

Semiclassical structure constants in the η -deformed $AdS_5 \times S^5$: Leading finite-size corrections

Plamen Bozhilov

*Institute for Nuclear Research and Nuclear Energy
Bulgarian Academy of Sciences
1784 Sofia, Bulgaria
bozhilov@inrne.bas.bg, bozhilov.p@gmail.com*

Abstract

We consider the leading finite-size effects on some structure constants for the η - deformed $AdS_5 \times S^5$ background in the framework of the semiclassical approach. The leading finite-size corrections are derived for the cases when we have two heavy string states represented by giant magnons and for two different choices of the light states corresponding to dilaton operator with nonzero momentum and primary scalar operators. Since the dual field theory is still unknown, the results obtained here must be considered as conjectures or as predictions from the string theory side.

1 Introduction

In the framework of the AdS/CFT duality [1] between string theories/M-theory on curved space-times with Anti-de Sitter subspaces and conformal field theories in different dimensions, an interesting issue to work on is to go beyond the spectral problem¹ and compute the corresponding correlation functions, if possible.

It was shown in [3] how to compute a two-point correlation function of operators, dual to classical spinning strings in a subspace of $AdS_5 \times S^5$. These investigations have been continued in [4], where the 2-point function of string vertex operators representing string state with large spin in AdS_5 have been obtained in the semiclassical approximation.

The computation of semiclassical three-point correlation functions in $AdS_5 \times S^5$ string theory, dual to $\mathcal{N} = 4$ SYM in four space-time dimensions, was started in [5]-[7] followed by many other papers.

A new development was to consider deformations of $AdS_5 \times S^5$ and obtain some three-point correlators (structure constants). This was done first for the γ -deformed [8] or TsT -transformed [9] $AdS_5 \times S^5$ dual to $\mathcal{N} = 1$ SYM. The first papers on this subject were [10, 11].

Recently, a new *integrable* deformation (η -deformation) of the $AdS_5 \times S^5$ superstring action has been discovered [12]². The bosonic part of the superstring sigma model Lagrangian on this η -deformed background was determined in [16]. From it one can extract the background metric g and the 2-form gauge potential b [17]. Then, one can find different string solutions on this background, e.g. the ones found in [17]-[20] (for other contributions on the subject see the references in [21]). In particular, giant magnon type solution, playing an essential role in the string theory-field theory duality, has been found [22]. Based on this solution, one can compute the corresponding vertices on it. Then, a natural task is to try to find the three-point correlation functions (structure constants) for two heavy string states represented by giant magnons and some light (supergravity) states. The first step in this direction was done in [23], where the finite-size effect was also taken into account for the case of two giant magnon states and dilaton operator with zero momentum. This result was extended in [21] for the cases of two giant magnon states and dilaton operator with nonzero momentum, primary scalar operators, and singlet scalar operators on higher string levels.

The semiclassical structure constants found in [21] are given in terms of several parameters and hypergeometric functions of two variables depending on them. However, it is important to know their dependence on the conserved string charge J_1 and the worldsheet momentum p , because namely these quantities are related to the corresponding operators in the dual field theory and the momentum of the magnon excitations in the dual spin-chain. Unfortunately, this can not be done exactly for the finite-size case due to the complicated dependence

¹For recent review see [2].

² η -deformed σ -models were defined and their integrability was proven in [13, 14]. See also [15].

between the above mentioned parameters and J_1, p . On the other hand, it is possible to find the leading finite-size corrections to such structure constants. Namely this will be our aim here.

This letter is organized as follows. In Sec. 2 we give a short review of the finite-size giant magnon solution on η -deformed $AdS_5 \times S^5$. In Sec. 3 and Sec. 4 we derive the leading finite-size effects on the structure constants for the cases when we have two heavy string states represented by finite-size giant magnons and for two different choices of the light states corresponding to dilaton operator with nonzero momentum and primary scalar operators. Sec. 5 is devoted to our concluding remarks.

2 Short review of the giant magnon solution on η -deformed $AdS_5 \times S^5$

Giant magnons live in the $R_t \times S_\eta^2$ subspace of the η -deformed $AdS_5 \times S^5$. The background seen by the string moving in the $R_t \times S_\eta^2$ subspace can be written as [17]

$$\begin{aligned} g_{tt} &= -1, & g_{\phi_1\phi_1} &= \sin^2 \theta, \\ g_{\theta\theta} &= \frac{1}{1 + \tilde{\eta}^2 \sin^2 \theta}, \end{aligned} \tag{2.1}$$

where $\tilde{\eta}$ is related to the deformation parameter η according to³

$$\tilde{\eta} = \frac{2\eta}{1 - \eta^2}. \tag{2.2}$$

By using a specific ansatz for the string embedding [24]

$$\begin{aligned} t(\tau, \sigma) &= \kappa\tau, & \phi_1(\tau, \sigma) &= \omega_1\tau + F(\xi), & \theta(\tau, \sigma) &= \theta(\xi), & \xi &= \alpha\sigma + \beta\tau, \\ \kappa, \omega_1, \alpha, \beta &= \text{constants}, \end{aligned}$$

one can find the following string solution [17]⁴

$$\chi(\xi) = \frac{\chi_\eta \chi_p \mathbf{dn}^2(x, m)}{\chi_p \mathbf{dn}^2(x, m) + \chi_\eta - \chi_p}, \tag{2.3}$$

³We changed the notation κ in [22] to $\tilde{\eta}$ because we use κ for other purposes.

