

Wigner Quasi-probability Distribution Function for Two-mode Entangled Coherent States Generated by Beam Splitter and Cavity QED

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Abstract

In this paper, the Wigner function of two-mode qubit and qutrit like entangled coherent states (ECS's) is investigated. We use the displacement operator together with parity operation to construct the superposition of two coherent states. By transmitting this superposition from 50-50 beam splitter the two-mode qubit like ECS is generated. Moreover, we introduce a controllable method for producing qutrit like ECS using atom-field interaction in cavity QED and beam splitter. We will show that the distances of peaks of Wigner functions for two-mode ECS's are entanglement sensitive and can be a witness for entanglement. To confirm the results we use concurrence measure to compare

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bipartite entanglement of ECS's with the behavior of peaks of Wigner functions. Moreover, we investigate decoherence effects on Wigner function, arising from transmitting ECS's through noisy channels.

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1 Introduction

Phase-space representations of quantum states have been important tools for exploring the connections between quantum and classical physics. In 1932, Wigner introduced a distribution function in mechanics that permitted a description of mechanical phenomena in a phase space [1, 2]. Some important characteristics of the spatial Wigner function of entangled photon pairs were analyzed in [3]. In [4] the negativity of the Wigner function was discussed as a measure of the non-classicality. Negativity of the Wigner function is the reason why the Wigner function can not be regarded as a real probability distribution but a quasi-probability distribution function and it is a good indication of the possibility of the occurrence of nonclassical properties. Wigner functions have been especially used for describing the quadratures of the electrical field with coherent and squeezed states or single photon states [5, 6, 7]. The four-dimensional chronocyclic Wigner function of the parametric downconversion state was discussed in [8].

In 1926, Schrodinger introduced coherent states [9] then advanced studies were done in [10, 11]. In recent years, there has been the considerable interest in studying multi-mode quantum states of radiation fields because they have widely role in quantum information theory. The problem of generating various quantum states of an electromagnetic field was discussed in [12, 13, 14, 15, 16, 17, 18, 19]. In [20, 21] it was proposed a scheme for producing a superposition of two arbitrary Glauber coherent states via parity and displacement operator. The other scheme for preparation of superposition of coherent states in cavity QED was studied in [22, 23] in which an atom either flying through or trapped within a cavity, is controlled

by the classical Stark effect. ECS's have many applications in quantum optics and quantum information processing [24, 25, 26, 27, 28, 29, 30, 31, 32, 33]. In [34, 35] the required conditions for the maximal entangled states of the form $|\psi\rangle = \mu|\alpha\rangle|\beta\rangle + \nu|\gamma\rangle|\delta\rangle$ have been studied and then have been generalized to the state $|\psi\rangle = \mu|\alpha\rangle|\beta\rangle + \lambda|\alpha\rangle|\delta\rangle + \rho|\gamma\rangle|\beta\rangle + \nu|\gamma\rangle|\delta\rangle$ in Ref. [36]. Generation of multipartite ECS's and entanglement of multipartite states constructed by linearly independent coherent states are investigated in [37, 38]. In [39] it was considered the production and entanglement properties of the generalized balanced N-mode Glauber coherent states of the form

$$|\Psi_N^{(d)}\rangle = \frac{1}{\sqrt{M_N^{(d)}}} \sum_{i=0}^{d-1} \mu_i \underbrace{|\alpha_i\rangle \cdots |\alpha_i\rangle}_{N \text{ modes}}, \quad (1.1)$$

which is a general form of the balanced two-mode entangled coherent state $|\Psi^{(2)}\rangle_{bal} = \frac{1}{\sqrt{M^{(2)}}}(|\alpha\rangle|\alpha\rangle + \mu|\beta\rangle|\beta\rangle)$ and $|\Psi^{(3)}\rangle_{bal} = \frac{1}{\sqrt{M^{(3)}}}(|\alpha\rangle|\alpha\rangle + \mu_1|\beta\rangle|\beta\rangle + \mu_2|\gamma\rangle|\gamma\rangle)$. By assumption that the coherent states are linearly independent, these states recast in two qubit and qutrit form respectively. Then the entanglement of this states was evaluated by concurrence measure.

Another problem which has been investigated extensively in quantum information processing is noise effect or decoherence which arise from the coupling of the system to its surroundings [38, 40, 41]. van Enk in [40] introduced the effect of noise on coherent states with the modes 1 or 2 after traveling through a noisy channel as $|\alpha\rangle_{1(2)}|0\rangle_E \rightarrow |\sqrt{\eta}\alpha\rangle_{1(2)}|\sqrt{1-\eta}\alpha\rangle_E$ where the second state now refers to the environment and η is the noise parameter, which gives the fraction of photons that survives the noisy channel. The effect of noise on entanglement between modes 1 and 2 in qubit and qutrit like ECS's was investigated.

In this paper we use the displacement operator together with parity operation to construct the superposition of two coherent states $|\alpha\rangle + \mu|\beta\rangle$. By transmitting this superposition from 50-50 beam splitter the two-mode qubit like ECS, $|\Psi^{(2)}\rangle$, is generated. Moreover we introduce a method for producing qutrit like ECS using atom-field interaction in cavity QED and beam splitter. The Wigner functions of both two-mode qubit and qutrit like ECS's yield some information about the quantum entanglement of ECS's. This information comes from the

separation of peaks of Wigner quasi-probability as a function of ECS's parameters (e.g., α, β, μ in state $|\alpha\rangle + \mu|\beta\rangle$). The results are compared with another useful measure such as concurrence which confirm the outcomes. Moreover we investigate the noise effects on Wigner function. As a result Wigner function is affected by noise effect and the entanglement of ECS's is decreased after traveling through the noisy channel.

The outline of this paper is as follows: In section 2 we propose a method for generation of two-mode ECS's and investigate the Wigner quasi-probability distribution function. We compare the Wigner function with bipartite entanglement which is obtained by concurrence measure. Section 3 devoted to the qutrit like ECS generated via cavity QED and beam splitter. The effect of noise on Wigner function of ECS is studied in section 4. Our conclusions are summarized in section 5.

2 Generation of Two-mode Qubit like ECS

In this section, we consider how to generate the two-mode ECS. The first problem is the generation of the superposition of two number coherent states like as $|\alpha\rangle + \mu|\beta\rangle$, (up to normalization factors). Various attempts proposed scheme for generating of discrete superposition of coherent states [13, 14, 15, 38, 42, 43, 44].

