

ON MOTIVIC JOYCE-SONG FORMULA FOR THE BEHREND FUNCTION IDENTITIES

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ABSTRACT. We prove the motivic version of Joyce-Song formula for the Behrend function identities proposed in [24]. The main method we use is Nicaise's motivic integration for formal schemes and Cluckers-Loeser's motivic constructible functions. As an application we prove that there is a Poisson algebra homomorphism from the motivic Hall algebra of the abelian category of coherent sheaves on a Calabi-Yau threefold Y to the motivic quantum torus of Y , thus generalizing the integration map of Joyce-Song in [27] and Bridgeland in [10] to the motivic level. Such an integration map has applications in the wall crossing of motivic Donaldson-Thomas invariants.

CONTENTS

1. Introduction	2
1.1. Background on Donaldson-Thomas theory	2
1.2. The motivic Donaldson-Thomas invariants	3
1.3. The motivic Joyce-Song formula.	4
1.4. Applications	8
1.5. Outline	10
Convention	10
Acknowledgments	10
2. The motivic Milnor fibre and the motivic volumes.	10
2.1. Grothendieck group of varieties.	10
2.2. Motivic integration on rigid varieties	12
3. Proof of the Conjecture	18
3.1. A motivic Blow-up formula	18
3.2. Techniques on the motivic constructible functions of Cluckers and Loeser	20
3.3. The proof of Formula (1) in Conjecture 1.2	23
3.4. The proof of Formula (2) in Conjecture 1.2	28
4. The Poisson algebra homomorphism	30
4.1. Motivic Hall algebras	30
4.2. Algebraic d -critical locus	32
4.3. The integration map	38
4.4. The proof of Theorem 4.16	42
	45
References	45

1. INTRODUCTION

1.1. Background on Donaldson-Thomas theory.

. **(1.1.1)** Let Y be a smooth Calabi-Yau threefold or a smooth threefold Deligne-Mumford stack. The Donaldson-Thomas invariants of Y count stable coherent sheaves on Y . The goal was achieved by R. Thomas in [49], who constructed a perfect obstruction theory E^\bullet in the sense of Li-Tian [35], and Behrend-Fantechi [3] on the moduli space X of stable sheaves over Y . If X is proper, then the virtual dimension of X is zero, and the integral

$$\mathrm{DT}_Y = \int_{[X]^{\mathrm{virt}}} 1$$

is the Donaldson-Thomas invariant of Y . Of course the Donaldson-Thomas invariants are defined for any projective threefold, and in general the virtual dimension is not zero and one should integrate some cohomology classes over the virtual fundamental cycle. Here we only restrict to the case of Calabi-Yau threefolds. Donaldson-Thomas invariants have been proved to have deep connections to Gromov-Witten theory and provided more deep understanding of the curve counting invariants, see [38], [39], [46], etc.

. **(1.1.2)** In the Calabi-Yau threefold case, in [1] Behrend proves that the moduli scheme X of stable sheaves on Y admits a symmetric obstruction theory which is defined by him in the same paper [1]. Also Behrend in the same paper constructs a canonical integer-valued constructible function

$$\nu_X : X \rightarrow \mathbb{Z}$$

on X , which we call the *Behrend function* of X . If X is proper, then in [1, Theorem 4.18] Behrend proves that

$$\mathrm{DT}_Y = \int_{[X]^{\mathrm{virt}}} 1 = \chi(X, \nu_X),$$

where $\chi(X, \nu_X)$ is the weighted Euler characteristic weighted by the Behrend function. Same result for a proper Deligne-Mumford stack X with a symmetric perfect obstruction theory is conjectured by Behrend in [1], and is proved in [25].

. **(1.1.3)** The perfect obstruction theory on the moduli scheme requires that we only can count stable coherent sheaves on Y . In order to count semi-stable sheaves on the abelian category $\mathcal{A} := \mathrm{Coh}(Y)$ of coherent sheaves on Y , Joyce-Song in [27] developed a theory of generalized Donaldson-Thomas invariants. Let \mathfrak{M} be the moduli stack of coherent sheaves on \mathcal{A} , which is an Artin stack locally of finite type. Then in [27], Joyce-Song generalized the definition of the Behrend function to \mathfrak{M} :

$$\nu_{\mathfrak{M}} : \mathfrak{M} \rightarrow \mathbb{Z}.$$

We can understand the Behrend function $\nu_{\mathfrak{M}}$ as follows: if there is a finite 1-morphism

$$f : X \rightarrow \mathfrak{M}$$

from a κ -scheme X to \mathfrak{M} , then $f^* \nu_{\mathfrak{M}} = (-1)^n \nu_X$, where n is the relative dimension. For any $E_1, E_2 \in \text{Coh}(Y)$, Joyce-Song in [27, §5.2] proves the following formula of the Behrend function identities:

(1)

$$\nu_{\mathfrak{M}}(E_1 \oplus E_2) = (-1)^{\chi(E_1, E_2)} \nu_{\mathfrak{M}}(E_1) \nu_{\mathfrak{M}}(E_2).$$

Here, $\chi(E_1, E_2) = \sum_i (-1)^i \dim \text{Ext}^i(E_1, E_2)$ is the Euler form.

(2)

$$\begin{aligned} & \int_{F \in \mathbb{P}(\text{Ext}^1(E_2, E_1))} \nu_{\mathfrak{M}}(F) d\chi - \int_{F \in \mathbb{P}(\text{Ext}^1(E_1, E_2))} \nu_{\mathfrak{M}}(F) d\chi \\ &= (\dim(\text{Ext}^1(E_2, E_1)) - \dim(\text{Ext}^1(E_1, E_2))) \nu_{\mathfrak{M}}(E_1 \oplus E_2). \end{aligned}$$

Here for the integral $\int_{F \in \mathbb{P}(\text{Ext}^1(E_2, E_1))} \nu_{\mathfrak{M}}(F) d\chi$, we understand it as the weighted Euler characteristic. The Formulas (1), (2) are essential to the wall-crossing of Donaldson-Thomas invariants as studied in [27], and [9], since they imply that the morphism from the motivic Hall algebra of \mathcal{A} to the ring of functions of the quantum torus is a Poisson algebra homomorphism. Then the wall-crossing techniques can be applied to get relations between generalized Donaldson-Thomas invariants.

. **(1.1.4)** Let $D^b(\mathcal{A}) := D^b(\text{Coh}(Y))$ be the bounded derived category of coherent sheaves on Y . An object $E \in D^b(\mathcal{A})$ is called semi-Schur if $\text{Ext}^{<0}(E, E) = 0$. It is very interesting to study these formulas for semi-Schur objects in the derived category $D^b(\text{Coh}(Y))$ of coherent sheaves on Y . Note that in [15] V. Bussi uses the (-1) -shifted symplectic structure on the moduli stack \mathfrak{M} of coherent sheaves to prove such Behrend function identities, where her proof relies on the local structure of the moduli stack in [28]. In [24], we use Berkovich spaces to prove these formulas.

1.2. The motivic Donaldson-Thomas invariants.

. **(1.2.1)** As in §(1.1.2) the Donaldson-Thomas invariant is the weighted Euler characteristic $\chi(X, \nu_X)$ for a Donaldson-Thomas moduli scheme X . One can ask if there exists a global defined perverse sheaf \mathcal{F} such that

$$\chi(X, \mathcal{F}) = \chi(X, \nu_X).$$

Such an idea is true if the moduli scheme X is the critical locus of a global regular function (or holomorphic) function $f : M \rightarrow \kappa$ on a higher dimensional smooth scheme M . Then the value of Behrend function ν_X is given by

$$\nu_X(P) = (-1)^{\dim(M)} (1 - \chi(\mathbb{F}_P)),$$

where \mathbb{F}_P is the Milnor fiber of f at $P \in X$. The sheaf \mathcal{F} is the perverse sheaf $\varphi_f[-1]$ of vanishing cycles of f and it is known that

$$\chi(X, \varphi_f[-\dim(M)]|_P) = \nu_X(P).$$

Let $i : M_0 \hookrightarrow M$ be the inclusion, where $M_0 = f^{-1}(0)$. The vanishing cycle sheaf φ_f is defined by

$$i^*\mathbb{C} \rightarrow \psi_f(\mathbb{C}) \rightarrow \varphi_f(\mathbb{C}) \xrightarrow{[1]} \dots$$

where ψ_f is the nearby cycle, and it is seen that the vanishing cycle supports on the critical locus of f . The nearby cycle can be understood as the nearby Milnor fiber. Thus it is interesting to lift the Donaldson-Thomas invariants to the motivic level of cycles.

. **(1.2.2)** Let $\mathcal{M}_\kappa = K(\text{Var}_\kappa)[\mathbb{L}^{-1}]$ be the motivic ring which will be reviewed in §2.1, where $K(\text{Var}_\kappa)$ is the Grothendieck ring of varieties. Similarly, let $\hat{\mu} = \varprojlim \mu_n$ and let $\mathcal{M}_\kappa^{\hat{\mu}} = K^{\hat{\mu}}(\text{Var}_\kappa)[\mathbb{L}^{-1}]$ be the equivariant motivic ring, where $K^{\hat{\mu}}(\text{Var}_\kappa)$ is the equivariant Grothendieck ring of varieties. The motivic Donaldson-Thomas theory on any Calabi-Yau three category was developed by Kontsevich-Soibelman in [31]. For instance, Kontsevich-Soibelman defined a motivic weight $MF(E) \in \mathcal{M}_\kappa^{\hat{\mu}}$ for any derived object $E \in D^b(\mathcal{A})$, which is given by the motivic Milnor fiber of E . Then Kontsevich-Soibelman prove that there exists an algebra homomorphism from the motivic Hall algebra $H(\mathcal{A})$ on the abelian category \mathcal{A} to the motivic quantum torus based on a conjecture about the motivic Milnor fibers. Then using the homomorphism Kontsevich-Soibelman prove a wall crossing formula for their motivic Donaldson-Thomas invariants.

The Kontsevich-Soibelman conjecture on the motivic Milnor fiber has been proved by Le in [32], [33] using the method of motivic integration.

. **(1.2.3)** The degree zero motivic Donaldson-Thomas invariants for any smooth projective threefold Y was studied by Behrend, Bryan and Szendroi in [2]. The essential point is the case of $Y = \mathbb{C}^3$, where the degree zero Donaldson-Thomas moduli space is the Hilbert scheme $\text{Hilb}^n(\mathbb{C}^3)$ of n -points on \mathbb{C}^3 which is the critical locus of a regular on a smooth higher dimensional variety. In this case the motivic Donaldson-Thomas invariants are the motive of vanishing cycles of the regular function.

1.3. The motivic Joyce-Song formula.

. **(1.3.1)** We follow the proposal of Joyce-Song in [27] to study the motivic Donaldson-Thomas invariants. In [24] we study the Joyce-Song formula using Berkovich spaces [4], and find that the techniques there can be generalized to the motivic level. In the paper [24, §6], we make the

conjecture for the motivic version of the Joyce-Song formulas. We briefly review the conjecture.

. **(1.3.2)** Since the moduli stack of derived objects has not been constructed yet, we fix a Bridgeland stability condition on $D^b(\text{Coh}(Y))$ and the heart \mathcal{A} of the corresponding bounded t -structure is an abelian category. Any object $E \in \mathcal{A}$ satisfies the condition that $\text{Ext}^{<0}(E, E) = 0$. Hence it is semi-Schur. The moduli stack \mathfrak{M} of objects in \mathcal{A} can be constructed, which is an Artin stack locally of finite type. For an arbitrary object E in the heart \mathcal{A} , we assume that E is Schur or stable under some stability condition, i.e. $\text{Ext}^i(E, E) = 0$ only except for $i = 1, 2$. There is a cyclic dg Lie algebra $R\text{Hom}(E, E)$ corresponding to E . On the cohomology $L_E := \text{Ext}^*(E, E)$ there is a cyclic L_∞ -algebra structure coming from the transfer theorem. In [23], [24], we define the Euler characteristic $\chi(E)$ of E by the Euler characteristic of the cyclic L_∞ -algebra $\text{Ext}^*(E, E)$ or the dg Lie algebra $R\text{Hom}(E, E)$. Donaldson-Thomas invariants count stable objects in the derived category and this Euler characteristic is equal to the pointed Donaldson-Thomas invariant given by the point E in the moduli space.

. **(1.3.3)** If E is semi-Schur, the cyclic L_∞ -algebra $\text{Ext}^*(E, E)$ defines a potential function

$$f : \text{Ext}^1(E, E) \rightarrow \kappa$$

on $\text{Ext}^1(E, E)$, see [23]. In general, f is a formal power series.

In the case of coherent sheaves, Joyce-Song prove that f is actually holomorphic, see [27]. In [28], Joyce etc use (-1) -shifted symplectic structure of [47] on the moduli of stable sheaves over smooth Calabi-Yau threefolds to show that the local potential function is actually algebraic, i.e. a regular function. The Euler characteristic of the topological Milnor fiber associated with the regular function f gives the pointed Donaldson-Thomas invariant.

. **(1.3.4)** Let \mathbb{K} be a non-archimedean complete discretely valued field of characteristic zero. The ring of integers of \mathbb{K} is denoted by R , and the residue field is denoted by κ . Our main example is $R = \kappa[[t]]$, and the corresponding nonarchimedean field $\mathbb{K} = \kappa((t))$.

Associated with the formal potential function f , there is a generically smooth *special* formal R -scheme:

$$\hat{f} : \mathfrak{X} \rightarrow \text{spf}(R),$$

see [7], [44]. If f is a regular function, then (\mathfrak{X}, \hat{f}) is the t -adic completion of the morphism $f : \text{Ext}^1(E, E) \rightarrow \kappa = \text{Spec}(\kappa[t])$. The generic fiber \mathfrak{X}_η is a rigid \mathbb{K} -variety, or a Berkovich space in sense of [7]. There exists a specialization map

$$sp : \mathfrak{X}_\eta \rightarrow \mathfrak{X}_0$$

from the generic fiber to the reduction \mathfrak{X}_0 , which is a κ -variety. For any $y \in \mathfrak{X}_0$, the *Analytic Milnor Fiber* $\mathfrak{F}_y(f)$ of y is defined as

$$\mathfrak{F}_y(f) := sp^{-1}(y).$$

The analytic Milnor fiber $\mathfrak{F}_y(f)$ is an analytic subspace of \mathfrak{X}_η . If we let

$$\hat{f}_y : \mathfrak{X}_y := \text{spf}(\hat{\mathcal{O}}_{\mathfrak{X},y}) \rightarrow \text{spf}(R)$$

to be the formal completion of \mathfrak{X} along $y \in \mathfrak{X}_0$, then from [44] the analytic Milnor fiber $\mathfrak{F}_y(f)$ is the generic fiber of the formal scheme \mathfrak{X}_y .

. **(1.3.5)** The formal R -scheme (\mathfrak{X}, \hat{f}) is quasi-excellent in sense of Temkin [48]. Let

$$h : \mathfrak{Y} \longrightarrow \mathfrak{X}$$

be the resolution of singularities of the formal scheme \mathfrak{X} . Let $E_i, i \in \mathcal{I}$ be the set of irreducible components of the exceptional divisors of h . For any $I \subset \mathcal{I}$ let

$$E_I := \bigcap_{i \in I} E_i$$

and

$$E_I^\circ := E_I \setminus \bigcup_{j \notin I} E_j.$$

Let $m_I = \gcd(m_i)_{i \in I}$, where m_i are the multiplicities of the components E_i . Then there is an Galois cover

$$\tilde{E}_I^\circ \rightarrow E_I^\circ$$

with Galois group μ_{m_I} . Hence we get an $\hat{\mu}$ -action on \tilde{E}_I° . See §2.2.11 and [44] for more details on the resolution of singularities. The following definition is given in [23], [24].

Definition 1.1. *The motivic Milnor fiber of the object E is defined as follows:*

$$\mathcal{S}_0(E) := \mathcal{S}_0(\hat{f}) := \sum_{\emptyset \neq I \subset \mathcal{I}} (1 - \mathbb{L})^{|I|-1} [\tilde{E}_I^\circ \cap h^{-1}(0)].$$

It is clear that $\mathcal{S}_0(E) \in \mathcal{M}_\kappa^{\hat{\mu}}$. From [44], the motivic volume of the analytic Milnor fiber is given by the motivic Milnor fiber, which we review in §2.2.

Of course, if we have a formal subscheme $\mathfrak{Z} \subset \mathfrak{X}$, then we define $\mathcal{S}_\mathfrak{Z}(\hat{f})$ to be the motivic Milnor fiber of \mathfrak{Z} :

$$\mathcal{S}_\mathfrak{Z}(\hat{f}) := \sum_{\emptyset \neq I \subset \mathcal{I}} (1 - \mathbb{L})^{|I|-1} [\tilde{E}_I^\circ \cap h^{-1}(\mathfrak{Z})].$$

. **(1.3.6)** We introduce the following localized ring of motives:

$$\mathcal{M}_{X,\text{loc}} = \mathcal{M}_X[\mathbb{L}^{-1/2}, (\mathbb{L}^i - 1)^{-1}, i \in \mathbb{N}_{>0}]$$

and

$$\mathcal{M}_{X,\text{loc}}^{\hat{\mu}} = \mathcal{M}_X^{\hat{\mu}}[\mathbb{L}^{-1/2}, (\mathbb{L}^i - 1)^{-1}, i \in \mathbb{N}_{>0}].$$

Let $E_1, E_2, E_1 \oplus E_2$ be semi-Schur objects in the derived category of coherent sheaves over Y . We introduce the conjecture for the motivic version of Joyce-Song formulas in [24]. First we have:

$$\text{Ext}^1(E, E) = \text{Ext}^1(E_1, E_1) \oplus \text{Ext}^1(E_2, E_2) \oplus \text{Ext}^1(E_1, E_2) \oplus \text{Ext}^1(E_2, E_1).$$

The conjecture is a motivic version of the Joyce-Song formula for Behrend function identities in §(1.1.3).

Conjecture 1.2. (1)

$$(1 - \mathcal{S}_{((0,0))}(E_1 \oplus E_2)) = (1 - \mathcal{S}_0(E_1)) \cdot (1 - \mathcal{S}_0(E_2)).$$

(2)

$$\begin{aligned} & \int_{F \in \mathbb{P}(\text{Ext}^1(E_2, E_1))} (1 - \mathcal{S}_0(F)) - \int_{F \in \mathbb{P}(\text{Ext}^1(E_1, E_2))} (1 - \mathcal{S}_0(F)) \\ &= ([\mathbb{P}^{\dim \text{Ext}^1(E_2, E_1)}] - [\mathbb{P}^{\dim \text{Ext}^1(E_1, E_2)}]) \left(1 - \mathcal{S}_{f|_{\text{Ext}^1(E_1, E_1) \oplus \text{Ext}^1(E_2, E_2)}} \right)^0. \end{aligned}$$

where $\int_{\mathfrak{M}_0}(-) : \mathcal{M}_{\mathfrak{M}_0}^{\hat{\mu}} \rightarrow \mathcal{M}_{\kappa}^{\hat{\mu}}$ is the pushforward of motives.