⁴See [25] where it was shown that the bosonic spinning strings on the η -deformed $AdS_5 \times S^5$ background are naturally described as periodic solutions of a novel finite-dimensional integrable system which can be viewed as a deformation of the Neumann model.

$$\begin{aligned}
\phi_1(\tau, \sigma) = & \omega_1 \tau + \frac{1}{\tilde{\eta} \alpha \omega_1^2 (\chi_\eta - 1) \sqrt{(\chi_\eta - \chi_m) \chi_p}} \times \\
& \left\{ \left[\beta (\kappa^2 + \omega_1^2 (\chi_\eta - 1)) \right] \mathbf{F} \left(\arcsin \sqrt{\frac{(\chi_\eta - \chi_m)(\chi_p - \chi)}{(\chi_p - \chi_m)(\chi_\eta - \chi)}}, m \right) \right. \\
& \left. - \frac{(\chi_\eta - \chi_p) \beta \kappa^2}{1 - \chi_p} \mathbf{\Pi} \left(\arcsin \sqrt{\frac{(\chi_\eta - \chi_m)(\chi_p - \chi)}{(\chi_p - \chi_m)(\chi_\eta - \chi)}}, -\frac{(\chi_\eta - 1)(\chi_p - \chi_m)}{(1 - \chi_p)(\chi_\eta - \chi_m)}, m \right) \right\}.
\end{aligned} \tag{2.4}$$

Here $\chi = \cos^2 \theta$, where θ is the non-isometric angle on the deformed sphere S_η^2 , while ϕ_1 is the isometric angle on it. $\mathbf{dn}(x, m)$ is one of the Jacobi elliptic functions, \mathbf{F} and $\mathbf{\Pi}$ are the incomplete elliptic integrals of first and third kind.

$$\begin{aligned}
x &= \frac{\tilde{\eta} \alpha \omega_1 \sqrt{(\chi_\eta - \chi_m) \chi_p}}{\alpha^2 - \beta^2} \xi, \\
m &= \frac{(\chi_p - \chi_m) \chi_\eta}{(\chi_\eta - \chi_m) \chi_p},
\end{aligned}$$

and $\chi_\eta > \chi_p > \chi_m$ are the roots of the equation $d\chi/d\xi = 0$, given by

$$\chi_\eta = 1 + \frac{1}{\tilde{\eta}^2} \quad \chi_p = 1 - \frac{\beta^2}{\alpha^2} \kappa^2, \quad \chi_m = 1 - \kappa^2. \tag{2.5}$$

Now, let us present the expressions for the conserved charges (the string energy E_s and the angular momentum J_1) and also the worldsheet momentum p equal to the angular difference $\Delta\phi_1$, since we are going to use it [17]:

$$E_s = \frac{T}{\tilde{\eta}} \left(1 - \frac{\beta^2}{\alpha^2} \right) \frac{\kappa}{\omega_1} \int_{\chi_m}^{\chi_p} \frac{d\chi}{\sqrt{(\chi_\eta - \chi)(\chi_p - \chi)(\chi - \chi_m)\chi}}, \tag{2.6}$$

$$\begin{aligned}
J_1 = & \frac{T}{\tilde{\eta}} \left[\left(1 - \frac{\beta^2 \kappa^2}{\alpha^2 \omega_1^2} \right) \int_{\chi_m}^{\chi_p} \frac{d\chi}{\sqrt{(\chi_\eta - \chi)(\chi_p - \chi)(\chi - \chi_m)\chi}} \right. \\
& \left. - \int_{\chi_m}^{\chi_p} \frac{\chi d\chi}{\sqrt{(\chi_\eta - \chi)(\chi_p - \chi)(\chi - \chi_m)\chi}} \right], \tag{2.7}
\end{aligned}$$

$$\begin{aligned}
\Delta\phi_1 \equiv p = & \frac{1}{\tilde{\eta}} \left[\frac{\beta}{\alpha} \int_{\chi_m}^{\chi_p} \frac{d\chi}{\sqrt{(\chi_\eta - \chi)(\chi_p - \chi)(\chi - \chi_m)\chi}} \right. \\
& \left. - \frac{\beta \kappa^2}{\alpha \omega_1^2} \int_{\chi_m}^{\chi_p} \frac{d\chi}{(1 - \chi) \sqrt{(\chi_\eta - \chi)(\chi_p - \chi)(\chi - \chi_m)\chi}} \right]. \tag{2.8}
\end{aligned}$$

Solving the integrals in (2.6)-(2.8) and introducing the notations

$$v = -\frac{\beta}{\alpha}, \quad W = \frac{\kappa^2}{\omega_1^2}, \quad \epsilon = \frac{(\chi_\eta - \chi_p)\chi_m}{(\chi_\eta - \chi_m)\chi_p}, \quad (2.9)$$

we finally obtain

$$E_s = \frac{2T}{\tilde{\eta}} \frac{(1-v^2)\sqrt{W}}{\sqrt{(\chi_\eta - \chi_m)\chi_p}} \mathbf{K}(1-\epsilon), \quad (2.10)$$

