On some Frobenius groups with the same prime graph as the almost simple group $\mathbf{PGL(2, 49)}$

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Abstract

The prime graph of a finite group G is denoted by $\Gamma(G)$ whose vertex set is $\pi(G)$ and two distinct primes p and q are adjacent in $\Gamma(G)$, whenever G contains an element with order pq. We say that G is unrecognizable by prime graph if there is a finite group H with $\Gamma(H) = \Gamma(G)$, in while $H \ncong G$. In this paper, we consider finite groups with the same prime graph as the almost simple group $\mathrm{PGL}(2, 49)$. Moreover, we construct some Frobenius groups whose their prime graph coincide with $\Gamma(\mathrm{PGL}(2, 49))$, in particular, we get that $\mathrm{PGL}(2, 49)$ is unrecognizable by prime graph.

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1 Introduction

Let $\mathbb{N}$ denotes the set of natural numbers. If $n \in \mathbb{N}$, then we denote by $\pi(n)$, the set of all prime divisors of n. Let G be a finite group. The set $\pi(|G|)$ is denoted by $\pi(G)$. Also the set of element orders of G is denoted by $\pi_e(G)$. We denote by $\mu(S)$, the maximal numbers of $\pi_e(G)$ under the divisibility relation. The *prime graph* of G is a graph whose vertex set is $\pi(G)$ and two distinct primes p and q are joined by an edge (and we write $p \sim q$), whenever G contains an element of order pq. The prime graph of G is denoted by $\Gamma(G)$. A finite group G is called *unrecognizable by prime graph* if for every finite group H such that $\Gamma(H) = \Gamma(G)$, however $H \ncong G$.

In [10], it is proved that if p is a prime number which is not a Mersenne or Fermat prime and $p \neq 11$, 19 and $\Gamma(G) = \Gamma(\mathrm{PGL}(2, p))$, then G has a unique nonabelian composition factor which is isomorphic to $\mathrm{PSL}(2,p)$ and if $p=13$, then G has a unique nonabelian composition factor which is isomorphic to $\mathrm{PSL}(2,13)$ or $\mathrm{PSL}(2,27)$. In [3], it is proved that if $q=p^\alpha$, where p is a prime and $\alpha>1$, then $\mathrm{PGL}(2,q)$ is uniquely determined by its element orders. Also in [1], it is proved that if $q=p^\alpha$, where p is an odd prime and α is an odd natural number, then $\mathrm{PGL}(2,q)$ is uniquely determined by its prime graph. However, in this paper as the main result we prove that the almost simple group $\mathrm{PGL}(2,49)$ is unrecognizable by prime graph. Also, finally we put a question about the existence of Frobenius groups with the same prime graph as the almost simple groups $\mathrm{PGL}(2,q)$.

2 Preliminary Results

Lemma 2.1. *([17]) Let G be a finite group and $N \trianglelefteq G$ such that G/N is a Frobenius group with kernel F and cyclic complement C. If $(|F|,|N|)=1$ and F is not contained in $NC_G(N)/N$, then $p|C| \in \pi_e(G)$ for some prime divisor p of $|N|$.*

Lemma 2.2. *([8]) Let G be a finite group and $|\pi(G)|\geq 3$. If there exist prime numbers r, s, $t\in\pi(G)$, such that $\{tr,ts,rs\}\cap\pi_e(G)=\emptyset$, then G is non-solvable.*

Lemma 2.3. *([19, Theorem 18.6]) Let G be a nonsolvable Frobenius complement. Then G has a normal subgroup G_0 with $|G:G_0|=1$ or 2 such that $G_0=\mathrm{SL}(2,5)\times M$ with M a Z-group of order prime to 2, 3 and 5.*

Using [14, Theorem A], we have the following result:

Lemma 2.4. *Let G be a finite group with $t(G)\geq 2$. Then one of the following holds:*

(a) G is a Frobenius or 2-Frobenius group;

(b) there exists a nonabelian simple group S such that $S\leq \overline{G}:=G/N\leq Aut(S)$ for some nilpotent normal subgroup N of G.

Lemma 2.5. *([20]) Let $G=L_n^\varepsilon(q)$, $q=p^m$, be a simple group which acts absolutely irreducibly on a vector space W over a field of characteristic p. Denote $H=W\leftthreetimes G$. If $n=2$ and q is odd then $2p\in\pi_e(H)$.*

3 Main Results

Lemma 3.1. *There are infinitely many finite Frobenius group G such that $\Gamma(G)=\Gamma(\mathrm{PGL}(2,49))$.*

Proof. Let F be a finite field of characteristic 7. Also let there are some elements α and β included in F such that $\alpha^2 = -1$ and $\beta^2 = 5$. We know that such a finite filed exists and moreover there are infinitely many filed with these properties.