. **(1.3.7)** We give an explanation about the conjectural formulas. For any semi-Schur object $E \in D^b(\mathcal{A})$, $\mathcal{S}_0(E)$ is the motivic Milnor fiber of E , and $(1 - \mathcal{S}_0(E))$ is the analogue of motivic vanishing cycle. Let $E := E_1 \oplus E_2$. Let

$$\phi : \tilde{\mathfrak{X}} \rightarrow \mathfrak{X} := \widehat{\text{Ext}^1(E, E)} \rightarrow \text{spf}(R)$$

be the formal blow-up of \mathfrak{X} along the completion $\mathfrak{Y} \subset \mathfrak{X}$, where $\mathfrak{Y} = \widehat{V}$ is the formal completion of V , and $V := \text{Ext}^1(E_1, E_1) \oplus \text{Ext}^1(E_2, E_2) \oplus 0 \oplus \text{Ext}^1(E_2, E_1) \subset \widehat{\text{Ext}^1(E, E)}$. We denote by $\mathfrak{Z} := \widehat{\text{Ext}^1(E_1, E_2)} \subset \mathfrak{X}$. Let $\mathbb{P}(\mathfrak{Z}) := \mathbb{P}(\text{Ext}^1(E_1, E_2)) \subset \tilde{\mathfrak{X}}$ be the closed formal subscheme of $\tilde{\mathfrak{X}}$. The corresponding reduction scheme is denoted by $\mathbb{P}(\mathfrak{Z})_0 = \mathbb{P}(\text{Ext}^1(E_1, E_2))$. Since the motivic vanishing cycle is constructible, then the integration

$$\int_{F \in \mathbb{P}(\text{Ext}^1(E_2, E_1))} (1 - \mathcal{S}_0(F))$$

can be understood as the motivic cycle $\mathcal{S}_{\mathbb{P}(\mathfrak{Z})_0}(\tilde{f})$, where $\tilde{f} = \phi \circ \hat{f} : \tilde{\mathfrak{X}} \rightarrow \text{spf}(R)$ is the formal R -scheme $\tilde{\mathfrak{X}}$ of the composition $\phi \circ \hat{f}$.

Remark 1.3. *The Euler characteristic of the motivic Milnor fiber $\mathcal{S}_0(E)$ is, plus the correct sign, the value of the Behrend function $v_{\mathfrak{M}}$ on $E \in \mathfrak{M}$. Hence taking the Euler characteristic on the formulas (1), (2) in Conjecture 1.2, when putting the right signs, we get the Joyce-Song formula (1), (2) in §(1.1.3).*

. **(1.3.8)** Our main result in this paper is to prove the above conjecture.

Theorem 1.4. *Conjecture 1.2 is true in $\mathcal{M}_{\kappa, \text{loc}}^{\hat{\mu}}$.*

. **(1.3.9)** Recall that for any semi-Schur object $E \in D^b(\text{Coh}(Y))$, we have a super potential function $f : \text{Ext}^1(E, E) \rightarrow \kappa$, which is from the cyclic L_∞ -algebra structure on $\text{Ext}^*(E, E)$. Taking completion we get a special formal scheme $\hat{f} : \hat{\mathfrak{X}} := \widehat{\text{Ext}^1(E, E)} \rightarrow \text{spf}(R)$. We use Nicaise's motivic integration for formal schemes in [44], and the method of Cluckers-Loeser [17] of motivic constructible functions as studied in [33] to prove the above formula. Actually our idea is motivated by Le's study of Kontsevich-Soibelman Conjecture in [32], [33].

It turns out that the positive techniques of motivic constructible functions in [17] used here is that it is convenient to show that the motivic volume of an annulus in the analytic Milnor fiber space is zero, which helps us to prove the Formula (1) in the conjecture. We hope that such an idea may help to study the motivic Donaldson-Thomas invariants under a torus action, parallel to the work of Maulik in [40].

. **(1.3.10)** The Formula (1) is similar to Kontsevich-Soibelman Conjecture 4.2 in [31], which is proved by Le in [33] using the same method in [17]. The Conjecture 4.2 in [31] plays an important role in the wall crossing of motivic Donaldson-Thomas theory of Kontsevich and Soibelman. We follow the proposal of Joyce, and Conjecture 1.2 is essential for the study of the wall crossing of the motivic Donaldson-Thomas invariants by defining a global motive for the Donaldson-Thomas moduli scheme in [14].

1.4. Applications.

. **(1.4.1)** One application of the motivic Joyce-Song formulas in Conjecture 1.2 is to prove a Poisson algebra homomorphism from the motivic Hall algebra $H(\mathcal{A})$ to the motivic quantum torus $\overline{\mathcal{M}}_{\kappa, \text{loc}}^{\hat{\mu}}[\Gamma]$, thus generalizing the Lie algebra homomorphism in [27, Theorem 5.14], and the Poisson algebra homomorphism in [10, Theorem 5.2] to the motivic level. Here the ring $\overline{\mathcal{M}}_{\kappa, \text{loc}}^{\hat{\mu}}[\Gamma]$ is roughly defined as follows. The ring $\mathcal{M}_{\kappa, \text{loc}}^{\hat{\mu}}[\Gamma]$ is a formal power series ring over $\mathcal{M}_{\kappa, \text{loc}}^{\hat{\mu}}$ generated by symbols x^α for $\alpha \in \Gamma$, where Γ

is the effective classes of the numerical K-group of Y . The ring $\overline{\mathcal{M}}_{\kappa, \text{loc}}^{\hat{\mu}}[\Gamma]$ is the quotient of the ring $\mathcal{M}_{\kappa, \text{loc}}^{\hat{\mu}}[\Gamma]$ modulo the relations

$$Y(Q_1 \otimes Q_2) - Y(Q_1) \odot Y(Q_2)$$

for quadratic forms Q_1, Q_2 and $Y(Q_i)$ are the motive of the quadratic forms for $i = 1, 2$. This is related to the triangle property of the orientation data in [31] and have applications to the wall crossing of motivic Donaldson-Thomas invariants.

. **(1.4.2)** The motivic Hall algebra $H(\mathcal{A})$ is a $K(\text{Var}_{\mathfrak{M}})[\mathbb{L}^{-1}]$ -module. We define a submodule of $H(\mathcal{A})$ by the elements $[X \rightarrow \mathfrak{M}]$ such that X is an algebraic d -critical locus in the sense of [28]. We call it the d -critical elements of $H(\mathcal{A})$ and denote it by $H_{d\text{-Crit}}(\mathcal{A})$. Then let

$$H_{\text{ssc}, d\text{-Crit}}(\mathcal{A}) = H_{d\text{-Crit}}(\mathcal{A}) / (\mathbb{L} - 1)H_{d\text{-Crit}}(\mathcal{A}).$$

We define the integration map

$$I : H_{\text{ssc}, d\text{-Crit}}(\mathcal{A}) \rightarrow \overline{\mathcal{M}}_{\kappa}^{\hat{\mu}}[\Gamma]$$

by taking the global motivic sheaf \mathcal{S}_X^{ϕ} for the algebraic d -critical locus X . By [12], if the algebraic d -critical locus (X, s) has an orientation, which is a root line bundle $K_{X, s}^{1/2}$ for the canonical line bundle $K_{X, s}$, then there exists a global motivic sheaf $\mathcal{S}_X^{\phi} \in \overline{\mathcal{M}}_X^{\hat{\mu}}$, where $\overline{\mathcal{M}}_X^{\hat{\mu}}$ is defined similarly to $\overline{\mathcal{M}}_{\kappa}^{\hat{\mu}}$ by considering the motives of quadratic forms over X . The sheaf \mathcal{S}_X^{ϕ} , when restricted to the local critical chart of X , is the perverse sheaf of vanishing cycles times the motive of a quadratic form over X . In this paper we always assume that there exists an orientation. Please see §4 for more details. The algebra $H_{\text{ssc}, d\text{-Crit}}(\mathcal{A})$ is called the semi-classical part of the Hall algebra and has a Poisson bracket, see §(4.3.6). We also define a Poisson bracket on the ring $\overline{\mathcal{M}}_{\kappa}^{\hat{\mu}}[\Gamma]$, see §(4.3.8). We prove that the integration map I is a Poisson algebra homomorphism.

. **(1.4.3)** Another application is to apply the Poisson algebra homomorphism to prove the motivic DT/PT-correspondence, and the flop-formula for the motivic Donaldson-Thomas invariants. In [46], Pandharipande-Thomas define another curve-counting invariants: the stable pair invariants, and conjecture that the stable pair invariants are the same as the Donaldson-Thomas invariants of the moduli space of ideal sheaves. The DT/PT-correspondence conjecture was prove by Bridgeland in [9] using the idea of the Hall algebra identities and the Poisson algebra homomorphism from the motivic Hall algebra to the ring of functions on the quantum torus, i.e. the classical part of the motivic quantum torus. The Euler characteristic level of this conjecture and the flop formula were proved by Toda in [50], [51] using Joyce's wall crossing formula for changing the Bridgeland stability conditions. Calabrese [16] proved the

flop formula using similar idea in [9]. Using the motivic integration map in this paper we should be able to prove the motivic version of the DT/PT-correspondence and the flop formula by the Hall algebra identity method of Bridgeland in [9], see [26].

1.5. Outline.

. **(1.5.1)** The outline of the paper is as follows. The materials about motivic integration are reviewed in §2, where in §2.1 we review the Grothendieck ring of varieties, and in §2.2 we briefly talk about the motivic integration of rigid varieties from formal scheme models following [44]. In §3 we prove Theorem 1.4. Here in §3.1 we prove a motivic blow-up formula for the motivic Milnor fibers, generalizing the one in [27, §4.1]; In §3.2 we review the techniques of motivic constructible functions in [17]; in §3.3 we prove Formula (1) in Conjecture 1.2; and in §3.4 we prove Formula (2) in Conjecture 1.2. Combining sections §3.3 and §3.4, Theorem 1.4 is proved. Section §4 serves as the proof of the Poisson algebra homomorphism from the motivic Hall algebra to the motivic quantum torus, where in §4.1 we introduce the motivic Hall algebra $H(\mathcal{A})$ for the abelian category \mathcal{A} ; in §4.2 we briefly review the notion of algebraic d -critical locus of Joyce in [28]; in §4.3 we define the integration map; and in §4.4 we prove that the integration map is a Poisson algebra homomorphism.

Convention. Throughout the paper we work over an algebraically closed field κ so that the nonarchimedean field is $\kappa((t))$ and its ring of integers is $R = \kappa[[t]]$. For the applications in §4, we consider the schemes and stacks over $\kappa = \mathbb{C}$, the field of complex numbers.

For a Berkovich analytic space \mathfrak{F} , we use $\chi(\mathfrak{F})$ to represent the Euler characteristics the étale cohomology of \mathfrak{F} . We use \mathbb{L} to represent the Lefschetz motive $[\mathbb{A}_\kappa^1]$.

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2. THE MOTIVIC MILNOR FIBRE AND THE MOTIVIC VOLUMES.

2.1. Grothendieck group of varieties.

. **(2.1.1)** In this section we briefly review the Grothendieck group of varieties. Let S be an algebraic variety over κ . Let Var_S be the category of S -varieties.

. **(2.1.2)** Let $K_0(\text{Var}_S)$ be the Grothendieck group of S -varieties. By definition $K_0(\text{Var}_S)$ is an abelian group with generators given by all the varieties $[X]$'s, where $X \rightarrow S$ are S -varieties, and the relations are $[X] = [Y]$, if X is isomorphic to Y , and $[X] = [Y] + [X \setminus Y]$ if Y is a Zariski closed subvariety of X . Let $[X], [Y] \in K_0(\text{Var}_S)$, and define $[X][Y] = [X \times_S Y]$. Then we have a product on $K_0(\text{Var}_S)$. Let \mathbb{L} represent the class of $[\mathbb{A}_\kappa^1 \times S]$. Let $\mathcal{M}_S = K_0(\text{Var}_S)[\mathbb{L}^{-1}]$ be the ring by inverting the class \mathbb{L} in the ring $K_0(\text{Var}_S)$.

If S is a point $\text{Spec}(\kappa)$, we write $K_0(\text{Var}_\kappa)$ for the Grothendieck ring of κ -varieties. One can take the map $\text{Var}_\kappa \rightarrow K_0(\text{Var}_\kappa)$ to be the universal Euler characteristic. After inverting the class $\mathbb{L} = [\mathbb{A}_\kappa^1]$, we get the ring \mathcal{M}_κ .

. **(2.1.3)** We introduce the equivariant Grothendieck group defined in [19]. Let μ_n be the cyclic group of order n , which can be taken as the algebraic variety $\text{Spec}(\kappa[x]/(x^n - 1))$. Let $\mu_{md} \rightarrow \mu_n$ be the map $x \mapsto x^d$. Then all the groups μ_n form a projective system. Let

$$\varprojlim_n \mu_n$$

be the direct limit.

Suppose that X is a S -variety. The action $\mu_n \times X \rightarrow X$ is called a *good* action if each orbit is contained in an affine subvariety of X . A good $\hat{\mu}$ -action on X is an action of $\hat{\mu}$ which factors through a good μ_n -action for some n .

The *equivariant Grothendieck group* $K_0^{\hat{\mu}}(\text{Var}_S)$ is defined as follows: The generators are S -varieties $[X]$ with a good $\hat{\mu}$ -action; and the relations are: $[X, \hat{\mu}] = [Y, \hat{\mu}]$ if X is isomorphic to Y as $\hat{\mu}$ -equivariant S -varieties, and $[X, \hat{\mu}] = [Y, \hat{\mu}] + [X \setminus Y, \hat{\mu}]$ if Y is a Zariski closed subvariety of X with the $\hat{\mu}$ -action induced from that on X , if V is an affine variety with a good $\hat{\mu}$ -action, then $[X \times V, \hat{\mu}] = [X \times \mathbb{A}_\kappa^n, \hat{\mu}]$. The group $K_0^{\hat{\mu}}(\text{Var}_S)$ has a ring structure if we define the product as the fibre product with the good $\hat{\mu}$ -action. Still we let \mathbb{L} represent the class $[S \times \mathbb{A}_\kappa^1, \hat{\mu}]$ and let $\mathcal{M}_S^{\hat{\mu}} = K_0^{\hat{\mu}}(\text{Var}_S)[\mathbb{L}^{-1}]$ be the ring obtained from $K_0^{\hat{\mu}}(\text{Var}_S)$ by inverting the class \mathbb{L} .

If $S = \text{Spec}(\kappa)$, then we write $K_0^{\hat{\mu}}(\text{Var}_S)$ as $K_0^{\hat{\mu}}(\text{Var}_\kappa)$, and $\mathcal{M}_S^{\hat{\mu}}$ as $\mathcal{M}_\kappa^{\hat{\mu}}$. Let $s \in S$ be a geometric point. Then we have natural maps $K_0^{\hat{\mu}}(\text{Var}_S) \rightarrow K_0^{\hat{\mu}}(\text{Var}_\kappa)$ and $\mathcal{M}_S^{\hat{\mu}} \rightarrow \mathcal{M}_\kappa^{\hat{\mu}}$ given by the correspondence $[X, \hat{\mu}] \mapsto [X_s, \hat{\mu}]$.

. **(2.1.4)** Let S be a scheme. Following [14], we need to define a new product \odot on $\mathcal{M}_S^{\hat{\mu}}$. The following definition is due to [14, Definition 2.3].

Definition 2.1. Let $[X, \hat{\sigma}], [Y, \hat{\tau}]$ be two elements in $K_0^{\hat{\mu}}(\text{Var}_S)$ or $\mathcal{M}_S^{\hat{\mu}}$. Then there exists $n \geq 1$ such that the $\hat{\mu}$ -actions $\hat{\sigma}, \hat{\tau}$ on X, Y factor through μ_n -actions σ_n, τ_n . Define J_n to be the Fermat curve

$$J_n = \{(t, u) \in (\mathbb{A}^1 \setminus \{0\})^2 : t^n + u^n = 1\}.$$

Let $\mu_n \times \mu_n$ act on $J_n \times (X \times_S Y)$ by

$$(\alpha, \alpha') \cdot ((t, u), (v, w)) = ((\alpha \cdot t, \alpha' \cdot u), (\sigma_n(\alpha)(v), \tau_n(\alpha')(w))).$$

Write $J_n(X, Y) = (J_n \times (X \times_S Y)) / (\mu_n \times \mu_n)$ for the quotient κ -scheme, and define a μ_n -action v_n on $J_n(X, Y)$ by

$$v_n(\alpha)((t, u), v, w)(\mu_n \times \mu_n) = ((\alpha \cdot t, \alpha \cdot u), v, w)(\mu_n \times \mu_n).$$

Let \hat{v} be the induced good $\hat{\mu}$ -action on $J_n(X, Y)$, and set

$$[X, \hat{\sigma}] \odot [Y, \hat{\tau}] = (\mathbb{L} - 1) \cdot [(X \times_S Y / \mu_n, \hat{\iota})] - [J_n(X, Y), \hat{v}]$$

in $K_0^{\hat{\mu}}(\text{Var}_S)$ or $\mathcal{M}_S^{\hat{\mu}}$. This defines a commutative, associative product on $K_0^{\hat{\mu}}(\text{Var}_S)$ or $\mathcal{M}_S^{\hat{\mu}}$.

Consider the Lefschetz motive $\mathbb{L} = [\mathbb{A}_{\kappa}^1]$. As in [14], we define $\mathbb{L}^{\frac{1}{2}}$ in $K_0^{\hat{\mu}}(\text{Var}_S)$ or $\mathcal{M}_S^{\hat{\mu}}$ by:

$$\mathbb{L}^{\frac{1}{2}} = [S, \hat{\iota}] - [S \times \mu_2, \hat{\rho}],$$

where $[S, \hat{\iota}]$ with trivial $\hat{\mu}$ -action $\hat{\iota}$ is the identity in $K_0^{\hat{\mu}}(\text{Var}_S)$ or $\mathcal{M}_S^{\hat{\mu}}$, and $S \times \mu_2$ is the two copies of S with the nontrivial $\hat{\mu}$ -action $\hat{\rho}$ induced by the left action of μ_2 on itself, exchanging the two copies of S . Then $\mathbb{L}^{\frac{1}{2}} \odot \mathbb{L}^{\frac{1}{2}} = \mathbb{L}$.

2.2. Motivic integration on rigid varieties.

. **(2.2.1)** Let \mathbb{K} be a non-archimedean complete discretely valued field of characteristic zero. The ring of integers of \mathbb{K} is denoted by R , and the residue field is denoted by κ . For instance, $R = \kappa[[t]]$ and $\mathbb{K} = \kappa((t))$ is the fractional field of R . We fix a uniformizing parameter π in R throughout the paper, and in the case that $R = \kappa[[t]]$, $\pi = t$.