$$J_1 = \frac{2T}{\tilde{\eta}\sqrt{(\chi_\eta - \chi_m)\chi_p}} \left[(1-v^2W - \chi_\eta) \mathbf{K}(1-\epsilon) \right. \\ \left. + (\chi_\eta - \chi_p) \mathbf{\Pi} \left(\frac{\chi_p - \chi_m}{\chi_\eta - \chi_m}, 1-\epsilon \right) \right], \quad (2.11)$$

$$\Delta\phi_1 \equiv p = \frac{2}{\tilde{\eta}\sqrt{(\chi_\eta - \chi_m)\chi_p}} \times \quad (2.12) \\ \left\{ \frac{vW}{(\chi_\eta - 1)(1 - \chi_p)} \left[(\chi_\eta - \chi_p) \mathbf{\Pi} \left(-\frac{(\chi_\eta - 1)(\chi_p - \chi_m)}{(\chi_\eta - \chi_m)(1 - \chi_p)}, 1-\epsilon \right) \right. \right. \\ \left. \left. - (1 - \chi_p) \mathbf{K}(1-\epsilon) \right] - v \mathbf{K}(1-\epsilon) \right\},$$

where \mathbf{K} and $\mathbf{\Pi}$ are the complete elliptic integrals of first and third kind.

The above results are for finite-size giant magnons. If they are of infinite size, one must set $\epsilon = 0$.

The spectrum of a string moving on η -deformed $AdS_5 \times S^5$ was considered in [22]. This was done by treating the corresponding worldsheet theory as integrable field theory. In particular, it was found that the dispersion relation for the infinite-size giant magnons [26] on this background, in the large string tension limit when $g \rightarrow \infty$ is given by (the relation between the string tension T and g is $T = g\sqrt{1 + \tilde{\eta}^2}$)

$$E = \frac{2g\sqrt{1 + \tilde{\eta}^2}}{\tilde{\eta}} \operatorname{arcsinh} \left(\tilde{\eta} \sin \frac{p}{2} \right). \quad (2.13)$$

The result (2.13) has been extended in [17] to the case of finite-size giant magnons. The corresponding dispersion relation is the following

$$E_s - J_1 = 2g\sqrt{1 + \tilde{\eta}^2} \left[\frac{1}{\tilde{\eta}} \operatorname{arcsinh} \left(\tilde{\eta} \sin \frac{p}{2} \right) - \frac{(1 + \tilde{\eta}^2) \sin^3 \frac{p}{2}}{4\sqrt{1 + \tilde{\eta}^2 \sin^2 \frac{p}{2}}} \epsilon \right], \quad (2.14)$$

where

$$\epsilon = 16 \exp \left[- \left(\frac{J_1}{g} + \frac{2\sqrt{1+\tilde{\eta}^2}}{\tilde{\eta}} \operatorname{arcsinh} \left(\tilde{\eta} \sin \frac{p}{2} \right) \right) \sqrt{\frac{1+\tilde{\eta}^2 \sin^2 \frac{p}{2}}{(1+\tilde{\eta}^2) \sin^2 \frac{p}{2}}} \right]. \quad (2.15)$$

3 Giant magnons on η -deformed $AdS_5 \times S^5$ and dilaton operator

The case of finite-size giant magnons and dilaton with zero momentum ($j = 0$) has been considered in [23]. Here we will be interested in the case when $j > 0$.

In [21] it was found that the normalized semiclassical structure constants for the case under consideration are given by

$$\begin{aligned} \mathcal{C}_\eta^{d,j} &= \frac{2\pi^{\frac{3}{2}} c_\Delta^{d,j} \Gamma(2 + \frac{j}{2}) (1 - v^2 \kappa^2)^{\frac{j-1}{2}}}{\Gamma(\frac{5+j}{2}) \sqrt{\kappa^2(1 + \tilde{\eta}^2 \kappa^2)}} \times \\ &\left[(1 - v^2 \kappa^2) F_1 \left(\frac{1}{2}, \frac{2+j}{2}, -\frac{1+j}{2}; 1; \frac{\tilde{\eta}^2(1-v^2)\kappa^2}{1 + \tilde{\eta}^2 \kappa^2}, \frac{(1 + \tilde{\eta}^2)(1-v^2)\kappa^2}{(1 + \tilde{\eta}^2 \kappa^2)(1-v^2 \kappa^2)} \right) \right. \\ &\left. - (1 - \kappa^2) F_1 \left(\frac{1}{2}, \frac{j}{2}, \frac{1-j}{2}; 1; \frac{\tilde{\eta}^2(1-v^2)\kappa^2}{1 + \tilde{\eta}^2 \kappa^2}, \frac{(1 + \tilde{\eta}^2)(1-v^2)\kappa^2}{(1 + \tilde{\eta}^2 \kappa^2)(1-v^2 \kappa^2)} \right) \right], \end{aligned} \quad (3.1)$$

where $F_1(a, b_1, b_2; c; z_1, z_2)$ is one of the hypergeometric functions of two variables (*Appell* F_1).

