

# Analytical study on holographic superfluid in AdS soliton background

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## Abstract

We analytically study the holographic superfluid phase transition in the AdS soliton background by using the variational method for the Sturm-Liouville eigenvalue problem. By investigating the holographic s-wave and p-wave superfluid models in the probe limit, we observe that the spatial component of the gauge field will hinder the phase transition. Moreover, we note that, different from the AdS black hole spacetime, in the AdS soliton background the holographic superfluid phase transition always belongs to the second order and the critical exponent of the system takes the mean-field value in both s-wave and p-wave models. Our analytical results are found to be in good agreement with the numerical findings.

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## I. INTRODUCTION

As we know, the phenomenology of conventional superconductors is extremely well explained by Bardeen-Cooper-Schrieffer (BCS) theory [1] and its extensions [2]. However, these theories fail to describe the core mechanism governing the high-temperature superconductor systems which is one of the unsolved mysteries in modern condensed matter physics. Interestingly, the anti-de Sitter/conformal field theories (AdS/CFT) correspondence [3–5], which can map strongly coupled non-gravitational physics to a weakly coupled perturbative gravitational problem, might provide some meaningful theoretical insights to understand the physics of high  $T_c$  superconductors from the gravitational dual [6–9]. The main idea is that the spontaneous  $U(1)$  symmetry breaking by bulk black holes can be used to construct gravitational duals of the transition from normal state to superconducting state in the boundary theory, which exhibits the behavior of the superconductor [10, 11]. In addition to the bulk AdS black hole spacetime, it was found that a holographic model can be constructed in the bulk AdS soliton background to describe the insulator and superconductor phase transition [12].

In general, the studies on the gravitational dual models of the superconductorlike transition focus on the vanishing spatial components of the  $U(1)$  gauge field on the AdS boundary. Considering that the supercurrent in superconducting materials is a well studied phenomenon in condensed matter systems, the authors of Refs. [13, 14] constructed a holographic superfluid solution by performing a deformation of the superconducting black hole, i.e., turning on a spatial component of the gauge field that only depends on the radial coordinate. It was found that the second-order superfluid phase transition can change to the first order when the velocity of the superfluid component increases relative to the normal component. Interestingly, the holographic superfluid phase transition remains second order for all allowed fractions of superfluid density in the strongly-backreacted regime at low charge  $q$  [15]. However, in the case of the fixed supercurrent, the superfluid phase transition is always of the first order for any nonzero supercurrent [16–18]. In Ref. [19], the effect of the scalar field mass on the superfluid phase transition was investigated and it was observed that the Cave of Winds exists for some special mass in the superfluid model. In order to explore the effect of the vector field on the superfluid phase transition, a holographic p-wave superfluid model in the AdS black holes coupled to a Maxwell complex vector field was introduced [20, 21] and it was revealed that the translating superfluid velocity from second order to first order increases with the increase of the mass squared of the vector field. On the other hand, from the perspective of the QNM analysis, the question of stability of holographic superfluids with finite superfluid

velocity was revisited and it was suggested that there might exist a spatially modulated phase slightly beyond the critical temperature [22, 23].

The aforementioned works on the holographic superfluid models concentrated on the AdS black hole configuration. More recently, the authors of Refs. [24, 25] extended the investigation to the soliton spacetime and investigated numerically the holographic s-wave superfluid model in the AdS soliton background. It was found that, in the probe limit, the first-order phase transition cannot be brought by introducing the spatial component of the vector potential of the gauge field in the AdS soliton background, which is different from the black hole spacetime [25]. In order to back up numerical results and further reveal the properties of the holographic superfluid model in the probe limit, in this work we will use the analytical Sturm-Liouville (S-L) method, which was first proposed in [26, 27] and later generalized to study holographic insulator/superconductor phase transition in [28], to analytically investigate the holographic s-wave superfluid model in the AdS soliton background. Considering that the increasing interest in study of the Maxwell complex vector field model [29–40], we will also extend the investigation to the holographic p-wave superfluid model in the AdS soliton background, which has not been constructed as far as we know. Besides to be used to check numerical computation, the analytical study can clearly disclose some general features for the effects of the spatial component of the gauge field on the holographic superfluid model in the AdS soliton background.

The structure of this work is as follows. In Sec. II we will investigate the holographic s-wave superfluid model in the AdS soliton background. In particular, we calculate the critical chemical potential of the system as well as the relations of condensed values of operators and the charge density with respect to  $(\mu - \mu_c)$ . In Sec. III we extend the discussion to the p-wave case which has not been constructed as far as we know. We will conclude in the last section with our main results.

## II. HOLOGRAPHIC S-WAVE SUPERFLUID MODEL

We start with the five-dimensional Schwarzschild-AdS soliton in the form

$$ds^2 = -r^2 dt^2 + \frac{dr^2}{f(r)} + f(r) d\varphi^2 + r^2(dx^2 + dy^2), \quad (1)$$

where  $f(r) = r^2(1 - r_s^4/r^4)$  with the tip of the soliton  $r_s$  which is a conical singularity in this solution. We can remove the singularity by imposing a period  $\beta = \pi/r_s$  for the coordinate  $\varphi$ . As a matter of fact, this soliton can be obtained from a five-dimensional AdS Schwarzschild black hole by making use of two Wick rotations.