. **(2.2.2)** Let $\mathfrak{X} \rightarrow \text{spf}(R)$ be a separated generically smooth formal scheme over R of topologically of finite type. We call such types of R -formal schemes *stft* R -formal schemes. A *stft* R -formal scheme \mathfrak{X} is obtained by gluing finite open covers by affine *stft* formal R -schemes. Each affine *stft* formal R -scheme is of the form

$$\text{spf}(A) \rightarrow \text{spf}(R)$$

for a topologically of finite type R -algebra A , which is isomorphic to an algebra of the form $R\{x_1, \dots, x_m\}/I$ for some integer $m > 0$ and some ideal I , where $R\{x_1, \dots, x_m\}$ is the algebra of converging power series over R .

The *special fiber* \mathfrak{X}_0 for an affine *stft* formal R -scheme $\mathfrak{X} = \mathrm{spf}(A)$ is the κ -scheme $\mathfrak{X}_0 = \mathrm{Spec}(A_0)$, where A_0 is the κ -algebra A/J with J the largest ideal of definition. In general the affine covers of $\mathrm{Spec}(A_0)$ glue to give the κ -scheme \mathfrak{X}_0 for any *stft* formal R -scheme \mathfrak{X} .

. **(2.2.3)** Let \mathfrak{X} be a generically smooth *stft* formal R -scheme. The generic fiber \mathfrak{X}_η is rigid \mathbb{K} -variety. The construction is obtained by gluing the constructions on affine charts. If $\mathfrak{X} = \mathrm{spf}(A)$ is affine, recall that from [45, §4.8, §4.9], there is a specialization map

$$sp : |\mathfrak{X}_\eta| \rightarrow |\mathfrak{X}| = |\mathfrak{X}_0|$$

such that if \mathfrak{U} is any open formal subscheme of \mathfrak{X} , then $sp^{-1}(\mathfrak{U})$ is an admissible open in \mathfrak{X}_η . Thus the generic fibers \mathfrak{U}_i of an affine open covers of a *stft* formal R -scheme \mathfrak{X} can be glued along the generic fibers of the intersections $\mathfrak{U}_i \cap \mathfrak{U}_j$ to obtain a rigid \mathbb{K} -variety \mathfrak{X}_η . The specialization maps glue to give a continuous map

$$(2.2.4) \quad sp : |\mathfrak{X}_\eta| \rightarrow |\mathfrak{X}| = |\mathfrak{X}_0|.$$

In the language of Berkovich analytic spaces, the analytification of \mathfrak{X}_η is a Berkovich analytic space over the nonarchimedean field \mathbb{K} in sense of Berkovich [4]. We still denote it by \mathfrak{X}_η . The construction is also obtained by gluing the constructions on affine charts. For $\mathfrak{X} = \mathrm{spf}(A)$ is affine, let $\mathcal{A} = A \otimes_R \mathbb{K}$ then $\mathfrak{X}_\eta = \mathcal{M}(\mathcal{A})$ is the spectrum of the affinoid \mathbb{K} -algebra \mathcal{A} , which consists of all bounded multiplicative semi-norms $x : \mathcal{A} \rightarrow \mathbb{R}_+$. The affine Berkovich spaces $\mathcal{M}(\mathcal{A})$ glue to give us the Berkovich analytic space \mathfrak{X}_η .

Finally we recall the following result in [45]. The construction of the generic fiber is functorial. A morphism of *stft* formal R -schemes $h : \mathfrak{Y} \rightarrow \mathfrak{X}$ induces a morphism of rigid \mathbb{K} -varieties $h_\eta : \mathfrak{Y}_\eta \rightarrow \mathfrak{X}_\eta$ and the square

$$(2.2.5) \quad \begin{array}{ccc} \mathfrak{Y}_\eta & \xrightarrow{h_\eta} & \mathfrak{X}_\eta \\ sp \downarrow & & \downarrow sp \\ \mathfrak{Y} & \xrightarrow{h} & \mathfrak{X} \end{array}$$

commutes. Thus there is a functor

$$(\cdot)_\eta : (\mathrm{stft} - \mathrm{For}/R) \rightarrow (\mathrm{sqc} - \mathrm{Rig}/\mathbb{K}) : \mathfrak{X} \mapsto \mathfrak{X}_\eta$$

from the category of *stft* formal R -schemes to the category of separated, quasi-compact rigid \mathbb{K} -varieties.

. **(2.2.6)** Let \mathfrak{X} be a generically smooth *stft* formal R -scheme. We follow the construction of Nicaise-Sebag, Nicaise in [42], [44] for the definition of the motivic integration of a gauge form ω on \mathfrak{X}_η , which takes values in $\mathcal{M}_{\mathfrak{X}_0}$.

We briefly recall the method to define the motivic integration $\int_{\mathfrak{X}} |\omega|$. First we have

$$\mathfrak{X} = \varinjlim_m \mathcal{X}_m,$$

where $\mathcal{X}_m = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}} \otimes_R R_m)$ and $R_m = R/(\pi)^{m+1}$. In Greenberg [20], the functor

$$\mathcal{Y} \mapsto \mathrm{Hom}_{R_m}(\mathcal{Y} \times_{\kappa} R_m, \mathcal{X}_m)$$

from the category of κ -schemes to the category of sets is presented by a κ -scheme

$$\mathrm{Gr}_m(\mathcal{X}_m)$$

of finite type such that

$$\mathrm{Gr}_m(\mathcal{X}_m)(A) = \mathcal{X}_m(A \otimes_{\kappa} R_m)$$

for any κ -algebra A . The projective limit $\varinjlim_m \mathcal{X}_m$ is denoted by $\mathrm{Gr}(\mathfrak{X})$. The functor Gr respects open and closed immersions and fiber products, and sends affine topologically of finite type formal R -schemes to affine κ -schemes. The motivic integration of a gauge form ω is defined by using the stable cylindrical subsets of $\mathrm{Gr}(\mathfrak{X})$, introduced by Loeser-Sebag in [36], and Nicaise-Sebag in [43].

Let $\mathbf{C}_{0,\mathfrak{X}}$ be the set of stable cylindrical subsets of $\mathrm{Gr}(\mathfrak{X})$ of some level. If $A \subset \mathbf{C}_{0,\mathfrak{X}}$ is a cylinder, and we have a function

$$\alpha : A \rightarrow \mathbb{Z} \cup \{\infty\}$$

such that $\alpha^{-1}(m) \subset \mathbf{C}_{0,\mathfrak{X}}$. Then

$$\int_A [\mathbb{A}_{\mathfrak{X}_0}^1]^{-\alpha} d\tilde{\mu} := \sum_{m \in \mathbb{Z}} \tilde{\mu}(\alpha^{-1}(m)) \cdot [\mathbb{A}_{\mathfrak{X}_0}^1]^{-m},$$

where

$$\tilde{\mu} : \mathbf{C}_{0,\mathfrak{X}} \rightarrow \mathcal{M}_{\mathfrak{X}_0}$$

is the unique additive morphism defined in [32, Proposition 5.1] by

$$\tilde{\mu}(A) = [\pi_m(A)] \cdot [\mathbb{A}_{\mathfrak{X}_0}^1]^{-(m+1)d}$$

for A a stable cylinder of level m , d is the relative dimension of \mathfrak{X} , and $\pi_m : \mathrm{Gr}(\mathfrak{X}) \rightarrow \mathrm{Gr}(\mathcal{X}_m)$ is the canonical projection.

Let ω be a gauge form on \mathfrak{X}_{η} , in [36], the authors constructed an integral-valued function

$$\mathrm{ord}_{\pi,\mathfrak{X}}(\omega)$$

on $\mathrm{Gr}(\mathfrak{X})$ that takes the role of α before. The motivic integration $\int_{\mathfrak{X}} |\omega|$ is defined to be

$$(2.2.7) \quad \int_{\mathfrak{X}} |\omega| := \int_{\mathrm{Gr}(\mathfrak{X})} [\mathbb{A}_{\mathfrak{X}_0}^1]^{-\mathrm{ord}_{\pi,\mathfrak{X}}(\omega)} d\tilde{\mu} \in \mathcal{M}_{\mathfrak{X}_0}.$$

From [36], [42], the forgetful map

$$\int : \mathcal{M}_{\mathfrak{X}_0} \rightarrow \mathcal{M}_{\kappa}$$

defined by

$$\int_{\mathfrak{X}} |\omega| \mapsto \int_{\mathfrak{X}_\eta} |\omega| := \int \int_{\mathfrak{X}} |\omega|$$

only depends on \mathfrak{X}_η , not on \mathfrak{X} .

. **(2.2.8)** In [44] Nicaise generalizes the motivic integration construction to generically smooth special formal R -schemes. A special formal R -scheme \mathfrak{X} is a separated Noetherian adic formal scheme endowed with a structural morphism $\mathfrak{X} \rightarrow \mathrm{spf}(R)$, such that \mathfrak{X} is a finite union of open formal subschemes which are formal spectra of special R -algebras. From Berkovich [7], a topological R -algebra A is special, iff A is topologically R -isomorphic to a quotient of the special R -algebra

$$R\{T_1, \dots, T_m\}[[S_1, \dots, S_n]] = R[[S_1, \dots, S_n]]\{T_1, \dots, T_m\}.$$

The Noetherian adic formal scheme \mathfrak{X} has the largest ideal of definition J . The closed subscheme of \mathfrak{X} defined by J is denoted by \mathfrak{X}_0 , which is a reduced Noetherian κ -scheme.

. **(2.2.9)** We briefly review the motivic integration of Nicaise in [44].

Definition 2.2. Let \mathfrak{X} be a special formal R -scheme. By a *Néron smoothening* we mean a morphism of special formal R -schemes $\mathfrak{Y} \rightarrow \mathfrak{X}$, such that \mathfrak{Y} is adic smooth over R and $\mathfrak{Y}_\eta \rightarrow \mathfrak{X}_\eta$ is an open embedding satisfying $\mathfrak{Y}_\eta(\tilde{K}) = \mathfrak{X}_\eta(\tilde{\mathbb{K}})$ for any finite unramified extension $\tilde{\mathbb{K}}$ of \mathbb{K} .

In [44, §2], Nicaise proves that a Néron smoothening of \mathfrak{X} exists and is given by the dilatation of \mathfrak{X} . Then \mathfrak{Y} is a stft formal R -scheme.

Definition 2.3. Let \mathfrak{X} be a generically smooth special formal R -scheme. We define

$$\int_{\mathfrak{X}} |\omega| := \int_{\mathfrak{Y}} |\omega|$$

and

$$\int_{\mathfrak{X}_\eta} |\omega| := \int_{\mathfrak{Y}_\eta} |\omega|$$

for a gauge form ω on \mathfrak{X}_η .

. **(2.2.10)** We recall the motivic volume of \mathfrak{X}_η in [44]. For $m \geq 1$, let $\mathbb{K}(m) := \mathbb{K}[T]/(T^m - \pi)$ be a totally ramified extension of degree m of \mathbb{K} , and $R(m) := R[T]/(T^m - \pi)$ the normalization of R in $\mathbb{K}(m)$. If \mathfrak{X} is a formal R -scheme, we define

$$\mathfrak{X}(m) := \mathfrak{X} \times_R R(m)$$

and

$$\mathfrak{X}_\eta(m) := \mathfrak{X}_\eta \times_{\mathbb{K}} \mathbb{K}(m).$$

If ω is a gauge form on \mathfrak{X}_η , we denote by $\omega(m)$ the pullback of ω via the natural morphism $\mathfrak{X}_\eta(m) \rightarrow \mathfrak{X}_\eta$.

Definition 2.4. Let \mathfrak{X} be a generically smooth special formal R -scheme. Let ω be a gauge form on \mathfrak{X}_η . Then the volume Poincaré series of (\mathfrak{X}, ω) is defined to be

$$S(\mathfrak{X}, \omega; T) := \sum_{d>0} \left(\int_{\mathfrak{X}(d)} |\omega(d)| \right) T^d \in \mathcal{M}_{\mathfrak{X}_0}[[T]].$$

. (2.2.11)

Definition 2.5. Let \mathfrak{X} be a generically smooth flat R -formal scheme. A resolution of singularities of \mathfrak{X} is a proper morphism $h : \mathfrak{Y} \rightarrow \mathfrak{X}$ of flat special formal R -schemes such that h induces an isomorphism on generic fibers, and such that \mathfrak{Y} is regular (meaning the local ring at points is regular), with a special fiber a strict normal crossing divisor \mathfrak{Y}_s . We say that the resolution h is tame if \mathfrak{Y}_s is a tame normal crossing divisor.

By Temkin's resolution of singularities for quasi-excellent schemes of characteristic zero in [48], any affine generically smooth flat special formal R -scheme $\mathfrak{X} = \text{spf}(A)$ admits a resolution of singularities by means of admissible blow-ups.

In general for any generically smooth R -formal scheme \mathfrak{X} , suppose that there is a resolution of singularities

$$(2.2.12) \quad h : \mathfrak{Y} \longrightarrow \mathfrak{X}$$

Let E_i , $i \in \mathcal{I}$, be the set of irreducible components of the exceptional divisors of the resolution. For $I \subset \mathcal{I}$, we set

$$E_I := \bigcap_{i \in I} E_i$$

and

$$E_I^\circ := E_I \setminus \bigcup_{j \notin I} E_j.$$

Let m_i be the multiplicity of the component E_i , which means that the special fiber of the resolution is

$$\sum_{i \in \mathcal{I}} m_i E_i.$$

Let $m_I = \gcd(m_i)_{i \in I}$. Let U be an affine Zariski open subset of \mathfrak{Y} , such that, on U , $f \circ h = uv^{m_I}$, with u a unit in U and v a morphism from U to $\mathbb{A}_{\mathbb{C}}^1$. The restriction of $E_I^\circ \cap U$, which we denote by $\tilde{E}_I^\circ \cap U$, is defined by

$$\{(z, y) \in \mathbb{A}_{\mathbb{C}}^1 \times (E_I^\circ \cap U) \mid z^{m_I} = u^{-1}\}.$$

The E_I° can be covered by the open subsets U of Y . We can glue together all such constructions and get the Galois cover

$$\tilde{E}_I^\circ \longrightarrow E_I^\circ$$

with Galois group μ_{m_I} . Remember that $\hat{\mu} = \varprojlim \mu_n$ is the direct limit of the groups μ_n . Then there is a natural $\hat{\mu}$ action on \tilde{E}_I° . Thus we get $[\tilde{E}_I^\circ] \in \mathcal{M}_{X_0}^{\hat{\mu}}$.

. (2.2.13) Using resolution of singularities, in [44, Theorem 7.12], Nicaise proves the following result:

Theorem 2.6. *Let \mathfrak{X} be a generically smooth special formal R -scheme of pure relative dimension d . Then we have a structural morphism $f : \mathfrak{X} \rightarrow \mathrm{spf}(R)$. Suppose that \mathfrak{X} has a resolution of singularities $\mathfrak{X}' \rightarrow \mathfrak{X}$ with special fiber $\mathfrak{X}'_s = \sum_{i \in I} N_i E_i$.*

Let ω be a \mathfrak{X} -bounded gauge form on \mathfrak{X}_η , where the definition of bounded gauge form is given by Nicaise in [44, Definition 2.11]. Then for any integer $m > 0$,

$$\int_{\mathfrak{X}(m)} |\omega(m)| = \mathbb{L}^{-d} \sum_{\emptyset \neq J \subset I} (\mathbb{L} - 1)^{|J|-1} [\tilde{E}_J^\circ] \left(\sum_{\substack{k_i \geq 1, i \in J \\ \sum_{i \in J} k_i N_i = d}} \mathbb{L}^{-\sum_i k_i \mu_i} \right) \in \mathcal{M}_{\mathfrak{X}_0}^{\mu_m}.$$

Furthermore, from [44, Corollary 7.13] we have:

Proposition 2.7. *With the same notations and conditions as in Theorem 2.6, the volume Poincaré series $S(\mathfrak{X}, \omega; T)$ is rational over $\mathcal{M}_{\mathfrak{X}_0}$. In fact, let $\mu_i := \mathrm{ord}_{E_i} \omega$, then*

$$S(\mathfrak{X}, \omega; T) = \mathbb{L}^{-d} \sum_{\emptyset \neq J \subset I} (\mathbb{L} - 1)^{|J|-1} [\tilde{E}_J^\circ] \prod_{i \in J} \frac{\mathbb{L}^{-\mu_i} T^{N_i}}{1 - \mathbb{L}^{-\mu_i} T^{N_i}} \in \mathcal{M}_{\mathfrak{X}_0}^{\hat{\mu}}[[T]].$$

The limit

$$S(\mathfrak{X}, \hat{\mathbb{K}}^s) := - \lim_{T \rightarrow \infty} S(\mathfrak{X}, \omega; T) := \mathbb{L}^{-d} \mathcal{S}_f$$

is called the *motivic volume* of \mathfrak{X} , where

$$\mathcal{S}_f = \sum_{\emptyset \neq J \subset I} (\mathbb{L} - 1)^{|J|-1} [\tilde{E}_J^\circ].$$

And

$$\begin{aligned} S(\mathfrak{X}_\eta, \hat{\mathbb{K}}^s) &:= - \lim_{T \rightarrow \infty} S(\mathfrak{X}_\eta, \omega; T) = - \lim_{T \rightarrow \infty} \sum_{m \geq 1} \left(\int_{\mathfrak{X}_\eta} |\omega(m)| \right) T^m \\ &= \mathbb{L}^{-d} \int_{\mathfrak{X}_0} \mathcal{S}_f \in \mathcal{M}_{\mathfrak{X}_0}^{\hat{\mu}} \end{aligned}$$

is called the *motivic volume* of \mathfrak{X}_η .

. (2.2.14) Let (\mathfrak{X}, f) be a generically smooth formal R -scheme. From Proposition 2.7, the motivic vanishing cycle \mathcal{S}_f belongs to $\mathcal{M}_{\mathfrak{X}_0}^{\hat{\mu}}$. For any point $x \in \mathfrak{X}_0$, let

$$\mathcal{S}_{f,x} = \sum_{\emptyset \neq J \subset I} (\mathbb{L} - 1)^{|J|-1} [\tilde{E}_J^\circ \cap h^{-1}(x)],$$

where $h : \mathfrak{X}' \rightarrow \mathfrak{X}$ is the resolution of singularities. We call $\mathcal{S}_{f,x}$ the motivic Milnor fiber of $x \in \mathfrak{X}_0$.

. **(2.2.15)** In summary, if we let $K(\text{GBSRig}_{\mathbb{K}})$ be the Grothendieck ring of the category of gauge bounded smooth rigid \mathbb{K} -varieties. Here for an object \mathfrak{X}_η in $\text{GBSRig}_{\mathbb{K}}$ we understand that the rigid variety \mathfrak{X}_η comes from the generic fiber of a generically smooth special formal R -scheme $f : \mathfrak{X} \rightarrow \text{spf}(R)$ with gauge bounded form ω . The Grothendieck ring

$$K(\text{GBSRig}_{\mathbb{K}}) := \bigoplus_{d \geq 0} K(\text{GBSRig}_{\mathbb{K}}^d)$$

is defined in [32, §5.2].