This expression can be rewritten as (according to (2.9) $W = \kappa^2$ for $\omega_1 = 1$)

$$\begin{aligned} \mathcal{C}_\eta^{d,j} &= \frac{2\pi^{\frac{3}{2}} c_\Delta^{d,j} \Gamma(2 + \frac{j}{2}) (1 - v^2 W)^{\frac{j-1}{2}}}{\Gamma(\frac{5+j}{2}) \sqrt{W(1 + \tilde{\eta}^2 W)}} \times \\ &\left[(1 - v^2 W) F_1 \left(\frac{1}{2}, \frac{2+j}{2}, -\frac{1+j}{2}; 1; \frac{\tilde{\eta}^2(1-v^2)W}{1 + \tilde{\eta}^2 W}, 1 - \epsilon \right) \right. \\ &\left. - (1 - W) F_1 \left(\frac{1}{2}, \frac{j}{2}, \frac{1-j}{2}; 1; \frac{\tilde{\eta}^2(1-v^2)W}{1 + \tilde{\eta}^2 W}, 1 - \epsilon \right) \right], \end{aligned} \quad (3.2)$$

$$\epsilon = \frac{(1 - W)(1 + \tilde{\eta}^2 v^2 W)}{(1 - v^2 W)(1 + \tilde{\eta}^2 W)}. \quad (3.3)$$

In order to find the leading finite-size effect on (3.2) one has to consider the limit $\epsilon \rightarrow 0$. By using the following expansions for the parameters

$$v = v_0 + (v_1 + v_2 \log \epsilon)\epsilon, \quad W = 1 + W_1 \epsilon, \quad (3.4)$$

one obtains [17] (by using (2.5) and (2.12))

$$W_1 = -\frac{(1 - v_0^2)(1 + \tilde{\eta}^2)}{1 + \tilde{\eta}^2 v_0^2}, \quad (3.5)$$

$$v_1 = \frac{v_0(1 - v_0^2) [1 - \log 16 + \tilde{\eta}^2 (2 - v_0^2(1 + \log 16))]}{4(1 + \tilde{\eta}^2 v_0^2)}, \quad (3.6)$$

$$v_2 = \frac{1}{4}v_0(1 - v_0^2), \quad (3.7)$$

where v_0 can be written as

$$v_0 = \frac{\cot \frac{\Delta\phi_1}{2}}{\sqrt{\tilde{\eta}^2 + \csc^2 \frac{\Delta\phi_1}{2}}} = \frac{\cot \frac{p}{2}}{\sqrt{\tilde{\eta}^2 + \csc^2 \frac{p}{2}}}. \quad (3.8)$$

Applying these results for the parameters in the ϵ -expansion of (3.2), one receives the following expression

$$\begin{aligned} \mathcal{C}_\eta^{d,j} &\approx \frac{c_\Delta^{d,j} \pi \Gamma(2 + \frac{j}{2}) \Gamma(\frac{j}{2})}{\Gamma(\frac{5+j}{2}) \Gamma(\frac{1+j}{2})} (1 + \tilde{\eta}^2)^{j/2} \left(\tilde{\eta}^2 + \csc^2 \frac{p}{2} \right)^{-\frac{1}{2}(1+j)} \\ &\left[\frac{(2 + \tilde{\eta}^2(1 - \cos p)) \csc^2 \frac{p}{2}}{\tilde{\eta}^2} \left({}_2F_1 \left(\frac{1}{2}, \frac{j}{2}; \frac{1+j}{2}; \frac{1}{1 + \frac{\csc^2 \frac{p}{2}}{\tilde{\eta}^2}} \right) \right. \right. \\ &\left. \left. - {}_2F_1 \left(-\frac{1}{2}, \frac{j}{2}; \frac{1+j}{2}; \frac{1}{1 + \frac{\csc^2 \frac{p}{2}}{\tilde{\eta}^2}} \right) \right) \right. \\ &\left. + 4 \left(\left(\frac{1}{\tilde{\eta}^2} \left(2(2 + \tilde{\eta}^2(1 - \cos p)) {}_2F_1 \left(\frac{1}{2}, \frac{j}{2}; \frac{1+j}{2}; \frac{1}{1 + \frac{\csc^2 \frac{p}{2}}{\tilde{\eta}^2}} \right) \right. \right. \right. \right. \\ &\left. \left. \left. - (4 + \tilde{\eta}^2(6 - j + 2\tilde{\eta}^2 - (2 + j + 2\tilde{\eta}^2) \cos p)) {}_2F_1 \left(-\frac{1}{2}, \frac{j}{2}; \frac{1+j}{2}; \frac{1}{1 + \frac{\csc^2 \frac{p}{2}}{\tilde{\eta}^2}} \right) \right) \right) \right) \right) \\ &\left. + \left(j(1 + \cos p) {}_2F_1 \left(-\frac{1}{2}, \frac{j}{2}; \frac{1+j}{2}; \frac{1}{1 + \frac{\csc^2 \frac{p}{2}}{\tilde{\eta}^2}} \right) \right) f \right) \exp(-f) \Big], \end{aligned} \quad (3.9)$$

where

$$f = \left(\frac{J_1}{g} + \frac{2\sqrt{1+\tilde{\eta}^2}}{\tilde{\eta}} \operatorname{arcsinh} \left(\tilde{\eta} \sin \frac{p}{2} \right) \right) \sqrt{\frac{1+\tilde{\eta}^2 \sin^2 \frac{p}{2}}{(1+\tilde{\eta}^2) \sin^2 \frac{p}{2}}} \quad (3.10)$$

and ${}_2F_1(a, b; c; z)$ is the Gauss' hypergeometric function.

As one can see from (3.9), the leading finite-size correction to $\mathcal{C}_\eta^{d,j}$ is exponentially small for $J_1 \gg g$.

Now, let us consider two particular cases as examples.