In order to construct the holographic s-wave model of superfluidity in the AdS soliton background, we

consider a Maxwell field and a charged complex scalar field coupled via the action

$$S = \int d^5x \sqrt{-g} \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - |\nabla_\mu \psi - iq A_\mu \psi|^2 - m^2 |\psi|^2 \right), \quad (2)$$

where  $q$  and  $m$  represent the charge and mass of the scalar field  $\psi$  respectively. Taking the ansatz of the matter fields as

$$\psi = \psi(r), \quad A_\mu dx^\mu = A_t(r) dt + A_\varphi(r) d\varphi, \quad (3)$$

where both a time component  $A_t$  and a spatial component  $A_\varphi$  of the vector potential have been introduced in order to consider the possibility of DC supercurrent, we can get the equations of motion in the probe limit

$$\begin{aligned} \psi'' + \left( \frac{3}{r} + \frac{f'}{f} \right) \psi' - \frac{1}{f} \left( m^2 + \frac{q^2 A_\varphi^2}{f} - \frac{q^2 A_t^2}{r^2} \right) \psi &= 0, \\ A_t'' + \left( \frac{1}{r} + \frac{f'}{f} \right) A_t' - \frac{2q^2 \psi^2}{f} A_t &= 0, \\ A_\varphi'' + \frac{3}{r} A_\varphi' - \frac{2q^2 \psi^2}{f} A_\varphi &= 0, \end{aligned} \quad (4)$$

where the prime denotes the derivative with respect to  $r$ . From the equation of motion for  $\psi$ , we can obtain the effective mass of the scalar field

$$m_{eff}^2 = m^2 + \frac{q^2 A_\varphi^2}{f} - \frac{q^2 A_t^2}{r^2}, \quad (5)$$

which implies that the increasing  $m^2$  and  $A_\varphi$  or decreasing  $A_t$  will hinder the s-wave superfluid phase transition.

We will get the consistent result in the following calculation.

In order to solve above equations, we have to impose the appropriate boundary conditions at the tip  $r = r_s$  and the boundary  $r \rightarrow \infty$ . At the tip  $r = r_s$ , the fields behave as

$$\begin{aligned} \psi &= \tilde{\psi}_0 + \tilde{\psi}_1 (r - r_s) + \tilde{\psi}_2 (r - r_s)^2 + \dots, \\ A_t &= \tilde{A}_{t0} + \tilde{A}_{t1} (r - r_s) + \tilde{A}_{t2} (r - r_s)^2 + \dots, \\ A_\varphi &= \tilde{A}_{\varphi 1} (r - r_s) + \tilde{A}_{\varphi 2} (r - r_s)^2 + \dots, \end{aligned} \quad (6)$$

where  $\tilde{\psi}_i$ ,  $\tilde{A}_{ti}$  and  $\tilde{A}_{\varphi i}$  ( $i = 0, 1, 2, \dots$  and  $\tilde{A}_{\varphi 0} = 0$ ) are the integration constants, and we have imposed the Neumann-like boundary conditions to render the physical quantities finite [12]. Obviously, we can find a constant nonzero gauge field  $A_t(r_s)$  at  $r = r_s$ , which is in strong contrast to that of the holographic superfluid model in the AdS black hole background where  $A_t(r_+) = 0$  at the horizon [13, 14, 25].

At the asymptotic AdS boundary  $r \rightarrow \infty$ , we have asymptotic behaviors

$$\psi = \frac{\psi_-}{r^{\Delta_-}} + \frac{\psi_+}{r^{\Delta_+}}, \quad A_t = \mu - \frac{\rho}{r^2}, \quad A_\varphi = S_\varphi - \frac{J_\varphi}{r^2}, \quad (7)$$

where  $\Delta_{\pm} = 2 \pm \sqrt{4 + m^2}$  is the conformal dimension of the scalar operator dual to the bulk scalar field,  $\mu$  and  $S_{\varphi}$  are the chemical potential and superfluid velocity, while  $\rho$  and  $J_{\varphi}$  are the charge density and current in the dual field theory, respectively. Note that, provided  $\Delta_{-}$  is larger than the unitarity bound, both  $\psi_{-}$  and  $\psi_{+}$  can be normalizable and they will be used to define operators in the dual field theory according to the AdS/CFT correspondence,  $\psi_{-} = \langle O_{-} \rangle$ ,  $\psi_{+} = \langle O_{+} \rangle$ , respectively. We can impose boundary conditions that either  $\psi_{-}$  or  $\psi_{+}$  vanishes [11, 41].

Interestingly, from Eq. (4) we can get the useful scaling symmetries

$$r \rightarrow \lambda r, \quad (t, \varphi, x, y) \rightarrow \frac{1}{\lambda}(t, \varphi, x, y), \quad (q, \psi) \rightarrow (q, \psi), \quad (A_t, A_{\varphi}) \rightarrow \lambda(A_t, A_{\varphi}), \quad (8)$$

where  $\lambda$  is a real positive number. Using these symmetries, we can obtain the transformation of the relevant quantities

$$(\mu, S_{\varphi}) \rightarrow \lambda(\mu, S_{\varphi}), \quad (\rho, J_{\varphi}) \rightarrow \lambda^3(\rho, J_{\varphi}), \quad \psi_i \rightarrow \lambda^{\Delta_i} \psi_i, \quad (9)$$

with  $i = +$  or  $i = -$ . We can use them to set  $q = 1$  and  $r_s = 1$  when performing numerical calculations and check the analytical expressions in this section.

Applying the S-L method to analytically study the properties of the holographic s-wave model of superfluidity in AdS soliton background, we will introduce a new variable  $z = r_s/r$  and rewrite Eq. (4) into

$$\psi'' + \left( \frac{f'}{f} - \frac{1}{z} \right) \psi' + \left[ \frac{1}{z^2 f} \left( \frac{q A_t}{r_s} \right)^2 - \frac{1}{z^4 f^2} \left( \frac{q A_{\varphi}}{r_s} \right)^2 - \frac{m^2}{z^4 f} \right] \psi = 0, \quad (10)$$

$$A_t'' + \left( \frac{1}{z} + \frac{f'}{f} \right) A_t' - \frac{2q^2 \psi^2}{z^4 f} A_t = 0, \quad (11)$$

$$A_{\varphi}'' - \frac{1}{z} A_{\varphi}' - \frac{2q^2 \psi^2}{z^4 f} A_{\varphi} = 0, \quad (12)$$

with  $f = (1 - z^4)/z^2$ . Here and hereafter in this section the prime denotes the derivative with respect to  $z$ .

### A. Critical chemical potential

It has been shown numerically that [12, 42], adding the chemical potential to the AdS soliton, the solution is unstable to develop a hair for the chemical potential bigger than a critical value, i.e.,  $\mu > \mu_c$ . For lower chemical potential  $\mu < \mu_c$ , the scalar field is zero and it can be interpreted as the insulator phase since in this model the normal phase is described by an AdS soliton where the system exhibits a mass gap. Therefore,

there is a phase transition when  $\mu \rightarrow \mu_c$  and the AdS soliton reaches the superconductor (or superfluid) phase for larger  $\mu$ .

