Multiple periodic solutions for two classes of nonlinear difference systems involving classical (ϕ_1, ϕ_2)-Laplacian

Xingyong Zhang [*], Liben Wang

Department of Mathematics, Faculty of Science, Kunming University of Science and Technology,

Kunming, Yunnan, 650500, P.R. China

Abstract: In this paper, we investigate the existence of multiple periodic solutions for two classes of nonlinear difference systems involving (ϕ_1, ϕ_2)-Laplacian. First, by using an important critical point theorem due to B. Ricceri, we establish an existence theorem of three periodic solutions for the first nonlinear difference system with (ϕ_1, ϕ_2)-Laplacian and two parameters. Moreover, for the second nonlinear difference system with (ϕ_1, ϕ_2)-Laplacian, by using the Clark's Theorem, we obtain a multiplicity result of periodic solutions under a symmetric condition. Finally, two examples are given to verify our theorems.

Keywords: Difference equations; Periodic solutions; Multiplicity; Variational approach

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1. Introduction and main results

Let $\mathbb{R}$ denote the real numbers, $\mathbb{Z}$ the integers, Given $a < b$ in $\mathbb{Z}$. Let $\mathbb{Z}[a, b] = \{a, a+1, ..., b\}$. Let $T > 1$ and N be fixed positive integers.

Firstly, in this paper, we are concerned with the existence of three periodic solutions for the following nonlinear difference system:

$$\begin{cases} \mu\Delta\Big[\rho_1(t-1)\phi_1\big(\Delta u_1(t-1)\big)\Big] - \mu\rho_3(t)\phi_3(u_1(t)) + \nabla_{u_1}W\big(t, u_1(t), u_2(t)\big) = 0 \\ \mu\Delta\Big[\rho_2(t-1)\phi_2\big(\Delta u_2(t-1)\big)\Big] - \mu\rho_4(t)\phi_4(u_2(t)) + \nabla_{u_2}W\big(t, u_1(t), u_2(t)\big) = 0, \end{cases} \tag{1.1}$$

where $\mu \in \mathbb{R}$, $\rho_i : \mathbb{R} \to \mathbb{R}^+$, ϕ_i, $i = 1, 2, 3, 4$ satisfy the following conditions:

(ρ) *ρ_i are T-periodic and $\min_{t\in\mathbb{Z}[1,T]} \rho_i(t) > 0$, $i = 1, 2, 3, 4$;*

$(\mathcal{A}1)$ *$\phi_i : \mathbb{R}^N \to \mathbb{R}^N$ are homeomorphisms such that $\phi_i(0) = 0$, $\phi_i = \nabla\Phi_i$, with $\Phi_i \in C^1(\mathbb{R}^N, [0, +\infty))$ strictly convex and $\Phi_i(0) = 0$, $i = 1, 2, 3, 4$.*

Remark Assumption $(\mathcal{A}1)$ is given in [2], where it is used to characterize the classical homeomorphism.

*E-mail address: zhangxingyong1@163.com

Moreover, assume that

($\mathcal{A}2$) *$W(t,x_1,x_2) = F(t,x_1,x_2) - \lambda G(t,x_1,x_2) + \nu H(t,x_1,x_2)$, where $\lambda, \nu \in \mathbb{R}$, $F, G, H : \mathbb{Z}\times\mathbb{R}^N\times\mathbb{R}^N \longrightarrow \mathbb{R}^N, (t,x_1,x_2) \longrightarrow F(t,x_1,x_2)$, $(t,x_1,x_2) \longrightarrow G(t,x_1,x_2)$, $(t,x_1,x_2) \longrightarrow H(t,x_1,x_2)$ are T-periodic in t for all $(x_1,x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ and continuously differentiable in (x_1,x_2) for every $t \in \mathbb{Z}[1,T]$.*

It is well known that variational methods have been important tools to study the existence and multiplicity of solutions for various difference systems. Lots of contributions has been obtained (for example, see [2], [5]-[16], [18]). Recently, in [2] and [30], by using a variational approach, Mawhin investigated the following second order nonlinear difference systems with ϕ-Laplacian:

$$\Delta\phi[\Delta u(n-1)] = \nabla_u F[n, u(n)] + h(n) \quad (n \in \mathbb{Z}), \tag{1.2}$$

where $\phi = \nabla\Phi$, Φ strictly convex, is a homeomorphism of $\mathbb{R}^N$ onto the ball $B_a \subset \mathbb{R}^N$ or of B_a onto $\mathbb{R}^N$. The assumption about ϕ implies three cases: firstly, classical homeomorphism if $\phi : \mathbb{R}^N \to \mathbb{R}^N$, for example, $\phi(0) = 0$, $\phi(x) = |x|^{p-2}x$ for some $p > 1$ and all $x \in \mathbb{R}^N/\{0\}$; secondly, bounded homeomorphism if $\phi : \mathbb{R}^N \to B_a$ $(a < +\infty)$, for example, $\phi(x) = \frac{x}{\sqrt{1+|x|^2}} \in B_1$ for all $x \in \mathbb{R}^N$; finally, singular homeomorphism if $\phi : B_a \subset \mathbb{R}^N \to \mathbb{R}^N$, for example, $\phi(x) = \frac{x}{\sqrt{1-|x|^2}}$ for all $x \in B_1$. Under some reasonable assumptions, by using variational approach, Mawhin obtained system (1.2) has at least one T-periodic solution or $N+1$ geometrically distinct T-periodic solutions.

However, to the best of our knowledge, except for recent works in [25] and [26] which are made by our first author and his cooperator named Yun Wang, there are no people to investigate the existence and multiplicity of solutions for system involving classical (ϕ_1, ϕ_2)-Laplacian. In [25], Wang and our first author investigated the multiplicity of T-periodic solutions for the following nonlinear difference system:

$$\begin{cases} \Delta\phi_1\big(\Delta u_1(t-1)\big) = \nabla_{u_1} F\big(t, u_1(t), u_2(t)\big) + h_1(t) \\ \Delta\phi_2\big(\Delta u_2(t-1)\big) = \nabla_{u_2} F\big(t, u_1(t), u_2(t)\big) + h_2(t), \end{cases} \tag{1.3}$$

where $F : \mathbb{Z}\times\mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}$ and $\phi_m, m = 1,2$ satisfy the following condition:

($\mathcal{A}$) ϕ_i is a homeomrphism from $\mathbb{R}^N$ onto $B_a \subset \mathbb{R}^N (a \in (0,+\infty])$, such that $\phi_i(0) = 0, \phi_i = \nabla\Phi_i$, with $\Phi_i \in C^1(\mathbb{R}^N, [0,+\infty])$ strictly convex and $\Phi_i(0) = 0$, $m = 1,2$.

Assumption ($\mathcal{A}$) implies that Φ_i, $i = 1,2$ are the classical homeomorphisms or the bounded homeomorphisms. They investigated the case that $F(t,x_1,x_2)$ is periodic on r_1 components of variables $x_1^{(1)}, \cdots, x_N^{(1)}$ and r_2 components of variables $x_1^{(2)}, \cdots, x_N^{(2)}$, where $1 \le r_1 \le N$ and $1 \le r_2 \le N$. By using a critical point theorem in [1] and a generalized saddle point theorem in [27], they obtain that system (1.3) has at least $r_1 + r_2 + 1$ geometrically distinct T-periodic solutions. Their results generalize those corresponding to classical homeomorphism and bounded homeomorphism in [30].

In [26], our first author and Wang investigated the existence of homoclinic solutions for the following nonlinear difference systems involving classical (ϕ_1, ϕ_2)-Laplacian:

$$\begin{cases} \Delta\phi_1\big(\Delta u_1(t-1)\big) + \nabla_{u_1} V\big(t, u_1(t), u_2(t)\big) = f_1(t) \\ \Delta\phi_2\big(\Delta u_2(t-1)\big) + \nabla_{u_2} V\big(t, u_1(t), u_2(t)\big) = f_2(t), \end{cases} \tag{1.4}$$

where $t \in \mathbb{Z}$, $u_m(t) \in \mathbb{R}^N$, $m = 1, 2$, $V(t, x_1, x_2) = -K(t, x_1, x_2) + W(t, x_1, x_2)$, $K, W : \mathbb{Z} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ and $\phi_m, m = 1, 2$ satisfy assumption $(\mathcal{A}1)$. They first improve some inequalities in [29]. Then by using a linking theorem in [28], some new existence results of homoclinic solutions for system (1.4) are obtained when W has super p-linear growth and K has sub p-linear growth.

Inspired by [2], [3], [25], [26] and [30], in this paper, we are interested in the existence of three T-periodic solutions for system (1.1). By using an important three critial point theorem established by B. Ricceri in [3], we investigate the existence of three T-periodic solutions for system (1.1), as stated in the following.

Define

$$\begin{aligned} I(u) &= \sum_{t=1}^{T} \left[\rho_1(t)\Phi_1(\Delta u_1(t)) + \rho_2(t)\Phi_2(\Delta u_2(t)) + \rho_3(t)\Phi_3(u_1(t)) + \rho_4(t)\Phi_4(u_2(t))\right], \\ \Psi(u) &= -\sum_{t=1}^{T} F(t, u_1(t), u_2(t)), \quad \Phi(u) = \sum_{t=1}^{T} G(t, u_1(t), u_2(t)), \\ \Gamma(u) &= -\sum_{t=1}^{T} H(t, u_1(t), u_2(t)), \qquad u \in E, \end{aligned}$$

where the definitions of E and its norm are in section 2 below.

