Holographic thermalization from non relativistic branes

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Abstract

In this paper, based on the fundamental principles of Gauge/gravity duality and considering a *global quench*, we probe the physics of thermalization for a special class of strongly coupled non relativistic QFTs by computing the entanglement entropy of the plasma. The isometry group of such QFTs is comprised of the generators of the Schrödinger algebra which could be precisely realized as an isometry group of the killing generators of an asymptotically Schrödinger Dp brane space time. In our analysis, we note that during the pre local stages of the thermal equilibrium the entanglement entropy has a faster growth in time compared to its relativistic cousin. However, it shows a linear growth during the post local stages of thermal equilibrium where the so called tsunami velocity associated with the linear growth of the entanglement entropy saturates to that of its value corresponding to the relativistic scenario. Finally, we explore the saturation region and it turns out that one must constraint certain parameters of the theory in a specific way in order to have a discontinuous transitions at the point of saturation.

1 Overview and Motivation

One of the most recent as well as promising developments that took place during the last couple of years is the understanding of the non equilibrium dynamics of strongly interacting QFTs at finite temperatures. For QFTs at equilibrium, it is the RG flow that helps us to identify various universal features those are insensitive to the microscopic detail of the system. For systems out of equilibrium, such notion of universality does not hold good and in fact one does not have much tools to deal with such systems when the system itself is strongly coupled. Under such circumstances, one of the primary goals of the AdS/CFT duality [1]-[2] turns out to be to explore the universal features among such strongly coupled systems in order to understand various practical experimental consequences.

In order to study non equilibrium dynamics of strongly interacting QFTs, one typically starts with a system that is primarily in some global equilibrium state which could be its ground state at $T = 0$ or some thermally excited state at finite temperatures. The

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natural next step would be to put this QFT out of its global equilibrium by turning on sources for some (relevant) operator in the theory namely,

$$S_{QFT} \rightarrow S_{QFT} + \int d^D x \varrho(x)\mathcal{O}(x). \tag{1}$$

For sources those are *localized* in time ($\varrho(\mathfrak{t})$) and act homogeneously on space ($\mathbf{x}$), the corresponding non equilibrium dynamics is termed as *global quench*. On the other hand, for sources those are actually localized in space time, the corresponding dynamics is termed as *local quench*. In this paper we would be considering the former scenario.

During the process of thermalization, the so called thermodynamic variables such as pressure, temperature etc. are not valid entities in order to describe the non equilibrium dynamics of the system and one might try to probe such systems in terms of various *non local* observable like the entanglement entropy, equal time two point correlation functions and Wilson loops. However, for strongly coupled systems such observable are hard to compute directly from the field theory perspective and therefore one needs to rely on various holographic techniques [3]-[30].

The process that we describe in this paper, essentially corresponds to a rapid thermalization followed by a global quench where one injects a uniform density of matter for a very short interval of time ($\delta\mathfrak{t} \sim 0$). Holographically, such processes are described in terms of the process of black hole formation during the gravitational collapse of a thin shell of matter in an AdS-Vaidya space time namely [3],

$$ds^2 = \frac{L^2}{u^2}(-f(u,\upsilon)d\upsilon^2 - 2d\upsilon du + d\mathbf{x}^2) \tag{2}$$

where, $\upsilon \sim \mathfrak{t} - u$ is the so called in going null coordinate that typically plays the role of time for the theory living on the boundary ($u \sim 0$). In order to describe a rapid thermalization, the function $f(u,\upsilon)$ typically takes the form,

$$f(u,\upsilon) = 1 - \Theta(\upsilon)h(u) \tag{3}$$

where, $\Theta(\upsilon)$ is the so called step function, such that for $\upsilon < 0$, in the dual gravity picture we have a pure AdS space time which corresponds to a ground state of the dual field theory, whereas on the other hand, for $\upsilon > 0$, the corresponding dual gravity picture encodes a black hole space time which in the language of the boundary field theory could be understood as describing a dual CFT in its thermally excited state [3].

From previous analysis [3]-[6], it turns out that: (1) The entanglement growth during the pre local stages of equilibrium is non linear namely, $\Delta\mathcal{S}_{EE} \sim \mathfrak{t}^{1+\frac{1}{z}}$ where, z is the dynamic critical exponent, (2) During the post local stages of equilibrium, the entanglement entropy exhibits a linear growth (in time) as if the correlations are carried out by free streaming of quasi particles [31]-[32] and, (3) The saturation regime. Keeping the spirit of these intriguing facts, the purpose of the present article is to explore each of the above mentioned issues in a non relativistic (Galilean) context where the underlying symmetry corresponding to these special class of strongly coupled QFTs is characterized by the Schrödinger (Sch) isometry group which might be regarded as the non relativistic generalization of the so called relativistic conformal group.

The precise motivation behind exploring thermalization in the context of Schrödinger isometry group is the celebrated AdS/cold atom correspondence [33]-[34] which states that the isometry group of fermions at unitarity is described by the Schrödinger algebra that could be realized geometrically in a dual gravitational set up [35]-[47]. The question that naturally arises in this context is that what is the fate of thermalization for such a system of fermions at unitarity and this issue has never been addressed in the literature to the best of our knowledge. The aim of the present analysis is to fill up this gap and provide some theoretical understanding of such phenomena for a system of cold atoms that might have some practical consequences in the near future.

The upshot of our analysis is the following:

• The entanglement entropy during its pre local stages of equilibrium grows at a faster rate compared to its corresponding growth in the relativistic scenario [3]. Specifically, we note that, $\Delta\mathcal{S}_{EE}^{(Sch)} \sim \mathfrak{t}^{\frac{5}{2}}$.

• We show that, like in the previous example(s) [3], it is indeed possible to find a *critical* extremal surface that dominates the area functional during the post local stages of equilibrium which yields a linear growth for the entanglement entropy. We also compute the corresponding tsunami velocity ($\mathfrak{v}_s^{(Sch)}$) and it turns out that, $\mathfrak{v}_s^{(Sch)} \leq \mathfrak{v}_s^{(Rel)}$ where, $\mathfrak{v}_s^{(Rel)}$ is the tsunami velocity that is computed for the relativistic conformal theory using the standard holographic techniques [3].

• Finally, we explore the saturation region and it turns out that the transition at the point of saturation could be a *discontinuous* transition if we restrict certain parameters of the theory in a specific way.

2 Thermalization

2.1 Basic set up

We start our analysis with a formal introduction to the non relativistic set up in the bulk which is essentially described in terms of a five dimensional *uncharged* Schrödinger black brane (Sch_5) solution of the following form [45],

$$\begin{aligned} ds_5^2 &= \frac{\mathcal{K}^{1/3}L^2}{u^2}\left[-\frac{f}{\mathcal{K}}d\tau^2 - \frac{f\beta^2L^4}{u^2\mathcal{K}}(d\tau+d\chi)^2 + \frac{d\chi^2}{\mathcal{K}} + \frac{du^2}{f} + dx^2 + dy^2\right] \\ f(u) &= 1-\left(\frac{u}{u_H}\right)^4, \quad \mathcal{K}(u) = 1+\frac{\beta^2L^4u^2}{u_H^4} \end{aligned} \tag{4}$$

where, the horizon of the black brane is located at $u=u_H$, whereas, on the other hand the boundary of the spacetime is located at $u=0$. Here, β is a dimensionful quantity (with dimension, $[\beta]=\frac{1}{L}$) associated with the Null Melvin Twist and is related to the particle number in the dual field theory. The coordinates, τ and χ are related to the isometry directions of the original type IIB supergravity solution in 10 dimensions [38].