For $j = 1$, (3.9) simplifies to

$$\begin{aligned} \mathcal{C}_\eta^{d,1} \approx & \frac{3c_\Delta^{d,1} \pi \sqrt{1+\tilde{\eta}^2}}{4\tilde{\eta}^2 (\tilde{\eta}^2 + \csc^2 \frac{p}{2})} \left\{ (2 + \tilde{\eta}^2(1 - \cos p)) \csc^2 \frac{p}{2} \times \right. \\ & \left[\mathbf{K} \left(\frac{1}{1 + \frac{\csc^2 \frac{p}{2}}{\tilde{\eta}^2}} \right) - \mathbf{E} \left(\frac{1}{1 + \frac{\csc^2 \frac{p}{2}}{\tilde{\eta}^2}} \right) \right] \\ & + \left[4 \left(2(2 + \tilde{\eta}^2(1 - \cos p)) \mathbf{K} \left(\frac{1}{1 + \frac{\csc^2 \frac{p}{2}}{\tilde{\eta}^2}} \right) \right. \right. \\ & \left. \left. - 4(4 + 5\tilde{\eta}^2 + 2\tilde{\eta}^4 - \tilde{\eta}^2(3 + 2\tilde{\eta}^2) \cos p) \mathbf{E} \left(\frac{1}{1 + \frac{\csc^2 \frac{p}{2}}{\tilde{\eta}^2}} \right) \right) \right] \\ & \left. + 4\tilde{\eta}^2(1 + \cos p) \mathbf{E} \left(\frac{1}{1 + \frac{\csc^2 \frac{p}{2}}{\tilde{\eta}^2}} \right) f \right] \exp(-f) \left. \right\}. \end{aligned} \quad (3.11)$$

For $j = 2$, (3.9) reduces to

$$\begin{aligned}
\mathcal{C}_\eta^{d,2} \approx & \frac{32c_\Delta^{d,2}(1+\tilde{\eta}^2)}{15\tilde{\eta}^3(\tilde{\eta}^2+\csc^2\frac{p}{2})^{3/2}} \left\{ (2+\tilde{\eta}^2(1-\cos p)) \csc^2\frac{p}{2} \times \right. \\
& \left[\sqrt{\tilde{\eta}^2+\csc^2\frac{p}{2}} \operatorname{arctanh}\left(\frac{\tilde{\eta}}{\sqrt{\tilde{\eta}^2+\csc^2\frac{p}{2}}}\right) - \tilde{\eta} {}_2F_1\left(-\frac{1}{2}, 1; \frac{3}{2}; \frac{1}{1+\frac{\csc^2\frac{p}{2}}{\tilde{\eta}^2}}\right) \right] \\
& + 8 \left[\sqrt{\tilde{\eta}^2+\csc^2\frac{p}{2}} (2+\tilde{\eta}^2(1-\cos p)) \operatorname{arctanh}\left(\frac{\tilde{\eta}}{\sqrt{\tilde{\eta}^2+\csc^2\frac{p}{2}}}\right) \right. \\
& \left. - \tilde{\eta}(2+2\tilde{\eta}^2+\tilde{\eta}^4-\tilde{\eta}^2(2+\tilde{\eta}^2)\cos p) {}_2F_1\left(-\frac{1}{2}, 1; \frac{3}{2}; \frac{1}{1+\frac{\csc^2\frac{p}{2}}{\tilde{\eta}^2}}\right) \right. \\
& \left. \left. + \tilde{\eta}^3(1+\cos p) {}_2F_1\left(-\frac{1}{2}, 1; \frac{3}{2}; \frac{1}{1+\frac{\csc^2\frac{p}{2}}{\tilde{\eta}^2}}\right) f \right] \exp(-f) \right\}. \tag{3.12}
\end{aligned}$$

4 Giant magnons on η -deformed $AdS_5 \times S^5$ and primary scalar operators

It was shown in [21] that the normalized semiclassical structure constants for the case at hand are given by

$$\begin{aligned}
\mathcal{C}_\eta^{pr,j} = & \frac{2\pi^{\frac{3}{2}}c_\Delta^{pr,j}\Gamma(\frac{j}{2})(1-v^2\kappa^2)^{\frac{j-1}{2}}}{\Gamma(\frac{1+j}{2})\sqrt{\kappa^2(1+\tilde{\eta}^2\kappa^2)}} \left\{ \left[1 - \frac{(1+jv^2)\kappa^2}{1+j} \right] \times \right. \\
& F_1\left(\frac{1}{2}, \frac{j}{2}, \frac{1-j}{2}; 1; \frac{\tilde{\eta}^2(1-v^2)\kappa^2}{1+\tilde{\eta}^2\kappa^2}, \frac{(1+\tilde{\eta}^2)(1-v^2)\kappa^2}{(1+\tilde{\eta}^2\kappa^2)(1-v^2\kappa^2)}\right) \\
& \left. - (1-v^2\kappa^2)F_1\left(\frac{1}{2}, \frac{2+j}{2}, -\frac{1+j}{2}; 1; \frac{\tilde{\eta}^2(1-v^2)\kappa^2}{1+\tilde{\eta}^2\kappa^2}, \frac{(1+\tilde{\eta}^2)(1-v^2)\kappa^2}{(1+\tilde{\eta}^2\kappa^2)(1-v^2\kappa^2)}\right) \right\}, \tag{4.1}
\end{aligned}$$