Now we construct some linear groups as follow:

$$C := \langle \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \alpha & 0 \\ \alpha & \frac{\beta+1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rangle,$$

$$K := \langle \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \rangle.$$

By the above definition, $C \cong \langle x, y, z | x^3 = y^5 = z^2 = 1, x^z = z, y^z = y, (xy)^2 = z \rangle$. This implies that $C \cong \mathrm{SL}(2,5)$. Also we have $K \cong F \oplus F$, is a direct sum of additive group F by itself. This means K is isomorphic to a vector space of dimension 2 over F and so $|K| = |F|^2$. It is obvious that C belongs to the normalizer of K in $\mathrm{GL}(3, F)$.

Now we define $G := K \rtimes C$. Since K is an elementary abelian 7-group, it is easy to prove that C acts fixed point freely on K by conjugation. Hence G is a Frobenius group with kernel K and complement C. This implies that in the prime graph of G, 7 is an isolated vertex. Also by $\Gamma(\mathrm{SL}(2,5))$, we get that 2 is adjacent to 3 and 5 and there is no edge between 3 and 5 in $\Gamma(G)$. Therefore, $\Gamma(G)$ coincides to $\Gamma(\mathrm{PGL}(2,49))$, which completes the proof. □

Lemma 3.2. *Each following group G is an almost simple group related to the simple group S. Moreover, G has a prime graph which coincide with the prime graph of the almost simple group* $\mathrm{PGL}(2,49)$*:*

(1) $G = S_7$ and $S = A_7$.

(2) $G = U_4(3) \cdot 2$ and $S = U_4(3)$.

(3) $G = U_3(5)$ or $G = U_3(5) \cdot 2$ and $S = U_3(5)$.

Proof. Using [4], it is straightforward. □

Theorem 3.3. *Let G be a finite group with the prime graph as same as the prime graph of* $\mathrm{PGL}(2,49)$. *Then G is isomorphic to one of the following groups:*

(1) A Frobenius group $K \rtimes C$, such that K is a 7-group and C contains a subgroup C_0 whose index in C is at most 2 and C_0 is isomorphic to $\mathrm{SL}(2,5)$.

(2) On of the almost simple group: S_7, $U_4(3)\cdot 2$, $U_3(5)\cdot 2$ *or* $\mathrm{PGL}(2,49)$.

(3) The simple group: $U_3(5)$.

In particular, $\mathrm{PGL}(2,49)$ *is unrecognizable by prime graph.*

Proof. By [18, Lemma 7], it follows that $\mu(\mathrm{PGL}(2,49))=\{7,48,50\}$. Hence, the connected components of the prime graph of $\mathrm{PGL}(2,49)$ are exactly $\{7\}$ and $\{2,3,5\}$. Also by $\mu(\mathrm{PGL}(2,49))$, there is no edge between 3 and 5 in $\Gamma(\mathrm{PGL}(2,49))$. Now since $\Gamma(G)=\Gamma(\mathrm{PGL}(2, 49))$, we deduce that these relations hold in the prime graph of G.

First we claim that G is not solvable. On the contrary, let G be a solvable group. So there is a Hall $\{3,5,7\}$-subgroup in G, say H. On the other hand $\{3,5,7\}$ is an independent subset of $\Gamma(G)$, which is a contradiction by Lemma 2.2. Therefore, G is not solvable and so by Lemma 2.4, either G is a Frobenius group or there is a nonabelian simple group S such that $S\leq\bar{G}:=G/\mathrm{Fit}(G)\leq\mathrm{Aut}(S)$.

Let G be a Frobenius group with kernel K and complement C. By Lemma 2.3, we know that K is nilpotent and $\pi(C)$ is a connected component of the prime graph of G. Hence we conclude that $\pi(K)=\{7\}$ and $\pi(C)=\{2,3,5\}$, since 7 is an isolated vertex in $\Gamma(G)$. Hence if C is solvable, then G is a solvable which is a contradiction by the above argument.

Thus we suppose that C is non-solvable. Then by Lemma 2.3, the complement C has a normal subgroup C_0 with index at most 2 which is isomorphic to $\mathrm{SL}(2,5)\times M$, where $\pi(M)\cap\{2,3,5\}=\emptyset$. On the other hand, by the previous argument, we know that $\pi(C)=\{2,3,5\}$. This implies that $M=1$ and so $C_0\cong\mathrm{SL}(2,5)$. Also by Lemma 3.1, we know that this such Frobenius complement exists. Hence G can be isomorphic to a Frobenius group $K:C$, where K is a 7-subgroup and C contains a subgroup isomorphic to $\mathrm{SL}(2,5)$ whose index is at most 2, Therefore if G is a Frobenius group, then we get Case (1).