At the critical chemical potential  $\mu_c$ , the scalar field  $\psi = 0$ . Thus, below the critical point Eq. (11) reduces to

$$A_t'' + \left( \frac{1}{z} + \frac{f'}{f} \right) A_t' = 0, \quad (13)$$

which leads to a general solution

$$A_t = \mu + c_1 \ln \left( \frac{1+z^2}{1-z^2} \right), \quad (14)$$

where  $c_1$  is an integration constant. Obviously, the second term is divergent at the tip  $z = 1$  if  $c_1 \neq 0$ . Considering the Neumann-like boundary condition (6) for the gauge field  $A_t$  at the tip  $z = 1$ , we have to set  $c_1 = 0$  to keep  $A_t$  finite, i.e., in this case  $A_t$  will be a constant. Thus, we can get the physical solution  $A_t(z) = \mu$  to Eq. (13) if  $\mu < \mu_c$ . This is consistent with the numerical results in Figs. 1 and 2 which plot the condensates of the operator  $\langle O_i \rangle = \psi_i$  and charge density  $\rho$  with respect to the chemical potential  $\mu$  for different values of the dimensionless parameter  $k = S_\varphi/\mu$ .

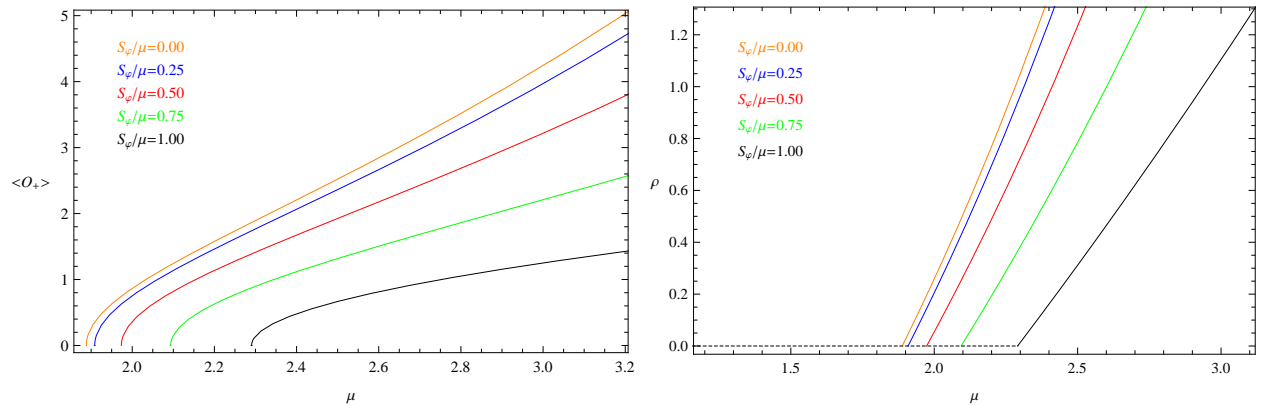


FIG. 1: (Color online) The condensate of the operator  $\langle O_+ \rangle$  and charge density  $\rho$  with respect to the chemical potential  $\mu$  for different values of the dimensionless parameter  $k = S_\varphi/\mu$  in the holographic s-wave model of superfluidity by using the numerical shooting method. In each panel, the five lines from left to right correspond to increasing  $S_\varphi/\mu$ , i.e.,  $S_\varphi/\mu = 0.00$  (orange), 0.25 (blue), 0.50 (red), 0.75 (green) and 1.00 (black) respectively. We choose  $m^2 = -15/4$  and scale  $q = 1$  and  $r_s = 1$  in the numerical computation.

Similarly, from Eq. (12) we have

$$A_\varphi'' - \frac{1}{z} A_\varphi' = 0, \quad (15)$$

which results in a solution

$$A_\varphi = S_\varphi(1 - z^2), \quad (16)$$

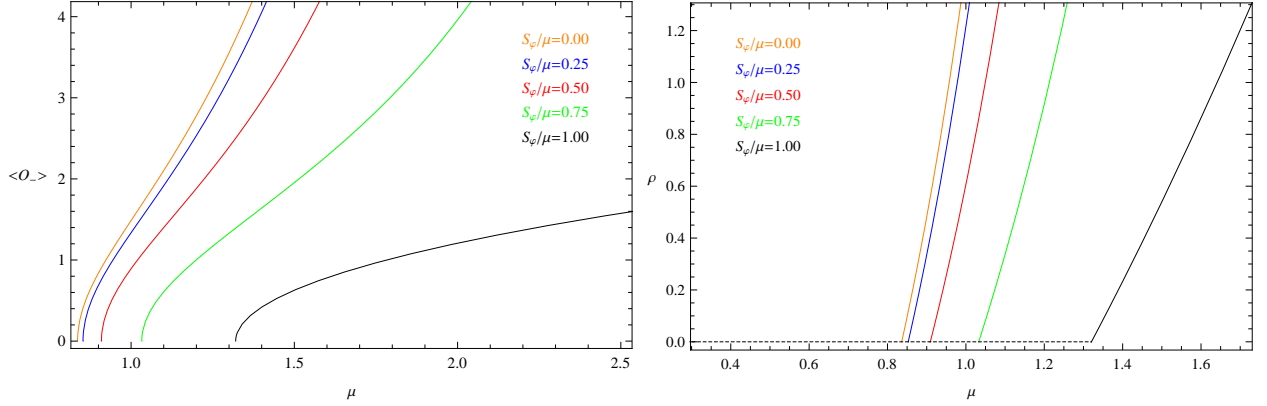


FIG. 2: (Color online) The condensate of the operator  $\langle O_- \rangle$  and charge density  $\rho$  with respect to the chemical potential  $\mu$  for different values of the dimensionless parameter  $k = S_\varphi/\mu$  in the holographic s-wave model of superfluidity by using the numerical shooting method. In each panel, the five lines from left to right correspond to increasing  $S_\varphi/\mu$ , i.e.,  $S_\varphi/\mu = 0.00$  (orange),  $0.25$  (blue),  $0.50$  (red),  $0.75$  (green) and  $1.00$  (black) respectively. We choose  $m^2 = -15/4$  and scale  $q = 1$  and  $r_s = 1$  in the numerical computation.

which is consistent with the boundary condition  $A_\varphi(1) = 0$  given in (6).