2.1 Generation of Two-mode Qubit like ECS

In [20, 21], the authors use the displacement operator together with parity operation to construct the unitary operation

$$\hat{U}(\lambda, \alpha) = e^{i\lambda\hat{D}(\alpha)\hat{\Pi}}, \quad (2.2)$$

where λ is real number, $\hat{\Pi} = \cos(\pi\hat{a}^\dagger\hat{a})$ is a Hermitian and unitary operator with property $\hat{\Pi}|n\rangle = (-1)^n|n\rangle$ where $|n\rangle$ refers to photon number state and $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ is usual displacement operator. In order to obtain a linear combination of two arbitrary Glauber

coherent states, it is enough to use another displacement operator, $\hat{D}(\beta)$:

$$\hat{D}(\beta)\hat{U}(\lambda, \alpha)|0\rangle = \cos \lambda|\beta\rangle + i \sin \lambda|\alpha + \beta\rangle. \quad (2.3)$$

Using $\hat{V}(\alpha, \beta, \lambda) = \hat{D}(\alpha)\hat{U}(\lambda, \beta - \alpha)$, one may recast the above state in a convenient form as follows

$$\hat{V}(\alpha, \beta, \lambda)|0\rangle = \cos \lambda|\alpha\rangle + i \sin \lambda e^{iIm(\alpha\beta^*)}|\beta\rangle. \quad (2.4)$$

We next use polarizing beam splitter (PBS). The polarizing beam splitter is commonly made by cementing together two birefringent materials like calcite or quartz, and has the property of splitting a light beam into its orthogonal linear polarizations. The beam splitter interaction given by the unitary transformation

$$\hat{B}_{i-1,i}(\theta) = \exp[\theta(\hat{a}_{i-1}^\dagger \hat{a}_i - \hat{a}_i^\dagger \hat{a}_{i-1})], \quad (2.5)$$

which \hat{a}_{i-1} , \hat{a}_i , \hat{a}_{i-1}^\dagger and \hat{a}_i^\dagger are the annihilation and creation operators of the field mode $i - 1$ and i , respectively. Using Baker-Hausdorf formula, the action of the beam splitter on two modes $i - 1$ and i , can be expressed as

$$\hat{B}_{i-1,i}(\theta) \begin{pmatrix} \hat{a}_{i-1} \\ \hat{a}_i \end{pmatrix} \hat{B}_{i-1,i}^\dagger(\theta) = \begin{pmatrix} \hat{a}'_{i-1} \\ \hat{a}'_i \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \hat{a}_{i-1} \\ \hat{a}_i \end{pmatrix}. \quad (2.6)$$

For 50 – 50 beam splitter, i.e. $\theta = \pi/4$ the above equation reduces to

$$\hat{B}_{1,2}(\pi/4)|\alpha'\rangle_1|0\rangle_2 = \left|\frac{\alpha'}{\sqrt{2}}\right\rangle_1 \left|\frac{\alpha'}{\sqrt{2}}\right\rangle_2. \quad (2.7)$$

This result says that like classical light wave where the incident intensity is evenly divided between the two output beams, e.g. half the incident average photon number, $\frac{|\alpha|^2}{2}$, emerges in each beam. Note that the output is not entangled. For producing ECS suppose that our input state is a superposition of two coherent states as $|\alpha'\rangle_1 + \mu|\beta'\rangle_1$. Following the procedure above, we may then, obtain the output state (see figure 1)

$$\hat{B}_{1,2}(\pi/4)(|\alpha'\rangle_1 + \mu|\beta'\rangle_1) \otimes |0\rangle_2 = \left|\frac{\alpha'}{\sqrt{2}}\right\rangle_1 \left|\frac{\alpha'}{\sqrt{2}}\right\rangle_2 + \mu \left|\frac{\beta'}{\sqrt{2}}\right\rangle_1 \left|\frac{\beta'}{\sqrt{2}}\right\rangle_2. \quad (2.8)$$

By renaming $\alpha \equiv \frac{\alpha'}{\sqrt{2}}$ and $\beta \equiv \frac{\beta'}{\sqrt{2}}$ we have $|\alpha\rangle_1|\alpha\rangle_2 + \mu|\beta\rangle_1|\beta\rangle_2$, which is a two-mode qubit like ECS.

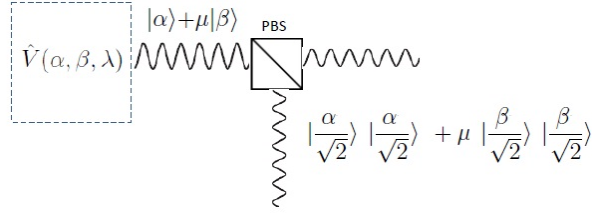


Figure 1: Experimental set up for generating ECS.

2.2 Wigner Function for Qubit like ECS's

One of the important quasi-probability distribution over phase space is the Wigner function. The Wigner function seems to be the earliest introduced of the phase-space quasi-probability distributions, making its debut in 1932. Wigner function is defined as [45]

$$W(\gamma) = \frac{1}{\pi^2} \int d^2\lambda C_W(\lambda) e^{\lambda^* \gamma - \lambda \gamma^*}, \quad (2.9)$$

in which $C_W(\lambda) = \text{Tr}(\rho D(\lambda))$. Here we investigate the Wigner function behaviour for qubit like ECS.

Let us consider qubit like ECS which was generated in previous section in the form

$$|\Psi^{(2)}\rangle = \frac{1}{\sqrt{M^{(2)}}} (|\alpha\rangle|\alpha\rangle + \mu|\beta\rangle|\beta\rangle), \quad (2.10)$$

in which the normalization factor is $M^{(2)} = 1 + \mu^2 + 2\mu \text{Re}(p^2)$ with $p = \langle\alpha|\beta\rangle$. Note that we used the superscript (2) for qubit-like states to distinguish it from that of qutrit like states in the next section. Wigner characteristic function reads

$$C_W^{(2)}(\lambda) = \frac{1}{M^{(2)}} \{ \langle\alpha|D(\lambda)|\alpha\rangle + \mu^2 \langle\beta|D(\lambda)|\beta\rangle + \mu p (\langle\beta|D(\lambda)|\alpha\rangle + \langle\alpha|D(\lambda)|\beta\rangle) \}, \quad (2.11)$$

by substituting $C_W^{(2)}(\lambda)$ in Eq.(2.9), the corresponding Wigner function is obtained as

$$W^{(2)}(\gamma) = \frac{2}{\pi M^{(2)}} \{ e^{-2|\gamma-\alpha|^2} + \mu^2 e^{-2|\gamma-\beta|^2} + \mu e^{-(|\alpha|^2+|\beta|^2)} e^{-2|\gamma|^2} (e^{2(\gamma\beta^*+\gamma^*\alpha)} + e^{2(\gamma\alpha^*+\gamma^*\beta)}) \}. \quad (2.12)$$