Theorem 1.1. *Suppose that (ρ), $(\mathcal{A}1)$, $(\mathcal{A}2)$ and the following conditions hold:*

$(\mathcal{A}3)$ there exist positive constants c_i $(i = 1, 2, 3, 4)$, $\theta > 1$ such that

$$(\phi_i(x) - \phi_i(y), x - y) \geq c_i|x - y|^{\theta}, \forall\ x, y \in \mathbb{R}^N, i = 1, 2, 3, 4,$$

where $(\cdot, \cdot)$ stands for the usual product in $\mathbb{R}^N$;

$(\mathcal{A}4)$ $\lim_{|x|\to\infty} \Phi_i(x) = +\infty$ and there exist positive constants $l \geq \theta$, d_i and m_i such that $\Phi_i(x) \leq d_i|x|^l + m_i$ for all $x \in \mathbb{R}^N$, $(i = 1, 2, 3, 4)$;

$(\mathcal{A}5)$ for all $t \in \mathbb{Z}[1, T]$ and all $\lambda > 0$, there exists $C_0(\lambda) \in \mathbb{R}$ such that for all $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$,

$$\lim_{|x_1|+|x_2|\to\infty} \frac{F(t, x_1, x_2)}{|x_1|^l + |x_2|^l} = +\infty, \quad \lambda G(t, x_1, x_2) \geq F(t, x_1, x_2) + C_0(\lambda);$$

$(\mathcal{A}6)$ $\sum_{t=1}^{T} G(t, 0, 0) = 0$.

Then for each $r > 0$, for each $\mu > \max\{0, \mu^*(I, \Psi, \Phi, r)\}$, and for each compact interval $[a, b] \subset]0, \beta(\mu I + \Psi, \Phi, r)[$, there exists a number $\rho > 0$ with the following property: for every $\lambda \in [a, b]$, there exists $\delta > 0$ such that, for each $\nu \in [0, \delta]$, system (1.1) has at least three T-periodic solutions in E whose norms are less than ρ, where

$$\beta(\mu I + \Psi, \Phi, r) = \sup_{u\in\Phi^{-1}(]r,+\infty[)} \frac{\mu I(u) + \Psi(u) - \inf\limits_{\Phi^{-1}(]-\infty,r])}(\mu I + \Psi)}{r - \Phi(u)}$$

$$\mu^*(I,\Psi,\Phi,r)=\inf\left\{\frac{\Psi(u)-\gamma+r}{\eta_r-I(u)}:u\in E,\Phi(u)<r,I(u)<\eta_r\right\}$$
$$\gamma=\inf_E(\Psi(u)+\Phi(u)),\quad \eta_r=\inf_{u\in\Phi^{-1}(r)}I(u).$$

Inspired by [3], we have the following corollary:

Corollary 1.1. *Suppose that (ρ), $(\mathcal{A}1)$-$(\mathcal{A}4)$ and $(\mathcal{A}6)$ hold. If*

$(\mathcal{A}5)'$ there exists $s>l$ such that for every $t\in\mathbb{Z}[1,T]$,

$$\lim_{|x_1|+|x_2|\to\infty}\frac{F(t,x_1,x_2)}{|x_1|^l+|x_2|^l}=+\infty,\qquad \lim_{|x_1|+|x_2|\to\infty}\frac{F(t,x_1,x_2)}{|x_1|^s+|x_2|^s}<+\infty$$

and

$$\lim_{|x_1|+|x_2|\to\infty}\frac{G(t,x_1,x_2)}{|x_1|^s+|x_2|^s}=+\infty,$$

then the conclusion of Theorem 1.1 holds.

Remark 1.1. *There exist examples satisfying $(\mathcal{A}1)$-$(\mathcal{A}6)$ in Theorem 1.1. For example, let $T>1$ and N be fixed integer. Let $\theta\geq 2$ and $q_i\geq 2$, $i=1,2,3,4$. Assume that $\phi_1(y)=|y|^{\theta-2}y+|y|^{q_1-2}y$, $\phi_2(y)=|y|^{\theta-2}y+|y|^{q_2-2}y$, $\phi_3(y)=|y|^{\theta-2}y+|y|^{q_3-2}y$, $\phi_4(y)=|y|^{\theta-2}y+|y|^{q_4-2}y$, ρ_i are T-periodic and satisfy $\rho_i>0$ for all $t\in\mathbb{Z}[1,T]$, $i=1,2,3,4$. Then $\Phi_1(y)=\frac{|y|^\theta}{\theta}+\frac{|y|^{q_1}}{q_1}$, $\Phi_2(y)=\frac{|y|^\theta}{\theta}+\frac{|y|^{q_2}}{q_2}$, $\Phi_3(y)=\frac{|y|^\theta}{\theta}+\frac{|y|^{q_3}}{q_3}$, $\Phi_4(y)=\frac{|y|^\theta}{\theta}+\frac{|y|^{q_4}}{q_4}$.*

Note that

$$(|x|^{\theta-2}x-|y|^{\theta-2}y,x-y)\geq c|x-y|^\theta$$

for all $x,y\in\mathbb{R}^N$, $\theta\geq 2$ and some $c>0$ (see [17]). Hence,

$$\begin{aligned}&(\phi_1(x)-\phi_1(y),x-y)\\ =\;&(|x|^{\theta-2}x+|x|^{q_1-2}x-|y|^{\theta-2}y-|y|^{q_1-2}y,x-y)\\ =\;&(|x|^{\theta-2}x-|y|^{\theta-2}y,x-y)+(|x|^{q_1-2}x-|y|^{q_1-2}y,x-y)\\ \geq\;&(|x|^{\theta-2}x-|y|^{\theta-2}y,x-y)\\ \geq\;&c|x-y|^\theta,\quad\forall\ x,y\in\mathbb{R}^N.\end{aligned}$$

Similarly, we have

$$(\phi_i(x)-\phi_i(y),x-y)\geq c_i|x-y|^\theta,\quad\forall\ x,y\in\mathbb{R}^N,i=2,3,4$$

for some $c_i>0$, $i=2,3,4$. So $(\mathcal{A}3)$ holds.

Take $l=\max\{\theta,q_1,q_2,q_3,q_4\}$ and let

$$\begin{aligned}F(t,x_1,x_2)&=\left[(\cos^2\frac{\pi t}{T}+2)|x_1|^l+(|\cos\frac{\pi t}{T}|+2)|x_2|^l\right]\ln(|x_1|^2+|x_2|^2+1)\\ G(t,x_1,x_2)&=(|\sin\frac{\pi t}{T}|+2)(|x_1|^l+|x_2|^l)^2\ln(|x_1|^2+|x_2|^2+1)\\ H(t,x_1,x_2)&=(\cos^2\frac{\pi t}{T}+2)\sin(|x_1|^2+|x_2|^2+2)\end{aligned}$$

$$W(t,x_1,x_2) \quad = \quad F(t,x_1,x_2) - \lambda G(t,x_1,x_2) + \nu H(t,x_1,x_2).$$

Then it is easy to obtain that ($\mathcal{A}2$) and ($\mathcal{A}6$) hold and Φ_i satisfy ($\mathcal{A}1$) and ($\mathcal{A}4$), $i = 1,2,3,4$. Moreover,

$$\lim_{|x_1|+|x_2|\to\infty} \frac{F(t,x_1,x_2)}{|x_1|^l + |x_2|^l} \geq 2 \lim_{|x_1|+|x_2|\to\infty} \ln(|x_1|^2 + |x_2|^2 + 1) = +\infty$$

and for all $\lambda > 0$,

$$\begin{aligned}
\lim_{|x_1|+|x_2|\to\infty} \frac{\lambda G(t,x_1,x_2)}{F(t,x_1,x_2)} &= \lim_{|x_1|+|x_2|\to\infty} \frac{\lambda\left(|\sin\frac{\pi t}{T}| + 2\right)(|x_1|^l + |x_2|^l)^2}{\left[(\cos^2\frac{\pi t}{T} + 2)|x_1|^l + (|\cos\frac{\pi t}{T}| + 2)|x_2|^l\right]} \\
&\geq \lim_{|x_1|+|x_2|\to\infty} \frac{2\lambda(|x_1|^l + |x_2|^l)^2}{3\left(|x_1|^l + |x_2|^l\right)} \\
&= \lim_{|x_1|+|x_2|\to\infty} \frac{2\lambda}{3}\left(|x_1|^l + |x_2|^l\right) = +\infty.
\end{aligned}$$

Hence, ($\mathcal{A}5$) holds.

Moreover, in this paper, we are also concerned with the multiplicity of T-periodic solutions for the following nonlinear difference system:

$$\begin{cases} \Delta\left(\gamma_1(t-1)\phi_1\big(\Delta u_1(t-1)\big)\right) - \gamma_3(t)\phi_3(|u_1(t)|) + \nabla_{u_1} F\big(t,u_1(t),u_2(t)\big) = 0 \\ \Delta\left(\gamma_2(t-1)\phi_2\big(\Delta u_2(t-1)\big)\right) - \gamma_4(t)\phi_4(|u_2(t)|) + \nabla_{u_2} F\big(t,u_1(t),u_2(t)\big) = 0, \end{cases} \tag{1.5}$$

where $\gamma_i : \mathbb{R} \to \mathbb{R}^+$ satisfy the following conditions:

(γ) γ_i *are T-periodic and* $\min_{t\in\mathbb{Z}[1,T]} \gamma_i(t) > 0$, $i = 1,2,3,4$,

and ϕ_i, $i = 1,2,3,4$ satisfy the assumption ($\mathcal{A}1$) and the following condition:

(ϕ) *there exist positive constants $p > 1, q > 1$, a_i, b_i, $i = 1,2,3,4$ such that*

$$a_i|x|^q \leq \Phi_i(x) \leq b_i|x|^q, i = 1,3, \quad \forall\, x \in \mathbb{R}^N$$

and

$$a_i|x|^p \leq \Phi_i(x) \leq b_i|x|^p, i = 2,4, \quad \forall\, x \in \mathbb{R}^N.$$

Moreover, $F : \mathbb{Z}\times\mathbb{R}^N\times\mathbb{R}^N \longrightarrow \mathbb{R}, (t,x_1,x_2) \longrightarrow F(t,x_1,x_2)$ is T-periodic in t for all $(x_1,x_2) \in \mathbb{R}^N\times\mathbb{R}^N$ and continuously differentiable in (x_1,x_2) for every $t \in \mathbb{Z}[1,T]$.

