# ON A CONJECTURE OF A LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTION

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Abstract

In this short note we prove a conjecture for the interval $(0, 1)$, related to a logarithmically completely monotonic function, presented in [5]. Then, we extend by proving a more generalized theorem. At the end we pose an open problem on a logarithmically completely monotonic function involving $q$-Digamma function.

**Key words**: completely monotonic, logarithmically completely monotonic

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## 1. Introduction

Recall from [14, Chapter XIII], [18, Chapter 1] and [19, Chapter IV] that a function $f$ is said to be completely monotonic on an interval $I$ if $f$ has derivatives of all orders on $I$ and satisfies

$$0 \leq (-1)^n f^{(n)}(x) < \infty \tag{1.1}$$

for $x \in I$ and $n \geq 0$. The celebrated Bernstein-Widder's Theorem (see [18, p. 3, Theorem 1.4] or [19, p. 161, Theorem 12b]) characterizes that a necessary and sufficient condition that $f$ should be completely monotonic for $0 < x < \infty$ is that

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t), \tag{1.2}$$

where $\alpha(t)$ is non-decreasing and the integral converges for $0 < x < \infty$. This expresses that a completely monotonic function $f$ on $[0, \infty)$ is a Laplace transform of the measure $\alpha$.

It is common knowledge that the classical Euler's gamma function $\Gamma(x)$ may be defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \tag{1.3}$$

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, is called psi function or digamma function.

An alternative definition of the gamma function $\Gamma(x)$ is

$$\Gamma(x) = \lim_{p\to\infty} \Gamma_p(x), \tag{1.4}$$

where

$$\Gamma_p(x) = \frac{p!p^x}{x(x+1)\cdots(x+p)} = \frac{p^x}{x(1+x/1)\cdots(1+x/p)} \tag{1.5}$$

for $x > 0$ and $p \in \mathbb{N}$. See [3, p. 250]. The $p$-analogue of the psi function $\psi(x)$ is defined as the logarithmic derivative of the $\Gamma_p$ function, that is,

$$\psi_p(x) = \frac{d}{dx}\ln\Gamma_p(x) = \frac{\Gamma_p'(x)}{\Gamma_p(x)}. \tag{1.6}$$

The function $\psi_p$ has the following properties (see [10, p. 374, Lemma 5] and [12, p. 29, Lemma 2.3]).

(1) It has the following representations

$$\psi_p(x) = \ln p - \sum_{k=0}^{p}\frac{1}{x+k} = \ln p - \int_0^\infty \frac{1-e^{-(p+1)t}}{1-e^{-t}}e^{-xt}dt. \tag{1.7}$$

(2) It is increasing on $(0,\infty)$ and $\psi_p'$ is completely monotonic on $(0,\infty)$.

In [2, pp. 374–375, Theorem 1], it was proved that the function

$$\theta_\alpha(x) = x^\alpha[\ln x - \psi(x)] \tag{1.8}$$

is completely monotonic on $(0,\infty)$ if and only if $\alpha \le 1$.

For the history, backgrounds, applications and alternative proofs of this conclusion, please refer to [4], [15, p. 8, Section 1.6.6] and closely-related references therein.

A positive function $f$ is said to be *logarithmically completely monotonic* [10] on an open interval $I$, if $f$ satisfies

$$(-1)^n[\ln f(x)]^{(n)} \ge 0, (x\in I, n = 1,2,\ldots). \tag{1.9}$$

If the inequality (1.2) is strict, then $f$ is said to be *strictly logarithmically completely monotonic*. Let $C$ and $L$ denote the set of completely monotonic functions and the set of logarithmically completely monotonic functions, respectively. The relationship between completely monotonic functions and logarithmically completely monotonic functions can be presented [10] by $L \subset C$.

## 2. Main results

In [5] has been posed the following conjecture.

**Conjecture 2.1.** *The function*

$$q(t) := t^{t(\psi(t)-\log t)-\gamma} \tag{2.1}$$

*is logarithmically completely monotonic on* $(0,\infty)$.

**Theorem 2.2.** *The function*

$$q(t) := t^{t(\psi(t)-\log t)-\gamma} \tag{2.2}$$

*is logarithmically completely monotonic on* $(0,1)$.

*Proof.* One easily finds that

$$\log q(t) = -t\cdot(\log t - \psi(t))\log t - \gamma\cdot\log t \tag{2.3}$$

Let $h(t) = -\gamma\cdot\log t, g(t) = -\log t; f(t) = t\cdot(\log t - \psi(t))$. Alzer [2] proved that the function $f(t) = t\cdot(\log t - \psi(t))$ is strictly completely monotonic on $(0,\infty)$. The functions $g(t) = -\log t$ and $h(t) = -\gamma\cdot\log t$ are also strictly completely monotonic on $(0,1)$. We complete the proof by recalling the results from [19].

1) The product of two completely monotone functions is completely monotonic function.

2) A non-negative finite linear combination of completely monotone functions is completely monotonic function. □

We extend the previous result to the following theorem.

**Theorem 2.3.** *The function*

$$q_p(t) := t^{t\cdot\left(\psi_p(t)-\log\frac{pt}{t+p+1}\right)-\gamma} \tag{2.4}$$

*is logarithmically completely monotonic on* $(0,1)$.

*Proof.* One easily finds that

$$\log q_p(t) = -t\big(\log\frac{pt}{t+p+1} - \psi(t)\big)\log t - \gamma\cdot\log t \tag{2.5}$$

Let $h(t) = -\gamma\cdot\log t, g(t) = -\log t; f_p(t) = t\cdot\big(\log\dfrac{pt}{t+p+1} - \psi_p(t)\big)$.

Krasniqi and Qi [11] proved that the function $f_p(t) = t\cdot\big(\log\frac{pt}{t+p+1} - \psi_p(t)\big)$ is strictly completely monotonic on $(0,\infty)$. The functions $g(t) = -\log t$ and $h(t) = -\gamma\cdot\log t$ are also strictly completely monotonic on $(0,1)$.

By refering the same results from [19] as in previous proof, we complete the proof. □

**Remark 2.4.** *Letting* $p\to\infty$ *in Theorem 2.3 , we obtain Theorem 2.2 .*

At the end we pose the following open problem:

**Problem 2.5.** *Let* $\psi_q(t)$ *be q-Digamma function. Find the family of functions* $\theta(t)$ *such that*

$$q(t) := t^{t\cdot(\psi_q(t)-\log\theta(t))-\gamma} \tag{2.6}$$

*is logarithmically completely monotonic on* $(0,\infty)$.

**Remark 2.6.** *This is a corrected version of paper* [9]

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