For simplicity, let us assume that α and β are real numbers and $\mu = 1$. Setting $\gamma = x + iy$, we plot diagram of $W^{(2)}(x, y)$ as a function of x and y for given α and β . Figure 2 shows that

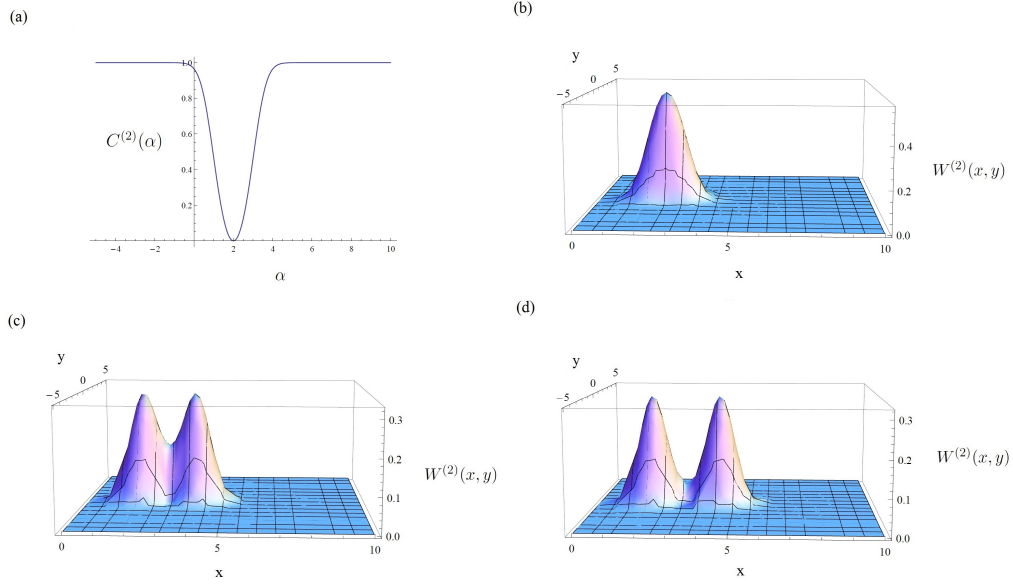


Figure 2: (Color online) (a) Concurrence of $|\Psi^{(2)}\rangle$ as a function of α for a given $\beta = 2$ and $W^{(2)}(x, y)$ as function of x and y for $\alpha = 2$: (b) $\beta = 2$, (c) $\beta = 4$ and (d) $\beta = 4.5$.

by considering α and β as real numbers, the Wigner function moves along x axis (real part of γ) while if we regard that α and β are imaginary numbers, the Wigner function moves along y axis (imaginary part of γ). Moreover, the study of Wigner function reveals information on the entanglement between modes in two-mode ECS, Eq.(2.10). For more details, one can do a comparison between (a), (b), (c) and (d) in figure 2 which shows that if $\alpha = \beta$ the state is separable and there is one peak (figure (2b)). For $\alpha \neq \beta$ the state is entangled. As a result, the separation of two peaks can be a witness for amount of entanglement of $|\Psi^{(2)}\rangle$. For clarity we assume that $y = 0$ and solve the equation $\frac{dW^{(2)}(x)}{dx} = 0$:

$$(\alpha - x)e^{-2(\alpha-x)^2} + (\beta - x)e^{-2(\beta-x)^2} + (\alpha + \beta - 2x)e^{-(\alpha-x)^2 - (\beta-x)^2} = 0. \quad (2.13)$$

Clearly, $x = \alpha = \beta$ is one solution for Eq.(2.13), which means that the diagram of $W^{(2)}(\gamma)$ assumes one peak at point $x = \alpha$. Whereas if $\alpha \neq \beta$ and considering $u = e^{-(x-\alpha)^2 + (x-\beta)^2}$,

Eq.(2.13) is rewritten as

$$(\alpha - x)u^2 + (-2x + \alpha + \beta)u + \beta - x = 0, \quad (2.14)$$

which has two solutions $u = -1$ and $u = \frac{x-\beta}{\alpha-x}$. The former must be ignored as u is always positive. The equation $u = e^{-(x-\alpha)^2+(x-\beta)^2} = \frac{x-\beta}{\alpha-x}$ is a transcendent equation and it can not be solved in an algebraic way. However, if we plot both sides of this equation for a given α and β , then their intersections are required solutions (see figure 3). Figure 3 shows that by

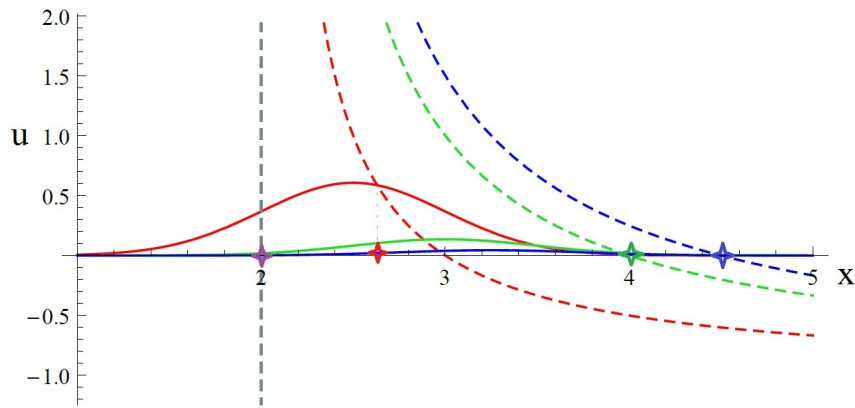


Figure 3: (Color online) $u = e^{-(x-\alpha)^2+(x-\beta)^2}$ (full line) and $u = \frac{x-\beta}{\alpha-x}$ (dashed line) as function of x for given $\alpha = 2$: (a) $\beta = 3$ (Red), (b) $\beta = 4$ (Green) and (c) $\beta = 4.5$ (Blue).

