Lemma 6.3, we can pick  $C > 0$  independent of  $r$  where

$$\begin{aligned} \int_{B_{r/4}} \|W^{\frac{1}{2}}(x)D\vec{u}(x)\|^2 dx &\leq \frac{C}{r^2} \int_{B_{r/2} \setminus B_{r/4}} |W^{\frac{1}{2}}(x)(\vec{u} - \vec{u}_{B_{r/2} \setminus B_{r/4}})|^2 dx \\ &\leq C \int_{B_r \setminus B_{r/8}} \|W^{\frac{1}{2}}(x)D\vec{u}(x)\|^2 dx \end{aligned}$$

which means

$$(C+1) \int_{B_{r/8}} \|W^{\frac{1}{2}}(x)D\vec{u}(x)\|^2 dx \leq C \int_{B_r} \|W^{\frac{1}{2}}(x)D\vec{u}(x)\|^2 dx$$

or

$$\int_{B_{r/8}} \|W^{\frac{1}{2}}(x)D\vec{u}(x)\|^2 dx \leq \delta \int_{B_r} \|W^{\frac{1}{2}}(x)D\vec{u}(x)\|^2 dx$$

where  $\delta = \frac{C}{C+1}$ ,

Finally, if  $2^{-3k-3}R < r \leq 2^{-3k}R$  and  $\gamma = -\frac{\ln \delta}{3 \ln 2}$  then

$$\int_{B_r} \|W^{\frac{1}{2}}(x)D\vec{u}(x)\|^2 dx \leq 2^\gamma \left(\frac{r}{R}\right)^\gamma \int_{B_R} \|W^{\frac{1}{2}}(x)D\vec{u}(x)\|^2 dx$$

$\square$



We will now prove Theorem 1.6.

*Proof of Theorem 1.6.* The proof is a modification of some ideas in [20]. First, note that for  $\epsilon$  small enough and  $J \subset I_x$  we have by our Poincare inequality for  $d = 2$  and decay of solutions that

$$\begin{aligned} & \frac{1}{|J|^{1+\epsilon}} \int_J |\vec{u}(x) - \vec{u}_J| dx \\ & \leq \left( \frac{1}{|J|^{1-\epsilon}} \int_J \|W^{-\frac{1}{2}}(x)\|^2 dx \right)^{\frac{1}{2}} \left( \frac{1}{|J|^{1+3\epsilon}} \int_J |W^{\frac{1}{2}}(x)(\vec{u}(x) - \vec{u}_J)| dx \right)^{\frac{1}{2}} \\ & \lesssim \left( \frac{1}{|J|^{1-\epsilon}} \int_J \|W^{-\frac{1}{2}}(x)\|^2 dx \right)^{\frac{1}{2}} \end{aligned}$$

To finish the proof, Let  $J_x$  and  $J_y$  be cubes of side length  $|x - y|$  and centered at  $x$  and  $y$  respectively. Since  $\vec{u}$  is locally integrable, let

$$\vec{\mathcal{U}}(x) = \lim_{k \rightarrow \infty} \frac{1}{|J_x^k|} \int_{J_x^k} \vec{u}(s) ds$$

where  $J_x^k = 2^{-k} J_x$ . Then note that by the Lebesgue differentiation theorem we have  $\vec{\mathcal{U}}$  coincides with  $\vec{u}$  a.e. and

$$\begin{aligned} |\vec{\mathcal{U}}(x) - \vec{u}_{J_x}| & \leq \sum_{k=0}^{\infty} |\vec{u}_{J_x^{k+1}} - \vec{u}_{J_x^k}| \\ & \lesssim \sum_{k=0}^{\infty} \frac{1}{|J_x^{k-1}|} \int_{J_x^{k-1}} |\vec{u}(s) - \vec{u}_{J_x^{k-1}}| ds \\ & \lesssim C_x \sum_{k=0}^{\infty} |J_x^{k-1}|^\epsilon \\ & \lesssim C_x |J_x|^\epsilon \end{aligned}$$

Similarly estimating  $|\vec{\mathcal{U}}(y) - \vec{u}_{J_y}|$  and  $|\vec{u}_{J_y} - \vec{u}_{J_x}|$  gives us that

$$|\vec{\mathcal{U}}(x) - \vec{\mathcal{U}}(y)| \lesssim C_{x,y} |x - y|^\epsilon$$

□.