In order to proceed further, our next step would be to search for the corresponding Eddington-Finkelstein like representation of the above black brane configuration (4). In order to do so, we introduce the light cone time v as,

$$dv = d\tau - \sqrt{\frac{\mathcal{K}}{1+\frac{\beta^2L^4}{u^2}}}\frac{du}{f}. \tag{5}$$

Substituting (5) into (4) we obtain,

$$ds_5^2 = \frac{\mathcal{K}^{1/3}L^2}{u^2}\left[-\frac{\tilde{f}}{\mathcal{K}}dv^2 - 2h(u)dvdu - \frac{2f\beta^2L^4}{u^2\mathcal{K}}dvd\chi - \frac{2\beta^2L^4}{u^2\mathcal{K}h}dud\chi + c(u)d\chi^2 + dx^2 + dy^2\right] \quad (6)$$

where,

$$\begin{aligned}
\tilde{f}(u) &= f(u)\left(1+\frac{\beta^2L^4}{u^2}\right) \equiv f(u)g(u); \quad g(u) = \left(1+\frac{\beta^2L^4}{u^2}\right)\\
h(u) &= \sqrt{\frac{g(u)}{\mathcal{K}(u)}}\\
c(u) &= \frac{1}{\mathcal{K}(u)}\left(1-\frac{f(u)\beta^2L^4}{u^2}\right).
\end{aligned} \quad (7)$$

In the limit of the vanishing time interval ($\Delta t \sim 0$) associated with external sources at the boundary, the width of the collapsing shell in the bulk eventually goes to zero [3]. Under such circumstances, the functions defined above (6) could be replaced as,

$$\begin{aligned}
f(u,v) &= 1-\Theta(v)\left(\frac{u}{u_H}\right)^4\\
\mathcal{K}(u,v) &= 1+\Theta(v)\left(\frac{\beta^2L^4u^2}{u_H^4}\right)\\
\tilde{f}(u,v) &= f(u,v)g(u)\\
h(u,v) &= \sqrt{\frac{g(u)}{\mathcal{K}(u,v)}}\\
c(u,v) &= \frac{1}{\mathcal{K}(u,v)}\left(1-\frac{f(u,v)\beta^2L^4}{u^2}\right).
\end{aligned} \quad (8)$$

where, $\Theta(v)$ is the so called step function, such that for, $v<0$ we are left with a pure Schrödinger space time of the form,

$$ds_5^2 = \frac{L^2}{u^2}\left[-g(u)dv^2 - 2\sqrt{g}dvdu - \frac{2\beta^2L^4}{u^2}dvd\chi - \frac{2\beta^2L^4}{u^2\sqrt{g}}dud\chi + \left(1-\frac{\beta^2L^4}{u^2}\right)d\chi^2 + d\mathbf{x}^2\right] \quad (9)$$

whereas, on the other hand, for $v>0$ the space time turns out to be precisely that of the Sch_5 black brane configuration (6).

2.2 Area functional: Preliminaries

The purpose of the present section is to explore the so called holographic entanglement entropy (HEE) [48]-[50] as a candidate that probes the dynamics of thermal quench in a strongly coupled non relativistic plasma living on the conformal boundary of Sch_5 space time. In order to compute the HEE, we consider a (rectangular) strip times χ like region at the boundary namely [48]-[50],

$$-\mathfrak{R}/2 \le x \le \mathfrak{R}/2; \quad 0 \le y \le L_y; \quad 0 \le \chi \le L_\chi \quad (10)$$

and compute the area functional associated with the extremal surface that extends into the bulk. Note that, here $L_\chi (= \int d\chi)$, is the length scale associated with the χ direction[40].

Since in the present example, we are dealing with a time dependent scenario, therefore it would be fare enough to use the so called covariant proposal for HEE [51]. This in turn suggests that one should parametrize the so called extremal surface in terms of the functions, $\upsilon(x)$ and $u(x)$. The boundary conditions that one should impose under such circumstances are given by the following set of constraints,

$$u(\mathfrak{R}/2) = 0, \quad \upsilon(\mathfrak{R}/2) = \mathfrak{t}, \quad u'(0) = \upsilon'(0) = 0$$
$$u(0) = u_T, \quad \upsilon(0) = \upsilon_T \tag{11}$$

where, u_T and υ_T correspond to the values of the functions at the tip of the extremal surface in the bulk.

2.2.1 Equations of motion

In order to evaluate the equations of for the extremal surface in the bulk, we first note down the induced metric on the extremal surface,

$$ds^2_{ES} = \frac{\mathcal{K}^{1/3}L^2}{u^2}\left[\left(1 - \frac{\tilde{f}(u,\upsilon)}{\mathcal{K}(u,\upsilon)}\upsilon'^2 - 2h(u,\upsilon)\upsilon'u'\right)dx^2 - \frac{2\beta^2 L^4}{u^2\mathcal{K}}\left(f\upsilon' + \frac{u'}{h}\right)dx d\chi + c(u,\upsilon)d\chi^2 + dy^2\right]. \tag{12}$$

Using (12), the corresponding area functional finally turns out to be,

$$\mathcal{A} = L_y L_\chi L^3 \int_{-\mathfrak{R}/2}^{\mathfrak{R}/2} dx \frac{\sqrt{\mathcal{K}(u,\upsilon)\mathsf{Q}(u,\upsilon)}}{u^3} \tag{13}$$

where the function $\mathsf{Q}(u,\upsilon)$ could be formally expressed as,

$$\mathsf{Q}(u,\upsilon) = c(u,\upsilon) - \frac{\upsilon'^2 f(u,\upsilon)}{\mathcal{K}(u,\upsilon)}\left(g(u)c(u,\upsilon) + \frac{\beta^4 L^8 f(u,\upsilon)}{u^4\mathcal{K}(u,\upsilon)}\right) - \frac{\beta^4 L^8 u'^2}{u^4 h^2(u,\upsilon)\mathcal{K}^2(u,\upsilon)}$$
$$-2\upsilon' u'\left(h(u,\upsilon)c(u,\upsilon) + \frac{\beta^4 L^8 f(u,\upsilon)}{u^4\mathcal{K}^2(u,\upsilon)h(u,\upsilon)}\right). \tag{14}$$