increasing $\Delta = \alpha - \beta$ the separation of two maximum in Wigner function increases too. This comes from the fact that putting $x = \alpha$ in Eq. (2.13) yields $(\beta - \alpha)[e^{-2(\beta-\alpha)^2} + e^{-(\beta-\alpha)^2}] \simeq 0$ which is satisfied only for $|\alpha - \beta| \gg 1$. The result is that the separation of two peaks in Wigner function is a monotone function of Δ and for large $\alpha - \beta$, the separation of two peaks in Wigner function just depends on $\Delta = \alpha - \beta$. To confirm the above results we use another measure, the so called concurrence, which illustrates that the entanglement of $|\Psi^{(2)}\rangle$, depends solely on Δ . As in general two coherent states $|\alpha\rangle$ and $|\beta\rangle$ are linearly independent, they may span a two dimensional qubit like Hilbert space $\{|0\rangle, |1\rangle\}$, hence the two-mode coherent state $|\Psi^{(2)}\rangle$ can be recast in two qubit form [39]. For any pure state in the form

$|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$, the concurrence is defined as $C = 2|a_{00}a_{11} - a_{01}a_{10}|$ [46, 47]. Therefore the concurrence of Eq.(2.10) in term of Δ is rewritten as

$$C^{(2)}(\Delta) = \frac{1 - e^{-\Delta^2}}{1 + e^{-\Delta^2}}. \quad (2.15)$$

This equation shows that the concurrence is a monotone function of Δ . If the separation of two peaks, becomes large ($\Delta \rightarrow \infty$), the concurrence tends to its maximum ($C_{max}^{(2)} = 1$), while for small separation (i.e. $\Delta \rightarrow 0$) the concurrence tends to zero and the state becomes separable. Figure 4 shows the concurrence as a monotone function of separation of two peaks (see figure 4).

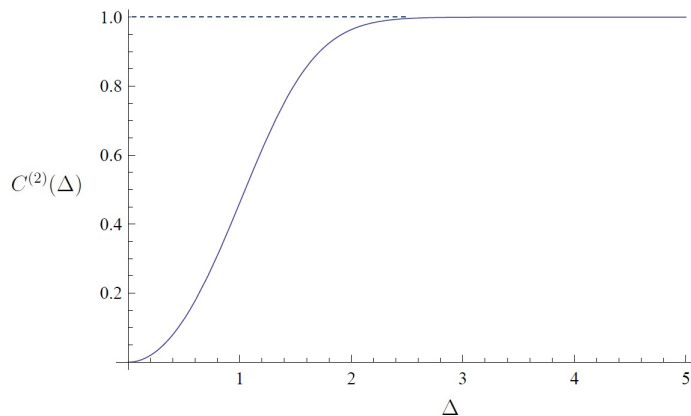


Figure 4: (Color online) Concurrence of $|\Psi^{(2)}\rangle$ as a function of separation of two peaks Δ .

3 Generation of Two-mode Qutrit like ECS

In the previous section, we introduced a method to produce superposition of even number of coherent states. These states lead to qubit like ECS. One may construct qutrit like ECS. To this aim, we introduce a controllable method for producing qutrit like ECS. Then the Wigner function and bipartite entanglement of this state is investigated.

3.1 Generation of Two-mode Qutrit like ECS

Here we use atom-field interaction in cavity QED and beam splitter to produce qutrit like ECS. The hamiltonian of Jaynes-Cummings model [22, 23] is defined as

$$\hat{H}_{int} = \omega_c \hat{a}^\dagger \hat{a} + \omega_a \hat{\sigma}_z + g(\hat{a}^\dagger \hat{\sigma}^- + \hat{a} \hat{\sigma}^+) + \varepsilon e^{-i\omega_L t} \hat{\sigma}^+ + \varepsilon^* e^{i\omega_L t} \hat{\sigma}^-, \quad (3.16)$$

where $\hat{a}(\hat{a}^\dagger)$ is the annihilation (creation) operator for the cavity field and for simplicity we assume that $\hbar = 1$. ω_c , ω_L and ω_a are the frequencies of the cavity, classical field and the atomic transition frequency between the excited state $|e\rangle$ and the ground state $|g\rangle$ respectively. g is the atom-cavity coupling constant. The complex amplitude is represented by ε . $\hat{\sigma}^\pm$ and $\hat{\sigma}_z$ are the atomic transition operators which are given by

$$\begin{aligned} \hat{\sigma}^+ &= |e\rangle\langle g|, \hat{\sigma}^- = |g\rangle\langle e|, \\ \hat{\sigma}_z &= |e\rangle\langle e| - |g\rangle\langle g|. \end{aligned} \quad (3.17)$$

In this method, an atom interact alternately with a (resonant) classical field and with the (dispersive) cavity field. Let us suppose that the atom is not affected by the cavity field and is initially resonant with the classical field so the interaction hamiltonian is given by $\hat{H}_{I1} = \varepsilon e^{-i\varphi} \hat{\sigma}^+ + \varepsilon^* e^{i\varphi} \hat{\sigma}^-$. If the atom is initially in the ground state $|g\rangle$ or $|e\rangle$, after the atom interacts with the classical field we have

$$\begin{aligned} |g\rangle &\rightarrow \frac{1}{\sqrt{1+|\varepsilon_k|^2}}(|g\rangle + \varepsilon_k |e\rangle), \\ |e\rangle &\rightarrow \frac{1}{\sqrt{1+|\varepsilon_k|^2}}(-\varepsilon_k^* |g\rangle + |e\rangle), \end{aligned} \quad (3.18)$$

respectively, where ε_k ($k = 0, 1, 2, \dots$) is an adjustable complex number controlled by the parameters of the classical field. Now if the atom is interacting dispersively with the cavity field and far away from the classical field, the effective hamiltonian of the atom-cavity system is given by $\hat{H}_{I2} = \frac{g^2}{\tilde{\Delta}} \hat{a}^\dagger \hat{a} \hat{\sigma}_z$, in which $\tilde{\Delta} = \omega_a - \omega_c$. We assume that the cavity field is initially in a coherent state $|\alpha\rangle$, thus the total system has the form $|\psi_1\rangle \equiv \frac{1}{\sqrt{1+|\varepsilon_0|^2}}(|g\rangle|\alpha\rangle + \varepsilon_0 |e\rangle|\alpha\rangle)$. After an interaction time $t = \frac{\pi\tilde{\Delta}}{2g^2}$, the atom-cavity system evolves to

$$|\psi_1\rangle \rightarrow \frac{1}{\sqrt{1+|\varepsilon_0|^2}}(|g\rangle|i\alpha\rangle + \varepsilon_0 |e\rangle|-i\alpha\rangle). \quad (3.19)$$