Let $K(\text{BSRig}_{\mathbb{K}})$ be the Grothendieck ring of the category $\text{BSRig}_{\mathbb{K}}$ of bounded smooth rigid \mathbb{K} -varieties, which is obtained from $K(\text{GBSRig}_{\mathbb{K}})$ by forgetting the gauge form. Then we can represent the above results in §(2.2.13) as follows:

Theorem 2.8. *There exists a homomorphism of additive groups:*

$$\text{MV} : K(\text{BSRig}_{\mathbb{K}}) \rightarrow \mathcal{M}_k^\mu$$

given by:

$$[\mathfrak{X}_\eta] \mapsto S(\mathfrak{X}_\eta, \widehat{\mathbb{K}}^s)$$

for a generically smooth special formal R -scheme \mathfrak{X} . Moreover, if \mathfrak{X} has relative dimension d , then

$$\text{MV}([\mathfrak{X}_\eta]) = \mathbb{L}^{-d} \cdot \int_{\mathfrak{X}_0} \mathcal{S}_f \in \mathcal{M}_k^\mu.$$

So MV is a morphism from the group $K(\text{BSRig}_{\mathbb{K}})$ to the group \mathcal{M}_k^μ .

Moreover, if $x \in \mathfrak{X}_0$ and let

$$\hat{f}_x : \text{spf}(\widehat{\mathcal{O}}_{\mathfrak{X},x}) \rightarrow \text{spf}(R)$$

be the formal completion of \mathfrak{X} along x , then the generic fiber $\text{spf}(\widehat{\mathcal{O}}_{\mathfrak{X},x})_\eta$ of the formal completion is the analytic Milnor fiber $\mathfrak{F}_x(\hat{f})$ defined in 1.3.4, and

$$\text{MV}([\mathfrak{F}_x(\hat{f})]) = \mathbb{L}^d \cdot \mathcal{S}_{f,x}.$$

3. PROOF OF THE CONJECTURE

3.1. A motivic Blow-up formula.

. **(3.1.1)** We prove a motivic blow-up formula for motivic Milnor fibers in the formal scheme setting, thus generalizing the one in [27, Theorem 4.11] for Behrend functions. Using Berkovich spaces, such a blow-up formula is proved in the Euler characteristic level in [24, Proposition 3.13].

(3.1.2) Let $f : \mathfrak{X} \rightarrow \mathrm{spf}(R)$ be a smooth special formal R -scheme and $\mathfrak{Z} \subset \mathfrak{X}$ a closed embedded formal subscheme. Let

$$\phi : \tilde{\mathfrak{X}} \rightarrow \mathfrak{X}$$

be the formal blow-up of \mathfrak{X} along \mathfrak{Z} . Set

$$\tilde{f} := f \circ \phi : \tilde{\mathfrak{X}} \rightarrow \mathrm{spf}(R).$$

For details of formal blow-up for *special* formal schemes see [44]. Let $y \in \mathfrak{Z} \cap \mathrm{Crit}(f)$, then $\phi^{-1}(y) = \mathbb{P}(T_y \mathfrak{X} / T_y \mathfrak{Z})$ is contained in $\mathrm{Crit}(\tilde{f})$.

Proposition 3.1. *We have the following formula*

$$\int_{\mathbb{P}(T_y \mathfrak{X} / T_y \mathfrak{Z})} \mathcal{S}_{\tilde{f}} = \mathcal{S}_{f,y} + [\mathbb{P}^{\dim(\mathfrak{X}) - \dim(\mathfrak{Z}) - 1} - 1] \cdot \mathcal{S}_{f|_{\mathfrak{Z},y}} \in \mathcal{M}_{\kappa}^{\hat{\mu}},$$

where $\int_{\mathbb{P}(T_y \mathfrak{X} / T_y \mathfrak{Z})}$ is understood as the pushforward from $\mathcal{M}_{\mathbb{P}(T_y \mathfrak{X} / T_y \mathfrak{Z})}^{\hat{\mu}} \rightarrow \mathcal{M}_{\kappa}^{\hat{\mu}}$.

Proof. It is enough to prove the formula for the case of affine formal schemes. Let $\mathfrak{X} = \mathrm{spf}(A)$, where $A = R\{T_1, \dots, T_m\}[[S_1, \dots, S_n]]$, and $\mathfrak{Z} = \mathrm{spf}(B)$, where $B = R\{T_1, \dots, T_l\}[[S_1, \dots, S_n]]$ for $l < m$. The ideal $I = (T_{l+1}, \dots, T_m) \subset A$ is an open ideal. The formal blow-up

$$\tilde{\mathfrak{X}} = \varinjlim_{n \in \mathbb{N}} \mathbf{Proj} \left(\bigoplus_{d=0}^{\infty} I^d \otimes_R (R/t^n) \right).$$

The morphism ϕ induces the following commutative diagram:

$$(3.1.3) \quad \begin{array}{ccc} \tilde{\mathfrak{X}}_{\eta} & \xrightarrow{\phi_{\eta}} & \mathfrak{X}_{\eta} \\ sp \downarrow & & \downarrow sp \\ \tilde{\mathfrak{X}}_0 & \xrightarrow{\phi_0} & \mathfrak{X}_0 \end{array}$$

where sp is the specialization map from the generic fiber to the special fiber. For any $y \in \mathfrak{Z}_0 \subset \mathfrak{X}_0$,

$$(sp \circ \phi_{\eta})^{-1}(y) = (\phi_0 \circ sp)^{-1}(y).$$

From the argument of $(sp \circ \phi_{\eta})^{-1}(y)$ and $(\phi_0 \circ sp)^{-1}(y)$ as in the proof of [24, Proposition 3.13], the left side is the preimage $sp^{-1}(\mathbb{P}(T_y \mathfrak{X} / T_y \mathfrak{Z}))$, and the the right side is

$$\left(\mathfrak{F}_y(\hat{f}|_{\mathfrak{Z}}) \times \mathbb{P}_{\mathbb{K}}^{\dim(\mathfrak{X}) - \dim(\mathfrak{Z}) - 1} \right) \sqcup (\mathfrak{F}_y(\hat{f}) \setminus \mathfrak{F}_y(\hat{f}|_{\mathfrak{Z}})).$$

Then applying the map MV in Theorem 2.8 we get that the left side is the motivic cycle $\int_{\mathbb{P}(T_y \mathfrak{X} / T_y \mathfrak{Z})} \mathcal{S}_{\tilde{f}}$, while the right side is

$$\mathcal{S}_{f,y} + [\mathbb{P}^{\dim(\mathfrak{X}) - \dim(\mathfrak{Z}) - 1} - 1] \cdot \mathcal{S}_{f|_{\mathfrak{Z},y}}.$$

□

3.2. Techniques on the motivic constructible functions of Cluckers and Loeser.

. (3.2.1) In this section we learn a little bit about Cluckers-Loeser's motivic constructible function theory in [17], which we will use to prove the conjecture. Le in [32] uses another method of Hrushovski-Kazhdan's ACVF theory in [22] to prove the Kontsevich-Soibelman conjecture on the motivic Milnor fiber. Later on he can use the theory of motivic constructible functions of Cluckers-Loeser to give a new proof, which is working over any field of characteristic zero. We adapt such a beautiful theory to our applications for the motivic Joyce-Song formula.

. (3.2.2) The theory of motivic constructible functions is motivated by the constructible functions for the Euler characteristic over reals. The idea of Cluckers-Loeser is to do integration (functions defined on) subobjects of $\kappa((t))^m$, or more wiser, integration on subobjects of

$$\kappa((t))^m \times \kappa^n \times \mathbb{Z}^r.$$

The theory is based on the Denef-Pas language \mathcal{L}_{DP} with the ring language for valued fields and residue fields and with the Presburger language for valued groups. Let \mathbb{T} be the theory of algebraic closed fields containing κ , as in [17, §16.2, 16.3], then $(\mathbb{K}((t)), \mathbb{K}, \mathbb{Z})$ is a model of \mathbb{T} . The primary definable \mathbb{T} -subassignment has the forms:

$$h[m, n, r](\mathbb{K}) := \mathbb{K}((t))^m \times \mathbb{K}^n \times \mathbb{Z}^r.$$

It can be taken as a functor

$$h[m, n, r] : \mathbb{K} \supset \kappa \rightarrow \text{Category of sets.}$$

Any formula φ in \mathcal{L}_{DP} with coefficients in $\kappa((t))$, and coefficients in κ , defines a subassignment $h_\varphi \subset h[m, n, r]$ by:

$$h_\varphi(\mathbb{K}) = \{x \in h[m, n, r](\mathbb{K}) \mid (\mathbb{K}, \mathbb{K}((t)), \mathbb{Z}) = \varphi(x)\}.$$

More generally, if $W = \mathfrak{X} \times X \times \mathbb{Z}^r$, with \mathfrak{X} a $\kappa((t))$ -variety, X a κ -variety, then

$$h_W(\mathbb{K}) := \mathfrak{X}(\mathbb{K}((t))) \times X(\mathbb{K}) \times \mathbb{Z}^r.$$

Definition 3.2. We define Def_κ to be the category of all the definable \mathbb{T} -subassignments

$$\mathbb{K} \mapsto h_\varphi(\mathbb{K}).$$

Let $S \subset \text{Def}_\kappa$ be any object. Let Def_S **or** $(\text{Def}_S(\mathcal{L}_{\text{DP}}, \mathbb{T}))$ be the category of objects of Def_κ over S . Define

$$\text{RDef}_S; \text{ or } (\text{RDef}_S(\mathcal{L}_{\text{DP}}, \mathbb{T}))$$

to be the subcategory of Def_S whose objects are subassignments of $S \times h_{\mathbb{A}_\kappa^n}$, for variable n , morphisms to S are the ones induced by the projection onto the S -factor.

We define the Grothendieck group for RDef_S .

Definition 3.3. *The Grothendieck group $K_0(\text{RDef}_S)$ is defined to be a free abelian group generated by symbols:*

$$[X \rightarrow S]$$

with $X \rightarrow S$ in RDef_S , modulo the relations:

$$[X \rightarrow S] = [Y \rightarrow S]$$

if $[X \rightarrow S]$ is isomorphic to $[Y \rightarrow S]$ in RDef_S , and

$$[X \cup Y \rightarrow S] + [X \cap Y \rightarrow S] = [X \rightarrow S] + [Y \rightarrow S]$$

for any definable \mathbb{T} -subassignments X and Y of $S \times h_{\mathbb{A}_\kappa^n}$ for some $n \in \mathbb{N}$.

. **(3.2.3)** Let $X \rightarrow S$ be an object in RDef_S and $m \in \mathbb{N}_{>0}$. Assume that $X = h_W$ with $W = \mathfrak{X} \times X \times \mathbb{Z}^r$. A good μ_m -action on X is a μ_m -action

$$\mu_m \times X \rightarrow X$$

on X such that each orbit intersected with h_X is contained in h_V with V an affine subvariety of X . A good $\hat{\mu}$ -action on X is a $\hat{\mu}$ -action on X that factors through a good μ_m -action on X for some $m \in \mathbb{N}_{>0}$.

Definition 3.4. *The monodromic Grothendieck group $K_0^{\hat{\mu}}(\text{RDef}_S)$ is a free abelian group generated by:*

$$[X \rightarrow S, \hat{\mu}]; ([X, \hat{\mu}])$$

with $X \rightarrow S$ in RDef_S , and X admits a $\hat{\mu}$ -action, with the relations in Definition 3.3, together with one more relation:

$$[X \times h_V, \hat{\mu}] = [X \times h_{\mathbb{A}_\kappa^n}, \hat{\mu}],$$

where V is the n -dimensional affine κ -space endowed with an linear $\hat{\mu}$ -action and \mathbb{A}_κ^n with trivial $\hat{\mu}$ -action for $n \in \mathbb{N}$.

The groups $K_0(\text{RDef}_S)$ and $K_0^{\hat{\mu}}(\text{RDef}_S)$ are rings with respect to the fiber product of subassignments in [17, §2.2].

. **(3.2.4)** We talk about the rings of motivic constructible functions. Let

$$A := \mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}, (1 - \mathbb{L}^i)^{-1}, i > 0],$$

where \mathbb{L} is the Lefschetz motive of the affine line \mathbb{A}_κ^1 . For $S \in \text{Def}_\kappa$, let $\mathcal{P}(S)$ be the subring of the ring of functions

$$S \rightarrow A$$

generated by:

- (1) all constant functions into A ;
- (2) all definable functions $S \rightarrow \mathbb{Z}$;
- (3) all functions of the form \mathbb{L}^α , where $\alpha : S \rightarrow \mathbb{Z}$ is a definable function.

This is called the ring of Presburger functions as in [17].

Let $\mathcal{P}^0(S)$ be the subring of $\mathcal{P}(S)$ generated by $\mathbb{L} - 1$ and by character function 1_Y for all definable subassignments Y of S .

Definition 3.5. *The ring of constructible motivic functions $\mathcal{C}(S)$ on S and the monodromic one $\mathcal{C}^{\hat{\mu}}(S)$ are defined as:*

$$\mathcal{C}(S) := K_0(\mathrm{RDef}_S) \otimes_{\mathcal{P}^0(S)} \mathcal{P}(S); \quad \mathcal{C}^{\hat{\mu}}(S) := K_0^{\hat{\mu}}(\mathrm{RDef}_S) \otimes_{\mathcal{P}^0(S)} \mathcal{P}(S).$$

The following result can be found in [33], [17, §16.2, §16.3].

Proposition 3.6. *Let X be an algebraic variety. Then*

- (1) $K_0(\mathrm{RDef}_{h_X}) \cong K_0(\mathrm{Var}_X)$;
- (2) $\mathcal{C}(h_X) \cong \mathcal{M}_{X,\mathrm{loc}}$;
- (3) $K_0^{\hat{\mu}}(\mathrm{RDef}_{h_X}) \cong K_0^{\hat{\mu}}(\mathrm{Var}_X)$;
- (4) $\mathcal{C}^{\hat{\mu}}(h_X) \cong \mathcal{M}_{X,\mathrm{loc}}^{\hat{\mu}}$.

. (3.2.5) We talk about the rationality results of the motivic constructible functions, which we refer to [17, §4.4-5.7]. Let $S \in \mathrm{Def}_k$, let $r \in \mathbb{N}_{>0}$ and $T = (T_1, \dots, T_r)$ be variables. Denote by $\mathcal{C}(S)[[T]]$ to be the formal power series ring with coefficients in $\mathcal{C}(S)$. If $\alpha : S \rightarrow \mathbb{N}^r$ is a definable function, let

$$T^\alpha := \sum_{i \in \mathbb{N}^r} \mathbb{1}_{C_j} T^i,$$

where

$$C_j := \{x \in S \mid \alpha(x) = j\}.$$

Let $\mathcal{C}(S)\{T\}$ be the $\mathcal{C}(S)$ -subalgebra of $\mathcal{C}(S)[[T]]$ generated by the series T^α with $\alpha : S \rightarrow \mathbb{N}^r$ definable. Let Γ be the multiplicative set of polynomials in $\mathcal{C}(S)[[T]]$ generated by $1 - \mathbb{L}^a T^b$ with $(a, b) \in \mathbb{Z} \times \mathbb{N}^r, b \neq 0$. We denote by

$$\mathcal{C}(S)\{T\}_\Gamma$$

the localization of $\mathcal{C}(S)\{T\}$ with respect to Γ and by

$$\mathcal{C}(S)[[T]]_\Gamma$$

the image of the injective morphism of rings

$$\mathcal{C}(S)\{T\}_\Gamma \rightarrow \mathcal{C}(S)[[T]]_\Gamma.$$

Let us also consider the $\mathcal{C}(S)$ -module

$$\mathcal{C}(S)[[T, T^{-1}]].$$

It is a ring defined by the Hadamard product: for

$$f = \sum_{i \in \mathbb{Z}^r} a_i T^i; \quad g = \sum_{i \in \mathbb{Z}^r} b_i T^i,$$

$$f * g := \sum_{i \in \mathbb{Z}^r} a_i b_i T^i.$$

The subrings $\mathcal{X}(S)[[T]]_\Gamma$ and $\mathcal{X}(S)[[T, T^{-1}]]_\Gamma$ are stable by the Hadamard product.

Let $\varphi \in \mathcal{C}(S \times \mathbb{Z}^r)$ and $i \in \mathbb{Z}^r$, or $\varphi \in \mathcal{C}(S \times \mathbb{N}^r)$ and $i \in \mathbb{N}^r$, we denote by φ_i the restriction of φ to $S \times \{i\}$ and consider it as an element of $\mathcal{C}(S)$. Define:

$$M(\varphi) := \sum_{i \in \Delta} \varphi_i T^i,$$

which is a series in $\mathcal{C}(S)[[T, T^{-1}]]$, where $\Delta = \mathbb{Z}^r$ or \mathbb{N}^r depending on $\varphi \in \mathcal{C}(S \times \mathbb{Z}^r)$ or $\mathcal{C}(S \times \mathbb{N}^r)$.

Theorem 3.7. ([17], [33]) *The mapping*

$$\mathcal{C}^{\hat{\mu}}(h_X \times \mathbb{N}^r) \rightarrow \mathcal{M}_{X, \text{loc}}^{\hat{\mu}}[[T]]_{\Gamma}$$

defined by

$$\varphi \mapsto M(\varphi)$$

is an isomorphism of rings.

3.3. The proof of Formula (1) in Conjecture 1.2.

. **(3.3.1)** Recall that for coherent sheaves or semi-Schur objects $E_1, E_2, E := E_1 \oplus E_2 \in D^b(\text{Coh}(Y))$ for a smooth Calabi-Yau threefold Y , we have the following data:

- (1) $f : \text{Ext}^1(E, E) \rightarrow \kappa$;
- (2) $f_1 : \text{Ext}^1(E_1, E_1) \rightarrow \kappa$;
- (3) $f_2 : \text{Ext}^1(E_2, E_2) \rightarrow \kappa$,

where f, f_1, f_2 are the corresponding superpotential functions coming from the corresponding cyclic L_{∞} -algebras studied in [24].

We have:

$$\text{Ext}^1(E, E) = \text{Ext}^1(E_1, E_1) \oplus \text{Ext}^1(E_2, E_2) \oplus \text{Ext}^1(E_1, E_2) \oplus \text{Ext}^1(E_2, E_1)$$

and let (x, y, z, w) be the corresponding coordinates of $\text{Ext}^1(E, E)$. Let \mathbb{C}^* act on $\text{Ext}^1(E, E)$ by

$$\lambda \cdot (x, y, z, w) = (x, y, \lambda \cdot z, \lambda^{-1} \cdot w).$$

Then $\text{Ext}^1(E, E)^{\mathbb{C}^*} = \text{Ext}^1(E_1, E_1) \oplus \text{Ext}^1(E_2, E_2)$.

. **(3.3.2)** For any semi-Schur object $E \in D^b(\text{Coh}(Y))$, we have the motivic Milnor fiber $\mathcal{S}_0(E) \in \mathcal{M}_{\kappa}^{\hat{\mu}}$ defined in Definition 1.1. It is given by the generically smooth special formal R -scheme

$$\hat{f} := \hat{f}_E : \widehat{\text{Ext}^1(E, E)} \rightarrow \text{spf}(R)$$

which is the formal completion of $f := f_E : \text{Ext}^1(E, E) \rightarrow \kappa$ along the origin.