As  $\mu \rightarrow \mu_c$  from below the critical point, the scalar field equation (10) becomes

$$\psi'' + \left( \frac{f'}{f} - \frac{1}{z} \right) \psi' + \left[ \frac{1}{z^2 f} \left( \frac{q\mu}{r_s} \right)^2 - \frac{(1-z^2)^2}{z^4 f^2} \left( \frac{qS_\varphi}{r_s} \right)^2 - \frac{m^2}{z^4 f} \right] \psi = 0. \quad (17)$$

With the boundary condition (7), we assume  $\psi$  takes the form

$$\psi(z) \sim \frac{\langle O_i \rangle}{r_s^{\Delta_i}} z^{\Delta_i} F(z), \quad (18)$$

where the trial function  $F(z)$  obeys the boundary conditions  $F(0) = 1$  and  $F'(0) = 0$ . From Eq. (17), we arrive at

$$(TF')' + T \left[ U + V \left( \frac{q\mu}{r_s} \right)^2 - W \left( \frac{qS_\varphi}{r_s} \right)^2 \right] F = 0, \quad (19)$$

where we have defined

$$T = z^{2\Delta_i - 1} f, \quad U = \frac{\Delta_i}{z} \left( \frac{\Delta_i - 2}{z} + \frac{f'}{f} \right) - \frac{m^2}{z^4 f}, \quad V = \frac{1}{z^2 f}, \quad W = \frac{(1-z^2)^2}{z^4 f^2}. \quad (20)$$

According to the Sturm-Liouville eigenvalue problem [43], the minimum eigenvalue of  $\Lambda = q\mu/r_s$  can be obtained from variation of the following functional

$$\Lambda^2 = \left( \frac{q\mu}{r_s} \right)^2 = \frac{\int_0^1 T (F'^2 - UF^2) dz}{\int_0^1 T (V - k^2 W) F^2 dz}, \quad (21)$$

where we will assume the trial function to be  $F(z) = 1 - az^2$  with a constant  $a$ . When  $k = 0$ , Eq. (21) reduces to the case considered in [28] for the holographic s-wave insulator/superconductor phase transition, where the spatial component  $A_\varphi$  has been turned off.

For different values of  $k$  and  $m^2$  with the fixed operator  $\langle O_+ \rangle$  or  $\langle O_- \rangle$ , we can obtain the minimum eigenvalue of  $\Lambda^2$  and the corresponding value of  $a$ . As an example, we have  $\Lambda_{min}^2 = 3.650$  and  $a = 0.3214$  for  $k = 0.25$  with  $m^2 = -15/4$ , which gives the critical chemical potential  $\Lambda_c = \Lambda_{min} = 1.911$  for the operator  $\langle O_+ \rangle$ . In Table I, we present the critical chemical potential  $\Lambda_c = q\mu_c/r_s$  for chosen  $k$  with fixed mass of the scalar field by  $m^2 = -15/4$  in the holographic s-wave superfluid model. Obviously, the agreement of the analytical results derived from the S-L method with the numerical calculation shown in Table I is impressive.

TABLE I: The critical chemical potential  $\Lambda_c = q\mu_c/r_s$  obtained by the analytical S-L method (left column) and from numerical calculation (right column) with the chosen values of  $k = S_\varphi/\mu$  for the scalar operators  $\langle \mathcal{O}_- \rangle$  and  $\langle \mathcal{O}_+ \rangle$  in the holographic s-wave superfluid model. Here we fix the mass of the scalar field by  $m^2 = -15/4$ .

	$\langle \mathcal{O}_- \rangle$		$\langle \mathcal{O}_+ \rangle$	
$k = 0.00$	0.8368	0.8362	1.890	1.888
$k = 0.25$	0.8534	0.8528	1.911	1.909
$k = 0.50$	0.9096	0.9092	1.975	1.973
$k = 0.75$	1.032(7)	1.032(8)	2.094	2.067
$k = 1.00$	1.320(5)	1.320(3)	2.291	2.290

We see that, from Table I and Figs. 1 and 2, the critical chemical potential  $\Lambda_c = q\mu_c/r_s$  increases as the dimensionless parameter  $k = S_\varphi/\mu$  increases for the fixed mass of the scalar field, i.e., the critical chemical potential becomes larger with the increase of the superfluid velocity, which indicates that the spatial component of the gauge field to modeling the superfluid hinders the phase transition. This result is consistent with the observation obtained from the effective mass of the scalar field in Eq. (5), which implies that the increasing  $A_\varphi$  will hinder the s-wave superfluid phase transition.

## B. Critical phenomena

Now we are in a position to study the critical phenomena of this holographic s-wave superfluid system. Considering that the condensation of the scalar operator  $\langle O_i \rangle$  is so small near the critical point, we can expand  $A_t(z)$  in  $\langle O_i \rangle$  as

$$A_t(z) \sim \mu_c + \langle O_i \rangle \chi(z) + \dots, \quad (22)$$

where we have introduced the boundary condition  $\chi(1) = 0$  at the tip. Defining a function  $\xi(z)$  as

$$\chi(z) = \frac{2q^2\mu_c}{r_s^{2\Delta_i}} \langle O_i \rangle \xi(z), \quad (23)$$

we obtain the equation of motion for  $\xi(z)$

$$(Q\xi)' - z^{2\Delta_i-3} F^2 = 0, \quad (24)$$



with

$$Q(z) = zf(z). \quad (25)$$

According to the asymptotic behavior in Eq. (7) and Eq. (23), we will expand  $A_t$  when  $z \rightarrow 0$  as

$$A_t(z) \simeq \mu - \frac{\rho}{r_s^2} z^2 \simeq \mu_c + 2\mu_c \left( \frac{q\langle O_i \rangle}{r_s^{\Delta_i}} \right)^2 \left[ \xi(0) + \xi'(0)z + \frac{1}{2}\xi''(0)z^2 + \dots \right]. \quad (26)$$