When $\Phi_i(x) = \frac{1}{q}|x|^q$, $i = 1,3$ and $\Phi_i(x) = \frac{1}{p}|x|^p$, $i = 2,4$, system (1.5) can be seen as a discrete analogue of the following (q,p)-Laplacian differential systems:

$$\begin{cases} \dfrac{d\left(\gamma_1(t)|\dot u_1(t)|^{q-2}\dot u_1(t)\right)}{dt} - \gamma_3(t)|u_1(t)|^{q-2}u_1(t) + \nabla_{u_1} F\big(t,u_1(t),u_2(t)\big) = 0 \\ \dfrac{d\left(\gamma_2(t)|\dot u_2(t)|^{p-2}\dot u_2(t)\right)}{dt} - \gamma_4(t)|u_2(t)|^{p-2}u_2(t) + \nabla_{u_2} F\big(t,u_1(t),u_2(t)\big) = 0. \end{cases} \tag{1.6}$$

Recently, by using variational methods, system (1.6) has been investigated by some authors (for example, see [19]-[23]) and some interesting results on the existence and multiplicity of solutions have been obtained. However, to the best of our knowledge, there are no people to investigate the nonlinear difference system

(1.5). In this paper, inspired by [18]-[23], we are interested in the existence and multiplicity of T-periodic solutions for system (1.5). By using the Clark's theorem, we obtain the following theorem.

Theorem 1.2. *Suppose that (γ), (ϕ) and the following conditions hold:*

($\mathcal{F}0$) there exist $\alpha_1 \in [0, q)$, $\alpha_2 \in [0, p)$, $h_i : \mathbb{Z}[1, T] \to \mathbb{R}^+, i = 1, 2$ *and* $l : \mathbb{Z}[1, T] \to \mathbb{R}^+$ *such that*

$$F(t, x_1, x_2) \leq h_1(t)|x_1|^{\alpha_1} + h_2(t)|x_2|^{\alpha_2} + l(t).$$

($\mathcal{F}1$) $F(t, 0, 0) = 0$;

($\mathcal{F}2$) $F(t, -x_1, -x_2) = F(t, x_1, x_2)$;

($\mathcal{F}3$) there exist constants $\beta_i \in (1, \min\{q, p\})$, $M_i \in (0, \infty)$, $i = 1, 2$ *and* $\delta \in (0, 1)$ *such that*

$$F(t, x_1, x_2) \geq M_1|x_1|^{\beta_1} + M_2|x_2|^{\beta_2}, \quad \forall\ |x_1| < \delta, |x_2| < \delta.$$

Then system (1.5) has at least $2NT$ *distinct pairs of nonzero solutions.*

2. Preliminaries

At first, we make some preliminaries. Define

$$E_T = \{h := \{h(t)\}_{t\in\mathbb{Z}} | h(t + T) = h(t), h(t) \in \mathbb{R}^N, t \in \mathbb{Z}\}$$

and let $E = E_T \times E_T$. For $h \in E_T$, set

$$\|h\|_r = \left(\sum_{t=1}^{T} |h(t)|^r\right)^{1/r} \quad \text{and} \quad \|h\|_\infty = \max_{t\in\mathbb{Z}[1,T]} |h(t)|, \quad r > 1. \tag{2.1}$$

Obviously, we have

$$\|h\|_\infty \leq \|h\|_r \leq T^{1/r}\|h\|_\infty. \tag{2.2}$$

On E_T, we define

$$\|h\|_{E_T} = \left(\sum_{t=1}^{T} |\Delta h(t)|^\theta + \sum_{t=1}^{T} |h(t)|^\theta\right)^{1/\theta}$$

and

$$\|h\|_{[E_T]} = \left(\sum_{t=1}^{T} |\Delta h(t)|^l + \sum_{t=1}^{T} |h(t)|^l\right)^{1/l}$$

For $u = (u_1, u_2) \in E$, define

$$\|u\| = \|u_1\|_{E_T} + \|u_2\|_{E_T}.$$

Then E is a separable and reflexive Banach space. Moreover, $\|\cdot\|_{E_T}$ is equivalent to $\|\cdot\|_r (r > 1)$ and $\|\cdot\|_{[E_T]}$. Hence, there exist positive constants C_i $(i = 1, \cdots, 6)$ such that

$$C_1\|\cdot\|_{E_T} \leq \|\cdot\|_\theta \leq C_2\|\cdot\|_{E_T}, \tag{2.3}$$

$$C_3\|\cdot\|_{E_T}\le\|\cdot\|_l\le C_4\|\cdot\|_{E_T}, \tag{2.4}$$

$$C_5\|\cdot\|_{E_T}\le\|\cdot\|_{[E_T]}\le C_6\|\cdot\|_{E_T}. \tag{2.5}$$

Lemma 2.1 (see [25]) *Let* $L:\mathbb{Z}[1,T]\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N\longrightarrow\mathbb{R},(t,x_1,x_2,y_1,y_2)\longrightarrow L(t,x_1,x_2,y_1,y_2)$ *and assume that* L *is continuously differentiable in* (x_1,x_2,y_1,y_2) *for all* $t\in\mathbb{Z}[1,T]$*. Then the functional* $\varphi:E\to\mathbb{R}$ *defined by*

$$\varphi(u)=\varphi(u_1,u_2)=\sum_{t=1}^{T}L\big(t,u_1(t),u_2(t),\Delta u_1(t),\Delta u_2(t)\big)$$

is continuously differentiable on E *and for* $u,v\in E$,

$$\begin{aligned}&\langle\varphi'(u),v\rangle=\langle\varphi'(u_1,u_2),(v_1,v_2)\rangle\\ =&\sum_{t=1}^{T}\big[\big(D_{x_1}L(t,u_1(t),u_2(t),\Delta u_1(t),\Delta u_2(t)),v_1(t)\big)\\ &+\big(D_{y_1}L(t,u_1(t),u_2(t),\Delta u_1(t),\Delta u_2(t)),\Delta v_1(t)\big)\\ &+\big(D_{x_2}L(t,u_1(t),u_2(t),\Delta u_1(t),\Delta u_2(t)),v_2(t)\big)\\ &+\big(D_{y_2}L(t,u_1(t),u_2(t),\Delta u_1(t),\Delta u_2(t)),\Delta v_2(t)\big)\big].\end{aligned}$$

Let

$$\begin{aligned}L(t,x_1,x_2,y_1,y_2)&=\mu[\rho_1(t)\Phi_1(y_1)+\rho_2(t)\Phi_2(y_2)+\rho_3(t)\Phi_3(x_1)+\rho_4(t)\Phi_4(x_2)]\\&\quad-F(t,x_1,x_2)+\lambda G(t,x_1,x_2)-\nu H(t,x_1,x_2),\end{aligned}$$

where F,G,H: $\mathbb{Z}[1,T]\times\mathbb{R}^N\times\mathbb{R}^N\longrightarrow\mathbb{R}^N$ are continuously differentiable in $(x_1,x_2)\in\mathbb{R}^N\times\mathbb{R}^N$ for all $t\in\mathbb{Z}[1,T]$. Then

$$\begin{aligned}\varphi(u)=&\sum_{t=1}^{T}[\mu(\rho_1(t)\Phi_1(\Delta u_1(t))+\rho_2(t)\Phi_2(\Delta u_2(t))+\rho_3(t)\Phi_3(u_1(t))+\rho_4(t)\Phi_4(u_2(t)))\\ &-F(t,u_1(t),u_2(t))+\lambda G(t,u_1(t),u_2(t))-\nu H(t,u_1(t),u_2(t))].\end{aligned}\tag{2.6}$$

Obviously, when $(\mathcal{A}1)$ and $(\mathcal{A}2)$ hold, φ is continuously differentiable on E and for $\forall\ u,v\in E$, we have

$$\begin{aligned}&\langle\varphi'(u),v\rangle=\langle\varphi'(u_1,u_2),(v_1,v_2)\rangle\\ =&\sum_{t=1}^{T}[\mu\rho_1(t)(\phi_1(\Delta u_1(t)),\Delta v_1(t))+\mu\rho_2(t)(\phi_2(\Delta u_2(t)),\Delta v_2(t))\\ &+\mu\rho_3(t)(\phi_3(u_1(t)),v_1(t))+\mu\rho_4(t)(\phi_4(u_2(t)),v_2(t))]\\ &-\sum_{t=1}^{T}[(\nabla_{u_1}F(t,u_1(t),u_2(t)),v_1(t))+(\nabla_{u_2}F(t,u_1(t),u_2(t)),v_2(t))]\\ &+\lambda\sum_{t=1}^{T}[(\nabla_{u_1}G(t,u_1(t),u_2(t)),v_1(t))+(\nabla_{u_2}G(t,u_1(t),u_2(t)),v_2(t))]\\ &-\nu\sum_{t=1}^{T}[(\nabla_{u_1}H(t,u_1(t),u_2(t)),v_1(t))+(\nabla_{u_2}H(t,u_1(t),u_2(t)),v_2(t))].\end{aligned}\tag{2.7}$$

Lemma 2.2. *If $u \in E$ is a solution of Euler equation $\varphi'(u) = 0$, then u is a solution of system (1.1).*