We will now prove a version of Theorem 1.5 for nonlinear  $p$ -Laplacian systems. More precisely, assume  $\Omega \subseteq \mathbb{R}^d$  is a domain,  $W$  is a matrix  $A_p$  weight, and that  $\vec{u} \in H^{1,p}(\Omega, W)$  satisfies

$$\text{Div} \left[ \langle D\vec{u}G, D\vec{u} \rangle_{\text{tr}}^{\frac{p-2}{2}} D\vec{u}G \right] = -\text{Div} F \quad (6.4)$$

where  $G$  is some  $n \times n$  matrix function such that

(i)  $\langle \eta G(x), \eta \rangle_{\text{tr}} \gtrsim \|W^{1/p}(x)\eta\|^p, \quad \eta \in \mathcal{M}_n(\mathbb{C})$   
(ii')  $|\langle \eta G(x), \nu \rangle_{\text{tr}}| \lesssim C \|W^{1/p}(x)\eta\| \|W^{1/p}(x)\nu\|, \quad \eta, \nu \in \mathcal{M}_n(\mathbb{C})$   
and  $F \in L^{p'}(\Omega, W^{-\frac{p'}{p}})$ .

We will first prove the following Caccioppoli inequality, which is a matrixly degenerate version of the Caccioppoli inequality proved in the very recent paper [8] for uniformly elliptic  $p$ -Laplacian systems. Note that again for this Caccioppoli inequality we do not require that  $W$  is a matrix  $A_p$  weight.

**Lemma 6.5.** *Let  $p > 2$ , and let  $W$  and  $G$  satisfy (i) and (ii'). If  $\vec{u} \in H^{1,p}(\Omega, W)$  is a weak solution to*

$$\text{Div} \left[ \langle D\vec{u}G, D\vec{u} \rangle_{\text{tr}}^{\frac{p-2}{2}} D\vec{u}G \right] = -\text{Div} F$$

where  $F \in L^{p'}(\Omega, W^{-\frac{p'}{p}})$ , then for any ball  $B_r$  whose closure is contained in  $\Omega$  we have

$$\begin{aligned} \int_{B_{r/2}} \|W^{\frac{1}{p}}(x)D\vec{u}(x)\|^p dx &\lesssim \int_{B_r} \|W^{-\frac{1}{p}}(x)F(x)\|^{p'} dx \\ &+ \frac{1}{r^p} \int_{B_r \setminus B_{r/2}} |W^{\frac{1}{p}}(x)\vec{u}(x)|^p dx \end{aligned} \quad (6.5)$$

*Proof.* The proof is similar to the arguments in [8], p. 57 - 62. As in the proof of (6.1), pick some  $\eta \in C_c^\infty(B_r)$  where  $\eta \equiv 1$  on  $B_{\frac{r}{2}}, 0 \leq \eta \leq 1$  on  $B_r$ , and

$$|\nabla \eta| \leq \frac{4}{r} \chi_{B_r \setminus B_{\frac{r}{2}}}$$

Since  $\vec{u} \in H^{1,p}(\Omega, W)$  is a weak solution to (6.4) we have that

$$\int_{\Omega} [\langle D\vec{u}G, D\vec{u} \rangle_{\text{tr}}]^{\frac{p-2}{2}} \langle D\vec{u}G, D(\eta^p \vec{u}) \rangle_{\text{tr}} dx = - \int_{\Omega} \langle F, D(\eta^p \vec{u}) \rangle_{\text{tr}} dx$$