Again the atomic transition is resonant with the classical field but far away from the cavity field. After a given time and performing a measurement on the atom we have

$$|\psi'_1\rangle = \frac{1}{\sqrt{M^{(2)}}}(-\varepsilon_0\varepsilon_1^*|\alpha\rangle + |\alpha\rangle), \quad (3.20)$$

where $M^{(2)} = 1 + |\varepsilon_0\varepsilon_1^*|^2 - 2\text{Re}(\varepsilon_0\varepsilon_1^*)e^{-2|\alpha|^2}$. Further the phase shifter $\hat{\mathcal{P}} = e^{-i\frac{\pi}{2}\hat{a}^\dagger\hat{a}}$ transforms the state Eq.(3.20) as

$$\hat{\mathcal{P}}|\psi'_1\rangle = \frac{1}{\sqrt{M^{(2)}}}(-\varepsilon_0\varepsilon_1^*|\alpha\rangle + |-\alpha\rangle). \quad (3.21)$$

Similar to the first step $N = 1$, once again the atom interacts alternately with a (resonant) classical field and with the (dispersive) cavity field $N = 2$, then by performing a measurement on the atom in ground state, we can generate superpositions of three coherent states with adjustable weighting factors as the form

$$|\psi'_2\rangle = \frac{1}{\sqrt{M^{(3)}}}(|2\alpha\rangle - (\varepsilon_0\varepsilon_2^* + \varepsilon_0\varepsilon_1^*)|0\rangle - \varepsilon_1\varepsilon_2^*|-\alpha\rangle), \quad (3.22)$$

in which $M^{(3)}$ is normalization factor. In [22] the optimal values of ε_0 , ε_1 , and ε_2 determined to be -0.8200 , 2.1184 , and -0.4720 , respectively. Acting displacement operator $\hat{D}(\beta) = e^{\beta\hat{a}^\dagger - \beta^*\hat{a}}$ with property $\hat{D}(\beta)|\alpha\rangle = |\alpha + \beta\rangle$, onto state Eq.(3.22) leads to the following superposition of coherent states

$$\hat{D}(\beta)|\psi'_2\rangle = \frac{1}{\sqrt{2e^{-8\alpha^2} + 5.4e^{-2\alpha^2} + 3.8225}}(|2\alpha + \beta\rangle + 1.35|\beta\rangle + |-\alpha + \beta\rangle). \quad (3.23)$$

By transmitting this state Eq.(3.23) through 50 – 50 beam splitter we have

$$|\Phi^{(3)}\rangle = \frac{1}{\sqrt{M^{(3)}}}\left(|\frac{2\alpha + \beta}{\sqrt{2}}\rangle|\frac{2\alpha + \beta}{\sqrt{2}}\rangle + 1.35|\frac{\beta}{\sqrt{2}}\rangle|\frac{\beta}{\sqrt{2}}\rangle + |-\frac{2\alpha + \beta}{\sqrt{2}}\rangle|-\frac{2\alpha + \beta}{\sqrt{2}}\rangle\right). \quad (3.24)$$

This state is a two-mode qutrit like ECS. In the following, we concern the Wigner function of qutrit like ECS's. Firstly, we will consider the general state $|\Psi^{(3)}\rangle = \frac{1}{\sqrt{M^{(3)}}}(|\alpha\alpha\rangle + \mu_1|\beta\beta\rangle + \mu_2|\gamma\gamma\rangle)$ then as an especial example we will consider the state Eq.(3.24).

3.2 Wigner Function for Qutrit like ECS's

Let us consider two qutrit like ECS

$$|\Psi^{(3)}\rangle = \frac{1}{\sqrt{M^{(3)}}}(|\alpha\rangle|\alpha\rangle + \mu_1|\beta\rangle|\beta\rangle + \mu_2|\gamma\rangle|\gamma\rangle), \quad (3.25)$$

with $M^{(3)} = 1 + \mu_1^2 + \mu_2^2 + 2\mu_1 p_1^2 + 2\mu_1\mu_2 p_2^2 + 2\mu_2 p_3^2$ in which $p_1 = \langle\alpha|\beta\rangle$, $p_2 = \langle\beta|\gamma\rangle$ and $p_3 = \langle\alpha|\gamma\rangle$. Again for simplicity we assumed that $\alpha, \beta, \gamma, \mu_1$ and μ_2 are real parameters.

Using the definition Eq.(2.9) the Wigner function is obtained as

$$W^{(3)}(\delta) = \frac{2}{\pi M^{(3)}} \left\{ e^{-2|\delta-\alpha|^2} + \mu_1^2 e^{-2|\delta-\beta|^2} + \mu_2^2 e^{-2|\delta-\gamma|^2} + \mu_1 p_1 e^{-\frac{1}{2}(\alpha+\beta)^2} e^{-2|\delta|^2} (e^{2(\delta\beta+\delta^*\alpha)} + e^{2(\delta^*\beta+\delta\alpha)}) \right. \\ \left. + \mu_1\mu_2 p_2 e^{-\frac{1}{2}(\gamma+\beta)^2} e^{-2|\delta|^2} (e^{2(\delta\gamma+\delta^*\beta)} + e^{2(\delta^*\gamma+\delta\beta)}) + \mu_2 p_3 e^{-\frac{1}{2}(\alpha+\gamma)^2} e^{-2|\delta|^2} (e^{2(\delta\gamma+\delta^*\alpha)} + e^{2(\delta^*\gamma+\delta\alpha)}) \right\}. \quad (3.26)$$

Taking $\delta = x+iy$ and assuming $\mu_1 = \mu_2 = 1$, the Wigner function $W^{(3)}(x, y)$ can be represented diagrammatically as a function of x and y for given α, β and γ in the figure 5. One can see