. **(3.3.3)** We make the following notations:

$$X := \text{Ext}^1(E, E); \quad Z := \text{Ext}^1(E_1, E_1) \oplus \text{Ext}^1(E_2, E_2).$$

For $E = E_1 \oplus E_2$, let

$$\hat{f} : \mathfrak{X} := \widehat{X} \rightarrow \text{spf}(R)$$

be the formal completion of X along the origin. Then

$$\hat{f}|_Z : \mathfrak{Z} \rightarrow \text{spf}(R)$$

is the formal completion of $f|_Z$ along the origin. Let $d = \dim(X)$, $d_i := \dim_\kappa(\text{Ext}^1(E_i, E_i))$ for $i = 1, 2$ and $d_{12} = \dim_\kappa(\text{Ext}^1(E_1, E_2))$, $d_{21} = \dim_\kappa(\text{Ext}^1(E_2, E_1))$.

For the formal R -scheme \mathfrak{X} , we have the generic fiber:

$$\mathfrak{X}_\eta = \left\{ (x, y, z, w) \in \mathbb{A}_{\mathbb{K}}^{d, \text{an}} \mid \begin{array}{l} \text{val}(x) > 0, \text{val}(y) > 0; \\ \text{val}(z) > 0, \text{val}(w) > 0; \\ f(x, y, z, w) = t. \end{array} \right\}$$

Here $\text{val}(x) := \min_{1 \leq i \leq d_1} \{\text{val}(x_i)\}$, and $\text{val}(y), \text{val}(z), \text{val}(w)$ are similarly defined. We divide the generic fiber \mathfrak{X}_η into two parts:

$$\mathfrak{X}_\eta = X_0 \sqcup X_1,$$

where

$$X_0 = \{(x, y, z, w) \in \mathfrak{X}_\eta \mid z = 0 \text{ or } w = 0\}$$

and

$$X_1 = \{(x, y, z, w) \in \mathfrak{X}_\eta \mid z \neq 0, w \neq 0\} = \mathfrak{X}_\eta \setminus X_0.$$

Theorem 3.8. *We have:*

$$\mathcal{S}_{\hat{f}, 0} = \mathbb{L}^{d-1} \cdot \text{MV}([\mathfrak{X}_\eta])$$

and

$$\mathcal{S}_{\hat{f}|_Z, 0} = \mathbb{L}^{d-1} \cdot \text{MV}([X_0]),$$

where

$$\text{MV} : K(\text{BSRig}_{\mathbb{K}}) \rightarrow \mathcal{M}_{\kappa}^{\hat{\mu}}$$

is the homomorphism in Theorem 2.8.

Proof. The first formula is just from Theorem 2.8. We prove:

$$\mathcal{S}_{\hat{f}|_Z, 0} = \mathbb{L}^{d-1} \cdot \text{MV}([X_0]).$$

By the property of the potential function $f : X \rightarrow \kappa$, if $z = 0$ or $w = 0$, then $f(x, y, z, w) = f(x, y, 0, 0)$. Then we may write:

$$X_0 = Y_0 \times \mathfrak{Z}_\eta,$$

where

$$Y_0 = \{(z, w) \in \mathbb{A}_{\mathbb{K}}^{d_3+d_4, \text{an}} \mid \text{val}(z) > 0, \text{val}(w) > 0\}$$

and

$$\mathfrak{Z}_\eta = \left\{ (x, y, 0, 0) \in \mathbb{A}_{\mathbb{K}}^{d_1+d_2, \text{an}} \mid \begin{array}{l} \text{val}(x) > 0, \text{val}(y) > 0; \\ f(x, y, 0, 0) = t. \end{array} \right\}$$

Let $dz \wedge dw := dz_1 \wedge \cdots \wedge dz_{d_3} \wedge dw_1 \wedge \cdots \wedge dw_{d_4}$ be the standard gauge form on the open ball Y_0 . From Theorem 2.6,

$$\int_{Y_0(m)} |dz \wedge dw(m)| = \mathbb{L}^{-d_3-d_4}.$$

So

$$\begin{aligned} \text{MV}([X_0]) &= \text{MV}([Y_0 \times \mathfrak{Z}_\eta]) = - \lim_{T \rightarrow \infty} \sum_{m \geq 1} \left(\int_{Y_0(m) \times \mathfrak{Z}_\eta(m)} |dz \wedge dw \wedge \omega(m)| \right) T^m \\ &= -\mathbb{L}^{-d_3-d_4} \lim_{T \rightarrow \infty} \sum_{m \geq 1} \left(\int_{\mathfrak{Z}_\eta(m)} |\omega(m)| \right) T^m \\ &= \mathbb{L}^{1-d} \mathcal{S}_{\hat{f}|_Y, 0}. \end{aligned}$$

So

$$\mathbb{L}^{d-1} \cdot \text{MV}([X_0]) = \mathcal{S}_{\hat{f}|_Y, 0}.$$

□

. (3.3.4) We prove the following result:

Theorem 3.9. *We have*

$$\text{MV}([X_1]) = 0$$

in $\mathcal{M}_{\kappa, \text{loc}}^{\hat{\mu}}$.

Proof. First similar to [33, Theorem 5.1], we argue that $\text{MV}([X_1]) \in \mathcal{C}^{\hat{\mu}}(\mathbb{N}_{>0})$, with structural map:

$$\theta : (x, y, z, w) \mapsto \text{val}(z) + \text{val}(w).$$

From Theorem 2.7,

$$\text{MV}([X_1]) = - \lim_{T \rightarrow \infty} \sum_{m \geq 1} \left(\int_{X_1(m)} |\omega(m)| \right) T^m,$$

with ω a gauge form on X_1 . By choosing a formal model \mathfrak{X}_1 of X_1 and a Néron smoothening \mathfrak{X}' , according to [44, §4],

$$\begin{aligned} \int_{X_1(m)} |\omega(m)| &= \\ \int_{\mathfrak{X}'_0} \sum_{n \in \mathbb{Z}} [\{(x, y, z, w) \in \text{Gr}_l \mathfrak{X}'(m) \mid \text{ord}_{t^{1/m}, \mathfrak{X}'(m)}(\omega(m))(x, y, z, w) = n\} \rightarrow \mathfrak{X}'_0] \end{aligned}$$

So the correspondence

$$(x, y, z, w) \mapsto \text{ord}_{t^{1/m}}(z) + \text{ord}_{t^{1/m}}(w)$$

defines a mapping

$$\theta_m : \int_{X_1(m)} |\omega(m)| \rightarrow \mathbb{N}_{>0}$$

for each $m \in \mathbb{N}_{>0}$. All of these maps θ_m give a map:

$$\theta : \text{MV}([X_1]) \rightarrow \mathbb{N}_{>0}.$$

So $\text{MV}([X_1])$ can be taken as an element in $\mathcal{C}^{\text{ft}}(\mathbb{N}_{>0})$ with structure morphism θ .

Let $n \in \mathbb{N}_{>0}$, and $\theta_m^{-1}(n)$ is a definable subset of $\int_{X_1(m)} |\omega(m)|$ defined by

$$\text{val}(z) + \text{val}(w) = n$$

i.e.

$$\theta^{-1}(n) = \text{MV}([X_1, n])$$

where

$$X_{1,n} := \bigcup_{m \geq 1} \{(x, y, z, w) \in X_1 \mid \text{val}(z) + \text{val}(w) = \frac{n}{m}\}.$$

Lemma 3.10. *We have:*

$$\text{MV}\left(\left[\{(x, y, z, w) \in X_1 \mid \text{val}(z) + \text{val}(w) = \frac{n}{m}\}\right]\right) = 0.$$

Proof. Let

$$Y := \{(x, y, z, w) \in X_1 \mid \text{val}(z) + \text{val}(w) = \frac{n}{m}\}.$$

The linear analytic group $\mathbb{G}_m := \mathbb{G}_{m, \mathbb{K}^{\text{alg}}}$ acts on

$$Z := \mathbb{A}_{\mathbb{K}}^{d_1+d_2, \text{an}} \times \left(\mathbb{A}_{\mathbb{K}}^{d_3, \text{an}} \setminus \{0\}\right) \times \left(\mathbb{A}_{\mathbb{K}}^{d_4, \text{an}} \setminus \{0\}\right)$$

by

$$\lambda \cdot (x, y, z, w) = (x, y, \lambda z, \lambda^{-1} w)$$

for $\lambda \in \mathbb{G}_m$. Let

$$\pi : Z \rightarrow Z / \mathbb{G}_m$$

be the projection and $\bar{Y} \subset Z / \mathbb{G}_m$ be the image of Y under π . Hence

$$Y \rightarrow \bar{Y}$$

is a fibration with fiber

$$\{(x, y, \lambda z, \lambda^{-1} w) \mid -\text{val}(z) \leq \text{val}(\lambda) \leq \text{val}(w)\}$$

over the class $[(x, y, z, w)] \in \bar{Y}$. The fiber is isomorphic to the annulus:

$$A_{\frac{n}{m}} := \{\lambda \in \mathbb{G}_m \mid 0 \leq \text{val}(\lambda) \leq \frac{n}{m}\}.$$

So

$$[Y] = [\bar{Y}] \cdot [A_{\frac{n}{m}}] = [\bar{Y} \times A_{\frac{n}{m}}]$$

and

$$\begin{aligned} \text{MV}([\Upsilon]) &= - \lim_{T \rightarrow \infty} \sum_{l \geq 1} \left(\int_{\Upsilon(l)} |\omega(l)| \right) T^l \\ &= - \lim_{T \rightarrow \infty} \sum_{l \geq 1} \left(\int_{\overline{\Upsilon}(l)} |\overline{\omega}(l)| \right) \cdot \left(\int_{A_{\frac{n}{m}}(l)} |d\lambda(l)| \right) T^l. \end{aligned}$$

for some gauge form $\omega, \overline{\omega}$ on $\Upsilon, \overline{\Upsilon}$, respectively. By [37, Lemma 7.6]:

$$\text{MV}([\Upsilon]) = \left(\lim_{T \rightarrow \infty} \sum_{l \geq 1} \left(\int_{\overline{\Upsilon}(l)} |\overline{\omega}(l)| \right) T^l \right) \cdot \left(\lim_{T \rightarrow \infty} \sum_{l \geq 1} \left(\int_{A_{\frac{n}{m}}(l)} |d\lambda(l)| \right) T^l \right).$$

For the same reason as in [33, Lemma 5.2], the annulus $A_{\frac{n}{m}}$ is $B - B'$, where

$$B := \{\tau \in \mathbb{A}_{\mathbb{K}^{\text{alg}}}^{1, \text{an}} \mid \text{val}(\lambda) \geq 0\}; \quad B' := \{\tau \in \mathbb{A}_{\mathbb{K}^{\text{alg}}}^{1, \text{an}} \mid \text{val}(\lambda) \geq \frac{n}{m}\}.$$

The closed ball $B(l)$ of radius l has motivic volume:

$$\int_{B(l)} d\lambda(l) = 1.$$

In [33, Lemma 5.2] Le shows that

$$\int_{B'(l)} d\lambda(l) = \begin{cases} 0, & m \nmid l; \\ \mathbb{L}^{-me}, & l = me. \end{cases}$$

Hence

$$\begin{aligned} \lim_{T \rightarrow \infty} \sum_{l \geq 1} \left(\int_{A_{\frac{n}{m}}(l)} |d\lambda(l)| \right) T^l &= \lim_{T \rightarrow \infty} \left(\sum_{l \geq 1} T^l - \sum_{e \geq 1} \mathbb{L}^{-me} T^{me} \right) \\ &= \lim_{T \rightarrow \infty} \left(\frac{T}{1-T} - \frac{\mathbb{L}^{-e} T^e}{1 - \mathbb{L}^{-e} T^e} \right) \\ &= 0. \end{aligned}$$

□

Lemma 3.11. *For the mapping $\theta : \text{MV}([\mathbb{X}_1]) \rightarrow \mathbb{N}_{>0}$, and $n \in \mathbb{N}_{>0}$,*

$$\theta^{-1}(n) = 0 \in \mathcal{M}_{\kappa, \text{loc}}^{\hat{\mu}}.$$

Proof. Let

$$s_n : \mathbb{X}_{1,n} \rightarrow \mathbb{N}_{>0}$$

be the map

$$(x, y, z, w) \mapsto m, \text{ if } \text{val}(z) + \text{val}(w) = \frac{n}{m}.$$

Then $\theta^{-1}(n) = \text{MV}([\mathbb{X}_{1,n}]) \in \mathcal{C}^{\hat{\mu}}(\mathbb{N}_{>0})$ and there is a structural mapping

$$\tau_n : \theta^{-1}(n) \rightarrow \mathbb{N}_{>0}$$

induced by s_n . For any $m \in \mathbb{N}_{>0}$, from Lemma 3.10,

$$\tau_n^{-1}(m) = \text{MV} \left([\{(x, y, z, w) \in \mathbb{X}_1 \mid \text{val}(z) + \text{val}(w) = \frac{n}{m}\}] \right) = 0.$$

By Theorem 3.7,

$$M : \mathcal{C}^{\hat{\mu}}(\mathbb{N}_{>0}) \rightarrow \mathcal{M}_{\kappa, \text{loc}}^{\hat{\mu}}[[T]]_{\Gamma}$$

is an isomorphism of rings, so

$$M(\theta^{-1}(n)) = \sum_{m \geq 1} \tau_n^{-1}(m) T^m = 0$$

and

$$\theta^{-1}(n) = 0 \in \mathcal{C}^{\hat{\mu}}(\mathbb{N}_{>0}).$$

Hence

$$\theta^{-1}(n) = 0 \in \mathcal{M}_{\kappa, \text{loc}}^{\hat{\mu}}.$$

□

Now the theorem is proved by the following: from the isomorphism

$$M : \mathcal{C}^{\hat{\mu}}(\mathbb{N}_{>0}) \rightarrow \mathcal{M}_{\kappa, \text{loc}}^{\hat{\mu}}[[T]]_{\Gamma}$$

and Lemma 3.11

$$M(\text{MV}([\mathcal{X}_1])) = \sum_{m \geq 1} \theta^{-1}(n) T^m = 0.$$

So

$$\text{MV}([\mathcal{X}_1]) = 0.$$

□

. **(3.3.5)** Now the proof of Formula (1) in Conjecture 1.2 is obtained as follows: From the formula $\mathcal{S}_{\hat{f}, 0} = \mathbb{L}^{d-1} \cdot \text{MV}([\mathfrak{X}_{\eta}])$ in Theorem 3.8 and $\text{MV}([\mathcal{X}_1]) = 0$ in Theorem 3.9, since $\text{MV}([\mathfrak{X}_{\eta}]) = \text{MV}([\mathcal{X}_0]) + \text{MV}([\mathcal{X}_1])$, we have $\mathcal{S}_{\hat{f}, 0} = \mathbb{L}^{d-1} \cdot \text{MV}([\mathcal{X}_0])$. So by Theorem 3.8 again,

$$\mathcal{S}_{\hat{f}, 0} = \mathcal{S}_{\hat{f}|_Z, 0}.$$

The potential function

$$f|_Z = f_1 + f_2,$$

where f_1, f_2 are the potential functions on $Z_1 := \text{Ext}^1(E_1, E_1)$ and $Z_2 := \text{Ext}^2(E_2, E_2)$. Hence

$$\hat{f}|_Z : \mathfrak{Z} \rightarrow \text{spf}(R)$$

can be split into $\hat{f}|_Z = \hat{f}_1 + \hat{f}_2$ with $\hat{f}_i : \mathfrak{Z}_i \rightarrow \text{spf}(R)$ the formal completion of Z_i along the origin for $i = 1, 2$. By motivic Thom-Sebastiani theorem proved in [19] for regular functions and [34] for formal functions,

$$(1 - \mathcal{S}_{\hat{f}|_Z, 0}) = (1 - \mathcal{S}_{\hat{f}_1, 0}) \cdot (1 - \mathcal{S}_{\hat{f}_2, 0})$$

Note that $\mathcal{S}_0(E) = \mathcal{S}_{\hat{f}, 0}$, and $\mathcal{S}_0(E_1) = \mathcal{S}_{\hat{f}_1, 0}$, $\mathcal{S}_0(E_2) = \mathcal{S}_{\hat{f}_2, 0}$, we have

$$(1 - \mathcal{S}_0(E)) = (1 - \mathcal{S}_0(E_1)) \cdot (1 - \mathcal{S}_0(E_2)).$$

3.4. The proof of Formula (2) in Conjecture 1.2.

. **(3.4.1)** We use the same notations as in §(3.3.3). For the coherent sheaves or simple complexes $E = E_1 \oplus E_2$, E_1, E_2 , let

$$\hat{f} : \mathfrak{X} = \widehat{\text{Ext}^1(E, E)} \rightarrow \text{spf}(R)$$

be the special formal scheme as in §(3.3.2). The formula (2) in Conjecture 1.2 is equivalent to the following formula:

$$(3.4.2) \quad \int_{F \in \mathbb{P}(\text{Ext}^1(E_2, E_1))} \mathcal{S}_0(F) - \int_{F \in \mathbb{P}(\text{Ext}^1(E_1, E_2))} \mathcal{S}_0(F) \\ = ([\mathbb{P}(\text{Ext}^1(E_2, E_1))] - [\mathbb{P}(\text{Ext}^1(E_1, E_2))]) \cdot \mathcal{S}_{\hat{f}|_Z, 0}.$$

We prove Formula (3.4.2). Let

$$U = \{(x, y, z, w) \in \text{Ext}^1(E, E) | w \neq 0\}$$

and

$$V = \{(x, y, z, w) \in U | z = 0\}.$$

We consider the formal schemes

$$\hat{f}|_U : \mathfrak{U} \rightarrow \text{spf}(R), \quad \hat{f}|_V : \mathfrak{V} \rightarrow \text{spf}(R)$$

which are the formal completions along the origin. Consider the formal admissible blow-up

$$\phi : \tilde{\mathfrak{U}} \rightarrow \mathfrak{U}$$

with center $\mathfrak{V} \subset \mathfrak{U}$. Let $\tilde{f} := \hat{f} \circ \phi$ be the composition map.