The equations of motion, that readily follow from (13), could be formally expressed as,

$$u^3\sqrt{\mathcal{G}(u,\upsilon)}\partial_x\left(\frac{\upsilon' f\left(g(u)c(u,\upsilon) + \frac{f\beta^4L^8}{u^4\mathcal{K}}\right) + \mathcal{K}u'\left(h(u,\upsilon)c(u,\upsilon) + \frac{\beta^4L^8 f}{u^4\mathcal{K}^2 h}\right)}{u^3\sqrt{\mathcal{G}(u,\upsilon)}}\right) = \frac{1}{2}\mathcal{F}_\upsilon(u,\upsilon)$$
$$u^3\sqrt{\mathcal{G}(u,\upsilon)}\partial_x\left(\frac{\frac{\beta^4L^8u'}{u^4h^2\mathcal{K}(u,\upsilon)} + \mathcal{K}\upsilon'\left(h(u,\upsilon)c(u,\upsilon) + \frac{\beta^4L^8 f}{u^4\mathcal{K}^2 h}\right)}{u^3\sqrt{\mathcal{G}(u,\upsilon)}}\right) - \frac{3\mathcal{G}(u,\upsilon)}{u} = \frac{1}{2}\mathcal{F}_u(u,\upsilon)$$
$$\tag{15}$$

where,

$$\mathcal{G}(u,\upsilon) = \mathcal{K}(u,\upsilon)\mathsf{Q}(u,\upsilon),$$

$$\mathcal{F}_{\upsilon}(u,\upsilon)=\frac{\partial f}{\partial \upsilon}\upsilon'^2 g(u)c(u,\upsilon)-\frac{\partial \mathcal{K}}{\partial \upsilon}\mathsf{Q}(u,\upsilon)-\frac{\partial c}{\partial \upsilon}\mathcal{K}(u,\upsilon)-\frac{\partial \mathcal{K}}{\partial \upsilon}\frac{\upsilon'^2 f(u,\upsilon)}{\mathcal{K}(u,\upsilon)}g(u)c(u,\upsilon)$$
$$+\frac{\partial c}{\partial \upsilon}\upsilon'^2 g(u)f(u,\upsilon)+\frac{2f(u,\upsilon)\upsilon'^2\beta^4L^8}{u^4\mathcal{K}(u,\upsilon)}\left(\frac{\partial f(u,\upsilon)}{\partial \upsilon}-\frac{f(u,\upsilon)}{\mathcal{K}(u,\upsilon)}\frac{\partial \mathcal{K}(u,\upsilon)}{\partial \upsilon}\right)$$
$$-\frac{2u'^2\beta^4L^8}{u^4h^2(u,\upsilon)\mathcal{K}(u,\upsilon)}\left(\frac{1}{h}\frac{\partial h}{\partial \upsilon}+\frac{1}{\mathcal{K}}\frac{\partial \mathcal{K}}{\partial \upsilon}\right)+2\mathcal{K}(u,\upsilon)\upsilon'u'\frac{\partial}{\partial \upsilon}(h(u,\upsilon)c(u,\upsilon))$$
$$+\frac{2\upsilon'u'\beta^4L^8}{u^4\mathcal{K}(u,\upsilon)h(u,\upsilon)}\left(\frac{\partial f}{\partial \upsilon}-\frac{2f(u,\upsilon)}{\mathcal{K}(u,\upsilon)}\frac{\partial \mathcal{K}}{\partial \upsilon}-\frac{f(u,\upsilon)}{h(u,\upsilon)}\frac{\partial h}{\partial \upsilon}\right),$$

$$\mathcal{F}_{u}(u,\upsilon)=\frac{\partial f}{\partial u}\upsilon'^2 g(u)c(u,\upsilon)-\frac{\partial \mathcal{K}}{\partial u}\mathsf{Q}(u,\upsilon)-\frac{\partial c}{\partial u}\mathcal{K}(u,\upsilon)+\frac{\partial(c(u,\upsilon)g(u))}{\partial u}\upsilon'^2 f(u,\upsilon)$$
$$-\frac{\partial \mathcal{K}}{\partial u}\frac{\upsilon'^2 f(u,\upsilon)}{\mathcal{K}(u,\upsilon)}g(u)c(u,\upsilon)+\frac{2f(u,\upsilon)\upsilon'^2\beta^4L^8}{u^4\mathcal{K}(u,\upsilon)}\left(\frac{\partial f(u,\upsilon)}{\partial u}-\frac{f(u,\upsilon)}{\mathcal{K}(u,\upsilon)}\frac{\partial \mathcal{K}(u,\upsilon)}{\partial u}\right)$$
$$-\frac{4\upsilon'^2 f^2(u,\upsilon)\beta^4L^8}{\mathcal{K}(u,\upsilon)u^5}-\frac{2u'^2\beta^4L^8}{u^4h^2(u,\upsilon)\mathcal{K}(u,\upsilon)}\left(\frac{1}{h}\frac{\partial h}{\partial u}+\frac{1}{\mathcal{K}}\frac{\partial \mathcal{K}}{\partial u}+\frac{2}{u}\right)+2\mathcal{K}\upsilon'u'\frac{\partial}{\partial u}(h(u,\upsilon)c(u,\upsilon))$$
$$+\frac{2\upsilon'u'\beta^4L^8}{u^4\mathcal{K}(u,\upsilon)h(u,\upsilon)}\left(\frac{\partial f}{\partial u}-\frac{2f(u,\upsilon)}{\mathcal{K}(u,\upsilon)}\frac{\partial \mathcal{K}}{\partial u}-\frac{f(u,\upsilon)}{h(u,\upsilon)}\frac{\partial h}{\partial u}-\frac{4f(u,\upsilon)}{u}\right).$$

Our next step would be compute the first integral of motion. Since the integrand in (13) does not explicitly depend on x, therefore the first integral of motion turns out to be,

$$\frac{u^3\sqrt{\mathcal{G}(u,\upsilon)}}{\mathcal{K}(u,\upsilon)c(u,\upsilon)}=\mathsf{J}=const. \tag{16}$$

Before we proceed further, it is customary to note that the extremal surface that we consider in our analysis extends both in the pure Sch_5 as well as in the Sch_5 black brane regions. Moreover, by considering the reflection symmetry of the entire configuration around $x=0$, it is indeed sufficient to consider only the positive half ($x>0$) of the entangling region at the boundary. In the following, we discuss both $\upsilon<0$, as well as, $\upsilon>0$ regions separately.