or in our notations

$$\begin{aligned}
\mathcal{C}_\eta^{pr,j} = & \frac{2\pi^{\frac{3}{2}}c_\Delta^{pr,j}\Gamma(\frac{j}{2})(1-v^2W)^{\frac{j-1}{2}}}{\Gamma(\frac{1+j}{2})\sqrt{W(1+\tilde{\eta}^2W)}} \left\{ \left[1 - \frac{(1+jv^2)W}{1+j} \right] \times \right. \\
& F_1\left(\frac{1}{2}, \frac{j}{2}, \frac{1-j}{2}; 1; \frac{\tilde{\eta}^2(1-v^2)W}{1+\tilde{\eta}^2W}, 1-\epsilon\right) \\
& \left. - (1-v^2W)F_1\left(\frac{1}{2}, \frac{2+j}{2}, -\frac{1+j}{2}; 1; \frac{\tilde{\eta}^2(1-v^2)W}{1+\tilde{\eta}^2W}, 1-\epsilon\right) \right\}. \tag{4.2}
\end{aligned}$$

Taking the limit $\epsilon \rightarrow 0$ in (4.2) and using (3.4)-(3.8), one finds that to the leading order $\mathcal{C}_\eta^{pr,j}$ becomes

$$\begin{aligned}
\mathcal{C}_\eta^{pr,j} \approx & \frac{c_\Delta^{pr,j} \pi \Gamma(\frac{j}{2})}{\Gamma(\frac{1+j}{2})} \left\{ \left[\frac{2\pi^{1/2} \sqrt{1+\tilde{\eta}^2}}{j^2 \tilde{\eta}^2 (2+\tilde{\eta}^2(1-\cos p))} \left(\frac{1+\tilde{\eta}^2}{\tilde{\eta}^2 + \csc^2 p/2} \right)^{\frac{i-1}{2}} \right. \right. \\
& {}_2F_1 \left(\frac{1}{2}, -\frac{1+j}{2}; 1; 1 \right) \left(2(1+j) {}_2F_1 \left(\frac{3}{2}, \frac{j}{2}; \frac{1+j}{2}; \frac{\tilde{\eta}^2}{\tilde{\eta}^2 + \csc^2 p/2} \right) \right. \\
& \left. \left. - (2+j(2+\tilde{\eta}^2) - j\tilde{\eta}^2 \cos p) {}_2F_1 \left(\frac{1}{2}, \frac{j}{2}; \frac{1+j}{2}; \frac{\tilde{\eta}^2}{\tilde{\eta}^2 + \csc^2 p/2} \right) \right) \right. \\
& + \left[\left(\Gamma \left(\frac{j}{2} - 1 \right) \left(\frac{1+\tilde{\eta}^2}{\tilde{\eta}^2 + \csc^2 p/2} \right)^{\frac{1+j}{2}} (2+\tilde{\eta}^2(1-\cos p)) \right. \right. \\
& \times \left(- (1+\tilde{\eta}^2) \left(4+2j(1-j) + \tilde{\eta}^2 \left(4+2j(1-2j) - \frac{(2-j)j \cot^2 p/2}{\tilde{\eta}^2 + \csc^2 p/2} (1+f) \right) \right) \right) \\
& \times {}_2F_1 \left(-\frac{1}{2}, \frac{j}{2}; \frac{1+j}{2}; \frac{\tilde{\eta}^2}{\tilde{\eta}^2 + \csc^2 p/2} \right) + \left(4(1+\tilde{\eta}^2) + j(2(1-j) \right. \\
& \left. + \tilde{\eta}^2 \left(4(1-j) + \frac{(2-j) \cot^2 p/2}{\tilde{\eta}^2 + \csc^2 p/2} (1-f) \right) + \right. \\
& \left. \left. \frac{\tilde{\eta}^4 \cot^2 p/2 \left(6-4j + \frac{(j-2) \cot^2 p/2}{\tilde{\eta}^2 + \csc^2 p/2} (1+f) \right)}{\tilde{\eta}^2 + \csc^2 p/2} \right) \right) {}_2F_1 \left(\frac{1}{2}, \frac{j}{2}; \frac{1+j}{2}; \frac{\tilde{\eta}^2}{\tilde{\eta}^2 + \csc^2 p/2} \right) \left. \right] \\
& / \left(\Gamma \left(\frac{3+j}{2} \right) \tilde{\eta}^2 (1+\tilde{\eta}^2)^{3/2} \right) + \left(8\Gamma \left(1 + \frac{j}{2} \right) \cos^2 p/2 \left(\frac{1+\tilde{\eta}^2}{\tilde{\eta}^2 + \csc^2 p/2} \right)^{\frac{i+1}{2}} \right. \\
& \times \left(\frac{2 {}_2F_1 \left(\frac{1}{2}, \frac{j}{2}; \frac{1+j}{2}; \frac{\tilde{\eta}^2}{\tilde{\eta}^2 + \csc^2 p/2} \right)}{2+\tilde{\eta}^2(1-\cos p)} - {}_2F_1 \left(-\frac{1}{2}, \frac{j}{2}; \frac{1+j}{2}; \frac{\tilde{\eta}^2}{\tilde{\eta}^2 + \csc^2 p/2} \right) \right) \\
& \left. / \left(\Gamma \left(\frac{3+j}{2} \right) \sqrt{1+\tilde{\eta}^2} \right) f \right] \exp(-f) \left. \right\}.
\end{aligned} \tag{4.3}$$

As in the previously considered case, the leading finite-size correction to $\mathcal{C}_\eta^{pr,j}$ is exponentially small for $J_1 \gg g$.