Proof At first, for any $u = (u_1, u_2), v = (v_1, v_2) \in E$, we can obtain the following two equalities:

$$-\sum_{t=1}^{T} \Big(\Delta\Big[\rho_1(t-1)\phi_1(\Delta u_1(t-1))\Big], v_1(t)\Big) = \sum_{t=1}^{T}(\rho_1(t)\phi_1(\Delta u_1(t)), \Delta v_1(t)), \tag{2.8}$$

$$-\sum_{t=1}^{T} \Big(\Delta\Big[\rho_2(t-1)\phi_2(\Delta u_2(t-1))\Big], v_2(t)\Big) = \sum_{t=1}^{T}(\rho_2(t)\phi_2(\Delta u_2(t)), \Delta v_2(t)). \tag{2.9}$$

In fact, since $u_1(t) = u_1(t+T)$ and $v_1(t) = v_1(t+T)$ for all $t \in \mathbb{Z}$, then

$$\begin{aligned}
& -\sum_{t=1}^{T} \Big(\Delta\Big[\rho_1(t-1)\phi_1(\Delta u_1(t-1))\Big], v_1(t)\Big) \\
= & -\sum_{t=1}^{T}(\rho_1(t)\phi_1(\Delta u_1(t)), v_1(t)) + \sum_{t=1}^{T}(\rho_1(t-1)\phi_1(\Delta u_1(t-1)), v_1(t)) \\
= & -\sum_{t=1}^{T}(\rho_1(t)\phi_1(\Delta u_1(t)), v_1(t)) + \sum_{t=1}^{T-1}(\rho_1(t)\phi_1(\Delta u_1(t)), v_1(t+1)) + (\rho_1(0)\phi_1(\Delta u_1(0)), v_1(1)) \\
= & \sum_{t=1}^{T}(\rho_1(t)\phi_1(\Delta u_1(t)), \Delta v_1(t)) + (\rho_1(0)\phi_1(\Delta u_1(0)), v_1(1)) - (\rho_1(T)\phi_1(\Delta u_1(T)), v_1(T+1)) \\
= & \sum_{t=1}^{T}(\rho_1(t)\phi_1(\Delta u_1(t)), \Delta v_1(t)).
\end{aligned}$$

Hence, (2.8) holds. Similarly, it is easy to get (2.9). Since $\varphi'(u) = 0$, then for all $v = (v_1, 0) \in E$, (2.7) implies that

$$\begin{aligned}
& \sum_{t=1}^{T} [\mu(\rho_1(t)\phi_1(\Delta u_1(t)), \Delta v_1(t)) + \mu(\rho_3(t)\phi_3(u_1(t)), v_1(t))] \\
= & \sum_{t=1}^{T}(\nabla_{u_1} F(t, u_1(t), u_2(t)), v_1(t)) - \lambda \sum_{t=1}^{T}(\nabla_{u_1} G(t, u_1(t), u_2(t)), v_1(t)) \\
& +\nu \sum_{t=1}^{T}(\nabla_{u_1} H(t, u_1(t), u_2(t)), v_1(t))
\end{aligned} \tag{2.10}$$

Note that v_1 is arbitrary. Then (2.8) and (2.10) imply that

$$\mu\Delta\Big[\rho_1(t-1)\phi_1(\Delta u_1(t-1))\Big] - \mu\rho_3(t)\phi_3(u_1(t)) + \nabla_{u_1} W(t, u_1(t), u_2(t)) = 0.$$

Similarly, Let $v_1 = 0$. We can obtain that

$$\mu\Delta\Big[\rho_2(t-1)\phi_2(\Delta u_2(t-1))\Big] - \mu\rho_4(t)\phi_4(u_2(t)) + \nabla_{u_2} W(t, u_1(t), u_2(t)) = 0. \quad \square$$

To prove Theorem 1.1, we will use the following three critical points theorem due to Ricceri [3].

Theorem 2.1 (see [3]) *Let X be a reflexive real Banach space, $I : X \to \mathbb{R}$ a sequentially weakly lower semicontinuous, coercive, bounded on each bounded subset of X, C^1 functional whose derivative admits a continuous inverse on X^*; $\Psi, \Phi : X \to \mathbb{R}$ two C^1 functionals with compact derivative. Assume also that the functional $\Psi + \lambda\Phi$ is bounded below for all $\lambda > 0$ and that*

$$\liminf_{\|x\| \to +\infty} \frac{\Psi(x)}{I(x)} = -\infty. \tag{2.11}$$

Then, for each $r > \sup_M \Phi$, where M is the set of all global minima of I, for each $\mu > \max\{0, \mu^*(I, \Psi, \Phi, r)\}$, and for each compact interval $[a, b] \subset]0, \beta(\mu I + \Psi, \Phi, r)[$, there exists a number $\rho > 0$ with the following property: for every $\lambda \in [a, b]$ and every C^1 functional $\Gamma : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\nu \in [0, \delta]$, the equation

$$\mu I'(x) + \Psi'(x) + \lambda\Phi'(x) + \nu\Gamma'(x) = 0$$

has at least three solutions in X whose norms are less than ρ, where

$$\begin{aligned}
\beta(\mu I + \Psi, \Phi, r) &= \sup_{x\in\Phi^{-1}(]r,+\infty[)} \frac{\mu I(x) + \Psi(x) - \inf_{\Phi^{-1}(]-\infty,r])}(\mu I + \Psi)}{r - \Phi(x)} \\
\mu^*(I, \Psi, \Phi, r) &= \inf\left\{ \frac{\Psi(x) - \gamma + r}{\eta_r - I(x)} : x \in X, \Phi(x) < r, I(x) < \eta_r \right\} \\
\gamma &= \inf_X(\Psi(x) + \Phi(x)), \quad \eta_r = \inf_{x\in\Phi^{-1}(r)} I(x).
\end{aligned}$$

3. Proof of Theorem 1.1

For the sake of convenience, we denote

$$\rho_i^+ = \max_{t\in\mathbb{Z}[1,T]} \rho_i(t), \quad \rho_i^- = \min_{t\in\mathbb{Z}[1,T]} \rho_i(t), \quad i = 1, 2, 3, 4.$$

Proof of Theorem 1.1 We prove that φ defined by (2.6) satisfies all the assumptions of Theorem 2.1. Let $X = E$. Then E is a reflexive and separable Banach space. Since all the topologies are equivalent in the finite dimensional Banach space E, then for any sequence $\{u^n\} \subset E$, assume that

$$u^n \to u^* \text{ in } E \text{ as } n \to \infty, \tag{3.1}$$

that is,

$$\begin{aligned}
&\left(\sum_{t=1}^{T} |\Delta u_1^n(t) - \Delta u_1^*(t)|^\theta + \sum_{t=1}^{T} |u_1^n(t) - u_1^*(t)|^\theta\right)^{1/\theta} \\
&+\left(\sum_{t=1}^{T} |\Delta u_2^n(t) - \Delta u_2^*(t)|^\theta + \sum_{t=1}^{T} |u_2^n(t) - u_2^*(t)|^\theta\right)^{1/\theta} \\
= \quad & \|u^n - u^*\| \to 0, \text{ as } n \to \infty,
\end{aligned}$$

which implies that $\lim_{n\to\infty} |\Delta u_i^n(t) - \Delta u_i^*(t)| = 0$ and $\lim_{n\to\infty} |u_i^n(t) - u_i^*(t)| = 0, i = 1, 2$ for every $t \in \mathbb{Z}[1, T]$. Hence, it is easy to obtain that $(\mathcal{A}1)$ implies that I is continuous in E and then sequentially weakly lower semicontinuous. Moreover, obviously, $(\mathcal{A}1)$ and Lemma 2.1 imply that I is a C^1 functional and

$$\begin{aligned}
\langle I'(u), v\rangle &= \sum_{t=1}^{T} [\rho_1(t)(\phi_1(\Delta u_1(t)), \Delta v_1(t)) + \rho_2(t)(\phi_2(\Delta u_2(t)), \Delta v_2(t)) \\
&\quad + \rho_3(t)(\phi_3(u_1(t)), v_1(t)) + \rho_4(t)(\phi_4(u_2(t)), v_2(t))], \quad \text{for } u, v \in E.
\end{aligned}$$

It follows from $(\mathcal{A}3)$ that

$$\langle I'(u) - I'(v), u - v\rangle = \sum_{t=1}^{T} [\rho_1(t)(\phi_1(\Delta u_1(t)) - \phi_1(\Delta v_1(t)), \Delta u_1(t) - \Delta v_1(t))$$

$$
\begin{aligned}
&+\rho_2(t)(\phi_2(\Delta u_2(t)) - \phi_2(\Delta v_2(t)), \Delta u_2(t) - \Delta v_2(t)) \\
&+\rho_3(t)(\phi_3(u_1(t)) - \phi_3(v_1(t)), u_1(t) - v_1(t)) \\
&+\rho_4(t)(\phi_4(u_1(t)) - \phi_4(v_2(t)), u_2(t) - v_2(t))] \\
\geq\ & \sum_{t=1}^{T} \left[c_1\rho_1^-|\Delta u_1(t) - \Delta v_1(t)|^\theta + c_2\rho_2^-|\Delta u_2(t) - \Delta v_2(t)|^\theta \right.\\
&\quad \left. + c_3\rho_3^-|u_1(t) - v_1(t)|^\theta + c_4\rho_4^-|u_2(t) - v_2(t)|^\theta\right] \\
\geq\ & \min\{c_1\rho_1^-, c_3\rho_3^-\}\|u_1 - v_1\|_{E_T}^\theta + \min\{c_2\rho_2^-, c_4\rho_4^-\}\|u_2 - v_2\|_{E_T}^\theta \\
\geq\ & \frac{1}{2^{\theta-1}} \min\{c_1\rho_1^-, c_2\rho_2^-, c_3\rho_3^-, c_4\rho_4^-\} \left(\|u_1 - v_1\|_{E_T} + \|u_2 - v_2\|_{E_T}\right)^\theta \\
=\ & \frac{1}{2^{\theta-1}} \min\{c_1\rho_1^-, c_2\rho_2^-, c_3\rho_3^-, c_4\rho_4^-\}\|u - v\|^\theta, \quad \text{for } u, v \in E.
\end{aligned}
$$