From the coefficients of the  $z^0$  term in both sides of the above formula, we have

$$\frac{q\langle O_i \rangle}{r_s^{\Delta_i}} = \frac{1}{[2\mu_c \xi(0)]^{\frac{1}{2}}} (\mu - \mu_c)^{\frac{1}{2}}, \quad (27)$$

with

$$\xi(0) = c_2 - \int_0^1 \frac{1}{Q(z)} \left[ c_3 + \int_1^z x^{2\Delta_i-3} F(x)^2 dx \right] dz, \quad (28)$$

where  $c_2$  and  $c_3$  are the integration constants which can be determined by the boundary condition of  $\chi(z)$ . For example, for the case of  $k = 0.25$  with  $m^2 = -15/4$ , we have  $\langle O_+ \rangle \approx 1.776(\mu - \mu_c)^{1/2}$  when  $a = 0.3214$  (we have scaled  $q = 1$  and  $r_s = 1$  for simplicity), which is in good agreement with the numerical result shown in the left panel of Fig. 1. Note that our expression (27) is valid for all cases considered here, so near the critical point, both of the scalar operators  $\langle O_+ \rangle$  and  $\langle O_- \rangle$  satisfy  $\langle O_i \rangle \sim (\mu - \mu_c)^{1/2}$ . This analytical result shows that the phase transition of the holographic s-wave superfluid model belongs to the second order and the critical exponent of the system takes the mean-field value  $1/2$ , which can be used to back up the numerical findings as shown in Figs. 1 and 2.

Comparing the coefficients of the  $z^1$  term in Eq. (26), we observe that  $\xi'(0) \rightarrow 0$ , which agrees well with the following relation by making integration of both sides of Eq. (24)

$$\left[ \frac{\xi'(z)}{z} \right] \Big|_{z \rightarrow 0} = - \int_0^1 z^{2\Delta_i-3} F^2 dz. \quad (29)$$

Considering the coefficients of the  $z^2$  term in Eq. (26), we get

$$\frac{\rho}{r_s^2} = - \left( \frac{q\langle O_i \rangle}{r_s^{\Delta_i}} \right)^2 \mu_c \xi''(0) = \Gamma(k, m)(\mu - \mu_c), \quad (30)$$

with a prefactor

$$\Gamma(k, m) = \frac{1}{2\xi(0)} \int_0^1 z^{2\Delta_i-3} F^2 dz, \quad (31)$$

which is a function of the parameter  $k$  and scalar field mass  $m^2$ . For the case of  $k = 0.25$  and  $m^2 = -15/4$  with the operator  $\langle O_+ \rangle$ , as an example, we can obtain  $\rho = 1.323(\mu - \mu_c)$  when  $a = 0.3214$  (we have scaled  $q = 1$

and  $r_s = 1$  for simplicity), which is consistent with the result given in the right panel of Fig. 1. Obviously, the parameter  $k$  and mass of the scalar field  $m^2$  will not change the linear relation between the charge density and chemical potential near  $\mu_c$ , i.e.,  $\rho \sim (\mu - \mu_c)$ , which is in good agreement with the numerical results plotted in Figs. 1 and 2.

On the other hand, near the critical point Eq. (12) becomes

$$A_\varphi'' - \frac{1}{z}A_\varphi' - \frac{2S_\varphi(1-z^2)}{z^4f} \left( \frac{q\langle O_i \rangle z^{\Delta_i} F}{r_s^{\Delta_i}} \right)^2 = 0. \quad (32)$$

Thus, we finally arrive at

$$A_\varphi = S_\varphi(1-z^2) + S_\varphi \left( \frac{q\langle O_i \rangle}{r_s^{\Delta_i}} \right)^2 \int z \left[ \int \frac{2x^{2\Delta_i-5}(1-x^2)F(x)^2}{f(x)} dx \right] dz, \quad (33)$$

which obeys the boundary condition  $A_\varphi(1) = 0$  presented in (6) at the critical point. For example, for the case of  $k = 0.25$  with  $m^2 = -15/4$ , we obtain  $A_\varphi = S_\varphi[(1-z^2) - 0.07382\langle O_+ \rangle^2 z^2 + \dots]$  when  $a = 0.3214$  (we have scaled  $q = 1$  and  $r_s = 1$  for simplicity) for the operator  $\langle O_+ \rangle$ . Obviously, Eq. (33) is consistent with the behavior of  $A_\varphi$  in Eq. (16) at the critical point.

### III. HOLOGRAPHIC P-WAVE SUPERFLUID MODEL

Since the S-L method is effective to obtain the properties of the holographic s-wave model of superfluidity in the AdS soliton background, we will use it to investigate analytically the holographic p-wave model of superfluidity in the AdS soliton background which has not been constructed as far as we know.

Considering the Maxwell complex vector field model which was first proposed in [29, 30], we will build the holographic p-wave model of superfluidity in the AdS soliton background via the action

$$S = \frac{1}{16\pi G} \int d^5x \sqrt{-g} \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \rho_{\mu\nu}^\dagger \rho^{\mu\nu} - m^2 \rho_\mu^\dagger \rho^\mu + iq\gamma \rho_\mu \rho_\nu^\dagger F^{\mu\nu} \right), \quad (34)$$

where the tensor  $\rho_{\mu\nu}$  is defined by  $\rho_{\mu\nu} = D_\mu \rho_\nu - D_\nu \rho_\mu$  with the covariant derivative  $D_\mu = \nabla_\mu - iqA_\mu$ ,  $q$  and  $m$  are the charge and mass of the vector field  $\rho_\mu$ , respectively. Since we consider the case without external magnetic field in this work, the parameter  $\gamma$ , which describes the interaction between the vector field  $\rho_\mu$  and the gauge field  $A_\mu$ , will not play any role.