For the simplest case $j = 1$, (4.3) reduces to

$$\begin{aligned}
\mathcal{C}_\eta^{pr,1} &\approx 2 c_\Delta^{pr,1} \pi \frac{\sqrt{1+\tilde{\eta}^2}}{\tilde{\eta}^2} \left\{ 2 \mathbf{E} \left(\frac{\tilde{\eta}^2}{\tilde{\eta}^2 + \csc^2 p/2} \right) - \frac{2 + \tilde{\eta}^2 \sin^2 p/2}{1 + \tilde{\eta}^2 \sin^2 p/2} \right. \\
&\times \mathbf{K} \left(\frac{\tilde{\eta}^2}{\tilde{\eta}^2 + \csc^2 p/2} \right) + \csc^2 p/2 \left[\left(2(1 + \tilde{\eta}^2 \sin^2 p/2)(8 + 7\tilde{\eta}^2 + 2\tilde{\eta}^4 \right. \right. \\
&- \tilde{\eta}^2(5 + 2\tilde{\eta}^2) \cos p) \mathbf{E} \left(\frac{\tilde{\eta}^2}{\tilde{\eta}^2 + \csc^2 p/2} \right) - (16 + 18\tilde{\eta}^2 + 7\tilde{\eta}^4 \\
&- 2\tilde{\eta}^2(7 + 4\tilde{\eta}^2) \cos p + \tilde{\eta}^4 \cos 2p) \mathbf{K} \left(\frac{\tilde{\eta}^2}{\tilde{\eta}^2 + \csc^2 p/2} \right) / (\tilde{\eta}^2 + \csc^2 p/2)^2 \\
&+ \left. \left(8\tilde{\eta}^2 \cos^2 p/2 \left(\mathbf{K} \left(\frac{\tilde{\eta}^2}{\tilde{\eta}^2 + \csc^2 p/2} \right) - (1 + \tilde{\eta}^2 \sin^2 p/2) \mathbf{E} \left(\frac{\tilde{\eta}^2}{\tilde{\eta}^2 + \csc^2 p/2} \right) \right) \right) \right. \\
&\left. \times \sin^4 p/2 \right] / (1 + \tilde{\eta}^2 \sin^2 p/2)^2 f \left. \right\} \exp(-f). \tag{4.4}
\end{aligned}$$

5 Concluding Remarks

In this letter, in the framework of the semiclassical approach, we computed the leading finite-size effects on the normalized structure constants in some three-point correlation functions in the η -deformed $AdS_5 \times S^5$, expressed in terms of the conserved string angular momentum J_1 and the worldsheet momentum p . Namely, we found the leading finite-size corrections on the structure constants in three-point correlators of two heavy giant magnons' string states and the following two light states:

1. Dilaton operator with nonzero momentum ($j \geq 1$)
2. Primary scalar operators.

It is interesting to see what happens when we take the limit $\tilde{\eta} \rightarrow 0$ when the deformation disappears, and the dual field theory is known - $\mathcal{N} = 4$ SYM.

For the case of dilaton operator with nonzero momentum j one finds

$$\begin{aligned}
\mathcal{C}^{d,j} &\approx \frac{2c_\Delta^j \pi \Gamma(2 + j/2) \Gamma(j/2) (\sin p/2)^{1+j}}{(1 + j) \Gamma(\frac{1+j}{2}) \Gamma(\frac{5+j}{2})} \times \\
&[j - (2(4 + (1 - j)j(1 + \cos p)) - 2j(1 + j)(1 + \cos p)f) \exp(-f)], \tag{5.1}
\end{aligned}$$

where now (3.10) becomes

$$f = \frac{J_1}{g \sin p/2} + 2 = \frac{2\pi J_1}{\sqrt{\lambda} \sin p/2} + 2, \tag{5.2}$$

and λ is the t'Hooft coupling constant (the string tension T and the t'Hooft coupling in $\mathcal{N} = 4$ SYM are related to each other as $T = \frac{\sqrt{\lambda}}{2\pi}$).

For the case of primary scalar operators we derive

$$\mathcal{C}^{pr,j} \approx 16\pi c_{\Delta}^{pr,j} \frac{\Gamma(j/2-1)\Gamma(j/2)}{\Gamma(\frac{j-1}{2})\Gamma(\frac{j+3}{2})} \sin^{j+1}\left(\frac{p}{2}\right) \exp(-f), \quad (5.3)$$

where f is the same as in (5.2).

A possible issue to solve is to try to extend the results obtained here to the case of dyonic giant magnon states with two nonzero conserved angular momenta J_1 and J_2 . We hope to report on this in the near future.

Acknowledgements

This work is partially supported by the NSFB grant DFNI T02/6.