Since clearly

$$D(\eta^p \vec{u}) = (p-1)\eta^{p-2}(\vec{u} \otimes \nabla \eta) + \eta^{p-1}D(\eta \vec{u})$$

one can easily check that

$$\begin{aligned} \langle D\vec{u}G, D(\eta^p \vec{u}) \rangle_{\text{tr}} &= \eta^{p-2} [\langle D(\eta \vec{u})G, D(\eta \vec{u}) \rangle_{\text{tr}} - \langle (\vec{u} \otimes \nabla \eta)G, D(\eta \vec{u}) \rangle_{\text{tr}} \\ &\quad + [(p-1) \langle D(\eta \vec{u})G, \vec{u} \otimes \nabla \eta \rangle_{\text{tr}} - (p-1) \langle (\vec{u} \otimes \nabla \eta)G, \vec{u} \otimes \nabla \eta \rangle_{\text{tr}}] \\ &:= \eta^{p-2} \mathcal{A}(x, \vec{u}, \eta) \end{aligned}$$

and similarly

$$\begin{aligned} [\langle D\vec{u}G, D\vec{u} \rangle_{\text{tr}} \eta^2]^{\frac{p-2}{2}} &= [\langle D(\eta\vec{u})G, D(\eta\vec{u}) \rangle_{\text{tr}} - \langle D(\eta\vec{u})G, \vec{u} \otimes \nabla\eta \rangle_{\text{tr}} \\ &\quad - \langle (\vec{u} \otimes \nabla\eta)G, D(\eta\vec{u}) \rangle_{\text{tr}} + \langle (\vec{u} \otimes \nabla\eta)G, \vec{u} \otimes \nabla\eta \rangle_{\text{tr}}]^{\frac{p-2}{2}} \\ &:= [\mathcal{B}(x, \vec{u}, \eta)]^{\frac{p-2}{2}} \end{aligned}$$

so that

$$\int_{\Omega} \mathcal{A}(x, \vec{u}, \eta) [\mathcal{B}(x, \vec{u}, \eta)]^{\frac{p-2}{2}} dx = 0.$$

Furthermore, define

$$\mathcal{N}(\vec{u}, \eta) := \int_{\Omega} [\langle D(\eta\vec{u})G, D(\eta\vec{u}) \rangle_{\text{tr}}]^{\frac{p-2}{2}} \langle D(\eta\vec{u})G, D(\eta\vec{u}) \rangle_{\text{tr}} dx$$

so by condition (i)

$$\mathcal{N}(\vec{u}, \eta) \gtrsim \int_{\Omega} \|W^{\frac{1}{p}}(x)D(\eta\vec{u})\|^p dx = \int_{B_r} \|W^{\frac{1}{p}}(x)D(\eta\vec{u})\|^p dx. \quad (6.6)$$

We will now obtain suitable upper bound for  $|\mathcal{N}(\vec{u}, \eta)|$ . By the definitions of  $\mathcal{A}(x, \vec{u}, \eta)$  and  $\mathcal{B}(x, \vec{u}, \eta)$  we can write

$$\begin{aligned} \mathcal{N}(\vec{u}, \eta) &= \int_{\Omega} [\mathcal{B}(x, \vec{u}, \eta)]^{\frac{p-2}{2}} \langle D(\eta\vec{u})G, D(\eta\vec{u}) \rangle_{\text{tr}} dx \\ &\quad + \int_{\Omega} [\langle D(\eta\vec{u})G, \vec{u} \otimes \nabla\eta \rangle_{\text{tr}} + \langle (\vec{u} \otimes \nabla\eta)G, D(\eta\vec{u}) \rangle_{\text{tr}} \\ &\quad - \langle (\vec{u} \otimes \nabla\eta)G, \vec{u} \otimes \nabla\eta \rangle_{\text{tr}}]^{\frac{p-2}{2}} \langle D(\eta\vec{u})G, D(\eta\vec{u}) \rangle_{\text{tr}} dx \\ &= \int_{\Omega} [\mathcal{B}(x, \vec{u}, \eta)]^{\frac{p-2}{2}} \mathcal{A}(x, \vec{u}, \eta) dx + \int_{\Omega} [\mathcal{B}(x, \vec{u}, \eta)]^{\frac{p-2}{2}} \\ &\quad \times \left[ \langle (\vec{u} \otimes \nabla\eta)G, D(\eta\vec{u}) \rangle_{\text{tr}} - (p-1) \left\langle D(\eta\vec{u})G, \vec{u} \otimes \nabla\eta \right\rangle_{\text{tr}} \right. \\ &\quad \left. + (p-1) \langle (\vec{u} \otimes \nabla\eta)G, \vec{u} \otimes \nabla\eta \rangle_{\text{tr}} \right] dx \\ &\quad + \int_{\Omega} [\langle D(\eta\vec{u}), \vec{u} \otimes \nabla\eta \rangle_{\text{tr}} + \langle (\vec{u} \otimes \nabla\eta)G, D(\eta\vec{u}) \rangle_{\text{tr}} \\ &\quad - \langle (\vec{u} \otimes \nabla\eta)G, \vec{u} \otimes \nabla\eta \rangle_{\text{tr}}]^{\frac{p-2}{2}} \langle D(\eta\vec{u}), D(\eta\vec{u}) \rangle_{\text{tr}} dx \end{aligned}$$