As in Refs. [20, 21], we take the same ansatz for the gauge field  $A_\mu$  just as in Eq. (3) and assume the condensate to pick out the  $x$  direction as special

$$\rho_\mu dx^\mu = \rho_x(r) dx, \quad (35)$$

where we can set  $\rho_x(r)$  to be real by using the  $U(1)$  gauge symmetry. Thus, in the soliton background (1), we can obtain the equations of motion for the holographic p-wave superfluid model

$$\begin{aligned}\rho_x'' + \left(\frac{1}{r} + \frac{f'}{f}\right)\rho_x' - \frac{1}{f}\left(m^2 + \frac{q^2 A_\varphi^2}{f} - \frac{q^2 A_t^2}{r^2}\right)\rho_x &= 0, \\ A_t'' + \left(\frac{1}{r} + \frac{f'}{f}\right)A_t' - \frac{2q^2 \rho_x^2}{r^2 f}A_t &= 0, \\ A_\varphi'' + \frac{3}{r}A_\varphi' - \frac{2q^2 \rho_x^2}{r^2 f}A_\varphi &= 0,\end{aligned}\tag{36}$$

where the prime denotes the derivative with respect to  $r$ . Obviously, we find that the effective mass of the vector field has the same expression just as in (5), which means that the increasing  $m^2$  and  $A_\varphi$  or decreasing  $A_t$  will hinder the p-wave superfluid phase transition.

Analyzing the boundary conditions of the matter fields, we observe that  $A_t$  and  $A_\varphi$  have the same boundary conditions just as Eq. (6) for the tip  $r = r_s$  and Eq. (7) for the boundary  $r \rightarrow \infty$ . But for the vector field  $\rho_x$ , we find that at the tip

$$\rho_x = \tilde{\rho}_{x0} + \tilde{\rho}_{x1}(r - r_s) + \tilde{\rho}_{x2}(r - r_s)^2 + \dots,\tag{37}$$

with the integration constant  $\tilde{\rho}_{xi}$  ( $i = 0, 1, 2, \dots$ ), and at the asymptotic AdS boundary

$$\rho_x = \frac{\rho_{x-}}{r^{2-\Delta}} + \frac{\rho_{x+}}{r^\Delta},\tag{38}$$

with the characteristic exponent  $\Delta = 1 + \sqrt{1 + m^2}$ . According to the AdS/CFT correspondence,  $\rho_{x-}$  and  $\rho_{x+}$  are interpreted as the source and the vacuum expectation value of the vector operator  $\langle O_x \rangle$  in the dual field theory respectively. Since we require that the condensate appears spontaneously, we will impose boundary condition  $\rho_{x-} = 0$  in this work.

From Eq. (36), we can also have the useful scaling symmetries

$$r \rightarrow \lambda r, \quad (t, \varphi, x, y) \rightarrow \frac{1}{\lambda}(t, \varphi, x, y), \quad q \rightarrow q, \quad (\rho_x, A_t, A_\varphi) \rightarrow \lambda(\rho_x, A_t, A_\varphi),\tag{39}$$

and the transformation of the relevant quantities

$$(\mu, S_\varphi) \rightarrow \lambda(\mu, S_\varphi), \quad (\rho, J_\varphi) \rightarrow \lambda^3(\rho, J_\varphi), \quad \rho_{x+} \rightarrow \lambda^{1+\Delta}\rho_{x+},\tag{40}$$

which can be used to set  $q = 1$  and  $r_s = 1$  when performing numerical calculations and check the analytical expressions in this section.

For convenience in the following discussion, we will change the coordinate  $z = r_s/r$  and convert Eq. (36) to be

$$\rho_x'' + \left(\frac{1}{z} + \frac{f'}{f}\right)\rho_x' + \left[\frac{1}{z^2 f} \left(\frac{q A_t}{r_s}\right)^2 - \frac{1}{z^4 f^2} \left(\frac{q A_\varphi}{r_s}\right)^2 - \frac{m^2}{z^4 f}\right]\rho_x = 0,\tag{41}$$

$$A_t'' + \left(\frac{1}{z} + \frac{f'}{f}\right) A_t' - \frac{2}{z^2 f} \left(\frac{q\rho_x}{r_s}\right)^2 A_t = 0, \quad (42)$$

$$A_\varphi'' - \frac{1}{z} A_\varphi' - \frac{2}{z^2 f} \left(\frac{q\rho_x}{r_s}\right)^2 A_\varphi = 0. \quad (43)$$

Here and hereafter in this section the prime denotes the derivative with respect to  $z$ .

### A. Critical chemical potential

Similar to the analysis for the holographic s-wave model of superfluidity, if  $\mu \leq \mu_c$ , the vector field  $\rho_x$  is nearly zero, i.e.,  $\rho_x \simeq 0$ . Thus, we can obtain the physical solutions  $A_t(z) = \mu$  to Eq. (42) and  $A_\varphi = S_\varphi(1 - z^2)$  to Eq. (43) when  $\mu < \mu_c$ , which are the same forms just as in the holographic s-wave superfluid model. This analytical result is consistent with the numerical findings in Fig. 3 which plots the condensate of the operator  $\langle O_x \rangle = \rho_{x+}$  and charge density  $\rho$  with respect to the chemical potential  $\mu$  for different values of the dimensionless parameter  $k = S_\varphi/\mu$ .