So I' is uniformly monotone in E. By $(\mathcal{A}1)$ and $(\mathcal{A}3)$, we have

$$
(\phi_i(x), x) \geq c_i|x|^\theta, \quad \text{for all } x \in \mathbb{R}^N, \quad i = 1, 2, 3, 4. \tag{3.2}
$$

Hence, (3.2) implies that

$$
\begin{aligned}
& \frac{\langle I'(u), u\rangle}{\|u\|} \\
=\ & \frac{1}{\|u\|} \sum_{t=1}^{T} [\rho_1(t)(\phi_1(\Delta u_1(t)), \Delta u_1(t)) + \rho_2(t)(\phi_2(\Delta u_2(t)), \Delta u_2(t)) \\
& + \rho_3(t)(\phi_3(u_1(t)), u_1(t)) + \rho_4(t)(\phi_4(u_2(t)), u_2(t))] \\
\geq\ & \frac{1}{\|u\|} \left\{ \sum_{t=1}^{T} \left[c_1\rho_1^-|\Delta u_1(t)|^\theta + c_2\rho_2^-|\Delta u_2(t)|^\theta + c_3\rho_3^-|u_1(t)|^\theta + c_4\rho_4^-|u_2(t)|^\theta\right] \right\} \\
\geq\ & \min\{c_1\rho_1^-, c_2\rho_2^-, c_3\rho_3^-, c_4\rho_4^-\} \frac{\|u_1\|_{E_T}^\theta + \|u_2\|_{E_T}^\theta}{\|u_1\|_{E_T} + \|u_2\|_{E_T}} \\
\geq\ & \frac{1}{2^{\theta-1}} \min\{c_1\rho_1^-, c_2\rho_2^-, c_3\rho_3^-, c_4\rho_4^-\} \frac{(\|u_1\|_{E_T} + \|u_2\|_{E_T})^\theta}{\|u_1\|_{E_T} + \|u_2\|_{E_T}} \\
=\ & \frac{1}{2^{\theta-1}} \min\{c_1\rho_1^-, c_2\rho_2^-, c_3\rho_3^-, c_4\rho_4^-\}\|u\|^{\theta-1}
\end{aligned} \tag{3.3}
$$

for all $u \in E$. So $\lim\limits_{\|u\|\to\infty} \frac{\langle I'(u), u\rangle}{\|u\|} = +\infty$, that is, I' is coercive in E. Next, we show that I' is also hemicontinuous in E. Assume that $s \to s^*$, $s, s^* \in [0, 1]$. Note that

$$
|\langle I'(u + sv), w\rangle - \langle I'(u + s^*v), w\rangle| \leq \|I'(u + sv) - I'(u + s^*v)\|\|w\| \tag{3.4}
$$

for all $u, v, w \in E$. Then the continuity of I' implies that $\langle I'(u + sv), w\rangle \to \langle I'(u + s^*v), w\rangle$ as $s \to s^*$ for all $u, v, w \in E$. Hence, I' is hemicontinuous in E. Thus by Theorem 26.A in [4], we know that I' admits a continuous inverse in E.

Obviously, $(\mathcal{A}2)$ implies that Ψ, Φ and Γ are C^1 functionals. Next, we show that Ψ', Φ' and Γ' are compact. Assume that $\{u^n\} \subset E$ is bounded. Then there exists a constant D_1 such that $\|u^n\| \leq D_1$ and there exists a subsequence, still denoted by $\{u_n\}$, such that $u^n \rightharpoonup u^*$ for some $u^* \in E$. Furthermore, $u^n \to u^*$. By the continuity of Ψ', Φ' and Γ', it is clear that

$$
\|\Psi'(u^n) - \Psi'(u^*)\| \to 0, \quad \|\Phi'(u^n) - \Phi'(u^*)\| \to 0, \quad \|\Gamma'(u^n) - \Gamma'(u^*)\| \to 0, \quad \text{as } n \to \infty.
$$

Hence, Ψ', Φ' and Γ' are compact in E. It follows from $(\mathcal{A}5)$ that

$$\Psi(u)+\lambda\Phi(u)=\sum_{t=1}^{T}[\lambda G(t,u_1(t),u_2(t))-F(t,u_1(t),u_2(t))]\geq TC_0(\lambda),$$

which shows that $\Psi+\lambda\Phi$ is bounded below for all $\lambda>0$. Moreover, $(\mathcal{A}5)$ implies that for any positive constant D_1, there exists a positive constant $D_2(D_1)$, which depends on D_1, such that

$$F(t,x_1,x_2)\geq D_1(|x_1|^l+|x_2|^l)-D_2(D_1) \tag{3.5}$$

for all $(x_1,x_2)\in\mathbb{R}^N\times\mathbb{R}^N$ and $t\in\mathbb{Z}[1,T]$. Then (3.5), $(\mathcal{A}4)$, (2.4) and (2.5) imply that

$$\begin{aligned}
&\lim_{\|u\|\to\infty}\frac{\Psi(u)}{I(u)}\\
=&\lim_{\|u\|\to\infty}\frac{-\sum\limits_{t=1}^{T}F(t,u_1(t),u_2(t))}{\sum\limits_{t=1}^{T}[\rho_1(t)\Phi_1(\Delta u_1(t))+\rho_2(t)\Phi_2(\Delta u_2(t))+\rho_3(t)\Phi_3(u_1(t))+\rho_4(t)\Phi_4(u_2(t))]}\\
\leq&\lim_{\|u\|\to\infty}\frac{-D_1\sum\limits_{t=1}^{T}(|u_1(t)|^l+|u_2(t)|^l)+D_2(D_1)T}{\sum\limits_{t=1}^{T}[\rho_1(t)\Phi_1(\Delta u_1(t))+\rho_2(t)\Phi_2(\Delta u_2(t))+\rho_3(t)\Phi_3(u_1(t))+\rho_4(t)\Phi_4(u_2(t))]}\\
\leq&\lim_{\|u\|\to\infty}\frac{-D_1\sum\limits_{t=1}^{T}(|u_1(t)|^l+|u_2(t)|^l)}{\sum\limits_{t=1}^{T}\left[d_1\rho_1^+|\Delta u_1(t)|^l+d_2\rho_2^+|\Delta u_2(t)|^l+d_3\rho_3^+|u_1(t)|^l+d_4\rho_4^+|u_2(t)|^l+\sum\limits_{i=1}^{4}m_i\rho_i^+\right]}\\
&+\lim_{\|u\|\to\infty}\frac{D_2(D_1)T}{\min\{\rho_1^-,\rho_2^-,\rho_2^-,\rho_4^-\}\sum\limits_{t=1}^{T}[\Phi_1(\Delta u_1(t))+\Phi_2(\Delta u_2(t))+\Phi_3(u_1(t))+\Phi_4(u_2(t))]}\\
\leq&\lim_{\|u\|\to\infty}\frac{-D_1C_3^l(\|u_1\|_{E_T}^l+\|u_2\|_{E_T}^l)}{\max\{d_1\rho_1^+,d_2\rho_2^+,d_3\rho_3^+,d_4\rho_4^+\}(\|u_1\|_{[E_T]}^l+\|u_2\|_{[E_T]}^l)+\sum\limits_{t=1}^{T}\sum\limits_{i=1}^{4}m_i\rho_i^+}\\
\leq&\lim_{\|u\|\to\infty}\frac{-D_1C_3^l\frac{1}{2^{l-1}}(\|u_1\|_{E_T}+\|u_2\|_{E_T})^l}{\max\{d_1\rho_1^+,d_2\rho_2^+,d_3\rho_3^+,d_4\rho_4^+\}(\|u_1\|_{[E_T]}^l+\|u_2\|_{[E_T]}^l)+\sum\limits_{t=1}^{T}\sum\limits_{i=1}^{4}m_i\rho_i^+}\\
\leq&\lim_{\|u\|\to\infty}\frac{-D_1C_3^l\frac{1}{2^{l-1}}(\|u_1\|_{E_T}+\|u_2\|_{E_T})^l}{\max\{d_1\rho_1^+,d_2\rho_2^+,d_3\rho_3^+,d_4\rho_4^+\}(C_6^l\|u_1\|_{E_T}^l+C_6^l\|u_2\|_{E_T}^l)+\sum\limits_{t=1}^{T}\sum\limits_{i=1}^{4}m_i\rho_i^+}\\
\leq&\lim_{\|u\|\to\infty}\frac{-D_1C_3^l\frac{1}{2^{l-1}}(\|u_1\|_{E_T}+\|u_2\|_{E_T})^l}{\max\{d_1\rho_1^+,d_2\rho_2^+,d_3\rho_3^+,d_4\rho_4^+\}C_6^l(\|u_1\|_{E_T}+\|u_2\|_{E_T})^l+\sum\limits_{t=1}^{T}\sum\limits_{i=1}^{4}m_i\rho_i^+}\\
=&\frac{1}{2^{l-1}}\frac{-D_1C_3^l}{\max\{d_1\rho_1^+,d_2\rho_2^+,d_3\rho_3^+,d_4\rho_4^+\}C_6^l}.
\end{aligned}$$

By the arbitrary of D_1, we obtain that

$$\lim_{\|u\|\to\infty}\frac{\Psi(u)}{I(u)}=-\infty.$$

By $(\mathcal{A}1)$ and $(\mathcal{A}4)$, we know that Φ_i reaches its unique minimum at 0, $i=1,2,3,4$ (see [2]) and so I has unique global minima 0. Then $M = \{0\}$. By $(\mathcal{A}6)$, we have $\sup_M \Phi = 0$. Hence, by Theorem 2.1, the conclusion of Theorem 1.1 holds. □