so that

$$|\mathcal{N}(\vec{u}, \eta)| \leq \sum_{j=1}^7 I_j$$

where

$$\begin{aligned}
I_1 &:= \left| \int_{\Omega} \mathcal{A}(x, \vec{u}, \eta) [\mathcal{B}(x, \vec{u}, \eta)]^{\frac{p-2}{2}} dx \right| = \left| \int_{\Omega} \langle F, D(\eta^p \vec{u}) \rangle_{\text{tr}} dx \right| \\
I_2 &:= \int_{\Omega} |\mathcal{B}(x, \vec{u}, \eta)|^{\frac{p-2}{2}} | \langle (\vec{u} \otimes \nabla \eta) G, D(\eta \vec{u}) \rangle_{\text{tr}} | dx \\
I_3 &:= \int_{\Omega} |\mathcal{B}(x, \vec{u}, \eta)|^{\frac{p-2}{2}} | \langle D(\eta \vec{u}) G, \vec{u} \otimes \nabla \eta \rangle_{\text{tr}} | dx \\
I_4 &:= \int_{\Omega} |\mathcal{B}(x, \vec{u}, \eta)|^{\frac{p-2}{2}} | \langle (\vec{u} \otimes \nabla \eta) G, \vec{u} \otimes \nabla \eta \rangle_{\text{tr}} | dx \\
I_5 &:= \int_{\Omega} | \langle D(\eta \vec{u}) G, D(\eta \vec{u}) \rangle_{\text{tr}} |^{\frac{p-2}{2}} | \langle D(\eta \vec{u}) G, D(\eta \vec{u}) \rangle_{\text{tr}} | dx \\
I_6 &:= \int_{\Omega} | \langle D(\eta \vec{u}) G, D(\eta \vec{u}) \rangle_{\text{tr}} |^{\frac{p-2}{2}} | \langle D(\eta \vec{u}) G, D(\eta \vec{u}) \rangle_{\text{tr}} | dx \\
I_7 &:= \int_{\Omega} | \langle (\vec{u} \otimes \nabla \eta) G, \vec{u} \otimes \nabla \eta \rangle_{\text{tr}} |^{\frac{p-2}{2}} | \langle D(\eta \vec{u}) G, D(\eta \vec{u}) \rangle_{\text{tr}} | dx.
\end{aligned}$$

We finish the proof by bounding each of these terms. Let  $\epsilon > 0$ . First, we have

$$\begin{aligned}
I_1 &\leq \int_{\Omega} | \langle F, D(\eta^p \vec{u}) \rangle_{\text{tr}} | dx \\
&\leq (p-1) \int_{\Omega} |\eta|^{p-2} | \langle F, \vec{u} \otimes \nabla \eta \rangle_{\text{tr}} | dx + \int_{\Omega} |\eta|^{p-1} | \langle F, D(\eta \vec{u}) \rangle_{\text{tr}} | dx \\
&\leq \int_{\Omega} |\nabla \eta| \|W^{-\frac{1}{p}} F\| \|W^{\frac{1}{p}} \vec{u}\| dx + \int_{\Omega} \|W^{-\frac{1}{p}} F\| |D(\eta \vec{u})| dx \\
&\leq \left( \int_{B_r} \|W^{-\frac{1}{p}} F\|^{p'} dx \right)^{\frac{1}{p'}} \left( \int_{B_r} |\nabla \eta|^p |W^{\frac{1}{p}} \vec{u}|^p dx \right)^{\frac{1}{p}} \\
&\quad + \left( \int_{B_r} \|W^{-\frac{1}{p}} F\|^{p'} dx \right)^{\frac{1}{p'}} \left( \int_{B_r} |D(\eta \vec{u})|^p dx \right)^{\frac{1}{p}} \\
&\lesssim C(\epsilon) \int_{B_r} \|W^{-\frac{1}{p}} F\|^{p'} dx + \frac{1}{r^p} \int_{B_r \setminus B_{r/2}} |W^{\frac{1}{p}} \vec{u}|^p dx + \epsilon \int_{B_r} \|W^{\frac{1}{p}} D(\eta \vec{u})\|^p dx
\end{aligned}$$