2.2.2 Region I: $\upsilon < 0$

In the region $\upsilon<0$, the first one of the above set of equations (15) enormously simplifies to give,

$$\frac{1}{c(u)}\left(\upsilon'+\frac{u'}{\sqrt{g(u)}}\right)=\mathsf{E}. \tag{17}$$

Under such circumstances, using (17), one could further simplify (14) as,

$$\mathsf{Q}(u)=c(u)\left(1+u'^2\right). \tag{18}$$

Using (18), it is quite trivial to show that [1],

$$u'=-\frac{\sqrt{c(u)}}{u^3}\sqrt{\mathsf{J}^2-\frac{u^6}{c(u)}} \tag{19}$$

[1]Note that, here we have considered the fact that, $u'(x)<0$ throughout its entire domain of definition.

which could be further integrated to obtain,

$$x(u) = \int_{u}^{u_T} \frac{u'^3 du'}{\sqrt{c(u')}} \frac{1}{\sqrt{\mathsf{J}^2 - \frac{u'^6}{c(u')}}}. \tag{20}$$

Finally, from (17), it is also trivial to show that,

$$\upsilon = \upsilon_T + u_T\sqrt{1+\frac{\beta^2 L^4}{u_T^2}} - u\sqrt{1+\frac{\beta^2 L^4}{u^2}}. \tag{21}$$

2.2.3 Matching on the shell

Before we proceed further, let us define the values corresponding to u and x at the point of intersection with the null shell, $\upsilon = 0$ as u_c and x_c respectively which finally yields,

$$u_c\sqrt{1+\frac{\beta^2 L^4}{u_c^2}} = u_T\sqrt{1+\frac{\beta^2 L^4}{u_T^2}} + \upsilon_T. \tag{22}$$

On the other hand, taking derivatives on both sides of (21) we find[2],

$$u'_{-} = -\sqrt{g(u_c)}\ \upsilon'_{-} = -\frac{\sqrt{c(u_c)}}{u_c^3}\sqrt{\mathsf{J}^2 - \frac{u_c^6}{c(u_c)}}. \tag{23}$$

To find the derivatives on the other side of the null shell, our next task would be to integrate (15) across the null shell which finally enable us to establish the precise map between different kinematics on the both sides of the null shell.

In order to proceed further, let us first note that since we inject matter along the null direction υ, therefore the corresponding conjugate momentum must encounter a jump as we move from the region, $\upsilon < 0$ to the region, $\upsilon > 0$. On the other hand, the momentum conjugate to u must remain continuous across the null shell, $\upsilon = 0$ [5]. The second condition naturally implies that,

$$\mathsf{P}_{-}^{(u)} = \mathsf{P}_{0}^{(u)} = \mathsf{P}_{+}^{(u)} \tag{24}$$

where, $\mathsf{P}_{0}^{(u)}$ stands for the momentum (conjugate to u) exactly on the null shell, $\upsilon = 0$. Now, computing momenta corresponding to the regions, $\upsilon < 0$ and $\upsilon > 0$ and taking the limit, $\upsilon \to 0$ we finally arrive,

$$\upsilon'_{-} \;=\; \upsilon'_{+} + \frac{\beta^4 L^8}{u_c^4\sqrt{g(u_c)}}(u'_{+} - u'_{-}). \tag{25}$$

As a next step of our analysis, we first focus on the first equation in (15). After performing the integration in the vicinity of the null shell, $\upsilon = 0$ we find,

$$u'_{+} = u'_{-} - \frac{\mathcal{Z}(u_c)}{2\sqrt{g(u_c)}c(u_c)} \tag{26}$$

[2]Here, $-$ and $+$ subscripts refer to entities in the regions, $\upsilon < 0$ and $\upsilon > 0$ respectively.

where, the function $\mathcal{Z}(u_c)$ could be formally expressed as,

$$\mathcal{Z}(u_c) = u'_- \sqrt{g(u_c)} c(u_c) \left(\frac{u_c}{u_H}\right)^4 - \frac{\beta^2 L^4 u_c^2}{\upsilon'^{(0)} u_H^4} - \frac{\sqrt{g(u_c)}|u'_c| u_c^2 \beta^2 L^4}{u_H^4}. \tag{27}$$

Note that, here $\upsilon'^{(0)}$ and u'_c correspond to the derivatives exactly on the null shell, $\upsilon = 0$. Substituting (27) into (26), one could further simplify the resulting expression as,

$$u'_+ = \left(1 - \frac{1}{2}\left(\frac{u_c}{u_H}\right)^4\right) u'_- + \frac{\beta^2 L^4 u_c^2}{2\sqrt{g(u_c)} c(u_c) u_H^4} \left(\frac{1}{\upsilon'^{(0)}} + \sqrt{g(u_c)}|u'_c|\right). \tag{28}$$

Finally, after some trivial algebra it is indeed quite straightforward to show,

$$\mathcal{G}_+ - \mathcal{G}_- = \mathsf{Q}_+ - \mathsf{Q}_- = c(u_c)(u'^2_+ - u'^2_-). \tag{29}$$

2.2.4 Region II: $\upsilon > 0$

We now turn our attention towards the black brane sector of the entire space time configuration. As a first step of our analysis, using (23), (25) and (27) and considering the close vicinity of the null shell we find,

$$\mathsf{E} = -\frac{\sqrt{g(u_c)}}{2}|u'_-| \left(\frac{u_c}{u_H}\right)^4 + \frac{\beta^2 L^4 u_c^2}{2c(u_c) u_H^4} \left(\frac{1}{\upsilon'^{(0)}} + \sqrt{g(u_c)}|u'_c|\right). \tag{30}$$

Therefore, unlike the previous examples in the literature [3], the first integral of motion is not guaranteed to be negative for non relativistic background.

We now focus on the first equation in (15), which yields the following relation,

$$\frac{1}{\mathcal{K}(u) c(u)} \left(\frac{u'}{h(u)} + \upsilon' f(u)\right) = \mathsf{E} \tag{31}$$

which could be further simplified in order to obtain,

$$\upsilon' = \frac{1}{f(u)} \left(\mathsf{E} - \frac{u'}{h(u)}\right) - \frac{\beta^2 L^4 \mathsf{E}}{u^2}. \tag{32}$$

Using (16) and (32), it is indeed quite straightforward to find,

$$u'^2 = f(u) \left(\frac{\mathsf{J}^2}{u^6} - 1\right) + \mathsf{E}^2 \left(1 - \frac{f(u)\beta^2 L^4}{u^2}\right) \equiv \mathcal{H}(u) \tag{33}$$

which naturally yields,

$$x(u) = \int_u^{u_c} \frac{du}{\sqrt{\mathcal{H}(u)}}. \tag{34}$$

Using (33), it is indeed quite trivial to show that,

$$\frac{d\upsilon}{du} = -\frac{1}{f(u) h(u)} \left(1 + \frac{\mathsf{E} h(u)}{\sqrt{\mathcal{H}(u)}} \left(1 - \frac{f(u)\beta^2 L^4}{u^2}\right)\right). \tag{35}$$

Next, combining (20) and (34) we note,

$$\frac{\mathfrak{R}}{2}=\int_{u_c}^{u_T}\frac{u^3du}{\sqrt{c(u)}}\frac{1}{\sqrt{\mathsf{J}^2-\frac{u^6}{c(u)}}}+\int_0^{u_c}\frac{du}{\sqrt{\mathcal{H}(u)}} \tag{36}$$

where, we have assumed that $u(x)$ is a monotonically decreasing function of x [3].