Now we assume that G is neither Frobenius nor 2-Frobenius group. Hence by Lemma 2.4, there exists a nonabelian simple group S such that:

$$S\leq\overline{G}:=G/K\leq\mathrm{Aut}(S)$$

in which K is the Fitting subgroup of G. Since $\{2,7\}$ is an independent subset of $\Gamma(G)$, by Lemma 2.4, we conclude that $7\in\pi(S)$ and $7\notin\pi(K)\cup\pi(\bar{G}/S)$. Also we know that $\pi(S)\subseteq\pi(G)$. Since $\pi(G)=\{2,3,5,7\}$, so by [13, Table 8], we get that S is isomorphic to A_7, A_8, A_9, A_{10}, $S_6(2)$, $O_8^+(2)$, $L_3(2^2)$, $L_2(2^3)$, $U_3(3)$, $U_4(3)$, $U_3(5)$, $L_2(7)$, $S_4(7)$, $L_2(7^2)$ or J_2. Now we consider each possibility for the simple group S.

Let $S \cong L_2(7)$. Then $5 \in \pi(K)$, since $5 \notin (\pi(S) \cup \pi(\bar{G}/S))$. On the other hand S contains a $\{3,7\}$-subgroup H. Hence G has a subgroup isomorphic to $K_5 : H$ where K_5 is 5-group. On the other hand $K_5 : H$ is solvable and so there is an edge between to prime numbers in $\Gamma(K_5 : H)$, which is impossible since $\Gamma(K_5 : H)$ is a subgraph of $\Gamma(G)$. Thus $S \ncong L_2(7)$.

Let $S \cong L_2(2^3)$. In this case, $5 \in \pi(K)$. Also we know that S contains a Frobenius group isomorphic to $8 : 7$. Hence by Lemma 2.1, we get that G has an element order $5 \cdot 7$, which is a contradiction.

Let $S \cong A_8$, A_9 or A_{10}. Thus S consists an element of order $3 \cdot 5$, which contradicts to the prime graph of G.

Let $S \cong J_2$, $O_8^+(2)$ or $S_6(2)$. In this case S contains an element of order 15, which is a contradiction.

By Lemma 3.2, the finite group S can be isomorphic to each simple group A_7, $U_3(3)$, $U_4(3)$ and $U_3(5)$.

Let S be isomorphic to $\mathrm{PSL}_2(49)$. Hence $\mathrm{PSL}_2(49) \leq \bar{G} \leq \mathrm{Aut}(\mathrm{PSL}_2(49))$.

Let $\pi(K)$ contains a prime r such that $r \neq 7$. Since K is nilpotent, we may assume that K is a vector space over a field with r elements (analogous to the proof of Lemma **??**). Hence the prime graph of the semidirect product $K \rtimes \mathrm{PSL}_2(49)$ is a subgraph of $\Gamma(G)$. Let B be a Sylow 7-subgroup of $\mathrm{PSL}_2(49)$. We know that B is not cyclic. On the other hand $K \rtimes B$ is a Frobenius group such that $\pi(K \rtimes B) = \{r, 7\}$. Hence B should be cyclic which is a contradiction. This implies that $K = 1$, since $7 \notin \pi(K)$.

We know that $\mathrm{Aut}(\mathrm{PSL}_2(49)) \cong Z_2 \times Z_2$. Since in the prime graph of $\mathrm{PSL}_2(49)$ there is not any edge between 7 and 2, we get that $G \ncong \mathrm{PSL}_2(49)$. Also if $G = \mathrm{PSL}_2(49) : \langle\theta\rangle$, where θ is a field automorphism, then we get that 2 and 7 are adjacent in G, which is a contradiction. If $G = \mathrm{PSL}_2(49) : \langle\gamma\rangle$, where γ is a diagonal-field automorphism, then we get that G does not contain any element with order $2 \cdot 7$ (see [3, Lemm 12]), which is contradiction, since in $\Gamma(G)$, $2 \sim 7$. This argument shows that $G \cong \mathrm{PSL}_2(49)$, which completes the proof. □

Problem 3.4. *Let $G = PGL(2,q)$ be an almost simple group related to the simple group $PSL(2,q)$. Find all Frobenius group H such that $\Gamma(H) = \Gamma(G)$.*

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