by Hölder's inequality and Young's inequality with  $\epsilon$ .

Next we estimate  $|\mathcal{B}(x, \vec{u}, \eta)|$ . Note that by (ii') we immediately get

$$\begin{aligned}
|\mathcal{B}(x, \vec{u}, \eta)| &\leq \|W^{\frac{1}{p}} D(\eta \vec{u})\|^2 + 2 \|W^{\frac{1}{p}} D(\eta \vec{u})\| \|W^{\frac{1}{p}} (\vec{u} \otimes \nabla \eta)\| + \|W^{\frac{1}{p}} (\vec{u} \otimes \nabla \eta)\|^2 \\
&\leq \|W^{\frac{1}{p}} D(\eta \vec{u})\|^2 + 2 |\nabla \eta| \|W^{\frac{1}{p}} D(\eta \vec{u})\| \|W^{\frac{1}{p}} \vec{u}\| + |\nabla \eta|^2 \|W^{\frac{1}{p}} \vec{u}\|^2
\end{aligned}$$

$$\lesssim \|W^{\frac{1}{p}}D(\eta\vec{u})\|^2 + |\nabla\eta|^2|W^{\frac{1}{p}}\vec{u}|^2$$

so

$$|\mathcal{B}(x, \vec{u}, \eta)|^{\frac{p-2}{2}} \lesssim \|W^{\frac{1}{p}}D(\eta\vec{u})\|^{p-2} + |\nabla\eta|^{p-2}|W^{\frac{1}{p}}\vec{u}|^{p-2}. \quad (6.7)$$

Thus, by (ii') and (6.7) we have

$$\begin{aligned} I_2 &\leq \int_{\Omega} |\nabla\eta|^{p-1}|W^{\frac{1}{p}}\vec{u}|^{p-1}\|W^{\frac{1}{p}}D(\eta\vec{u})\| dx + \int_{\Omega} |\nabla\eta||W^{\frac{1}{p}}\vec{u}|\|W^{\frac{1}{p}}D(\eta\vec{u})\|^{p-1} dx \\ &\leq \left( \int_{\Omega} |\nabla\eta|^p|W^{\frac{1}{p}}\vec{u}|^p dx \right)^{\frac{1}{p'}} \left( \int_{\Omega} \|W^{\frac{1}{p}}D(\eta\vec{u})\|^p dx \right)^{\frac{1}{p}} \\ &\quad + \left( \int_{\Omega} |\nabla\eta|^p|W^{\frac{1}{p}}\vec{u}|^p dx \right)^{\frac{1}{p}} \left( \int_{\Omega} \|W^{\frac{1}{p}}D(\eta\vec{u})\|^p dx \right)^{\frac{1}{p'}} \\ &\leq \epsilon \int_{B_r} \|W^{\frac{1}{p}}D(\eta\vec{u})\|^p dx + \frac{C(\epsilon)}{r^p} \int_{B_r \setminus B_{r/2}} |W^{\frac{1}{p}}\vec{u}|^p dx. \end{aligned}$$

For some constant  $C(\epsilon)$  depending on  $\epsilon$ . By the symmetry of (ii') we have that  $I_3$  satisfies the same condition.