Proof of Corollary 1.1 It follows from $(\mathcal{A}5)'$ that there exist $D_3 > 0$ and $D_4 > 0$ such that for every $t \in \mathbb{Z}[1, T]$,

$$F(t, x_1, x_2) \leq D_3|x_1|^s + D_3|x_2|^s + D_4$$

and for any $D_5 > D_3$, there are a constant $D_6(D_5)$, which depends on D_5, such that

$$G(t, x_1, x_2) \geq D_5|x_1|^s + D_5|x_2|^s + D_6(D_5).$$

Obviously, for every $\lambda > 0$, we can find a sufficiently large $D_5(\lambda)$ such that $\lambda D_5(\lambda) > D_3$. Hence, we have

$$\lambda G(t, x_1, x_2) \geq D_3|x_1|^s + D_3|x_2|^s + \lambda D_6(D_5(\lambda)) \geq F(t, x_1, x_2) - D_4 + \lambda D_6(D_5(\lambda)).$$

So $(\mathcal{A}5)'$ implies $(\mathcal{A}5)$. □

4. Proof of Theorem 1.2

When the condition (γ) holds, on E_T, we define

$$\|u\|_{(E_{T,q})} = \left(\sum_{t=1}^{T} \gamma_1(t)|\Delta u_1(t)|^q + \sum_{t=1}^{T} \gamma_3(t)|u_1(t)|^q\right)^{1/q}$$

and

$$\|u\|_{(E_{T,p})} = \left(\sum_{t=1}^{T} \gamma_2(t)|\Delta u_2(t)|^p + \sum_{t=1}^{T} \gamma_4(t)|u_2(t)|^p\right)^{1/p}.$$

For $u = (u_1, u_2) \in E$, define

$$\|u\|_{(\infty)} = \|u_1\|_\infty + \|u_2\|_\infty.$$

Moreover, it is clear that E is homeomorphic to $\mathbb{R}^{2NT}$. Then there is a basis of E denoted by $\{e_1, e_2, ..., e_{2NT}\}$. For every $u \in E$, there exists a unique point $(\lambda_1, \lambda_2, ..., \lambda_{2NT}) \in \mathbb{R}^{2NT}$ such that

$$u = \sum_{i=1}^{2NT} \lambda_i e_i$$

and define

$$\|u\|_{(2)} = \left(\sum_{i=1}^{2NT} \lambda_i^2\right)^{\frac{1}{2}}.$$

Set

$$E_\delta = \{u \in E : \|u\|_{(2)} = \delta\}.$$

Since both E and E_T are finite-dimensional spaces, then $\|\cdot\|_{(\infty)}$ is equivalent to $\|\cdot\|_{(2)}$ on E, and both $\|\cdot\|_{(E_T,q)}$ and $\|\cdot\|_{(E_T,p)}$ are equivalent to $\|\cdot\|_\infty$ on E_T. Hence, there exist positive constants R_i $(i=1,2,\cdots,6)$ such that

$$R_1\|\cdot\|_{(2)} \le \|\cdot\|_{(\infty)} \le R_2\|\cdot\|_{(2)}, \tag{4.1}$$

$$R_3\|\cdot\|_\infty \le \|\cdot\|_{(E_T,q)} \le R_4\|\cdot\|_\infty, \tag{4.2}$$

$$R_5\|\cdot\|_\infty \le \|\cdot\|_{(E_T,p)} \le R_6\|\cdot\|_\infty. \tag{4.3}$$

In Lemma 2.1, let

$$L(t,x_1,x_2,y_1,y_2) = \gamma_1(t)\Phi_1(y_1) + \gamma_2(t)\Phi_2(y_2) + \gamma_3(t)\Phi_3(x_1) + \gamma_4(t)\Phi_4(x_2) - F(t,x_1,x_2),$$

where F: $\mathbb{Z}[1,T]\times\mathbb{R}^N\times\mathbb{R}^N \longrightarrow \mathbb{R}$ is continuously differentiable in $(x_1,x_2)\in\mathbb{R}^N\times\mathbb{R}^N$ for all $t\in\mathbb{Z}[1,T]$. Then

$$\begin{aligned}\varphi(u) \quad = \quad & \sum_{t=1}^{T}\gamma_1(t)\Phi_1(\Delta u_1(t)) + \sum_{t=1}^{T}\gamma_2(t)\Phi_2(\Delta u_2(t)) \\ & + \sum_{t=1}^{T}\gamma_3(t)\Phi_3(u_1(t)) + \sum_{t=1}^{T}\gamma_4(t)\Phi_4(u_2(t)) - \sum_{t=1}^{T}F(t,u_1(t),u_2(t)).\end{aligned} \tag{4.4}$$

and for $\forall\ u,v\in E$, we have

$$\begin{aligned}& \langle\varphi'(u),v\rangle \\ = \quad & \langle\varphi'(u_1,u_2),(v_1,v_2)\rangle \\ = \quad & \sum_{t=1}^{T}[\gamma_1(t)(\phi_1(\Delta u_1(t)),\Delta v_1(t)) + \gamma_2(t)(\phi_2(\Delta u_2(t)),\Delta v_2(t)) \\ & +\gamma_3(t)(\phi_3(u_1(t)),v_1(t)) + \gamma_4(t)(\phi_4(u_2(t)),v_2(t)) \\ & -\sum_{t=1}^{T}(\nabla_{u_1}F(t,u_1(t),u_2(t)),v_1(t)) - \sum_{t=1}^{T}(\nabla_{u_2}F(t,u_1(t),u_2(t)),v_2(t)).\end{aligned} \tag{4.5}$$

Similar to the argument of Lemma 2.2, it is easy to obtain the following Lemma:

Lemma 4.1 *If $u\in E$ is a solution of Euler equation $\varphi'(u)=0$, then u is a solution of system (1.5).*

Denote with θ the zero element of X and with Σ the family of sets $A\subset X\backslash\{\theta\}$ such that A is closed in X and symmetric with respect to θ, i.e. $u\in A$ implies $-u\in A$.

Theorem 4.1 (see [24], Theorem 9.1) *Let X be a real Banach space and φ be an even function belonging to $C^1(X,\mathbb{R})$ with $\varphi(\theta)=0$, bounded from below and satisfying (PS) condition. Suppose that there is a set $K\in\Sigma$ such that K is homeomorphic to S^{j-1} ($j-1$ dimension unit sphere) by an odd map and* $\sup_K\varphi<0$. *Then, φ has at least j distinct pairs of nonzero critical points.*

Proof of Theorem 1.2 It follows from (ϕ), (4.4), (2.2), (4.2), (4.3) and $(\mathcal{F}0)$ that

$$\varphi(u) \quad = \quad \sum_{t=1}^{T}\gamma_1(t)\Phi_1(\Delta u_1(t)) + \sum_{t=1}^{T}\gamma_2(t)\Phi_2(\Delta u_2(t))$$

$$
\begin{aligned}
& +\sum_{t=1}^{T}\gamma_3(t)\Phi_3(u_1(t))+\sum_{t=1}^{T}\gamma_4(t)\Phi_4(u_2(t))-\sum_{t=1}^{T}F(t,u_1(t),u_2(t)) \\
\geq\ & a_1\sum_{t=1}^{T}\gamma_1(t)|\Delta u_1(t)|^q+a_2\sum_{t=1}^{T}\gamma_2(t)|\Delta u_2(t)|^p \\
& +a_3\sum_{t=1}^{T}\gamma_3(t)|u_1(t)|^q+a_4\sum_{t=1}^{T}\gamma_4(t)|u_2(t)|^p-\sum_{t=1}^{T}F(t,u_1(t),u_2(t)) \\
\geq\ & \min\{a_1,a_3\}\|u_1\|^q_{(E_{T,q})}+\min\{a_2,a_4\}\|u_2\|^p_{(E_{T,p})} \\
& -\sum_{t=1}^{T}[h_1(t)|u_1(t)|^{\alpha_1}+h_2(t)|u_2(t)|^{\alpha_2}+l(t)] \\
\geq\ & \min\{a_1,a_3\}R_3^q\|u_1\|^q_\infty+\min\{a_2,a_4\}R_5^p\|u_2\|^p_\infty \\
& -\|u_1\|^{\alpha_1}_\infty\sum_{t=1}^{T}h_1(t)-\|u_2\|^{\alpha_2}_\infty\sum_{t=1}^{T}h_2(t)-\sum_{t=1}^{T}l(t) \qquad (4.6)
\end{aligned}
$$

for all $u\in E$. Since $\alpha_1\in[0,q)$ and $\alpha_2\in[0,p)$, it is easy to see that

$$
\varphi(u)\to+\infty,\ \text{ as } \|u\|_{(\infty)}=\|u_1\|_\infty+\|u_2\|_\infty\to\infty, \qquad (4.7)
$$

which implies that φ is bounded from below and any (PS) sequence $\{u_n\}$ is bounded. Hence φ satisfies (PS) condition. Obviously, $(\mathcal{F}1)$ and $(\mathcal{F}2)$ imply that $\varphi(0)=0$ and φ is even. Next, we prove that there exists a set $K\subset E$ such that K is homeomorphic to S^{2NT-1} by an odd map, and $\sup_K\varphi<0$. Note that $\delta<1$. For all $u=(u_1,u_2)\in E_\delta$ and $r>0$, by (4.1) we have