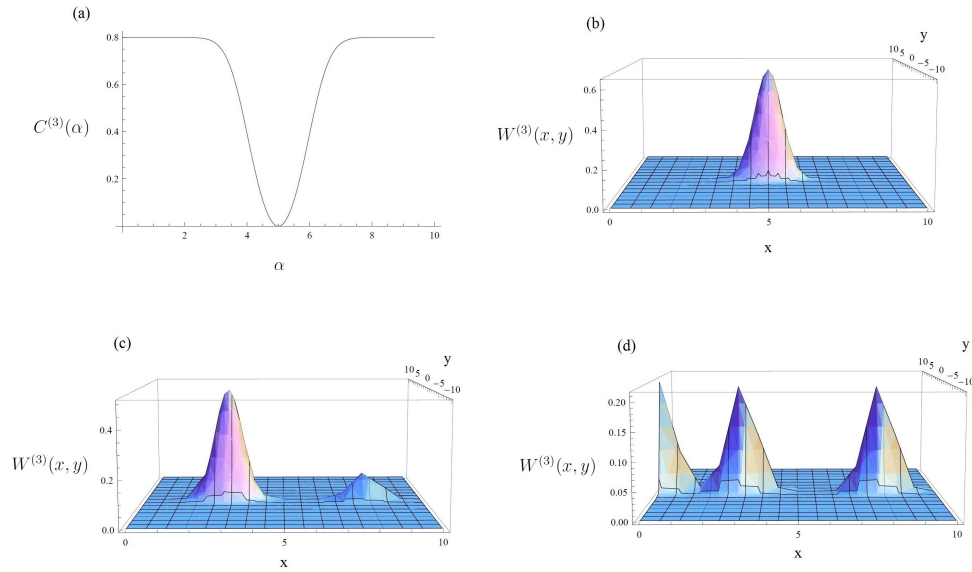


Figure 5: (Color online) (a) Concurrence of $|\Psi^{(3)}\rangle$ as a function of α for $\beta = \gamma = 5$ and $W^{(3)}(x, y)$ as function of x and y : (b) $\alpha = \beta = \gamma = 5$, (c) $\alpha = \beta = 3, \gamma = 8$, (d) $\alpha = 0, \beta = 3, \gamma = 8$.

that for $\alpha = \beta = \gamma = 5$ there is just one peak in diagram of $W^{(3)}(x, y)$ and the qutrit like ECS

is reduced to separable state ($C^{(3)}(\alpha) = 0$), while for $\alpha \neq \beta \neq \gamma$ there exist three peaks and the state is in general entangled. Figure 5 shows that the distance of peaks in Wigner function $W^{(3)}(x, y)$ depends on parameters Δ_i defined as

$$\begin{aligned}\Delta_1 &= |\alpha - \beta|, \\ \Delta_2 &= |\alpha - \gamma|, \\ \Delta_3 &= |\beta - \gamma|.\end{aligned}\tag{3.27}$$

which in turn implies that $W^{(3)}(x, y)$ is entanglement sensitive. To confirm this result we use the general concurrence measure for bipartite state $|\psi\rangle = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} a_{ij} |e_i \otimes e_j\rangle$ [48]. The norm of concurrence vector is obtained as $C = 2(\sum_{i < j} \sum_{k < l} |a_{ik}a_{jl} - a_{il}a_{jk}|^2)^{1/2}$, where d_1 and d_2 are dimensions of first and second part respectively. If the set $\{|\alpha\rangle, |\beta\rangle, |\gamma\rangle\}$ are linearly independent meaning they may span a three dimensional Hilbert space $\{|0\rangle, |1\rangle, |2\rangle\}$, hence two mode coherent state $|\Psi^{(3)}\rangle$ can be recast in two qutrit form. Therefore the concurrence in terms of the separation of peaks is obtained as

$$\begin{aligned}C^{(3)}(\Delta_1, \Delta_2, \Delta_3) &= \frac{2}{3+2e^{-\Delta_1^2}+2e^{-\Delta_2^2}+2e^{-\Delta_3^2}} \{3 + e^{-2\Delta_1^2} + e^{-2\Delta_2^2} + e^{-2\Delta_3^2} \\ &\quad + 2e^{-\Delta_2^2-\Delta_3^2} - 12e^{-\frac{1}{2}(\Delta_1^2+\Delta_2^2+\Delta_3^2)} + 2e^{-\Delta_1^2}(e^{-\Delta_2^2} + e^{-\Delta_3^2})\}^{1/2}.\end{aligned}\tag{3.28}$$

This equation shows that if all $\Delta_i \rightarrow 0$ ($i = 1, 2, 3$) the concurrence tends to zero and the state become non entangled and we have one peak (figure(5b)) while for $\Delta_i \rightarrow \infty$, the concurrence tends to its maximum value i.e. $C_{max} = 1.154$ and three peaks appear as in figure (5d). We note that the Wigner function is positive for all values of x and y . Another example for two-mode qutrit like ECS which is generated via atom-field interaction was given in Eq.(3.24). For this state, Wigner function is obtained as

$$\begin{aligned}W^{(3)}(\delta) &= \frac{2}{\pi M^{(3)}} \{e^{-2|\delta - \frac{2\alpha+\beta}{\sqrt{2}}|^2} + 1.8225e^{-2|\delta - \frac{\beta}{\sqrt{2}}|^2} + e^{-2|\delta - \frac{-2\alpha+\beta}{\sqrt{2}}|^2} \\ &\quad + 1.35e^{-2\alpha^2-2\alpha\beta-\beta^2-2|\delta|^2} (e^{\sqrt{2}(\delta\beta+\delta^*(2\alpha+\beta))} + e^{\sqrt{2}(\delta^*\beta+\delta(2\alpha+\beta))}) \\ &\quad + 1.35e^{-2\alpha^2+2\alpha\beta-\beta^2-2|\delta|^2} (e^{\sqrt{2}(\delta(-2\alpha+\beta)+\delta^*\beta)} + e^{\sqrt{2}(\delta^*(-2\alpha+\beta)+\delta\beta)}) \\ &\quad + e^{-4\alpha^2-\beta^2-2|\delta|^2} (e^{\sqrt{2}(\delta(-2\alpha+\beta)+\delta^*(2\alpha+\beta))} + e^{\sqrt{2}(\delta^*(-2\alpha+\beta)+\delta(2\alpha+\beta))})\}.\end{aligned}\tag{3.29}$$

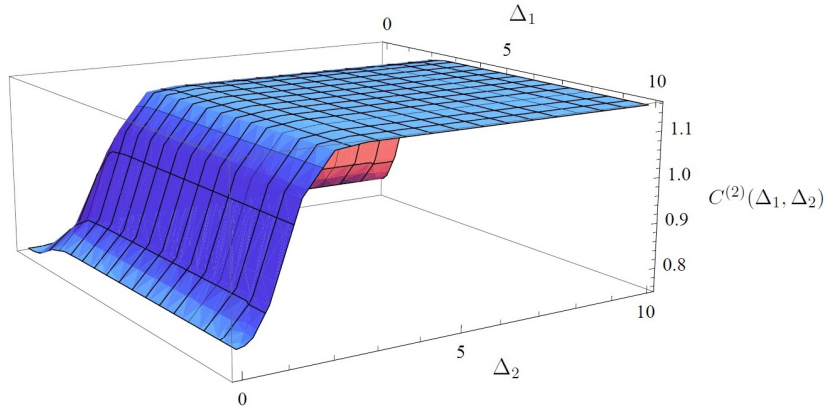


Figure 6: (Color online) Concurrence of $|\Psi^{(3)}\rangle$ as function of Δ_1 and Δ_2 in the range $[0, 10]$ for a given $\Delta_3 = 10$.