. **(3.4.3)** Let

$$\epsilon_{21} := (0, 0, 0, \epsilon_{21}) \in V.$$

Then from Theorem 3.1,

$$\int_{\mathbb{P}(T_{\epsilon_{21}}\mathfrak{U}/T_{\epsilon_{21}}\mathfrak{V})} \mathcal{S}_{\tilde{f}} = \mathcal{S}_{\hat{f}, \epsilon_{21}} + [\mathbb{P}^{\dim(\mathfrak{U}) - \dim(\mathfrak{V}) - 1} - 1] \cdot \mathcal{S}_{\hat{f}|_{\mathfrak{V}}, \epsilon_{21}} \in \mathcal{M}_{\kappa}^{\hat{\mu}},$$

where $\int_{\mathbb{P}(T_{\epsilon_{21}}\mathfrak{U}/T_{\epsilon_{21}}\mathfrak{V})}$ is understood as the pushforward from $\mathcal{M}_{\mathbb{P}(T_{\epsilon_{21}}\mathfrak{U}/T_{\epsilon_{21}}\mathfrak{V})}^{\hat{\mu}} \rightarrow \mathcal{M}_{\kappa}^{\hat{\mu}}$. Note that

$$V = \text{Ext}^1(E_1, E_1) \oplus \text{Ext}^1(E_2, E_2) \oplus W,$$

where $W = \{(x, y, z, w) \in X | x = y = z = 0, w \neq 0\}$. The property of the potential function $f : X \rightarrow \kappa$ implies that $f|_V = f_1 + f_2 + 0$, where $f_i : \text{Ext}^1(E_i, E_i) \rightarrow \kappa$ is the potential function for E_i with $i = 1, 2$. Hence the motivic Milnor fiber of $f|_V$ is the product of the motivic Milnor fiber of $f|_Z$ with the motive of a small ball in $\widehat{\text{Ext}^1(E_2, E_1)}$. Since the close ball has motive 1 from Theorem 2.6, we have:

$$\mathcal{S}_{\hat{f}|_{\mathfrak{V}}, \epsilon_{21}} = \mathcal{S}_{\hat{f}|_Z, 0}.$$

Hence integrating over all $[\epsilon_{21}] \in \mathbb{P}(\text{Ext}^1(E_2, E_1))$,

(3.4.4)

$$\begin{aligned} \int_{\mathbb{P}(\text{Ext}^1(E_2, E_1))} \mathcal{S}_{\hat{f}, \epsilon_{21}} &= \\ \int_{\mathbb{P}(\text{Ext}^1(E_1, E_2)) \times \mathbb{P}(\text{Ext}^1(E_2, E_1))} \mathcal{S}_{\tilde{f}} - [\mathbb{P}^{\dim(\text{Ext}^1(E_1, E_2))} - 1] \cdot [\mathbb{P}^{\dim(\text{Ext}^1(E_2, E_1))}] \mathcal{S}_{\hat{f}|_{Z, 0}}. \end{aligned}$$

Similarly let $\epsilon_{12} = (0, 0, \epsilon_{12}, 0) \in V'$, where $V' \subset U'$ are defined by:

$$U' = \{(x, y, z, w) \in \text{Ext}^1(E, E) | z \neq 0\}$$

and

$$V' = \{(x, y, z, w) \in U' | w = 0\}.$$

Then a similar argument gives:

(3.4.5)

$$\begin{aligned} \int_{\mathbb{P}(\text{Ext}^1(E_1, E_2))} \mathcal{S}_{\hat{f}, \epsilon_{12}} &= \\ \int_{\mathbb{P}(\text{Ext}^1(E_1, E_2)) \times \mathbb{P}(\text{Ext}^1(E_2, E_1))} \mathcal{S}_{\tilde{f}} - [\mathbb{P}^{\dim(\text{Ext}^1(E_2, E_1))} - 1] \cdot [\mathbb{P}^{\dim(\text{Ext}^1(E_1, E_2))}] \mathcal{S}_{\hat{f}|_{Z, 0}}. \end{aligned}$$

Then (3.4.4)-(3.4.5) we get:

$$\begin{aligned} &\int_{\mathbb{P}(\text{Ext}^1(E_2, E_1))} \mathcal{S}_{\hat{f}, \epsilon_{21}} - \int_{\mathbb{P}(\text{Ext}^1(E_1, E_2))} \mathcal{S}_{\hat{f}, \epsilon_{12}} \\ &= \left((-[\mathbb{P}(\text{Ext}^1(E_1, E_2))] + 1)[\mathbb{P}(\text{Ext}^1(E_2, E_1))] \right. \\ &\quad \left. + ([\mathbb{P}(\text{Ext}^1(E_2, E_1))] - 1)[\mathbb{P}(\text{Ext}^1(E_1, E_2))] \right) \cdot \mathcal{S}_{\hat{f}|_{Z, 0}} \\ &= ([\mathbb{P}(\text{Ext}^1(E_2, E_1))] - [\mathbb{P}(\text{Ext}^1(E_1, E_2))]) \cdot \mathcal{S}_{\hat{f}|_{Z, 0}}, \end{aligned}$$

which is the formula in (3.4.2). Hence Formula (2) in Conjecture 1.2 is proved. \square

4. THE POISSON ALGEBRA HOMOMORPHISM

In this section we study the Poisson algebra homomorphism from the motivic Hall algebra of the abelian category of coherent sheaves on the Calabi-Yau threefold Y to the motivic quantum torus.

4.1. Motivic Hall algebras.

. **(4.1.1)** In this section we review the definition and construction of motivic Hall algebra of Joyce and Bridgeland in [29], [10].

. (4.1.2) We define the Grothendieck ring of stacks of finite type.

Definition 4.1. *The Grothendieck ring of stacks $K(\text{St} / \kappa)$ is defined to be the κ -vector space spanned by isomorphism classes of Artin stacks of finite type over κ with affine stabilizers, modulo the relations:*

(1) *for every pair of stacks \mathcal{X}_1 and \mathcal{X}_2 a relation:*

$$[\mathcal{X}_1 \sqcup \mathcal{X}_2] = [\mathcal{X}_1] + [\mathcal{X}_2];$$

(2) *for any geometric bijection $f : \mathcal{X}_1 \rightarrow \mathcal{X}_2$, $[\mathcal{X}_1] = [\mathcal{X}_2]$;*

(3) *for any Zariski fibrations $p_i : \mathcal{X}_i \rightarrow \mathcal{Y}$ with the same fibers, $[\mathcal{X}_1] = [\mathcal{X}_2]$.*

Let $[\mathbb{A}^1] = \mathbb{L}$, the Lefschetz motive. If S is a stack of finite type over κ , we define the relative Grothendieck ring of stacks $K(\text{St} / S)$ as follows:

Definition 4.2. *The relative Grothendieck ring of stacks $K(\text{St} / S)$ is defined to be the κ -vector space spanned by isomorphism classes of morphisms*

$$[\mathcal{X} \xrightarrow{f} S],$$

with \mathcal{X} an Artin stack over S of finite type with affine stabilizers, modulo the following relations:

(1) *for every pair of stacks \mathcal{X}_1 and \mathcal{X}_2 a relation:*

$$[\mathcal{X}_1 \sqcup \mathcal{X}_2 \xrightarrow{f_1 \sqcup f_2} S] = [\mathcal{X}_1 \xrightarrow{f_1} S] + [\mathcal{X}_2 \xrightarrow{f_2} S];$$

(2) *for any diagram:*

$$\begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{g} & \mathcal{X}_2 \\ & \searrow f_1 & \swarrow f_2 \\ & S, & \end{array}$$

where g is a geometric bijection, then $[\mathcal{X}_1 \xrightarrow{f_1} S] = [\mathcal{X}_2 \xrightarrow{f_2} S]$;

(3) *for any pair of Zariski fibrations*

$$\mathcal{X}_1 \xrightarrow{h_1} \mathcal{Y}; \quad \mathcal{X}_2 \xrightarrow{h_2} \mathcal{Y}$$

with the same fibers, and $g : \mathcal{Y} \rightarrow S$, a relation

$$[\mathcal{X}_1 \xrightarrow{g \circ h_1} S] = [\mathcal{X}_2 \xrightarrow{g \circ h_2} S].$$

. (4.1.3) The motivic Hall algebra in [10] is defined as follows. Let \mathfrak{M} be the moduli stack of coherent sheaves on Y . It is an algebraic stack, locally of finite type over κ . The motivic Hall algebra is the vector space

$$H(\mathcal{A}) = K(\text{St} / \mathfrak{M})$$

equipped with a non-commutative product given by the role:

$$[\mathcal{X}_1 \xrightarrow{f_1} \mathfrak{M}] \star [\mathcal{X}_2 \xrightarrow{f_2} \mathfrak{M}] = [\mathcal{Z} \xrightarrow{b \circ h} \mathfrak{M}],$$

where h is defined by the following Cartesian square:

$$\begin{array}{ccccc} \mathcal{Z} & \xrightarrow{h} & \mathfrak{M}^{(2)} & \xrightarrow{b} & \mathfrak{M} \\ \downarrow & & \downarrow (a_1, a_2) & & \\ \mathcal{X}_1 \times \mathcal{X}_2 & \xrightarrow{f_1 \times f_2} & \mathfrak{M} \times \mathfrak{M} & & \end{array}$$

with $\mathcal{M}^{(2)}$ the stack of short exact sequences in \mathcal{A} , and the maps a_1, a_2, b send a short exact sequence

$$0 \rightarrow A_1 \rightarrow B \rightarrow A_2 \rightarrow 0$$

to sheaves A_1, A_2 , and B respectively. Then $H(\mathcal{A})$ is an algebra over $K(\text{St}/\kappa)$.

4.2. Algebraic d -critical locus.

. **(4.2.1)** We recall the algebraic (analytic) d -critical locus from [28]. We mainly focus on the algebraic version, but the analytic version is a slightly modification of the algebraic one.

The algebraic d -critical locus is a classical model for the -1 -shifted symplectic derived scheme as developed by PTVV in [47]. In the same paper [47], PTVV prove that the moduli space of stable coherent sheaves or simple complexes over Calabi-Yau threefolds admit the -1 -shifted symplectic derived structure, hence their underlying schemes have the algebraic d -critical locus structure. Thus the algebraic d -critical locus of Joyce provides the classical schemetical framework for the moduli space of stable simple complex over smooth Calabi-Yau threefolds.

. **(4.2.2)** To define the algebraic d -critical locus, we first recall the following theorem in [28]:

Theorem 4.3. ([28]) *Let X be a κ -scheme, which is locally of finite type. Then there exists a sheaf \mathcal{S}_X of κ -vector spaces on X , unique up to canonical isomorphism, which is uniquely characterized by the following two properties:*

(i) *Suppose that $R \subseteq X$ is Zariski open, U is a smooth κ -scheme, and $i : R \hookrightarrow U$ is a closed embedding. Then there is an exact sequence of sheaves of κ -vector spaces on R :*

$$0 \rightarrow I_{R,U} \rightarrow i^{-1}(\mathcal{O}_U) \xrightarrow{i^\#} \mathcal{O}_X|_R \rightarrow 0,$$

where $\mathcal{O}_X, \mathcal{O}_U$ are the structure sheaves of X and U , and $i^\#$ is the morphism of sheaves over R . There is also an exact sequence of sheaves of κ -vector spaces over R :

$$0 \rightarrow \mathcal{S}_X|_R \xrightarrow{I_{R,U}} \frac{i^{-1}(\mathcal{O}_U)}{I_{R,U}^2} \xrightarrow{d} \frac{i^{-1}(T^*U)}{I_{R,U} \cdot i^{-1}(T^*U)}$$

where d maps $f + I_{R,U}^2$ to $df + I_{R,U} \cdot i^{-1}(T^*U)$.

(ii) *If $R \subseteq S \subseteq X$ are Zariski open, and U, V are smooth κ -schemes, and*

$$i : R \hookrightarrow U$$

$$j : S \hookrightarrow V$$

are closed embeddings. Let

$$\Phi : U \rightarrow V$$

be a morphism with $\Phi \circ i = j|_R : R \rightarrow V$. Then the following diagram of sheaves on R commutes:

$$(4.2.3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{S}|_R & \xrightarrow{\iota_{S,V}|_R} & \frac{i^{-1}(\mathcal{O}_V)}{I_{S,V}^2}|_R & \xrightarrow{d} & \frac{j^{-1}(T^*V)}{I_{S,V} \cdot j^{-1}(T^*V)}|_R \longrightarrow 0 \\ & & \downarrow \text{id} & & \downarrow i^{-1}(\Phi^\#) & & \downarrow i^{-1}(d\Phi) \\ 0 & \longrightarrow & \mathcal{S}_X|_R & \xrightarrow{\iota_{R,U}} & \frac{i^{-1}(\mathcal{O}_U)}{I_{R,U}^2} & \xrightarrow{d} & \frac{i^{-1}(T^*U)}{I_{R,U} \cdot i^{-1}(T^*U)} \longrightarrow 0. \end{array}$$

Here $\Phi : U \rightarrow V$ induces

$$\Phi^\# : \Phi^{-1}(\mathcal{O}_V) \rightarrow \mathcal{O}_U$$

on U , and we have:

$$(4.2.4) \quad i^{-1}(\Phi^\#) : j^{-1}(\mathcal{O}_V)|_R = i^{-1} \circ \Phi^{-1}(\mathcal{O}_V) \rightarrow i^{-1}(\mathcal{O}_U),$$

a morphism of sheaves of κ -algebras on R . As $\Phi \circ i = j|_R$, then (4.2.4) maps to $I_{S,V}|_R \rightarrow I_{R,U}$, and $I_{S,V}^2|_R \rightarrow I_{R,U}^2$. Thus (4.2.4) induces the morphism in the second column of (4.2.3). Similarly, $d\Phi : \Phi^{-1}(T^*V) \rightarrow T^*U$ induces the third column of (4.2.3).

According to [28], there is a natural decomposition

$$\mathcal{S}_X = \mathcal{S}_X^0 \oplus \kappa_X$$

and κ_X is the constant sheaf on X and $\mathcal{S}_X \subset \mathcal{S}_X$ is the kernel of the composition:

$$\mathcal{S}_X \rightarrow \mathcal{O}_X \xrightarrow{i_X^\#} \mathcal{O}_{X^{\text{red}}}$$

with X^{red} the reduced κ -scheme of X , and $i_X : X^{\text{red}} \hookrightarrow X$ the inclusion.

Definition 4.4. An algebraic d-critical locus over the field κ is a pair (X, s) , where X is a κ -scheme, locally of finite type, and $s \in H^0(\mathcal{S}_X^0)$ for \mathcal{S}_X^0 in Theorem 4.3. These data satisfy the following conditions: for any $x \in X$, there exists a Zariski open neighbourhood R of x in X , a smooth κ -scheme U , a regular function $f : U \rightarrow \kappa$, and a closed embedding $i : R \hookrightarrow U$, such that $i(R) = \text{Crit}(f)$ as κ -subschemes of U , and $\iota_{R,U}(s|_R) = i^{-1}(f) + I_{R,U}^2$. We call the quadruple (R, U, f, i) a critical chart on (X, s) .

. (4.2.5) Some properties of (X, s) are as follows:

Theorem 4.5. [28] *Let (X, s) be an algebraic d -critical locus, and $(R, U, f, i), (S, V, g, j)$ be critical charts on (X, s) . Then for each $x \in R \cap S \subset X$ there exists subcharts*

$$(R', U', f', i') \subseteq (R, U, f, i),$$

$$(S', V', g', j') \subseteq (S, V, g, j)$$

with $x \in R' \cap S' \subseteq X$, a critical chart (T, W, h, k) on (X, s) , and embeddings

$$\Phi : (R', U', f', i') \hookrightarrow (T, W, h, k)$$

and

$$\Psi : (S', V', g', j') \hookrightarrow (T, W, h, k).$$

. (4.2.6) We introduce the canonical line bundle of (X, s) :

Theorem 4.6. [28, Theorem 2.28] *Let (X, s) be an algebraic d -critical locus, and $X^{\text{red}} \subset X$ the associated reduced κ -scheme. Then there exists a line bundle $K_{X,s}$ on X^{red} which we call the canonical Line bundle of (X, s) , that is natural up to canonical isomorphism, and is characterized by the following properties:*

(i) *If (R, U, f, i) is a critical chart on (X, s) , there is a natural isomorphism*

$$\iota_{R,U,f,i} : (K_{X,s})|_{R^{\text{red}}} \rightarrow i^*(K_U^{\otimes 2})|_{R^{\text{red}}}$$

where K_U is the canonical line bundle of U .

(ii) *Let $\Phi : (R, U, f, i) \hookrightarrow (S, V, g, j)$ be an embedding of critical charts on (X, s) . Then there is an isomorphism of line bundles on $\text{Crit}(f)^{\text{red}}$:*

$$J_\Phi : (K_U^{\otimes 2})|_{\text{Crit}(f)} \xrightarrow{\cong} \Phi|_{\text{Crit}(f)}^*(K_V^{\otimes 2}).$$

Since $i : R \rightarrow \text{Crit}(f)$ is an isomorphism as schemes with $\Phi \circ i = j|_R$, this gives

$$i|_{R^{\text{red}}}^*(J_\Phi) : i^*(K_U^{\otimes 2})|_{R^{\text{red}}} \xrightarrow{\cong} j^*(K_V^{\otimes 2})|_{R^{\text{red}}},$$

and we have:

$$\iota_{S,V,g,j}|_{R^{\text{red}}} = i|_{R^{\text{red}}}^*(J_\Phi) \circ \iota_{R,U,f,i} : (K_{X,s})|_{R^{\text{red}}} \rightarrow j^*(K_V^{\otimes 2})|_{R^{\text{red}}}.$$

. (4.2.7) We talk about the orientation data for d -critical locus in [28]:

Definition 4.7. *Let (X, s) be an algebraic d -critical locus, and $K_{X,s}$ the canonical line bundle of (X, s) . An orientation on (X, s) is a choice of square root line bundle $K_{X,s}^{1/2}$ for $K_{X,s}$ on X^{red} . I.e., an orientation of (X, s) is a line bundle L over X^{red} and an isomorphism $L^{\otimes 2} = L \otimes L \cong K_{X,s}$. A d -critical locus with an orientation will be called an oriented d -critical locus.*

Bussi, Brav and Joyce [12] prove the following interesting result: Let (\mathbf{X}, ω) be a (-1) -shifted symplectic derived scheme over κ in the sense of [47], and let $X := \mathbf{t}_0(\mathbf{X})$ be the associated classical κ -scheme of \mathbf{X} . Then X naturally extends to an algebraic d -critical locus (X, s) . The canonical line bundle $K_{X,s} \cong \det(\mathbb{L}_{\mathbf{X}})|_{X^{\text{red}}}$ is the determinant line bundle of the cotangent complex $\mathbb{L}_{\mathbf{X}}$ of \mathbf{X} .

. **(4.2.8)** One of the applications of the (-1) -shifted symplectic derived scheme or stack is on moduli problems. Let Y be a smooth Calabi-Yau threefold over κ , and X a classical moduli scheme of simple coherent sheaves in $\text{Coh}(Y)$, the abelian category of coherent sheaves on Y . Then in [47], the authors prove that there is a natural (-1) -shifted derived scheme structure \mathbf{X} on the moduli space X , such that if

$$i : X \hookrightarrow \mathbf{X}$$

is the inclusion, then the pullback $i^*\mathbb{L}_{\mathbf{X}}$ of the cotangent complex of \mathbf{X} is a perfect obstruction theory of X , thus from the result in [12], X has an algebraic d -critical locus structure.

Actually similar story holds for derived Artin stacks. Still let Y be a smooth Calabi-Yau threefold over κ , and \mathcal{M} a moduli stack of simple coherent sheaves in $\text{Coh}(Y)$. Then in [47], there is a natural (-1) -shifted derived stack structure \mathcal{M} on the moduli space \mathcal{M} , such that if

$$i : \mathcal{M} \hookrightarrow \mathcal{M}$$

is the inclusion, then the pullback $i^*\mathbb{L}_{\mathcal{M}}$ of the cotangent complex of \mathcal{M} is an obstruction theory of \mathcal{M} , thus from the result in [13], \mathcal{M} has an algebraic d -critical stack structure.