Similarly, using Hölder's inequality with respect to  $p/2$  we have

$$\begin{aligned} I_4 &\leq \int_{\Omega} |\nabla\eta|^p|W^{\frac{1}{p}}\vec{u}|^p dx + \int_{\Omega} |\nabla\eta|^2\|W^{\frac{1}{p}}D(\eta\vec{u})\|^{p-2}|W^{\frac{1}{p}}\vec{u}|^2 dx \\ &\lesssim \frac{1}{r^p} \int_{B_r \setminus B_{r/2}} |W^{\frac{1}{p}}\vec{u}|^p dx + \left( \frac{1}{r^p} \int_{B_r \setminus B_{r/2}} |W^{\frac{1}{p}}\vec{u}|^p dx \right)^{\frac{2}{p}} \left( \int_{B_r} |W^{\frac{1}{p}}D(\eta\vec{u})|^p dx \right)^{\frac{p-2}{p}} \\ &\leq \frac{C(\epsilon)}{r^p} \int_{B_r \setminus B_{r/2}} |W^{\frac{1}{p}}\vec{u}|^p dx + \epsilon \int_{B_r} |W^{\frac{1}{p}}D(\eta\vec{u})|^p dx. \end{aligned}$$

Likewise, Hölder's inequality with respect to  $2p/(p+2)$  gives

$$\begin{aligned} I_5 &\leq \int_{\Omega} |\nabla\eta|^{\frac{p-2}{2}}\|W^{\frac{1}{p}}D(\eta\vec{u})\|^{\frac{p+2}{2}}|W^{\frac{1}{p}}\vec{u}|^{\frac{p-2}{2}} \\ &\leq \left( \int_{\Omega} |\nabla\eta|^p|W^{\frac{1}{p}}\vec{u}|^p dx \right)^{\frac{2p}{p-2}} \left( \int_{\Omega} \|W^{\frac{1}{p}}D(\eta\vec{u})\|^p dx \right)^{\frac{2p}{p+2}} \\ &\lesssim \epsilon \int_{B_r} \|W^{\frac{1}{p}}D(\eta\vec{u})\|^p dx + \frac{C(\epsilon)}{r^p} \int_{B_r \setminus B_{r/2}} |W^{\frac{1}{p}}\vec{u}|^p dx \end{aligned}$$

and note that  $I_6$  is estimated in exactly the same way.

Finally,

$$I_7 \leq \int_{\Omega} |\nabla\eta|^{p-2}|W^{\frac{1}{p}}\vec{u}|^{p-2}\|W^{\frac{1}{p}}D(\nabla\vec{u})\|^2 dx$$

$$\begin{aligned}
&\leq \left( \int_{\Omega} |\nabla \eta|^p |W^{\frac{1}{p}} \vec{u}|^p dx \right)^{\frac{p-2}{p}} \left( \|W^{\frac{1}{p}} D(\eta \vec{u})\|^p dx \right)^{\frac{2}{p}} \\
&\leq \epsilon \int_{B_r} \|W^{\frac{1}{p}} D(\eta \vec{u})\|^p dx + \frac{C(\epsilon)}{r^p} \int_{B_r \setminus B_{r/2}} |W^{\frac{1}{p}} \vec{u}|^p dx.
\end{aligned}$$

Combining everything and setting  $\epsilon$  small enough we have

$$\begin{aligned}
\int_{B_{r/2}} \|W^{\frac{1}{p}} D(\eta \vec{u})\|^p dx &\leq \int_{B_r} \|W^{\frac{1}{p}} D(\eta \vec{u})\|^p dx \\
&\lesssim \int_{B_r} \|W^{-\frac{1}{p}} F\|^{p'} dx + \frac{1}{r^p} \int_{B_r \setminus B_{r/2}} |W^{\frac{1}{p}} \vec{u}|^p dx.
\end{aligned}$$

□

**Theorem 6.6.** *Let  $p > 2$  and let  $W, G, \vec{u}$ , and  $F$  satisfy the conditions of the previous Lemma. Then there exists  $q > p$  such that given  $B_{2r} \subset \Omega$  we have*

$$\begin{aligned}
\left( \frac{1}{|B_{r/2}|} \int_{B_{r/2}} \|W^{\frac{1}{p}}(x) D\vec{u}(x)\|^q dx \right)^{\frac{1}{q}} &\lesssim \left( \frac{1}{|B_r|} \int_{B_r} \|W^{\frac{1}{p}}(x) D\vec{u}(x)\|^p dx \right)^{\frac{1}{p}} \\
&\quad + \left( \frac{1}{|B_r|} \int_{B_r} \|W^{-\frac{1}{p}}(x) F(x)\|^{\frac{qp'}{p}} dx \right)^{\frac{1}{q}}.
\end{aligned}$$