Moreover, integrating (35) we find,

$$\mathsf{t}=\int_0^{u_c}\frac{du}{f(u)h(u)}\left(1+\frac{\mathsf{E}h(u)}{\sqrt{\mathcal{H}(u)}}\left(1-\frac{f(u)\beta^2L^4}{u^2}\right)\right). \tag{37}$$

Before we proceed further, the crucial point that is to be noted at this stage is the following: The integrand above in (37) clearly seems to be divergent near the horizon, $u\sim u_H$ due to the vanishing of the function $f(u)$ there. However, it turns out that,

$$\lim_{u\to u_H}\left(1+\frac{\mathsf{E}h(u)}{\sqrt{\mathcal{H}(u)}}\left(1-\frac{f(u)\beta^2L^4}{u^2}\right)\right)\approx 1+\frac{\mathsf{E}}{\sqrt{\mathcal{H}(u_H)}}. \tag{38}$$

Now, since $\mathcal{H}(u_H)=\mathsf{E}^2$, therefore the above integral (37) is finite iff, $\mathsf{E}<0$, which in turn suggests that the second term on the R.H.S. of (30) must be less than the corresponding leading term.

Our next natural task would be to find the area functional(s) corresponding to both $\upsilon<0$ as well as $\upsilon>0$ regions separately. The area functional (13) corresponding to $\upsilon<0$ region turns out to be,

$$\mathcal{A}_-=L_yL_\chi L^3\int_{\frac{u_c}{u_T}}^{1}\frac{d\xi}{u_T^2\xi^3}\frac{\sqrt{c(\xi)}}{\sqrt{1-\frac{c_T}{c(\xi)}\xi^6}} \tag{39}$$

where, we have defined a new variable, $\xi=\frac{u}{u_T}$ such that, $u_T=(c_T\mathsf{J}^2)^{1/6}$ corresponds to the so called terminal point along with, $c_T=c(u_T)$.

Finally, the area functional corresponding to $\upsilon>0$ region turns out to be,

$$\mathcal{A}_+=L_yL_\chi L^3\int_0^{\frac{u_c}{u_T}}\frac{d\xi}{\sqrt{c_T}u_T^2\xi^6}\frac{\sqrt{c(\xi)\mathcal{K}(\xi)}}{\sqrt{\mathcal{H}(\xi)}} \tag{40}$$

which eventually results in a total area functional of the form,

$$\mathcal{A}=\frac{L_yL_\chi L^3}{u_T^2}\left[\int_{\frac{u_c}{u_T}}^{1}\frac{d\xi}{\xi^3}\frac{\sqrt{c(\xi)}}{\sqrt{1-\frac{c_T}{c(\xi)}\xi^6}}+\int_0^{\frac{u_c}{u_T}}\frac{d\xi}{\sqrt{c_T}\xi^6}\frac{\sqrt{c(\xi)\mathcal{K}(\xi)}}{\sqrt{\mathcal{H}(\xi)}}\right]. \tag{41}$$

Clearly, the integral (41) is divergent near the UV scale of the boundary theory and therefore it should be regularized by means of a proper UV cut off. On top of it, in our analysis, we would be mostly interested to explore the behaviour of the area functional (41) after a *global* quantum quench when the system passes from the so called vacuum to the thermally excited state. As a result, the quantity that we would be mostly interested to compute is the change in the area functional namely, $\Delta\mathcal{A}=\mathcal{A}-\mathcal{A}_-^{(vac)}$ where,

$$\mathcal{A}_-^{(vac)}=L_yL_\chi L^3\int_0^{1}\frac{d\xi}{u_T^2\xi^3}\frac{\sqrt{c(\xi)}}{\sqrt{1-\frac{c_T}{c(\xi)}\xi^6}} \tag{42}$$

is the extremal (hyper)surface corresponding to the vacuum configuration.

During the rest of our analysis, we would mostly explore the behaviour of holographic entanglement entropy while the system thermalizes followed by a global quench. During such an evolution, one might encounter a competition between different time scales in the theory. In the case of a strip, the typical time scale for the entanglement entropy to saturate is, $\mathfrak{t}_{sat} \sim \mathfrak{R}/2$. When $\mathfrak{t}_{sat} \ll u_H$, the entanglement entropy saturates long before the system reaches local thermal equilibrium. On the other hand, for $\mathfrak{t}_{sat} \gg u_H$, the system reaches thermal equilibrium before the entanglement entropy saturates. In our analysis, however, we would be interested in the later situation while the correlations are of long range. Under such circumstances, one might encounter the following three scenarios: (1) For $\mathfrak{t} \ll u_H$, the corresponding configuration is termed as pre local equilibrium growth where the extremal surface cuts the null shell very close to the boundary, (2) For $\mathfrak{t} \sim u_H$, on the other hand, the corresponding configuration is termed as the post local equilibrium growth where the extremal surface starts intersecting the null shell at a scale that is comparable with u_H and (3) Finally, for $\mathfrak{t} \sim \mathfrak{t}_{sat}$, the extremal surface falls entirely within the black brane configuration. In the following, we explore all the above three cases one by one.

2.3 Pre-local equilibrium

We first focus on the behaviour of holographic entanglement entropy at early times namely, $\mathfrak{t} \ll u_H$ such that, $u_c/u_H \ll 1$. Note that here u_H could be identified as the time scale for local equilibrium growth. However, during such pre local equilibrium stage, one could safely take the limit, $u_c \to 0$ where the extremal surface crosses the null shell. Under such circumstances, the space time region corresponding to the black brane configuration turns out to be extremely small.

To start with, from (37), we note that,

$$\mathfrak{t} = \int_0^{\frac{u_c}{u_T}} \frac{u_T^2 \xi d\xi}{\beta L^2} \left(1 - \frac{\beta^3 L^6 \mathsf{E}}{\mathsf{J}} + \left(\frac{\mathsf{E} u_T^2 \beta L^2}{\mathsf{J}} - \frac{u_T^2}{2\beta^2 L^4}\left(1 - \frac{\beta^4 L^8}{u_H^4}\right)\right)\xi^2\right) + \mathcal{O}(\xi^5). \quad (43)$$

Performing the above integral (43), it is indeed quite trivial to find,

$$\mathfrak{t} \approx \frac{u_c^2}{2\beta L^2}\left(1 - \frac{\beta^3 L^6 \mathsf{E}}{\mathsf{J}} - \frac{u_c^2}{4\beta^2 L^4}\left(1 - \frac{2\mathsf{E}\beta^3 L^6}{\mathsf{J}}\right)\right) + \frac{\beta L^2}{8}\left(\frac{u_c}{u_H}\right)^4 \quad (44)$$

which could be inverted to obtain,

$$u_c \approx \frac{\sqrt{\mathfrak{t}}}{\sqrt{L}\mathfrak{a}} + \mathcal{O}(\mathfrak{t}^{3/2}) \quad (45)$$

where, $\mathfrak{a} = \frac{1}{2\beta L^2}\left(1 - \frac{\beta^3 L^6 \mathsf{E}}{\mathsf{J}}\right)$.