$$
\begin{aligned}
& M_1r^{\beta_1}\|u_1\|^{\beta_1}_\infty+M_2r^{\beta_2}\|u_2\|^{\beta_2}_\infty \\
=\ & M_1r^{\beta_1}R_2^{\beta_1}\left\|\frac{u_1}{R_2}\right\|^{\beta_1}_\infty+M_2r^{\beta_2}R_2^{\beta_2}\left\|\frac{u_2}{R_2}\right\|^{\beta_2}_\infty \\
\geq\ & \min\{M_1r^{\beta_1}R_2^{\beta_1},M_2r^{\beta_2}R_2^{\beta_2}\}\left(\left\|\frac{u_1}{R_2}\right\|^{\max\{\beta_1,\beta_2\}}_\infty+\left\|\frac{u_2}{R_2}\right\|^{\max\{\beta_1,\beta_2\}}_\infty\right) \\
\geq\ & 2^{1-\max\{\beta_1,\beta_2\}}\min\{M_1r^{\beta_1}R_2^{\beta_1},M_2r^{\beta_2}R_2^{\beta_2}\}\left(\left\|\frac{u_1}{R_2}\right\|_\infty+\left\|\frac{u_2}{R_2}\right\|_\infty\right)^{\max\{\beta_1,\beta_2\}} \\
=\ & 2^{1-\max\{\beta_1,\beta_2\}}\min\{M_1r^{\beta_1}R_2^{\beta_1},M_2r^{\beta_2}R_2^{\beta_2}\}\left(\frac{1}{R_2}\right)^{\max\{\beta_1,\beta_2\}}\|u\|^{\max\{\beta_1,\beta_2\}}_{(\infty)} \\
\geq\ & 2^{1-\max\{\beta_1,\beta_2\}}\min\{M_1r^{\beta_1}R_2^{\beta_1},M_2r^{\beta_2}R_2^{\beta_2}\}\left(\frac{1}{R_2}\right)^{\max\{\beta_1,\beta_2\}}R_1^{\max\{\beta_1,\beta_2\}}\|u\|^{\max\{\beta_1,\beta_2\}}_{(2)} \\
=\ & 2\min\{M_1r^{\beta_1}R_2^{\beta_1},M_2r^{\beta_2}R_2^{\beta_2}\}\left(\frac{R_1\delta}{2R_2}\right)^{\max\{\beta_1,\beta_2\}}. \qquad (4.8)
\end{aligned}
$$

Then for all $u=(u_1,u_2)\in E_\delta$ and $0<r<\frac{1}{R_2}$, by (ϕ), $(\mathcal{F}3)$, (2.2), (4.1)-(4.3) and (4.8) we have

$$
\begin{aligned}
\varphi(ru)\ =\ & \sum_{t=1}^{T}\gamma_1(t)\Phi_1(r\Delta u_1(t))+\sum_{t=1}^{T}\gamma_2(t)\Phi_2(r\Delta u_2(t)) \\
& +\sum_{t=1}^{T}\gamma_3(t)\Phi_3(ru_1(t))+\sum_{t=1}^{T}\gamma_4(t)\Phi_4(ru_2(t))-\sum_{t=1}^{T}F(t,ru_1(t),ru_2(t)) \\
\leq\ & b_1\sum_{t=1}^{T}\gamma_1(t)|r\Delta u_1(t)|^q+b_2\sum_{t=1}^{T}\gamma_2(t)|r\Delta u_2(t)|^p
\end{aligned}
$$

$$
\begin{aligned}
& +b_3\sum_{t=1}^{T}\gamma_3(t)|ru_1(t)|^q+b_4\sum_{t=1}^{T}\gamma_4(t)|ru_2(t)|^p-\sum_{t=1}^{T}F(t,ru_1(t),ru_2(t)) \\
\leq\ & \max\{b_1,b_3\}r^q\|u_1\|^q_{(E_T,q)}+\max\{b_2,b_4\}r^p\|u_2\|^p_{(E_T,p)} \\
& -M_1r^{\beta_1}\sum_{t=1}^{T}|u_1(t)|^{\beta_1}-M_2r^{\beta_2}\sum_{t=1}^{T}|u_2(t)|^{\beta_2} \\
\leq\ & \max\{b_1,b_3\}r^qR_4^q\|u_1\|^q_\infty+\max\{b_2,b_4\}r^pR_6^p\|u_2\|^p_\infty \\
& -M_1r^{\beta_1}\|u_1\|^{\beta_1}_\infty-M_2r^{\beta_2}\|u_2\|^{\beta_2}_\infty \\
\leq\ & \max\{b_1,b_3\}r^qR_4^qR_2^q\|u\|^q_{(2)}+\max\{b_2,b_4\}r^pR_6^pR_2^p\|u\|^p_{(2)} \\
& -2\min\{M_1r^{\beta_1}R_2^{\beta_1},M_2r^{\beta_2}R_2^{\beta_2}\}\left(\frac{R_1\delta}{2R_2}\right)^{\max\{\beta_1,\beta_2\}} \\
=\ & \max\{b_1,b_3\}r^q(R_4R_2\delta)^q+\max\{b_2,b_4\}r^p(R_6R_2\delta)^p \\
& -2\min\{M_1r^{\beta_1}R_2^{\beta_1},M_2r^{\beta_2}R_2^{\beta_2}\}\left(\frac{R_1\delta}{2R_2}\right)^{\max\{\beta_1,\beta_2\}}.
\end{aligned} \tag{4.9}
$$

Since $\beta_i\in(1,\min\{q,p\})$, $i=1,2$. Then (4.9) implies that there exist sufficiently small $r_0\in(0,1)$ and $\epsilon>0$ such that there exists sufficiently small $r_0\in(0,1)$ and $\epsilon>0$ such that $\varphi(r_0u)<-\epsilon$ for all $u\in E_\delta$. Set

$$
E_\delta^{r_0}=\{r_0u:u\in E_\delta\}\quad\text{and}\quad S^{2NT-1}=\left\{(\lambda_1,\lambda_2,\cdots,\lambda_{2NT})\in\mathbb{R}^{2NT}:\sum_{i=1}^{2NT}\lambda_i^2=1\right\}.
$$

Then $E_\delta^{r_0}\in\Sigma$ and

$$
\psi(u)<-\epsilon,\quad\forall\ u\in E_\delta^{r_0}. \tag{4.10}
$$

Define the map $\psi:E_\delta^{r_0}\to S^{2NT-1}$ by

$$
\psi(u)=\psi\left(\sum_{i=1}^{2NT}\lambda_ie_i\right)=\frac{1}{r_0\delta}(\lambda_1,\lambda_2,\cdots,\lambda_{2NT}).
$$

Then it is easy to see that ψ is an odd and homeomorphic map. Moreover, (4.10) implies that $\sup_{E_\delta^{r_0}}\varphi\leq-\epsilon<0$. Therefore, by Theorem 4.1, we obtain that system (1.5) has at least $2NT$ distinct pairs of solutions in E. $\square$

5. Examples

Example 5.1. We present an example to which Theorem 1.1 applies and make an estimate for the parameters in our result. Let $T=2$ and N be fixed integer. Assume that $\phi_1(y)=y+|y|^{\frac{1}{3}}y$, $\phi_2(y)=y+|y|^{\frac{1}{2}}y$, $\phi_3(y)=\phi_4(y)=2y$, ρ_i are 2-periodic and satisfy $\rho_i>0$ for all $t\in\mathbb{Z}[1,2]$ $(i=1,2,3,4)$. Then $\Phi_1(y)=\frac{|y|^2}{2}+\frac{|y|^{\frac{7}{3}}}{\frac{7}{3}}$, $\Phi_2(y)=\frac{|y|^2}{2}+\frac{|y|^{\frac{5}{2}}}{\frac{5}{2}}$, $\Phi_3(y)=\Phi_4(y)=|y|^2$. Let

$$
\begin{aligned}
F(t,x_1,x_2) &= |x_1|^3+|x_2|^3 \\
G(t,x_1,x_2) &= |x_1|^4+|x_2|^4 \\
H(t,x_1,x_2) &= (\cos^2\frac{\pi t}{2}+2)\sin(|x_1|^2+|x_2|^2+2)
\end{aligned}
$$

$$W(t,x_1,x_2) \quad = \quad F(t,x_1,x_2) - \lambda G(t,x_1,x_2) + \nu H(t,x_1,x_2).$$

Then

$$\begin{aligned}
I(u) &= \sum_{t=1}^{2}\left[\rho_1(t)\left(\frac{|\Delta u_1(t)|^2}{2} + \frac{|\Delta u_1(t)|^{\frac{7}{3}}}{\frac{7}{3}}\right) + \rho_2(t)\left(\frac{|\Delta u_2(t)|^2}{2} + \frac{|\Delta u_2(t)|^{\frac{5}{2}}}{\frac{5}{2}}\right)\right] \\
&\quad + \sum_{t=1}^{2}\left[\rho_3(t)|u_1(t)|^2 + \rho_4(t)|u_2(t)|^2\right], \\
\Psi(u) &= -\sum_{t=1}^{2}\left(|u_1(t)|^3 + |u_2(t)|^3\right), \quad \Phi(u) = \sum_{t=1}^{2}\left(|u_1(t)|^4 + |u_2(t)|^4\right), \\
\Gamma(u) &= -\sum_{t=1}^{T}(\cos^2\frac{\pi t}{2} + 2)\sin(|u_1(t)|^2 + |u_2(t)|^2 + 2), \qquad u \in E,
\end{aligned}$$

where the definition of E and its norm are in section 2. Take $\theta = 2$ and $l = \frac{5}{2}$. With a similar discussion as in Remark 1.1, we can prove that all conditions of Theorem 1.1 hold. Since, by a simply computation we have

$$\gamma = \inf_E(\Psi(u) + \Phi(u)) = \inf_E \sum_{t=1}^{2}\left(|u_1(t)|^4 + |u_2(t)|^4 - |u_1(t)|^3 - |u_2(t)|^3\right) = -\frac{27}{64}, \tag{5.1}$$

which is obtained when $|u_1(1)| = |u_1(2)| = |u_2(1)| = |u_2(2)| = \frac{3}{4}$. Moreover, for $r > 0$, we have

$$\begin{aligned}
\Phi^{-1}(r) &= \{u \in E : |u_1(1)|^4 + |u_1(2)|^4 + |u_2(1)|^4 + |u_2(2)|^4 = r\}, \\
\Phi^{-1}(]-\infty, r]) &= \{u \in E : |u_1(1)|^4 + |u_1(2)|^4 + |u_2(1)|^4 + |u_2(2)|^4 \leq r\}, \\
\Phi^{-1}(]r, +\infty[) &= \{u \in E : |u_1(1)|^4 + |u_1(2)|^4 + |u_2(1)|^4 + |u_2(2)|^4 > r\}.
\end{aligned}$$