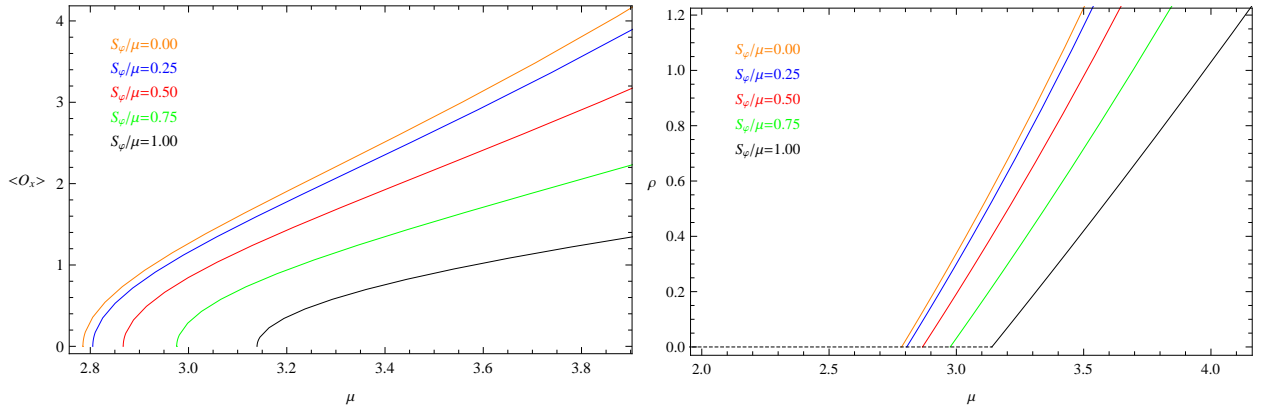


FIG. 3: (Color online) The condensate of the operator  $\langle O_x \rangle$  and charge density  $\rho$  with respect to the chemical potential  $\mu$  for different values of  $k = S_\varphi/\mu$  in the holographic p-wave model of superfluidity by using the numerical shooting method. In each panel, the five lines from left to right correspond to increasing  $S_\varphi/\mu$ , i.e.,  $S_\varphi/\mu = 0.00$  (orange), 0.25 (blue), 0.50 (red), 0.75 (green) and 1.00 (black) respectively. We choose  $m^2 = 5/4$  and scale  $q = 1$  and  $r_s = 1$  in the numerical computation.

As  $\mu \rightarrow \mu_c$ , the vector field equation (41) will become

$$\rho_x'' + \left(\frac{1}{z} + \frac{f'}{f}\right) \rho_x' + \left[ \frac{1}{z^2 f} \left(\frac{q\mu}{r_s}\right)^2 - \frac{(1-z^2)^2}{z^4 f^2} \left(\frac{qS_\varphi}{r_s}\right)^2 - \frac{m^2}{z^4 f} \right] \rho_x = 0. \quad (44)$$

Defining a trial function  $F(z)$  which matches the boundary behavior (38) for  $\rho_x$  [26]

$$\rho_x(z) \sim \frac{\langle O_x \rangle}{r_s^\Delta} z^\Delta F(z), \quad (45)$$

with the boundary conditions  $F(0) = 1$  and  $F'(0) = 0$ , from Eq. (44) we can get the equation of motion for

$F(z)$

$$(MF')' + M \left[ P + V \left( \frac{q\mu}{r_s} \right)^2 - W \left( \frac{qS_\varphi}{r_s} \right)^2 \right] F = 0, \quad (46)$$

with

$$M = z^{1+2\Delta} f, \quad P = \frac{\Delta}{z} \left( \frac{\Delta}{z} + \frac{f'}{f} \right) - \frac{m^2}{z^4 f}, \quad (47)$$

where  $V(z)$  and  $W(z)$  have been introduced in (20). Following the S-L eigenvalue problem [43], we deduce the eigenvalue  $\Lambda = q\mu/r_s$  minimizes the expression

$$\Lambda^2 = \left( \frac{q\mu}{r_s} \right)^2 = \frac{\int_0^1 M (F'^2 - PF^2) dz}{\int_0^1 M (V - k^2 W) F^2 dz}, \quad (48)$$

where we still assume the trial function to be  $F(z) = 1 - az^2$  with a constant  $a$ . When the dimensionless parameter  $k = 0$ , Eq. (48) reduces to the case considered in [37] for the holographic p-wave insulator/superconductor phase transition, where the spatial component  $A_\varphi$  has been turned off.

For different values of  $k$  and  $m^2$ , we can get the minimum eigenvalue of  $\Lambda^2$  and the corresponding value of  $a$ , for example,  $\Lambda_{min}^2 = 7.879$  and  $a = 0.3716$  for  $k = 0.25$  with  $m^2 = 5/4$ , which leads to the critical chemical potential  $\Lambda_c = \Lambda_{min} = 2.807$ . In Table II, we present the critical chemical potential  $\Lambda_c = q\mu_c/r_s$  for chosen  $k$ . In order to compare with numerical results given in Fig. 3, we fix the mass of the vector field by  $m^2 = 5/4$ . Obviously, the analytical results derived from S-L method are in very good agreement with the numerical computations.

TABLE II: The critical chemical potential  $\Lambda_c = q\mu_c/r_s$  for the vector operator  $\langle O_x \rangle$  obtained by the analytical S-L method and numerical shooting method with chosen  $k = S_\varphi/\mu$  for the fixed mass of the vector field  $m^2 = 5/4$  in the holographic p-wave superfluid model.

$k$	0.00	0.25	0.50	0.75	1.00
Analytical	2.787	2.807	2.868	2.976(1)	3.140
Numerical	2.785	2.805	2.867	2.975(8)	3.139

From Table II, we observe that, for the fixed mass of the vector field, the critical chemical potential  $\Lambda_c = q\mu_c/r_s$  becomes larger with the increase of  $k = S_\varphi/\mu$ , i.e., the critical chemical potential increases with the increase of the superfluid velocity. The fact implies that the spatial component of the gauge field to modeling the superfluid hinders the phase transition, which supports the observation obtained from the effective mass of the vector field in Eq. (36).

## B. Critical phenomena

Since the condensation of the vector operator  $\langle O_x \rangle$  is so small when  $\mu \rightarrow \mu_c$ , we can expand  $A_t(z)$  in small  $\langle O_x \rangle$  as

$$A_t(z) \sim \mu_c + \langle O_x \rangle \chi(z) + \dots, \quad (49)$$

with the boundary condition  $\chi(1) = 0$  at the tip. Introducing a function  $\xi(z)$  as

$$\chi(z) = \frac{2q^2\mu_c}{r_s^{2(1+\Delta)}} \langle O_x \rangle \xi(z), \quad (50)$$

we get the equation of motion for  $\xi(z)$

$$(Q\xi')' - z^{2\Delta-1}F^2 = 0, \quad (51)$$

where  $Q(z)$  has been defined in (25).