Our next task would be to compute the difference, $\Delta\mathcal{A} = \mathcal{A} - \mathcal{A}_-^{(vac)}$. Considering the limit $u_c/u_T \ll 1$, we finally obtain,

$$|\Delta\mathcal{A}^{(Sch)}| \approx \frac{L_y L_\chi L c_T u_c^5}{10 u_T^6 \beta} + \mathcal{O}((u_c/u_H)^4). \quad (46)$$

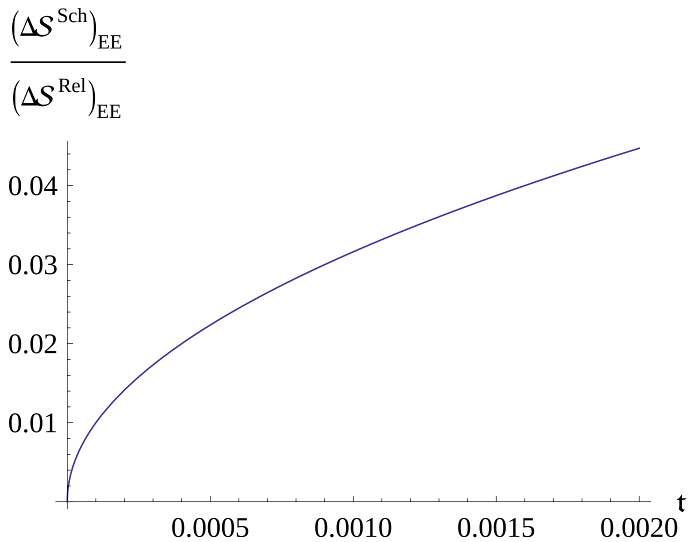

Figure 1: Holographic entanglement entropy ratio $\left(\frac{\Delta\mathcal{S}^{(Sch)}_{EE}}{\Delta\mathcal{S}^{(Rel)}_{EE}}\right)$ plot during pre-local equilibrium stage.

Using (45) and (46), we note that,

$$|\Delta\mathcal{A}^{(Sch)}| \sim \mathfrak{t}^{\frac{5}{2}} = |\Delta\mathcal{A}^{(Rel)}|\sqrt{\mathfrak{t}}. \tag{47}$$

Clearly, for non relativistic CFTs (with $z = 2$ fixed point) we observe a non trivial evolution of the area functional which is indeed different from their relativistic cousins [3]. From (47), we note that for QFTs with Schrödinger isometry group, the entanglement entropy grows at a faster rate compared to its relativistic counterpart namely,

$$\frac{\Delta\mathcal{S}^{(Sch)}_{EE}}{\Delta\mathcal{S}^{(Rel)}_{EE}} \sim \sqrt{\mathfrak{t}}. \tag{48}$$

From the Fig.(1), we note that the ratio (48) indeed exhibits a non linear growth in time during the pre-local stages of thermal evolution which indicates that the entanglement entropy corresponding to the non relativistic configuration evolves at a faster rate compared to its cousin entanglement entropy corresponding to the usual relativistic scenario.

2.4 Post-local equilibrium

The temperature scale that we focus in this section is $\frac{\mathfrak{R}}{2} \gg \mathfrak{t} \gg u_H$. The typical gravity picture in this region is the following: In this regime of time, the hyper-surface starts penetrating the null shell at a scale (u^*_c) that is bigger than the horizon radius itself namely, $u^*_c \geq u_H$. Those extremal surfaces for which $u_c < u^*_c$, reach the boundary. On the other hand, extremal surfaces for which $u_c > u^*_c$ never reach the boundary and hit the singularity. Such extremal surfaces are called critical extremal surfaces and the corresponding point of intersection (u^*_c) is known as the critical point [3].

In order to study critical extremal surface, we first argue that Eq.(33) could be thought of as describing the motion of a particle in one dimension with an effective potential, $\mathsf{V}_{eff}(u) \equiv \mathcal{H}(u)$. The minima of this potential is guaranteed to produce a stable orbit which is obtained by tuning the parameter u_c such that both the velocity as well as the acceleration of the particle turns out to be zero at that point. This yields,

$$\mathcal{H}'(u_m)|_{u_c=u_c^*} \sim 0, \quad \mathcal{H}(u_m)|_{u_c=u_c^*} = 0 \tag{49}$$

where, $u = u_m(\ll u_T)$ is some point in between u_T and u_H that minimizes the potential [3].

In the following, we further illustrate the above discussions and provide a detailed calculation in order to locate this critical extremal surface. From (28), we first note that for an increment,

$$u_c \to \tilde{u}_c = 2^{1/4}u_H(1+\varepsilon) \equiv u_s(1+\varepsilon), \quad \varepsilon > 0, \quad |\varepsilon| \ll 1 \tag{50}$$

the change in u'_+ turns out to be,

$$\Delta u'_+ \approx -4\varepsilon u'_- \left(1 + \frac{u'_+(u_s)}{2|u'_-|}\left(1 + \frac{\mathcal{X}'(u_s)}{2\mathcal{X}(u_s)}\right)\right) + \mathcal{O}(\varepsilon^2) \approx -4\varepsilon u'_- > 0 \tag{51}$$

where, $\mathcal{X}(u_s) = \frac{1}{2\sqrt{g(u_s)}c(u_s)}\left(\frac{1}{v'(0)} + \sqrt{g(u_s)}|u'_s|\right)$, and we have also considered the fact that, $u'_+(u_s) = 0$. Therefore, from the above discussion we note that the extremal surfaces those intersect the null shell at $u_c > u_s$, always move away from the boundary and never contribute to the entanglement entropy of the boundary theory.

Next, from (33), we note that the first term in $\mathcal{H}(u)$ is zero both at $u = u_H$ and $u = u_T/c_T^{1/6}$ and is negative for, $u_H < u < u_T/c_T^{1/6}$. Note that, in the large $u_T(\gg u_H)$ limit, this upper bound saturates exactly at u_T [3]. As a consequence of this, $\mathcal{H}(u)$ exhibits a minima in between which could be obtained from (49) as,

$$u_T^6 = \frac{c_T u_m^7 f'(u_m)}{6f(u_m)}\left(\frac{u_T^6}{c_T u_m^6} - 1 - \frac{\mathsf{E}^2\beta^2 L^4}{u_m^2}\right) + \frac{c_T u_m^4 \mathsf{E}^2\beta^2 L^4}{3}. \tag{52}$$

Substituting, $u_m \sim u_s = 2^{1/4}u_H$ into (52) we find,

$$u_T^{(s)6} = \frac{2^{7/2}c_T^{(s)}u_H^6}{3}\left(\frac{u_T^{(s)6}}{2^{3/2}c_T^{(s)}u_H^6} - 1 - \frac{\mathsf{E}^2\beta^2L^4}{\sqrt{2}u_H^2}\right) + \frac{2c_T^{(s)}u_H^4\mathsf{E}^2\beta^2L^4}{3}. \tag{53}$$

On the other hand, from the second condition in (49) we obtain,

$$\frac{u_T^{(s)6}}{2^{3/2}c_T^{(s)}u_H^6} = 1 + \mathsf{E}^2\left(1 + \frac{\beta^2L^4}{\sqrt{2}u_H^2}\right). \tag{54}$$

Substituting (54) into (53), we find,

$$u_T^{(s)} = \mathsf{E}^{1/3}\left(1 + \frac{2c_T^{(s)}u_H^4\beta^2L^4}{3}\right)^{1/6} \tag{55}$$

which clearly indicates the existence of the tip corresponding to a family of extremal surfaces that includes the critical extremal surface parametrized by, $u_c^* = u_m = u_s$.