Then

$$\begin{aligned}
\eta_r &= \inf_{u\in\Phi^{-1}(r)} I(u) \\
&= \inf_{u\in\Phi^{-1}(r)}\left\{\sum_{t=1}^{2}\left[\rho_1(t)\left(\frac{|\Delta u_1(t)|^2}{2} + \frac{|\Delta u_1(t)|^{\frac{7}{3}}}{\frac{7}{3}}\right) + \rho_2(t)\left(\frac{|\Delta u_2(t)|^2}{2} + \frac{|\Delta u_2(t)|^{\frac{5}{2}}}{\frac{5}{2}}\right)\right]\right. \\
&\qquad\left. + \sum_{t=1}^{2}\left[\rho_3(t)|u_1(t)|^2 + \rho_4(t)|u_2(t)|^2\right]\right\} \\
&\geq \inf_{u\in\Phi^{-1}(r)} \sum_{t=1}^{2}\left[\rho_3(t)|u_1(t)|^2 + \rho_4(t)|u_2(t)|^2\right] \\
&= \inf_{u\in\Phi^{-1}(r)}\left(\rho_3(1)|u_1(1)|^2 + \rho_3(2)|u_1(2)|^2 + \rho_4(1)|u_2(1)|^2 + \rho_4(2)|u_2(2)|^2\right) \\
&= \min\{\rho_3(1), \rho_3(2), \rho_4(1), \rho_4(2)\}\sqrt{r},
\end{aligned} \tag{5.2}$$

which can be obtained by using the Lagrange multiplier method. By (5.1), (5.2) and the fact that $\Phi(0) = I(0) = 0$, we have

$$\begin{aligned}
\mu^*(I, \Psi, \Phi, r) &= \inf\left\{\frac{\Psi(u) - \gamma + r}{\eta_r - I(u)} : u \in E, \Phi(u) < r, I(u) < \eta_r\right\} \\
&\leq \frac{\Psi(0) - \gamma + r}{\eta_r - I(0)} = \frac{\frac{27}{64} + r}{\eta_r} \leq \frac{\frac{27}{64} + r}{\min\{\rho_3(1), \rho_3(2), \rho_4(1), \rho_4(2)\}\sqrt{r}}.
\end{aligned} \tag{5.3}$$

When $\mu > \frac{r^{\frac{1}{4}}}{\min\{\rho_3(1),\rho_3(2),\rho_4(1),\rho_4(2)\}}$, we have

$$\begin{aligned}
& \inf_{\Phi^{-1}(]-\infty,r])}(\mu I+\Psi) \\
= & \inf_{\Phi^{-1}(]-\infty,r])} \mu\left\{\sum_{t=1}^{2}\left[\rho_1(t)\left(\frac{|\Delta u_1(t)|^2}{2}+\frac{|\Delta u_1(t)|^{\frac{7}{3}}}{\frac{7}{3}}\right)+\rho_2(t)\left(\frac{|\Delta u_2(t)|^2}{2}+\frac{|\Delta u_2(t)|^{\frac{5}{2}}}{\frac{5}{2}}\right)\right]\right. \\
& \left.+\sum_{t=1}^{2}\left[\rho_3(t)|u_1(t)|^2+\rho_4(t)|u_2(t)|^2\right]\right\}-\sum_{t=1}^{2}\left(|u_1(t)|^3+|u_2(t)|^3\right) \\
= & 0, \qquad (5.4)
\end{aligned}$$

which is obtained when $u_1(1)=u_1(2)=u_2(1)=u_2(2)=0$. When $\mu > \frac{r^{\frac{1}{4}}}{\min\{\rho_3(1),\rho_3(2),\rho_4(1),\rho_4(2)\}}$, we choose $u_0 : u_1(1)=u_1(2)=u_2(1)=u_2(2)=\mu[\rho_3(1)+\rho_3(2)+\rho_4(1)+\rho_4(2)]$, then

$$\begin{aligned}
& |u_1(1)|^4+|u_1(2)|^4+|u_2(1)|^4+|u_2(2)|^4 \\
= & 4\mu^4[\rho_3(1)+\rho_3(2)+\rho_4(1)+\rho_4(2)]^4 \\
> & \frac{4r[\rho_3(1)+\rho_3(2)+\rho_4(1)+\rho_4(2)]^4}{(\min\{\rho_3(1),\rho_3(2),\rho_4(1),\rho_4(2)\})^4} \\
> & r
\end{aligned}$$

which implies that $u_0 \in \Phi^{-1}(]r,+\infty[)$. Therefore, when $\mu > \frac{\max\{\frac{\frac{27}{64}+r}{\sqrt{r}},r^{\frac{1}{4}}\}}{\min\{\rho_3(1),\rho_3(2),\rho_4(1),\rho_4(2)\}}$, by (5.4) and the fact that $u_0 \in \Phi^{-1}(]r,+\infty[)$ we have

$$\begin{aligned}
\beta(\mu I+\Psi,\Phi,r) & = \sup_{u\in\Phi^{-1}(]r,+\infty[)}\frac{\mu I(u)+\Psi(u)-\inf_{\Phi^{-1}(]-\infty,r])}(\mu I+\Psi)}{r-\Phi(u)} \\
& = \sup_{u\in\Phi^{-1}(]r,+\infty[)}\frac{\mu I(u)+\Psi(u)}{r-\Phi(u)} \\
& \geq \frac{\mu I(u_0)+\Psi(u_0)}{r-\Phi(u_0)} \\
& = \frac{3\mu^3[\rho_3(1)+\rho_3(2)+\rho_4(1)+\rho_4(2)]^3}{4\mu^4[\rho_3(1)+\rho_3(2)+\rho_4(1)+\rho_4(2)]^4-r}. \qquad (5.5)
\end{aligned}$$

Hence, by Theorem 1.1, we obtain that for each $r>0$, for each $\mu > \frac{\max\{\frac{\frac{27}{64}+r}{\sqrt{r}},r^{\frac{1}{4}}\}}{\min\{\rho_3(1),\rho_3(2),\rho_4(1),\rho_4(2)\}}$, and for each compact interval $[a,b] \subset]0, \frac{3\mu^3[\rho_3(1)+\rho_3(2)+\rho_4(1)+\rho_4(2)]^3}{4\mu^4[\rho_3(1)+\rho_3(2)+\rho_4(1)+\rho_4(2)]^4-r}[$, there exists a number $\rho>0$ with the following property: for every $\lambda\in[a,b]$, there exists $\delta>0$ such that, for each $\nu\in[0,\delta]$, system (1.1) has at least three 2-periodic solutions in E whose norms are less than ρ.

Example 5.2. We present this example when Theorem 1.2 applies. Let $N=6$ and $T=4$. Assume that $\phi_1(x)=\phi_3(x)=|x|^3x$, $\phi_2(x)=\phi_4(x)=|x|x$. Consider the following nonlinear difference (ϕ_1,ϕ_2)-Laplacian system:

$$\begin{cases}
\Delta\left(\left[\sin^2\frac{\pi}{4}(t-1)+1\right]\phi_1\big(\Delta u_1(t-1)\big)\right)-\left[|\sin\frac{\pi}{4}t|+1\right]\phi_3(|u_1(t)|) \\
\qquad\qquad +\nabla_{u_1}F\big(t,u_1(t),u_2(t)\big)=0 \\
\Delta\left(\left[\cos^2\frac{\pi}{4}(t-1)+1\right]\phi_2\big(\Delta u_2(t-1)\big)\right)-\left[|\cos\frac{\pi}{4}t|+1\right]\phi_4(|u_2(t)|) \\
\qquad\qquad +\nabla_{u_2}F\big(t,u_1(t),u_2(t)\big)=0.
\end{cases} \qquad (5.6)$$

Then $\gamma_1(t) = \sin^2 \frac{\pi}{4}t + 1$, $\gamma_2(t) = \cos^2 \frac{\pi}{4}t + 1$, $\gamma_3(t) = |\sin \frac{\pi}{4}t| + 1$, $\gamma_4(t) = |\cos \frac{\pi}{4}t| + 1$. Obviously, the conditions (γ) and (ϕ) hold and γ_i, $i = 1, 2, 3, 4$ are T-periodic ($T = 4$).

If we assume that

$$F(t, x_1, x_2) = \left(\left|\sin \frac{\pi}{4}t\right| + 1\right) |x_1|^{\frac{3}{2}} + \left(\cos^2 \frac{\pi}{4}t + 1\right) |x_2|^2,$$

then, obviously, $(\mathcal{F}0)$, $(\mathcal{F}1)$ and $(\mathcal{F}2)$ hold and there exists enough small $\delta \in (0, 1)$ such that

$$\begin{aligned} F(t, x_1, x_2) &= \left(\left|\sin \frac{\pi}{4}t\right| + 1\right) |x_1|^{\frac{3}{2}} + \left(\cos^2 \frac{\pi}{4}t + 1\right) |x_2|^2 \\ &\geq |x_1|^{\frac{3}{2}} + |x_2|^2 \\ &\geq |x_1|^2 + |x_2|^{\frac{5}{2}}, \quad \forall\ |x_1| < \delta, |x_2| < \delta. \end{aligned} \tag{5.7}$$

Let $\beta_1 = 2$, $\beta_2 = \frac{5}{2}$ and $M_1 = M_2 = 1$. Then (5.7) implies that $(\mathcal{F}3)$ holds. Hence, by Theorem 1.2, we obtained that system (5.6) has at least 48 distinct pairs of 4-periodic solutions.

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