References

- [1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity”, *Adv. Theor. Math. Phys.* **2**, 231 (1998) [arXiv:hep-th/9711200];
S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory”, *Phys. Lett.* **B428**, 105 (1998) [arXiv:hep-th/9802109];
E. Witten, “Anti-de Sitter space and holography”, *Adv. Theor. Math. Phys.* **2**, 253 (1998) [arXiv:hep-th/9802150].
- [2] N. Beisert et al., “Review of AdS/CFT Integrability: An Overview”, *Lett. Math. Phys.* **99** 3 (2012), arXiv:1012.3982v5[hep-th].
- [3] Romuald A. Janik, Piotr Surowka, Andrzej Wereszczynski, “On correlation functions of operators dual to classical spinning string states”, *JHEP* 1005:030,2010, arXiv:1002.4613 [hep-th]
- [4] E.I. Buchbinder, A.A. Tseytlin, “On semiclassical approximation for correlators of closed string vertex operators in AdS/CFT”, *JHEP* 1008:057,2010, arXiv:1005.4516 [hep-th]
- [5] K. Zarembo, “Holographic three-point functions of semiclassical states”, *JHEP* 1009:030,2010, arXiv:1008.1059 [hep-th]
- [6] Miguel S. Costa, Ricardo Monteiro, Jorge E. Santos, Dimitrios Zoakos, “On three-point correlation functions in the gauge/gravity duality”, *JHEP* 1011:141,2010, arXiv:1008.1070 [hep-th]

- [7] R. Roiban and A. A. Tseytlin, “On semiclassical computation of 3-point functions of closed string vertex operators in $AdS_5 \times S^5$ ”
Phys. Rev. **D82** 106011 (2010) [arXiv:hep-th/1008.4921].
- [8] Oleg Lunin, Juan Maldacena, “Deforming field theories with $U(1) \times U(1)$ global symmetry and their gravity duals”, JHEP 0505 (2005) 033, arXiv:hep-th/0502086.
- [9] Sergey Frolov, “Lax Pair for Strings in Lunin-Maldacena Background”, JHEP 0505:069,2005, arXiv:hep-th/0503201.
- [10] D. Arnaudov, R.C. Rashkov, “Three-point correlators: Examples from Lunin-Maldacena background”, Phys. Rev. D 84, 086009 (2011), arXiv:1106.4298 [hep-th].
- [11] Changrim Ahn, Plamen Bozhilov, “Three-point Correlation Function of Giant Magnons in the Lunin-Maldacena background”, Phys.Rev. D84 (2011) 126011, arXiv:1106.5656 [hep-th].
- [12] F. Delduc, M. Magro, B. Vicedo, “An integrable deformation of the $AdS_5 \times S^5$ superstring action”, Phys. Rev. Lett. 112, 051601 (2014), arXiv:1309.5850 [hep-th]
- [13] C. Klimcik, “Yang-Baxter σ -models and dS/AdS T-duality”, JHEP 0212:051,2002, arXiv:hep-th/0210095
- [14] C. Klimcik, “On integrability of the Yang-Baxter σ -model”, J.Math.Phys.50:043508,2009, arXiv:0802.3518 [hep-th]
- [15] C. Klimcik, “ η and λ deformations as \mathcal{E} -models”, Nucl.Phys. B900 (2015) 259, arXiv:1508.05832 [hep-th]
- [16] G. Arutyunov, R. Borsato, S. Frolov, “ S -matrix for strings on η -deformed $AdS_5 \times S^5$ ”, JHEP 1404 (2014) 002, arXiv:1312.3542 [hep-th]
- [17] Changim Ahn and Plamen Bozhilov, “Finite-size giant magnons on η -deformed $AdS_5 \times S^5$ ”, Phys. Lett. **B737** 293 (2014), arXiv:1406.0628[hep-th].
- [18] Aritra Banerjee, Kamal L. Panigrahi, “On the Rotating and Oscillating strings in $(AdS_3S^3)_\kappa$ ”, JHEP 1409 (2014) 048, arXiv:1406.3642 [hep-th]
- [19] Takashi Kameyama, Kentaroh Yoshida, “A new coordinate system for q-deformed $AdS_5 \times S^5$ and classical string solutions”, J.Phys. A48 (2015) 7, 075401, arXiv:1408.2189 [hep-th]
- [20] Aritra Banerjee, Soumya Bhattacharya, Kamal L. Panigrahi, “Spiky strings in κ -deformed AdS”, JHEP 1506 (2015) 057, arXiv:1503.07447 [hep-th]
- [21] Plamen Bozhilov, “Some three-point correlation functions in the η -deformed $AdS_5 \times S^5$ ”, to appear in IJMPA.

- [22] G. Arutyunov, M. de Leeuw, S. van Tongeren “On the exact spectrum and mirror duality of the $(AdS_5 \times S^5)_\eta$ superstring”, arXiv:1403.6104 [hep-th]
- [23] Changrim Ahn, Plamen Bozhilov, “A HHL 3-point correlation function in the η -deformed $AdS_5 \times S^5$ ”, Phys. Lett. B743 (2015) 121-126, arXiv:1412.6668 [hep-th]
- [24] M. Kruczenski, J. Russo, A.A. Tseytlin, “Spiky strings and giant magnons on S^5 ”, JHEP **0610** 002 (2006), [arXiv:hep-th/0607044].
- [25] Gleb Arutyunov, Daniel Medina-Rincon, “Deformed Neumann model from spinning strings on $(AdS_5 \times S^5)_\eta$ ”, JHEP 1410 (2014) 50, arXiv:1406.2536 [hep-th]
- [26] D.M. Hofman and J. Maldacena, “Giant magnons”, J. Phys. **A 39** 13095-13118 (2006), [arXiv:hep-th/0604135]