Finally, in order to compute the area functional (41) in this region, we consider the expansion,

$$u_c = u_c^*(1-\lambda), \quad |\lambda| \ll 1. \tag{56}$$

Under such circumstances, the major contribution to the area functional (41) comes from the sector, $u \sim u_m \sim u_T \xi_m$. In this sector, one can expand,

$$\begin{aligned} \mathcal{H}(\xi) &= \mathcal{H}(\xi_m) + (\xi-\xi_m)\mathcal{H}'(\xi_m) + \frac{1}{2}(\xi-\xi_m)^2\mathcal{H}''(\xi_m) + \mathcal{O}((\xi-\xi_m)^3) \\ &\approx (\xi-\xi_m)\mathcal{H}'(\xi_m) + \frac{1}{2}(\xi-\xi_m)^2\mathcal{H}''(\xi_m). \end{aligned} \tag{57}$$

By virtue of (56), we note,

$$(\xi-\xi_m)\mathcal{H}'|_{\xi_c=\xi_c^*} \equiv (\xi_c-\xi_c^*)\mathcal{H}'(\xi_c) = -\xi_c^*\lambda\mathcal{H}'(\xi_c) \tag{58}$$

such that, $|\xi_c^*\lambda\mathcal{H}'(\xi_c)| \ll 1$.

Under such circumstances, from (37) one arrives at,

$$\begin{aligned} \mathfrak{t} &\approx -\int_{\xi\sim\xi_m} \frac{u_T\ \mathsf{E}^*\ d(\xi_m-\xi)}{f(\xi_m)\sqrt{-\xi_c^*\lambda\mathcal{H}'(\xi_c)+\frac{1}{2}(\xi_m-\xi)^2\mathcal{H}''(\xi_m)}}\left(1-\frac{f(\xi_m)\beta^2L^4}{\xi_m^2u_T^2}\right) \\ &\approx -\frac{u_T\ \mathsf{E}^*}{f(\xi_m)\sqrt{\frac{1}{2}\mathcal{H}''(\xi_m)}}\left(1-\frac{f(\xi_m)\beta^2L^4}{\xi_m^2u_T^2}\right)\ \log\lambda \end{aligned} \tag{59}$$

where,

$$\mathsf{E}^* = -\frac{1}{\sqrt{c(u_m)\mathcal{K}(u_m)}}\sqrt{-f(u_m)\left(\frac{u_T^6}{c_Tu_m^6}-1\right)}. \tag{60}$$

From (36), the length of the strip turns out to be,

$$\mathfrak{R} \approx \frac{u_T\sqrt{\mathfrak{b}}}{2}\text{Hypergeometric2F1}\left[\frac{1}{2},\frac{2}{3},\frac{5}{3},\mathfrak{b}\right] - \frac{2u_T}{\sqrt{\frac{1}{2}\mathcal{H}''(\xi_m)}}\ \log\lambda \tag{61}$$

where, we have set $\mathfrak{b} = \frac{c_T}{c(\xi_m)}$.

Finally, the area functional turns out to be,

$$\Delta\mathcal{A}^{(Sch)} \approx -\frac{L_yL_\chi L^3\sqrt{\mathcal{K}(\xi_m)}}{u_T^2\sqrt{\mathfrak{b}}\xi_m^6\sqrt{\frac{1}{2}\mathcal{H}''(\xi_m)}}\ \log\lambda. \tag{62}$$

Combining, (59) and (62) we finally obtain,

$$\Delta\mathcal{A}^{(Sch)} = \frac{L_yL_\chi L^3u_T^3f(u_m)}{u_m^6\mathsf{E}^*\sqrt{c_T}\sqrt{c(u_m)\mathcal{K}(u_m)}}\mathfrak{t}. \tag{63}$$

In the large $u_T(\gg u_m)$ limit, one could further simplify (63) as,

$$\Delta\mathcal{A}^{(Sch)} \approx \left(\frac{L_yL_\chi L^3}{u_H^3}\right)\mathfrak{v}_s^{(Sch)}\mathfrak{t} \tag{64}$$

where,

$$\begin{aligned}
\mathfrak{v}_s^{(Sch)} &= \frac{\sqrt{-f(u_m)}}{\sqrt{c_T}}\left(\frac{u_H}{u_m}\right)^3 \approx \mathfrak{v}_s^{(Rel)}\left(1-\left(\frac{u_H}{u_T}\right)^4\left(1-\frac{u_H^4}{\beta^2 L^4 u_T^2}\right)\right)\\
\mathfrak{v}_s^{(Rel)} &= \sqrt{-f(u_m)}\left(\frac{u_H}{u_m}\right)^3.
\end{aligned} \tag{65}$$

Here, $\mathfrak{v}_s^{(Sch)}$ is the so called tsunami velocity associated with Schrödinger Dp branes in the bulk and corresponds to the *linear* growth of the entanglement entropy (during the post local equilibrium stage) which is independent of the shape of the entangling region [3]. From (65), it is also evident that in the large $u_T(\gg u_H)$ limit, the tsunami velocity in a non relativistic set up gradually saturates to its corresponding value in the relativistic set up ($\mathfrak{v}_s^{(Rel)}$) [3].

2.5 Saturation

The situation that we consider in this section essentially corresponds to the fact that the extremal surface lies entirely outside the horizon of the black brane namely, $u_T \leq u_H$. In this region, the entanglement entropy is supposed get saturated to its equilibrium value and which essentially turns out to be the entropy corresponding to a thermal state. As a result, in this regime, the major contribution to the entanglement entropy arises from the geometry around the horizon of the black brane.