Diagram of concurrence and Wigner function for the state $|\Phi^{(3)}\rangle$ is shown in figure 7. Figure (7a) shows that only for $\alpha = 0$ the concurrence is zero and independent of β . This corresponds exactly with one peak in $W^{(3)}(x, y)$. By increasing α the width of the peak is increased so for $\alpha = 2$, we see three peaks in diagram (see figure (7c) and (7d)). Moreover by assuming $y = 0$ for given α and β , we draw a profile of $W^{(3)}(x)$ as a function of x for given α and β (see also figure 8). Figure 8 shows that if $\alpha = 0$ Wigner function has one peak and the state is separable and by increasing α , the peaks go away from each other and concurrence is raised.

4 Noise Effect on Wigner Function of ECS's

Let us assume that the mode 1 travel through a noisy channel characterized by

$$|\alpha\rangle_1|0\rangle_E \rightarrow |\sqrt{\eta}\alpha\rangle_1|\sqrt{1-\eta}\alpha\rangle_E, \quad (4.30)$$

where the second state now refers to the environment and η is the noise parameter, which gives the fraction of photons that survives the noisy channel [40, 41]. Here the effect of noise on Wigner function of qubit like ECS is investigated. Let us consider the state $|\Psi^{(2)}\rangle$. After

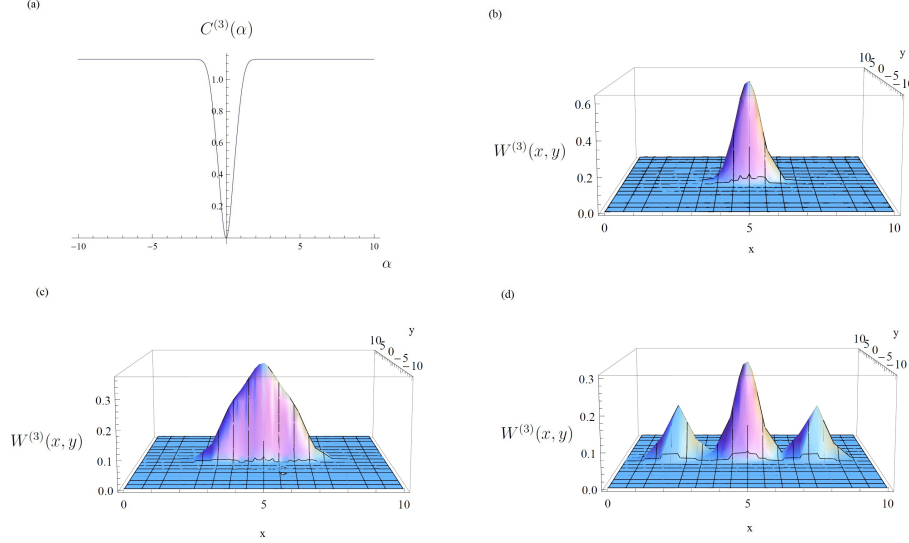


Figure 7: (Color online) (a) Concurrence of $|\Phi^{(3)}\rangle$ as a function of α and $W^{(3)}(x,y)$ as function of x and y for $\beta = 7$: (b) $\alpha = 0$, (c) $\alpha = 1$ and (d) $\alpha = 2$.

traveling through the noisy channel the state $|\Psi^{(2)}\rangle$ becomes

$$\begin{aligned}
 |\Psi'^{(2)}\rangle &= \frac{1}{\sqrt{M^{(2)}}} (|\sqrt{\eta}\alpha, \sqrt{\eta}\alpha\rangle_{1,2} |\sqrt{1-\eta}\alpha, \sqrt{1-\eta}\alpha\rangle_E \\
 &\quad + \mu |\sqrt{\eta}\beta, \sqrt{\eta}\beta\rangle_{1,2} |\sqrt{1-\eta}\beta, \sqrt{1-\eta}\beta\rangle_E).
 \end{aligned} \tag{4.31}$$

In order to study the noise effect on Wigner function we should trace out the mode 2 and environment mode E by partial trace, i.e. $\rho_1 = Tr_{2E}(|\Psi^{(2)}\rangle\langle\Psi^{(2)}|)$, then reduced density matrix in coherent basis $|\alpha\rangle$ and $|\beta\rangle$ reads

$$\rho_1 = \frac{1}{M^{(2)}} \{ |\sqrt{\eta}\alpha\rangle\langle\sqrt{\eta}\alpha| + \mu^2 |\sqrt{\eta}\beta\rangle\langle\sqrt{\eta}\beta| + \mu p^{2-\eta} (|\sqrt{\eta}\alpha\rangle\langle\sqrt{\eta}\beta| + |\sqrt{\eta}\beta\rangle\langle\sqrt{\eta}\alpha|) \}. \tag{4.32}$$

From Eq.(2.9) we find the Wigner function of qubit like ECS after traveling through a noisy channel as

$$W^{(2)}(\gamma) = \frac{2}{\pi M^{(2)}} \{ e^{-2|\gamma-\sqrt{\eta}\alpha|^2} + \mu^2 e^{-2|\gamma-\sqrt{\eta}\beta|^2} + \mu p^{2-\eta} e^{-\frac{\eta}{2}(\alpha+\beta)^2} e^{-2|\gamma|^2} (e^{2\sqrt{\eta}(\gamma^*\alpha+\beta\gamma)} + e^{2\sqrt{\eta}(\gamma^*\beta+\alpha\gamma)}) \}. \tag{4.33}$$

Setting $\lambda = x$ diagram of $W^{(2)}(x)$ as a function of x for given η, α, β and $\mu = 1$ is represented in figure (9b). On the other hand the state $|\Psi^{(2)}\rangle$ after traveling through the noisy channel

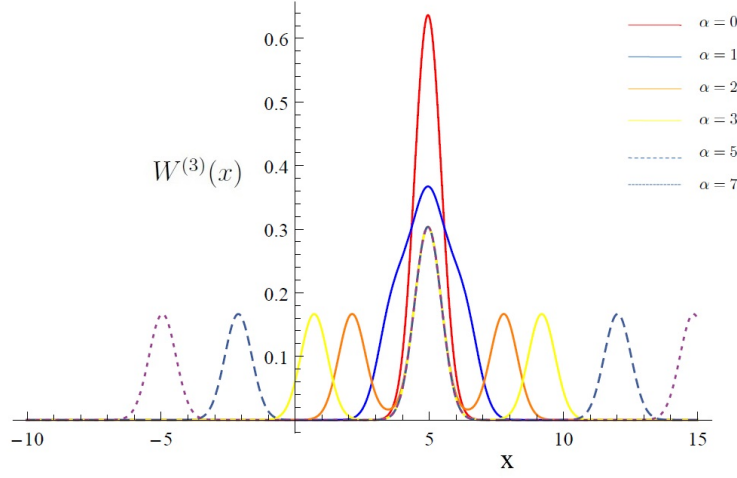


Figure 8: (Color online) Wigner function of $|\Phi^{(3)}\rangle$ as a function of x for $\beta = 7$.