*Proof.* The proof is very similar to the proof of Theorem 1.5. Let  $\epsilon > 0$  be chosen where Theorem 1.2 is true, so by (6.5)

$$\begin{aligned}
&\left( \frac{1}{|B_{r/2}|} \int_{B_{r/2}} \|W^{\frac{1}{p}} D\vec{u}\|^p dx \right)^{\frac{1}{p}} \\
&\lesssim \left( \frac{1}{|B_r|} \int_{B_r} \|W^{-\frac{1}{p}} F\|^{p'} dx \right)^{\frac{1}{p}} + \frac{1}{r} \left( \frac{1}{|B_r|} \int_{B_r} |W^{\frac{1}{p}}(\vec{u} - \vec{u}_{B_r})|^p dx \right)^{\frac{1}{p}} \\
&\lesssim \left( \frac{1}{|B_r|} \int_{B_r} \|W^{-\frac{1}{p}} F\|^{p'} dx \right)^{\frac{1}{p}} + \left( \frac{1}{|B_r|} \int_{B_r} \|W^{\frac{1}{p}} D\vec{u}\|^{p-\epsilon} dx \right)^{\frac{1}{p-\epsilon}}
\end{aligned}$$

However, setting

$$U(x) = \|W^{\frac{1}{p}}(x) D\vec{u}(x)\|^{p-\epsilon}, \quad G(x) = \|W^{-\frac{1}{p}}(x) F(x)\|^{\frac{p'(p-\epsilon)}{p}}, \quad \text{and } s = \frac{p}{p-\epsilon}$$

we have that

$$\frac{1}{|B_{r/2}|} \int_{B_{r/2}} (U(x))^s dx \lesssim \left( \frac{1}{|B_r|} \int_{B_r} U(x) dx \right)^s + \frac{1}{|B_r|} \int_{B_r} (G(x))^s dx.$$

Again Lemma 2.2 in [8] now says that there exists  $t > s = \frac{p}{p-\epsilon}$  where

$$\begin{aligned} & \left( \frac{1}{|B_{r/2}|} \int_{B_{r/2}} \|W^{\frac{1}{p}} D\vec{u}\|^{t(p-\epsilon)} dx \right)^{\frac{1}{t}} \\ & \lesssim \left( \frac{1}{|B_r|} \int_{B_r} |W^{\frac{1}{p}} D\vec{u}|^p dx \right)^{\frac{p-\epsilon}{p}} + \left( \frac{1}{|B_r|} \int_{B_r} \|W^{-\frac{1}{p}} F\|^{\frac{tp'(p-\epsilon)}{p}} dx \right)^{\frac{1}{t}}. \end{aligned}$$

Setting  $q = t(p - \epsilon) > p$  clearly completes the proof.  $\square$

Finally, note that (thanks to (6.5)) the same arguments used to prove Theorem 1.6 also prove the following

**Theorem 6.7.** *Let  $d = 2, p > 2$ , and let  $W$  and  $G$  satisfy (i) and (ii'). If  $\vec{u}$  is a weak solution*

$$\operatorname{Div} \left[ \langle D\vec{u}G, D\vec{u} \rangle_{tr}^{\frac{p-2}{2}} D\vec{u}G \right] = 0$$

*Then there exists  $\epsilon > 0$  such that for  $x, y \in \Omega$  with  $|x-y| < \frac{1}{2} \operatorname{dist}(\{x, y\}, \Omega^c)$ , we have*

$$|\vec{u}(x) - \vec{u}(y)| \lesssim C_{x,y} |x - y|^\epsilon$$

*where*

$$C_{x,y} = \left( \sup \frac{1}{|Q|^{1-\epsilon}} \int_Q \|W^{-\frac{1}{p}}(\xi)\|^{p'} d\xi \right)^{\frac{1}{p'}}$$

*and the supremum is over cubes  $Q \subset \Omega$  centered at either  $x$  or  $y$ , and having side length less than  $|x - y|$ .*

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