. **(4.2.9)** The orientation of d -critical locus or stack has application to motivic Donaldson-Thomas theory. Let $X := (X, s)$ be an algebraic d -critical locus, which is the moduli scheme of simple coherent sheaves or simple complexes over a smooth Calabi-Yau threefold Y . Let (R, U, f, i) be a critical chart of (X, s) . Then in [14], the authors associated with this local chart a perverse sheaf of vanishing cycle

$$(4.2.10) \quad \mathcal{S}_{U,f}^\phi \in \mathcal{M}_X^{\hat{d}}$$

such that

$$\mathcal{S}_{U,f}^\phi|_{X_c} = \mathbb{L}^{-\dim(U)/2} \odot ([U_c, i] - \mathcal{S}_{U,f-c})|_{X_c},$$

where $f : X \rightarrow \kappa$ is the function f restricted to X , and $X = \sqcup_{c \in \text{inf}(X)} X_c$ and $X_c = X \cap U_c$ with $U_c = f^{-1}(c) \in U$. We call $\mathcal{S}_{U,f}^\phi$ the *motivic vanishing cycle* of f .

. **(4.2.11)** As in [14], a principal \mathbb{Z}_2 -bundle $P \rightarrow X$ is a proper, surjective, étale morphism of κ -schemes $\pi : P \rightarrow X$ together with a free involution $\sigma : P \rightarrow P$ such that the orbits of \mathbb{Z}_2 are the fibers of π .

Let $\mathbb{Z}_2(X)$ be the abelian group of isomorphism classes $[P]$ of principal \mathbb{Z}_2 -bundles $P \rightarrow X$, with multiplication $[P] \cdot [Q] = [P \otimes_{\mathbb{Z}_2} Q]$ and the identity the trivial bundle $[X \times \mathbb{Z}_2]$. We know that $P \otimes_{\mathbb{Z}_2} P \cong X \times \mathbb{Z}_2$, so every element in $\mathbb{Z}_2(X)$ has order 1 or 2.

In [14], the authors define the motive of a principal \mathbb{Z}_2 -bundle $P \rightarrow X$ by:

$$Y(P) = \mathbb{L}^{-\frac{1}{2}} \odot ([X, \hat{\iota}] - [P, \hat{\rho}]) \in \mathcal{M}_X^{\hat{\mu}},$$

where $\hat{\rho}$ is the $\hat{\mu}$ -action on P induced by the μ_2 -action on P .

In [14], for any scheme Y , the authors define an ideal $I_Y^{\hat{\mu}}$ in $\mathcal{M}_Y^{\hat{\mu}}$ which is generated by

$$\phi_*(Y(P \otimes_{\mathbb{Z}_2} Q) - Y(P) \odot Y(Q))$$

for all morphisms $\phi : X \rightarrow Y$ and principal \mathbb{Z}_2 -bundles P, Q over X . Then define

$$\overline{\mathcal{M}}_Y^{\hat{\mu}} = \mathcal{M}_Y^{\hat{\mu}} / I_Y^{\hat{\mu}}.$$

Then $(\overline{\mathcal{M}}_Y^{\hat{\mu}}, \odot)$ is a commutative ring with \odot and there is a natural projection $\prod_Y^{\hat{\mu}} : \mathcal{M}_Y^{\hat{\mu}} \rightarrow \overline{\mathcal{M}}_Y^{\hat{\mu}}$.

. **(4.2.12)** Let (X, s) be an oriented d -critical locus. Recall the isomorphism in the canonical line bundle $K_{X,s}$ in Theorem 4.6. Let $Q_{R,U,f,i} \rightarrow R$ be the principal \mathbb{Z}_2 -bundle parameterizing local isomorphisms

$$\alpha : K_{X,s}^{1/2}|_{R^{\text{red}}} \rightarrow i^*(K_U)|_{R^{\text{red}}}$$

with $\alpha \otimes \alpha = \iota_{R,U,f,i}$, where

$$\iota_{R,U,f,i} : K_{X,s}|_{R^{\text{red}}} \rightarrow i^*(K_U^{\otimes 2})|_{R^{\text{red}}}$$

is the isomorphism in Theorem 4.6.

Theorem 4.8. ([14]) *If (X, s) is a finite type algebraic d -critical locus with a choice of orientation $K_{X,s}^{1/2}$, then there exists a unique motive*

$$\mathcal{S}_{X,s}^{\phi} \in \overline{\mathcal{M}}_X^{\hat{\mu}}$$

such that if (R, U, f, i) is a critical chart on (X, s) , then

$$\mathcal{S}_{X,s}^{\phi}|_R = i^*(\mathcal{S}_{U,f}^{\phi}) \odot Y(Q_{R,U,f,i}) \in \overline{\mathcal{M}}_R^{\hat{\mu}}$$

where

$$Y(Q_{R,U,f,i}) = \mathbb{L}^{-1/2} \odot ([R, \hat{\iota}] - [Q, \hat{\rho}]) \in \mathcal{M}_R^{\hat{\mu}}$$

is the motive of the principal \mathbb{Z}_2 -bundle defined in §2.5 of [14].

Remark 4.9. (1) *On the Donaldson-Thomas moduli scheme X over the Calabi-Yau threefold Y , the Behrend function*

$$v_X : X \rightarrow \mathbb{Z}$$

is a constructible function defined using the local Euler obstruction of the canonical cycle of X defined in §2 of [1]. In [1], Behrend proves

$$\chi(X, v_X) = \int_{[X]^{\text{virt}}} 1$$

if X is a proper scheme, where $[X]^{\text{virt}}$ is the virtual fundamental class of X defined by the perfect obstruction theory. This is the Donaldson-Thomas invariant.

- (2) *Let (X, s) be the corresponding algebraic d -critical locus of the moduli scheme X . If (X, s) is oriented, i.e. there exists a global square root $K_{X,s}^{1/2}$, then there exists $\mathcal{S}_X^\phi \in \overline{\mathcal{M}}_X^{\text{ft}}$ such that*

$$\chi(X, \mathcal{S}_X^\phi) = \chi(X, v_X),$$

thus categorifying the Donaldson-Thomas invariant.

- (3) *The orientation data $K_{X,s}^{1/2}$ and the triangle property of the motives of the quadratic forms Q were introduced by Kontsevich-Soibelman in [31] in the more general setting of the motivic Donaldson-Thomas invariants. Several cases of the square root line bundle have been proved in [18], [30], [21].*

. **(4.2.13)** In order to define the Poisson algebra homomorphism from the motivic Hall algebra to the motivic quantum torus, we need to modify the global motive \mathcal{S}_X^ϕ .

Let us now fix the moduli stack \mathfrak{M} as the stack of coherent sheaves on the abelian category of coherent sheaves \mathcal{A} of Y . From [10], [27], \mathfrak{M} is an Artin stack, locally of finite type.

Lemma 4.10. *Let (X, s) be a finite type algebraic d -critical locus, which is the moduli scheme of stable coherent sheaves over Y , or the coarse moduli scheme of moduli of semi-stable coherent sheaves on Y , then (X, s) is the coarse moduli scheme of an Artin stack \mathcal{X} of finite type, which is the underlying Artin stack of a (-1) -shifted derived Artin stack \mathcal{X} in sense of [13].*

Proof. We consider the algebraic d -critical locus $(X, s) \subset \mathfrak{M}$ such that it is the coarse moduli space of a moduli stack \mathcal{X} with fixed topological data. From [47] and [13], \mathcal{X} can be extended to a canonical (-1) -shifted derived Artin stack structure \mathcal{X} . \square

. **(4.2.14)** On the Artin stack \mathcal{X} , in [13], the authors define an algebraic d -critical stack structure (\mathcal{X}, s) on \mathcal{X} , similar to Definition 4.4. An oriented algebraic d -critical stack is the one (\mathcal{X}, s) such that there exists a global square root line bundle $K_{\mathcal{X},s}^{1/2}$. Let $t : X \rightarrow \mathcal{X}$ be a morphism from an algebraic d -critical locus to the algebraic d -critical stack (\mathcal{X}, s) , then in Theorem 5.14 of [13], there exists a

$$\mathcal{S}_{\mathcal{X}}^{\phi} \in \overline{\mathcal{M}}_{\mathcal{X},\text{loc}}^{\hat{\mu}}$$

such that

$$t^* \mathcal{S}_{\mathcal{X}}^{\phi} = \mathbb{L}^{n/2} \odot \mathcal{S}_X^{\phi} \in \overline{\mathcal{M}}_{X,\text{loc}}^{\hat{\mu}},$$

where n is the relative dimension of the morphism t .

Remark 4.11. *Étale locally if the algebraic d -critical stack (\mathcal{X}, s) is given by the quotient stack*

$$[Q/E],$$

then Q is an algebraic d -critical locus, and the morphism t is given by:

$$t : Q \rightarrow [Q/E]$$

and

$$\mathcal{S}_{[Q/E]}^{\phi} = \mathbb{L}^{\dim(E)/2} \odot \mathcal{S}_Q^{\phi}$$

with \mathcal{S}_Q^{ϕ} the motivic vanishing cycle sheaf.

4.3. The integration map.

. **(4.3.1)** In this section we define the integration map from the motivic Hall algebra to the motivic quantum torus.

. **(4.3.2)** Recall that in §3 of [10], there exists maps of commutative rings:

$$K(\text{Sch}/\kappa) \rightarrow K(\text{Sch}/\kappa)[\mathbb{L}^{-1}] \rightarrow K(\text{St}/\kappa),$$

where $K(\text{Sch}/\mathbb{C})$ is the Grothendieck ring of schemes of finite type over κ . Since $H(\mathcal{A})$ is an algebra over $K(\text{St}/\kappa)$, define a $K(\text{Sch}/\kappa)[\mathbb{L}^{-1}]$ -module

$$H_{\text{reg}}(\mathcal{A}) \subset H(\mathcal{A})$$

to be the span of classes of maps $[X \xrightarrow{f} \mathfrak{M}]$ with X a scheme. An element of $H(\mathcal{A})$ is regular if it lies in $H_{\text{reg}}(\mathcal{A})$. Then from Theorem 5.1 of [10], the Hall algebra product preserves the regular elements in $H_{\text{reg}}(\mathcal{A})$.

. **(4.3.3)** For our purpose, we define a $K(\text{Sch} / \kappa)[\mathbb{L}^{-1}]$ -module

$$H_{d\text{-Crit}}(\mathcal{A}) \subset H(\mathcal{A})$$

to be the span of classes of maps $[X \xrightarrow{f} \mathfrak{M}]$ with (X, s) an algebraic d -critical locus in the sense of Joyce [28], reviewed in §4.2. Since X is a scheme, the module

$$H_{d\text{-Crit}}(\mathcal{A}) \subset H_{\text{reg}}(\mathcal{A}).$$

The following is a generalization of Theorem 5.1 of [10]:

Theorem 4.12. *The sub-module of d -critical elements of $H(\mathcal{A})$ is closed under the convolution product:*

$$H_{d\text{-Crit}}(\mathcal{A}) \star H_{d\text{-Crit}}(\mathcal{A}) \subset H_{d\text{-Crit}}(\mathcal{A})$$

and is a $K(\text{Sch} / \kappa)[\mathbb{L}^{-1}]$ -algebra. Moreover, the quotient

$$H_{\text{ssc}, d\text{-Crit}}(\mathcal{A}) = H_{d\text{-Crit}}(\mathcal{A}) / (\mathbb{L} - 1)H_{d\text{-Crit}}(\mathcal{A})$$

is a commutative $K(\text{Sch} / \kappa)$ -algebra.

Proof. The proof is similar to Theorem 5.1 of [10]. Let

$$a_i = [X_i \xrightarrow{f_i} \mathfrak{M}] \in H_{d\text{-Crit}}(\mathcal{A}); i = 1, 2$$

with both X_1 and X_2 are algebraic d -critical loci. Let E_i be the family of coherent sheaves on X_i corresponding to the map f_i . As in the proof of Theorem 5.1 of [10], stratify $X_1 \times X_2$ by locally closed sub varieties S_j , we have the following diagram:

$$\begin{array}{ccccccc} \mathcal{Z}_j & \longrightarrow & \mathcal{Z} & \xrightarrow{q} & \mathfrak{M}^{(2)} & \xrightarrow{h} & \mathfrak{M} \\ t_j \downarrow & & t \downarrow & & (a_1, a_2) \downarrow & & \\ S_j & \longrightarrow & X_1 \times X_2 & \xrightarrow{f_1 \times f_2} & \mathfrak{M} \times \mathfrak{M} & & \end{array}$$

where \mathcal{Z}_j is the fiber product. Let

$$V^k(x_1, x_2) = \text{Ext}_{\mathfrak{M}}^k(E_2|_{\{x_2\} \times \mathfrak{M}}, E_1|_{\{x_1\} \times \mathfrak{M}}); \quad (x_1, x_2) \in S$$

and $d_k(S_j) = \dim(\text{Ext}_{\mathfrak{M}}^k)$. Then from §7.1 of [10],

$$\mathcal{Z}_j \cong [Q_j / \kappa^{d_0(S_j)}],$$

where $Q_j = V^1(S_j)$ is the total space of the trivial vector bundle over S_j with fiber $V^1(x_1, x_2)$ over $(x_1, x_2) \in S_j$, and

$$a_1 \star a_2 = [\mathcal{Z} \xrightarrow{boh} \mathfrak{M}] = \sum_j \mathbb{L}^{-d_0(S_j)} [Q_j \xrightarrow{g_j} \mathfrak{M}].$$

So for us we only need to prove that Q_j is an algebraic d -critical locus. Since Q_j is the trivial vector bundle $S_j \times \kappa^{d_1(S_j)}$, by assuming that S_j is a strata

such that $d_1(S_j)$ is constant, the algebraic d -critical structure on S_j comes as follows: around a point $(x_1, x_2) \in S_j$, there exists an algebraic function

$$f_{E_1 \oplus E_2} : \text{Ext}^1(E_1 \oplus E_2, E_1 \oplus E_2) \rightarrow \kappa$$

such that $(x_1, x_2) \in \text{Crit}(f_{E_1 \oplus E_2})$. Or we can use [13, Corollary 5.17] to argue that $\mathcal{Z}_j = [Q_j / \kappa^{d_0(S_j)}]$ is a canonical truncation $\mathbf{t}_0(\mathcal{X})$ for a (-1) -shifted Artin stack \mathcal{X} , where \mathcal{X} is the derived moduli stack of coherent sheaves of the form $E_1 \oplus E_2$.

The second statement is the same as in Theorem 5.1 of [10], since split Q_j into the zero-section and the complement, we can write

$$(4.3.4) \quad a_1 \star a_2 = \sum_j \mathbb{L}^{-d_0(S_j)} \left([S_j \xrightarrow{k} \mathfrak{M}] + (\mathbb{L} - 1)[\mathbb{P}(Q_j) \xrightarrow{\delta_j} \mathfrak{M}] \right)$$

and

$$(4.3.5) \quad a_1 \star a_2 = \sum_j \mathbb{L}^{-d_0(S_j)} [S_j \xrightarrow{k} \mathfrak{M}] = [X_1 \times X_2 \xrightarrow{k} \mathfrak{M}] \pmod{(\mathbb{L} - 1)}.$$

So we are done. \square

. **(4.3.6)** The algebra $H_{\text{ssc}, d-\text{Crit}}(\mathcal{A})$ is called semi-classical Hall algebra for the elements of d -critical locus. In [10], Bridgeland also defines a Poisson bracket on $H(\mathcal{A})$ by:

$$\{f, g\} = \frac{f \star g - g \star f}{\mathbb{L} - 1}.$$

This bracket preserves the subalgebra $H_{d-\text{Crit}}(\mathcal{A})$.

. **(4.3.7)** We define the motivic quantum torus. Let $K(Y) = K(\mathcal{A})$ be the Grothendieck group of the category \mathcal{A} . Let $E, F \in k(\mathcal{A})$ and let

$$\chi(E, F) = \sum_i (-1)^i \dim_{\kappa} \text{Ext}^i(E, F).$$

So $\chi(-, -)$ is a bilinear form on $K(\mathcal{A})$, which is called the Euler form. The numerical Grothendieck group is the quotient

$$N(Y) = K(Y) / K(Y)^\perp,$$

where $K(Y)^\perp$ means the Euler form zero subgroup. Let $\Gamma \subset N(Y)$ denote the monoid of effective classes, which is to say the classes of the form $[E]$ with E a sheaf.

Remark 4.13. The stack \mathfrak{M} split into disjoint union of open and closed substacks

$$\mathfrak{M} = \bigsqcup_{\alpha \in \Gamma} \mathfrak{M}_\alpha$$

where $\mathfrak{M}_\alpha \subset \mathfrak{M}$ is the stack of objects of class $\alpha \in \Gamma$. And $\mathfrak{M}_\alpha \subset \mathfrak{M}$ implies that $K(\text{St} / \mathfrak{M}_\alpha) \subset K(\text{St} / \mathfrak{M})$.

Also the Hall algebra

$$H(\mathcal{A}) = \bigoplus_{\alpha \in \Gamma} H(\mathcal{A})_{\alpha}$$

and $H(\mathcal{A})$ is a graded algebra with respect to the convolution product.

(4.3.8) Let $\mathcal{M}_{\kappa}^{\hat{\mu}}$ be the ring of motives and consider

$$\mathcal{M}_{\kappa, \text{loc}}^{\hat{\mu}} = \mathcal{M}_{\kappa}^{\hat{\mu}}[\mathbb{L}^{-1}, \mathbb{L}^{-1/2}, (\mathbb{L}^i - 1)^{-1}, i \in \mathbb{N}_{\geq 0}].$$

Let $\overline{\mathcal{M}}_{\kappa, \text{loc}}^{\hat{\mu}}$ be the ring with the product \odot .