becomes

$$|\Psi'^{(2)}\rangle = \frac{1}{\sqrt{M^{(2)}}} (|\sqrt{\eta}\alpha, \sqrt{\eta}\alpha\rangle_{1,2} |\sqrt{1-\eta}\alpha, \sqrt{1-\eta}\alpha\rangle_E + |\sqrt{\eta}\beta, \sqrt{\eta}\beta\rangle_{1,2} |\sqrt{1-\eta}\beta, \sqrt{1-\eta}\beta\rangle_E), \quad (4.34)$$

In order to study the noise effect on entanglement between modes 1 and 2, we should trace out the environment mode E by partial trace, i.e. $\rho_{12} = \text{Tr}_E(|\Psi'^{(2)}\rangle\langle\Psi'^{(2)}|)$. By assumption $|\alpha\rangle$ and $|\beta\rangle$ are linearly independent the reduced density matrix in orthogonal basis $|0\rangle$ and $|1\rangle$ reads

$$\rho_{12}^{(2)} = \frac{1}{2 + 2p^2} \begin{pmatrix} a_{11} & a_{12} & a_{12} & a_{14} \\ a_{12} & a_{22} & a_{22} & a_{24} \\ a_{12} & a_{22} & a_{22} & a_{24} \\ a_{14} & a_{24} & a_{24} & a_{44} \end{pmatrix}, \quad (4.35)$$

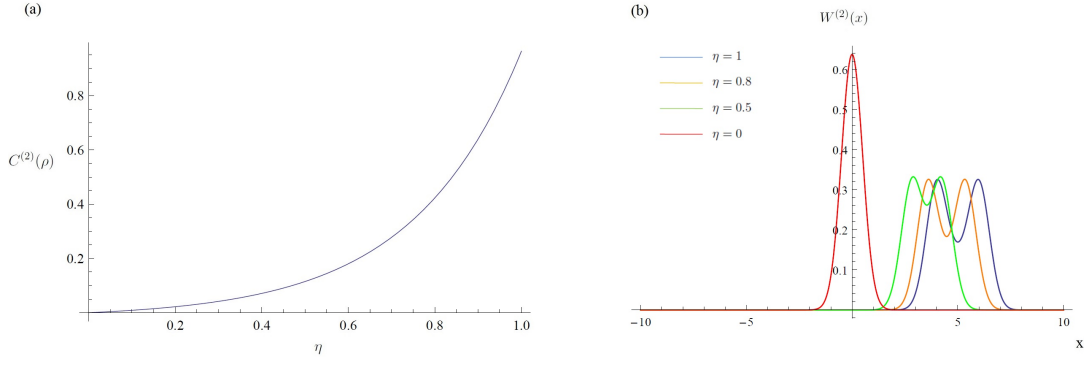


Figure 9: (Color online) (a) $C^{(2)}(\rho)$ as a function of η , (b) $W^{(2)}(x)$ as a function of x for a given η and $\alpha = 4, \beta = 6$.

where

$$\begin{aligned}
a_{11} &= 1 + 2p^2 + p^{4\eta}, \\
a_{12} &= p^{-\eta} \sqrt{1 - p^{2\eta}} (p^2 + p^{4\eta}), \\
a_{14} &= p^{2-2\eta} + p^{2\eta} - p^{4\eta} - p^2, \\
a_{22} &= p^{2\eta} - p^{4\eta}, \\
a_{24} &= p^\eta (1 - p^{2\eta})^{3/2}, \\
a_{44} &= (1 - p^{2\eta})^2,
\end{aligned} \tag{4.36}$$

in which $p = e^{-\frac{1}{2}(\alpha-\beta)^2}$. Clearly this state is a two qubit mixed state and one of the suitable measure to evaluate the amount of entanglement is concurrence. For any two-qubit mixed state, concurrence is defined as $C = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$ where the λ_i 's are the non-negative eigenvalues, in decreasing order, of the Hermitian matrix $R = \sqrt{\sqrt{\rho} \tilde{\rho} \sqrt{\rho}}$, with $\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)$ in which ρ^* is the complex conjugate of ρ when it is expressed in a standard basis and σ_y represents the usual second Pauli matrix in a local basis $\{|0\rangle, |1\rangle\}$ [46].

The concurrence is found as

$$C^{(2)}(\rho) = \frac{p^2(-1 + p^{-2\eta})}{1 + p^2}. \tag{4.37}$$

Figure (9a) demonstrate the behaviour of concurrence for $\rho_{12}^{(2)}$ as a function of η . Figure (9a) shows that by decreasing noise parameter η the concurrence of state $|\Psi^{(2)}\rangle$ is decreased. Figure

(9b) yields the same result, i.e. by decreasing the η two peaks close together and finally in $\eta = 0$ we have one peak, which in turn implies that the state reduces to separable state as one would expect for complete decoherence.

5 Conclusion

In summary, we introduced two methods for generation of two-mode qubit and qutrit like ECS's. We found that the Wigner functions of these ECS's reveals some information on the entanglement between modes in ECS's. For qubit like ECS, it was shown that if $\alpha = \beta$ the state is separable and there is one peak in Wigner function $W^{(2)}(x, y)$. For $\alpha \neq \beta$, by increasing β the distance of the peaks is also increased. The same results arise when we use concurrence measure. It was shown that concurrence is a monotone function of separation of two peaks, Δ , in which for Δ large enough concurrence tends to its maximum value ($C_{max}^{(2)} = 1$) while for small values ($\Delta \rightarrow 0$) the concurrence tends to zero. A similar result was discussed for qutrit like ECS, $|\Psi^{(3)}\rangle$. On the other hand we showed that the Wigner function $W^{(3)}(x, y)$ is entanglement sensitive. Finally we investigated the noise effects on the two-mode qubit like ECS. We recognized that by decreasing noise parameter η , the concurrence of state $|\Psi^{(2)}\rangle$ is decreased and at the same time two peaks on the Wigner function $W^{(2)}(x)$ approach each other. Ultimately for $\eta = 0$ (maximum noise) the state is separable and one peak appears in the profile of Wigner function.

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