Considering the asymptotic behavior of  $A_t$  and Eq. (50), near  $z \rightarrow 0$  we will expand  $A_t$  as

$$A_t(z) \simeq \mu - \frac{\rho}{r_s^2} z^2 \simeq \mu_c + 2\mu_c \left( \frac{q\langle O_x \rangle}{r_s^{1+\Delta}} \right)^2 \left[ \xi(0) + \xi'(0)z + \frac{1}{2}\xi''(0)z^2 + \dots \right]. \quad (52)$$

According to the coefficients of the  $z^0$  term in both sides of the above formula, we obtain

$$\frac{q\langle O_x \rangle}{r_s^{1+\Delta}} = \frac{1}{[2\mu_c\xi(0)]^{\frac{1}{2}}} (\mu - \mu_c)^{\frac{1}{2}}, \quad (53)$$

with

$$\xi(0) = c_2 - \int_0^1 \frac{1}{Q(z)} \left[ c_3 + \int_1^z x^{2\Delta-1} F(x)^2 dx \right] dz, \quad (54)$$

where  $c_2$  and  $c_3$  are the integration constants which can be determined by the boundary condition of  $\chi(z)$ . For example, for the case of  $k = 0.25$  with  $m^2 = 5/4$ , we have  $\langle O_x \rangle \approx 1.818(\mu - \mu_c)^{1/2}$  when  $a = 0.3716$  (we have scaled  $q = 1$  and  $r_s = 1$  for simplicity), which agrees well with the numerical result shown in the left panel of Fig. 3. Obviously, the expression (53) is valid for all cases considered here. Since the parameter  $k$  and mass of the vector field  $m^2$  will not alter Eq. (53) except for the prefactor, we can obtain the relation  $\langle O_x \rangle \sim (\mu - \mu_c)^{1/2}$  near the critical point, which shows that the phase transition of the holographic p-wave superfluid model belongs to the second order and the critical exponent of the system takes the mean-field value  $1/2$ . The analytic result supports the numerical findings obtained from the left panel of Fig. 3.

From the coefficients of the  $z^1$  term in Eq. (52), we find that  $\xi'(0) \rightarrow 0$ , which is consistent with the following relation by making integration of both sides of Eq. (51)

$$\left[ \frac{\xi'(z)}{z} \right] \Big|_{z \rightarrow 0} = - \int_0^1 z^{2\Delta-1} F^2 dz. \quad (55)$$

Comparing the coefficients of the  $z^2$  term in Eq. (52), we arrive at

$$\frac{\rho}{r_s^2} = - \left( \frac{q\langle O_x \rangle}{r_s^{1+\Delta}} \right)^2 \mu_c \xi''(0) = \Gamma(k, m)(\mu - \mu_c), \quad (56)$$

with

$$\Gamma(k, m) = \frac{1}{2\xi(0)} \int_0^1 z^{2\Delta-1} F^2 dz, \quad (57)$$

which is a function of the parameter  $k$  and vector field mass  $m^2$ . For the case of  $k = 0.25$  with  $m^2 = 5/4$ , as an example, we can obtain  $\rho = 1.013(\mu - \mu_c)$  when  $a = 0.3716$  (we have scaled  $q = 1$  and  $r_s = 1$  for simplicity), which is in good agreement with the result shown in the right panel of Fig. 3. Note that the parameter  $k$  and mass of the vector field  $m^2$  will not alter Eq. (56), we can obtain the linear relation between the charge density and chemical potential near  $\mu_c$ , i.e.,  $\rho \sim (\mu - \mu_c)$ , which can be used to back up the numerical result presented in the right panel of Fig. 3.

Similarly, considering Eq. (43) near the phase transition point, i.e.,

$$A_\varphi'' - \frac{1}{z} A_\varphi' - \frac{2S_\varphi(1-z^2)}{z^2 f} \left( \frac{q\langle O_x \rangle z^\Delta F}{r_s^{1+\Delta}} \right)^2 = 0, \quad (58)$$

we can solve it and get

$$A_\varphi = S_\varphi(1-z^2) + S_\varphi \left( \frac{q\langle O_x \rangle}{r_s^{1+\Delta}} \right)^2 \int z \left[ \int \frac{2x^{2\Delta-3}(1-x^2)F(x)^2}{f(x)} dx \right] dz, \quad (59)$$

which is consistent with the boundary condition  $A_\varphi(1) = 0$  at the critical point. For example, for the case of  $k = 0.25$  with  $m^2 = 5/4$ , we have  $A_\varphi = S_\varphi[(1-z^2) - 0.02450\langle O_x \rangle^2 z^2 + \dots]$  when  $a = 0.3716$  (we have scaled  $q = 1$  and  $r_s = 1$  for simplicity), which supports our numerical computation.

#### IV. CONCLUSIONS

We have applied the S-L method to study analytically the properties of the holographic superfluid models in the AdS soliton background in order to understand the influence of the spatial component of the gauge field on the superfluid phase transition. By investigating the s-wave (the scalar field) and p-wave (the vector field) models in the probe limit, we obtained analytically the critical chemical potentials which are perfectly in agreement with those obtained from numerical computations. We observed that the critical chemical potential increases with the increase of the superfluid velocity, which indicates that the spatial component of the gauge field hinders the phase transition. Moreover, we found that in the superfluid model the S-L method can present us analytical results on the critical exponent of condensation operator, the relation between the charge density

and chemical potential, and the behavior of the spatial component of the gauge field near the phase transition point. In particular, we analytically demonstrated that, different from the findings as shown in the AdS black hole background where the spatial component of the gauge field can determine the order of the superfluid phase transition, in the AdS soliton the first-order phase transition cannot be brought by the supercurrent, i.e., the holographic superfluid phase transition always belongs to the second order and the critical exponent of the system takes the mean-field value  $1/2$  in both s-wave and p-wave models. The analytical results can be used to back up the numerical findings in both holographic s-wave [25] and p-wave superfluid models in the AdS soliton background. Since the superfluid velocity provides richer physics in the superfluid phase transition in the AdS black hole background [13, 14], it would be of interest to generalize our study to the AdS black hole configuration and analytically discuss the effect of the spatial component of the gauge field on the system. We will leave it for further study.

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