Definition 4.14. Define

$$\overline{\mathcal{M}}_{\kappa, \text{loc}}^{\hat{\mu}}[\Gamma] = \bigoplus_{\alpha \in \Gamma} \overline{\mathcal{M}}_{\kappa, \text{loc}}^{\hat{\mu}} \cdot x^{\alpha}$$

to be the ring generated by symbols x^{α} for $\alpha \in \Gamma$, with product defined by:

$$x^{\alpha} \star x^{\beta} = \mathbb{L}^{\frac{1}{2} \cdot \chi(\alpha, \beta)} \cdot x^{\alpha + \beta}.$$

The ring is commutative since the Euler form is skew-symmetric. The Poisson bracket is given by:

$$\{x^{\alpha}, x^{\beta}\} = \mathbb{L}^{\frac{1}{2} \cdot \chi(\alpha, \beta)} \odot \frac{\sum_{i=0}^3 (-1)^{i+1} \mathbb{L}^{\dim \text{Ext}^i(\alpha, \beta)}}{\mathbb{L} - 1} \cdot x^{\alpha + \beta}.$$

We define the integration map. Let

$$(4.3.9) \quad I : H_{\text{SSC}, d-\text{Crit}}(\mathcal{A}) \rightarrow \overline{\mathcal{M}}_{\kappa, \text{loc}}^{\hat{\mu}}[\Gamma]$$

be the map defined by: for any element $[Z \rightarrow \mathfrak{M}] \in H_{\text{SSC}, d-\text{Crit}}(\mathcal{A})$, let

$$t : Z \rightarrow \mathcal{Z}$$

be the map from the algebraic d -critical locus Z to the corresponding d -critical Artin stack \mathcal{Z} . Then

$$I([Z \rightarrow \mathfrak{M}]) = \left(\int t^* \mathcal{S}_{\mathcal{Z}}^{\phi} \right) \cdot x^{\alpha} \in \overline{\mathcal{M}}_{\kappa, \text{loc}}^{\hat{\mu}}[\Gamma]$$

where $\int : \overline{\mathcal{M}}_{Z, \text{loc}}^{\hat{\mu}} \rightarrow \overline{\mathcal{M}}_{\kappa, \text{loc}}^{\hat{\mu}}$ is the pushforward of motives.

Remark 4.15. Let $v_{\mathcal{Z}}$ be the Behrend function on \mathcal{Z} which is the pullback $i^* v_{\mathfrak{M}}$ from $i : \mathcal{Z} \rightarrow \mathfrak{M}$. Then taking cohomology of the perverse sheaf $t^* \mathcal{S}_{\mathcal{Z}}^{\phi}$ we get the weighted Euler characteristic $\chi(Z, t^* v_{\mathcal{Z}})$. This is the map I in [10, Theorem 5.2].

Theorem 4.16. The map I in (4.3.9) is a Poisson algebra homomorphism.

Remark 4.17. Theorem 4.16 generalizes the result of Bridgeland in [10, Theorem 5.2] to the motivic level.

Remark 4.18. *The proof of Theorem 4.16 relies on the motivic Behrend function identities in Conjecture 1.2. The Euler characteristic level of these identities was originally proved for coherent sheaves by Joyce-Song [27]. These identities was recently proved by V. Bussi [15] using algebraic method and also works in characteristic p . In [24] we study these formulas using Berkovich spaces.*

4.4. The proof of Theorem 4.16.

. **(4.4.1)** For each algebraic d -critical locus (Z, s) such that Z factors through $[Z \xrightarrow{f} \mathfrak{M}_\alpha]$, the perverse sheaf \mathcal{S}_Z^ϕ of vanishing cycles is constructible. hence there exists a stratification $Z = \cup_i Z_i$ such that $\mathcal{S}_Z^\phi|_{Z_i}$ is given by the vanishing cycle of the function

$$f : U \rightarrow \kappa,$$

where we can take Z_i fits into a critical chart Z_i, U, f, i . So I is well-defined.

. **(4.4.2)** From Serre duality,

$$V^k(x_1, x_2) = V^{3-k}(x_2, x_1)^*.$$

Let $\hat{Q}_j = V^2(S_j)$ be the bundle over S_j whose fiber at (x_1, x_2) is $V^1(x_2, x_1)$. Let

$$g_j : Q_j \rightarrow \mathfrak{M}; \quad \hat{g}_j : \hat{Q}_j \rightarrow \mathfrak{M}$$

be the induced morphisms induced by taking the universal extensions. For

$$\begin{aligned} a_1 &= [X_1 \xrightarrow{f_1} \mathfrak{M}_{\alpha_1}], \\ a_2 &= [X_1 \xrightarrow{f_2} \mathfrak{M}_{\alpha_2}], \end{aligned}$$

we have:

$$(4.4.3) \quad I(a_i) = \left(\int t_i^* \mathcal{S}_{X_i}^\phi \right) \cdot x^{\alpha_i} \in \overline{\mathcal{M}}_{\kappa, \text{loc}}^{\hat{\mu}}[\Gamma],$$

where

$$t_i : X_i \rightarrow \mathcal{X}_i$$

are the smooth morphisms from the d -critical loci to the corresponding d -critical Artin stacks for $i = 1, 2$.

From the expression of $a_1 \star a_2$ in (4.3.5) in the proof of Theorem 4.12,

$$(4.4.4) \quad I(a_1 \star a_2) = \left(\int \sum_j t_j^* \mathcal{S}_{S_j}^\phi \right) \cdot x^{\alpha_1 + \alpha_2} = \left(\int t^* \mathcal{S}_{\mathcal{X}_1 \times \mathcal{X}_2}^\phi \right) \cdot x^{\alpha_1 + \alpha_2},$$

where $t_j : S_j \rightarrow \mathcal{S}_j$ is the morphism from the d -critical locus scheme to the d -critical stack \mathcal{S}_j , and so is the morphism

$$t := t_1 \times t_2 : X_1 \times X_2 \rightarrow \mathcal{X}_1 \times \mathcal{X}_2.$$

From [13], we have:

$$t^* \mathcal{S}_{\mathcal{X}_1 \times \mathcal{X}_2}^\phi = \mathbb{L}^{n/2} \odot \mathcal{S}_{X_1 \times X_2}^\phi = \mathbb{L}^{n/2} \cdot \mathcal{S}_{U, f}^\phi \odot Y(Q_{X, U, f, i}),$$

where $(X = X_1 \times X_2, U, f, i)$ is the local critical chart of $X_1 \times X_2$,

$$\mathcal{S}_{U,f}^\phi = \mathbb{L}^{-\dim(U)/2}(\mathbb{1}_U - \mathcal{S}_{U,f}),$$

n is the relative dimension of the smooth morphism t , and $Y(Q_{X,U,f,i})$ is the motive of a quadratic form, parameterizing the local isomorphism of the canonical line bundles as in §(4.2.12).

Over a point $(x_1, x_2) \in X_1 \times X_2 \subset U$, the dimension $\dim(U) = \dim \text{Ext}^1(E_1 \oplus E_2, E_1 \oplus E_2)$ and $n = \dim \text{Ext}^0(E_1 \oplus E_2, E_1 \oplus E_2)$, where E_1, E_2 are the coherent sheaves corresponding to x_1, x_2 respectively. When restricted to $X_1 \times X_2 \subset U$, the quadratic form $Q_{X,U,f,i}$ split into the product

$$Q_{X,U,f,i} = Q_{X_1,U_1,f_1,i_1} \otimes Q_{X_2,U_2,f_2,i_2}$$

and

$$Y(Q_{X,U,f,i}) = Y(Q_{X_1,U_1,f_1,i_1}) \odot Y(Q_{X_2,U_2,f_2,i_2}) \in \overline{\mathcal{M}}_{X_1 \times X_2}^{\hat{n}}.$$

Here (X_1, U_1, f_1, i_1) and (X_2, U_2, f_2, i_2) are the critical charts of $x_1 \in X_1$ and $x_2 \in X_2$ respectively. For coherent sheaves E_1, E_2 , the first formula in Conjecture 1.2 is:

$$(1 - \mathcal{S}_0(E_1 \oplus E_2)) = (1 - \mathcal{S}_0(E_1)) \cdot (1 - \mathcal{S}_0(E_2)).$$

And this formula of motivic Milnor fibers holds for every point on $X_1 \times X_2$. Hence we calculate (let $E := E_1 \oplus E_2$):

$$\begin{aligned} \mathbb{L}^{n/2} \odot \mathcal{S}_{X_1 \times X_2}^\phi &= \mathbb{L}^{\dim \text{Ext}^0(E,E)/2} \cdot \mathbb{L}^{-\dim \text{Ext}^1(E,E)/2} \odot (\mathbb{1} - \mathcal{S}_{U,f}) \cdot Y(Q_{X,U,f,i}) \\ &= \mathbb{L}^{\chi(E_1,E_2)/2} \odot \mathbb{L}^{\dim \text{Ext}^0(E_1,E_1)/2} \mathbb{L}^{-\dim \text{Ext}^1(E_1,E_1)/2} (\mathbb{1} - \mathcal{S}_{U_1,f_1}) \cdot \\ &\quad \mathbb{L}^{\dim \text{Ext}^0(E_2,E_2)/2} \mathbb{L}^{-\dim \text{Ext}^1(E_2,E_2)/2} (\mathbb{1} - \mathcal{S}_{U_2,f_2}) \cdot Y(Q_{X_1,U_1,f_1,i_1}) \odot Y(Q_{X_2,U_2,f_2,i_2}) \\ &= \mathbb{L}^{\chi(E_1,E_2)/2} \odot \mathbb{L}^{\dim \text{Ext}^0(E_1,E_1)/2} \cdot \mathcal{S}_{X_1}^\phi \cdot \mathbb{L}^{\dim \text{Ext}^0(E_2,E_2)/2} \cdot \mathcal{S}_{X_2}^\phi \\ &= \mathbb{L}^{\chi(E_1,E_2)/2} \cdot t_1^* \mathcal{S}_{\mathcal{X}_1}^\phi \cdot t_2^* \mathcal{S}_{\mathcal{X}_2}^\phi. \end{aligned}$$

So

$$(4.4.5) \quad t^* \mathcal{S}_{\mathcal{X}_1 \times \mathcal{X}_2}^\phi = \mathbb{L}^{n/2} \odot \mathcal{S}_{X_1 \times X_2}^\phi = \mathbb{L}^{\chi(a_1, a_2)/2} \cdot t_1^* \mathcal{S}_{\mathcal{X}_1}^\phi \cdot t_2^* \mathcal{S}_{\mathcal{X}_2}^\phi.$$

Comparing the formulas in (4.4.3), (4.4.4) and (4.4.5), we have

$$I(a_1 \star a_2) = I(a_1) \star I(a_2).$$

. (4.4.6) Now we prove that the map I preserves the Poisson bracket. From the definition of the Poisson bracket:

$$\{a_1, a_2\} = \frac{a_1 \star a_2 - a_2 \star a_1}{\mathbb{L} - 1}.$$

Modulo $\mathbb{L} - 1$, from (4.3.5),

$$(4.4.7) \quad \{a_1, a_2\} = \sum_j \left(\frac{\mathbb{L}^{d_3(S_j)} - \mathbb{L}^{d_0(S_j)}}{\mathbb{L} - 1} \right) [S_j \xrightarrow{k} \mathfrak{M}] + [\mathbb{P}(Q_j) \xrightarrow{\hat{g}_j} \mathfrak{M}] - [\mathbb{P}(\widehat{Q}_j) \xrightarrow{\widehat{g}_j} \mathfrak{M}].$$

Recall that in (4.4.3),

$$I(a_i) = \left(\int t_i^* \mathcal{S}_{\mathcal{X}_i}^\phi \right) \cdot x^{\alpha_i} \in \overline{\mathcal{M}}_{\kappa, \text{loc}}^{\hat{\mu}}[\Gamma].$$

From the definition of motivic quantum torus, we first calculate the Poisson bracket of $I(a_i)$:

(4.4.8)

$$\begin{aligned} & \{I(a_1), I(a_2)\} \\ &= \mathbb{L}^{\frac{1}{2} \cdot \chi(\alpha_1, \alpha_2)} \cdot \frac{\sum_{i=0}^3 (-1)^{i+1} \mathbb{L}^{\dim \text{Ext}^1(\alpha_1, \alpha_2)}}{\mathbb{L} - 1} \cdot \left(\int t_1^* \mathcal{S}_{\mathcal{X}_1}^\phi \right) \cdot \left(\int t_2^* \mathcal{S}_{\mathcal{X}_2}^\phi \right) \cdot x^{\alpha_1 + \alpha_2}. \end{aligned}$$

Still let

$$\begin{aligned} t_j &: S_j \rightarrow \mathcal{S}_j \\ q_j &: Q_j \rightarrow \mathcal{Q}_j \\ \widehat{q}_j &: \widehat{Q}_j \rightarrow \widehat{\mathcal{Q}}_j \end{aligned}$$

be the smooth morphisms from the d -critical loci to the d -critical Artin stacks. Then from (4.4.7),

$$\begin{aligned} (4.4.9) \quad I(\{a_1, a_2\}) &= \sum_j \frac{\mathbb{L}^{d_3(S_j)} - \mathbb{L}^{d_0(S_j)}}{\mathbb{L} - 1} \left(\int t_j^* \mathcal{S}_{S_j}^\phi \right) \cdot x^{\alpha_1 + \alpha_2} \\ &\quad + \left(\left(\int_{\mathbb{P}(Q_j)} q_j^* \mathcal{S}_{Q_j}^\phi \right) - \left(\int_{\mathbb{P}(\widehat{Q}_j)} \widehat{q}_j^* \mathcal{S}_{\widehat{Q}_j}^\phi \right) \right) \cdot x^{\alpha_1 + \alpha_2}. \end{aligned}$$

Note that (still let $E := E_1 \oplus E_2$)

$$\begin{aligned} t_j^* \mathcal{S}_{S_j}^\phi &= \mathbb{L}^{\frac{1}{2} \dim \text{Ext}^0(E, E)} \cdot \mathbb{L}^{-\frac{1}{2} \dim \text{Ext}^1(E, E)} \cdot (1 - \mathcal{S}_{S_j}) \\ q_j^* \mathcal{S}_{Q_j}^\phi &= \mathbb{L}^{\frac{1}{2} \dim \text{Ext}^0(E, E)} \cdot \mathbb{L}^{-\frac{1}{2} \dim \text{Ext}^1(E, E)} \cdot (1 - \mathcal{S}_{Q_j}) \\ \widehat{q}_j^* \mathcal{S}_{\widehat{Q}_j}^\phi &= \mathbb{L}^{\frac{1}{2} \dim \text{Ext}^0(E, E)} \cdot \mathbb{L}^{-\frac{1}{2} \dim \text{Ext}^1(E, E)} \cdot (1 - \mathcal{S}_{\widehat{Q}_j}). \end{aligned}$$

Here for simplicity of the notations we assume that the vanishing cycles $(1 - \mathcal{S}_{S_j})$, $(1 - \mathcal{S}_{Q_j})$ and $(1 - \mathcal{S}_{\widehat{Q}_j})$ have already contained the quadratic forms in the definition of global motives in Theorem 4.8. The reason is that the quadratic forms on $\mathbb{P}(Q_j)$ and $\mathbb{P}(\widehat{Q}_j)$ are the same as the quadratic forms on X since Q_j and \widehat{Q}_j are trivial vector bundles over S_j .

The Formula (2) in Conjecture 1.2 says that

$$\left(\int_{\mathbb{P}(Q_j)} (1 - \mathcal{S}_{Q_j}) - \int_{\mathbb{P}(\widehat{Q}_j)} (1 - \mathcal{S}_{\widehat{Q}_j}) \right) = ([\mathbb{P}(Q_j)] - [\mathbb{P}(\widehat{Q}_j)]) \cdot (1 - \mathcal{S}_{X_1}) \cdot (1 - \mathcal{S}_{X_2}).$$

So using

$$t_i^* \mathcal{S}_{\mathcal{X}_i}^\phi = \mathbb{L}^{\frac{1}{2} \dim \text{Ext}^0(E_i, E_i)} \cdot \mathcal{S}_{X_i}^\phi = \mathbb{L}^{\frac{1}{2} \dim \text{Ext}^0(E_i, E_i)} \cdot \mathbb{L}^{-\frac{1}{2} \dim \text{Ext}^1(E_i, E_i)} \cdot (1 - \mathcal{S}_{X_i}),$$

and note that

$$\sum_j (1 - \mathcal{S}_{S_j}) = (1 - \mathcal{S}_{X_1}) \cdot (1 - \mathcal{S}_{X_2}),$$

we have:

$$\begin{aligned} I(\{a_1, a_2\}) &= \frac{\mathbb{L}^{d_3(S_j)} - \mathbb{L}^{d_0(S_j)}}{\mathbb{L} - 1} \mathbb{L}^{\frac{1}{2}\chi(\alpha_1, \alpha_2)} \cdot \left(\int t_1^* \mathcal{S}_{\mathcal{X}_1}^\phi \right) \cdot \left(\int t_2^* \mathcal{S}_{\mathcal{X}_2}^\phi \right) \cdot x^{\alpha_1 + \alpha_2} \\ &\quad + \left([\mathbb{P}^{\dim \text{Ext}^1(E_1, E_2)}] - [\mathbb{P}^{\dim \text{Ext}^1(E_2, E_1)}] \right) \cdot \mathbb{L}^{\frac{1}{2}\chi(\alpha_1, \alpha_2)}. \\ &\quad \left(\int t_1^* \mathcal{S}_{\mathcal{X}_1}^\phi \right) \cdot \left(\int t_2^* \mathcal{S}_{\mathcal{X}_2}^\phi \right) \cdot x^{\alpha_1 + \alpha_2} \\ &= \left(\frac{\mathbb{L}^{\dim \text{Ext}^3(E_1, E_2)} - \mathbb{L}^{\dim \text{Ext}^0(E_1, E_2)}}{\mathbb{L} - 1} + \frac{\mathbb{L}^{\dim \text{Ext}^1(E_1, E_2)} - \mathbb{L}^{\dim \text{Ext}^1(E_2, E_1)}}{\mathbb{L} - 1} \right) \cdot \\ &\quad \mathbb{L}^{\frac{1}{2}\chi(\alpha_1, \alpha_2)} \cdot \left(\int t_1^* \mathcal{S}_{\mathcal{X}_1}^\phi \right) \cdot \left(\int t_2^* \mathcal{S}_{\mathcal{X}_2}^\phi \right) \cdot x^{\alpha_1 + \alpha_2} \\ &= \mathbb{L}^{\frac{1}{2}\chi(\alpha_1, \alpha_2)} \cdot \frac{\sum_{i=0}^3 (-1)^{i+1} \mathbb{L}^{\dim \text{Ext}^i(\alpha_1, \alpha_2)}}{\mathbb{L} - 1} \cdot \left(\int t_1^* \mathcal{S}_{\mathcal{X}_1}^\phi \right) \cdot \left(\int t_2^* \mathcal{S}_{\mathcal{X}_2}^\phi \right) \cdot x^{\alpha_1 + \alpha_2} \\ &= \{I(a_1), I(a_2)\}. \end{aligned}$$

The last equality is from (4.4.8). The proof is complete. \square

Remark 4.19. When letting $\mathbb{L}^{\frac{1}{2}} = (-1)$ we get the semi-classical limit of Kontsevich-Soibelman in [31]. Note that

$$\lim_{\mathbb{L}^{\frac{1}{2}} \rightarrow (-1)} \frac{\mathbb{L}^n - \mathbb{L}^m}{\mathbb{L} - 1} = n - m$$

for any $n, m \in \mathbb{N}_{>0}$. So when taking the semi-classical limit we get

$$I(\{a_1, a_2\}) = (-1)^{\chi(\alpha_1, \alpha_2)} \cdot \chi(\alpha_1, \alpha_2) \cdot \chi(X_1, f_1^* \nu_{\mathfrak{M}}) \cdot \chi(X_2, f_2^* \nu_{\mathfrak{M}}) \cdot x^{\alpha_1 + \alpha_2},$$

which is the Poisson bracket in [10].

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