In order to proceed further we set,

$$u_c = u_T(1-\lambda), \quad u_T \sim u_H. \tag{66}$$

Using (66), it is trivial to show,

$$\frac{\mathfrak{R}}{2} \approx \int_0^{1-\lambda} \frac{u_H d\xi}{\sqrt{\mathcal{H}(\xi)}}. \tag{67}$$

On the other hand, from (41) we note that,

$$\mathcal{A} \approx \frac{L_y L_\chi L^3}{\sqrt{c_T} u_H^2}\int_0^{1-\lambda}\frac{d\xi}{\xi^6}\frac{\sqrt{c(\xi)\mathcal{K}(\xi)}}{\sqrt{\mathcal{H}(\xi)}}. \tag{68}$$

Using (67), one could further rewrite (68) as,

$$\begin{aligned}
\mathcal{A}_{sat}^{(Sch)} &\approx \frac{L_y L_\chi L^3 \mathfrak{R}}{2\sqrt{c_T}u_H^3} + \frac{L_y L_\chi L^3}{\sqrt{c_T}u_H^2}\int_{\frac{\epsilon}{u_H}}^1 \frac{d\xi}{\xi^6}\frac{\sqrt{c(\xi)\mathcal{K}(\xi)}}{\sqrt{\mathcal{H}(\xi)}}\left(1-\frac{\xi^6}{\sqrt{c(\xi)\mathcal{K}(\xi)}}\right)\\
&\approx \frac{L_y L_\chi L^3 \mathfrak{R}}{2\sqrt{c_T}u_H^3} + \frac{\beta L_y L_\chi L^5}{3\epsilon^3} + ...
\end{aligned} \tag{69}$$

where, ϵ is the so called UV cut-off of the theory.

We now focus towards estimating the saturation time ($\mathfrak{t}_{sat}$) for the entanglement entropy in the theory. To do that, we first consider the case of *continuous* transitions where the entanglement entropy is continuous across the saturation time, $\mathfrak{t} = \mathfrak{t}_{sat}$.

From (32), we first note that,

$$\upsilon'^{(0)}|_{u_c\sim u_H} \sim \frac{1}{f(u_H)} \to \infty \tag{70}$$

which together with the fact, $u_c' \sim u_T' \sim 0$ naturally yields,

$$\mathsf{E}|_{u_c\sim u_H} \approx 0. \tag{71}$$

As a next step of our analysis, we expand the function $\mathcal{H}(\xi)$ about $\xi \sim \xi_H$,

$$\mathcal{H}(\xi) \approx -\lambda\xi_H\mathcal{H}'(\xi_H) + \frac{1}{2}(\xi-\xi_H)^2\mathcal{H}''(\xi_H). \tag{72}$$

Substituting (72) into (67), we finally obtain,

$$\frac{\mathfrak{R}}{2} \approx -u_H \int_{\xi\sim\xi_c} \frac{d(\xi_H-\xi)}{(\xi_H-\xi)}\sqrt{\frac{2}{\mathcal{H}''(\xi_H)}} \approx -\frac{\sqrt{2}u_H^2}{\beta L^2\sqrt{|f''(\xi_H)|}}\log\lambda. \tag{73}$$

Finally, the saturation time turns out to be,

$$\mathfrak{t}_{sat} \approx -\int_{\xi\sim\xi_c} \frac{u_H d(\xi_H-\xi)}{|f'(\xi_H)|(\xi_H-\xi)} = -\frac{u_H}{|f'(\xi_H)|}\log\lambda. \tag{74}$$

Comparing (73) and (74), we finally arrive at,

$$\mathfrak{t}_{sat} \approx \left(\frac{\sqrt{6}\beta L^2}{8u_H}\right)\mathfrak{R}. \tag{75}$$

In case of *discontinuous* transition, there is no straightforward formula for the saturation time. However, considering the fact that the area functional is still continuous across the transition point, one might try to have a rough estimate on the characteristic time scale for saturation. Considering linear growth all the way upto saturation point we find,

$$\mathfrak{t}_l \approx \frac{\mathfrak{R}}{2\sqrt{c_T}\mathfrak{v}_s^{(Sch)}} + \mathcal{O}(u_H/\mathfrak{R}). \tag{76}$$

Comparing (75) and (76) we find,

$$\frac{\mathfrak{t}_l}{\mathfrak{t}_{sat}} \approx \frac{4u_H}{\sqrt{6}\beta L^2}\left(\frac{u_m}{u_H}\right) > 1 \tag{77}$$

which thereby suggests that in the case of discontinuous saturation, the linear growth might persists for a longer time compared to that of the case of a continuous saturation [3].

As a final goal of our analysis, we would like to explore the conditions for a saturation to be continuous in the context of non relativistic quench. For the saturation to be continuous, the entity, $\mathfrak{t}-\mathfrak{t}_{sat}$ must be negative all the way to, $u_c - u_T \to 0$ [3].

To start with we note that,

$$u_c = u_T(1-\lambda), \quad u_T = u_B(1-\varsigma), \quad |\lambda|, |\varsigma| \ll 1 \tag{78}$$

where we consider large entangling region such that, $u_B \sim u_H$.

Under such circumstances, from (30) we note that,

$$\mathsf{E} \approx -\frac{\sqrt{6\lambda g(u_T)}}{2}. \tag{79}$$

Finally, the difference turns out to be,

$$\mathfrak{t} - \mathfrak{t}_{sat} \approx -\frac{\mathfrak{K} u_T \sqrt{6 g(u_T)}}{2}\sqrt{\lambda} + \mathcal{O}(\lambda) \tag{80}$$

where,

$$\mathfrak{K} = \int_0^1 \frac{d\xi}{f(\xi)}\left(1 - \frac{f(\xi)\beta^2 L^4}{\xi^2 u_T^2}\right)\frac{\sqrt{c_T}\xi^3}{\sqrt{f(\xi)(1-c_T\xi^6)}}. \tag{81}$$

Clearly, the saturation is continuous for $\mathfrak{K} > 0$ and on the other hand it is discontinuous for $\mathfrak{K} < 0$. Computing the integral above in (81), we find,

$$\mathfrak{K} \approx \frac{\beta L^2}{2u_H}\left(1 - \frac{2\beta^2 L^4}{u_H^2}\left(1 - \frac{\beta^2 L^4}{6u_H^2}\right)\right). \tag{82}$$

If we now set the dimensionless quantity, $\frac{\beta^2 L^4}{u_H^2} \equiv \mathfrak{n}$, where $\mathfrak{n}$ is some real number ($\mathfrak{n} \in \mathbb{R}$), then the condition for the saturation to be discontinuous turns out to be,

$$\frac{\beta^2 L^4}{u_H^2} \equiv \mathfrak{n} \leq 3 + \sqrt{6} \tag{83}$$

which thereby imposes an important constraint on the parameters of the theory.

3 Summary and final remarks

We now summarize the key findings of our analysis. It turns out that during the process of thermalization, for a certain class of strongly coupled non relativistic QFTs (with underlying Schrödinger isometry group), the evolution of the entanglement entropy could be classified into following three categories :

• The pre local equilibrium growth where the entanglement entropy exhibits a faster evolution (compared to its relativistic cousins) with time namely, $\Delta\mathcal{S}_{EE}^{(Sch)} \sim \mathfrak{t}^{\frac{5}{2}}$.

• During the post local stages of the equilibrium, the most dominant contribution to the area functional appears form the critical extremal surface which yields a linear growth for the entanglement entropy of the system. It turns out that the corresponding tsunami velocity always saturates the bound [3].

• Finally, we note that at the point of saturation, the entanglement entropy exhibits a discontinuity if we constraint certain parameters of the dual gravitational theory in a specific way.

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