Renormalized coupling constants for 3D scalar $\lambda\phi^4$ field theory and pseudo-ϵ expansion

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Abstract

Renormalized coupling constants g_{2k} that enter the critical equation of state and determine nonlinear susceptibilities of the system possess universal values g^*_{2k} at the Curie point. They are calculated, along with the ratios $R_{2k} = g_{2k}/g_4^{k-1}$, for the three-dimensional scalar $\lambda\phi^4$ field theory within the pseudo-ϵ expansion approach. Pseudo-ϵ expansions for g^*_6, g^*_8, R^*_6, and R^*_8 are derived in the five-loop approximation, numerical estimates are presented for R^*_6 and R^*_8. The higher-order coefficients of the pseudo-ϵ expansions for the sextic coupling are so small that simple Padé approximants turn out to be sufficient to yield very good numerical results. Their use gives $R^*_6 = 1.650$ while the most recent lattice estimate is $R^*_6 = 1.649(2)$. For the octic coupling pseudo-ϵ expansions are less favorable from the numerical point of view. Nevertheless, Padé–Borel resummation leads in this case to $R^*_8 = 0.890$, the number differing only slightly from the values $R^*_8 = 0.871$, $R^*_8 = 0.857$ extracted from the lattice and field-theoretical calculations.

Key words: *Nonlinear susceptibilities, effective coupling constants, Ising model, renormalization group, pseudo-ϵ expansion*

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The critical behavior of the systems undergoing continuous phase transitions and described by the three-dimensional (3D) Euclidean scalar $\lambda\phi^4$ field theory is characterized by a set of universal parameters including, apart from critical exponents, renormalized effective coupling constants g_{2k} and their ratios $R_{2k} = g_{2k}/g_4^{k-1}$. These ratios enter the small magnetization expansion of free energy and determine, along with renormalized quartic coupling constant g_4, the nonlinear susceptibilities of various orders:

$$F(z,m) - F(0,m) = \frac{m^3}{g_4}\left(\frac{z^2}{2} + z^4 + R_6 z^6 + R_8 z^8 + ...\right), \tag{1}$$

$$\chi_4 = \left.\frac{\partial^3 M}{\partial H^3}\right|_{H=0} = -24\frac{\chi^2}{m^3} g_4, \tag{2}$$

$$\chi_6 = \left.\frac{\partial^5 M}{\partial H^5}\right|_{H=0} = -6!\frac{\chi^3 g_4^2}{m^6}(R_6 - 8), \tag{3}$$

$$\chi_8 = \left.\frac{\partial^7 M}{\partial H^7}\right|_{H=0} = -8!\frac{\chi^4 g_4^3}{m^9}(R_8 - 24R_6 + 96), \tag{4}$$

where $z = M\sqrt{g_4/m^{1+\eta}}$ is a dimensionless magnetization, renormalized mass $m \sim (T - T_c)^\nu$ being the inverse correlation length, χ is a linear susceptibility while χ_4, χ_6, and χ_8 are nonlinear susceptibilities of fourth, sixth, and eighth orders.

For the 3D Ising model or, equivalently, for 3D scalar $\lambda\phi^4$ field theory the nonlinear susceptibilities and the scaling equation of state are intensively studied theoretically. During last four decades renormalized coupling constants g_{2k} and the ratios R_{2k} were evaluated by a number of analytical and numerical methods [1–32]. Estimating the universal critical values g_4^*, g_6^* and R_6^* by means of the field-theoretical renormalization group (RG) approach in physical dimensions has shown that RG technique enables one to get accurate numerical estimates for these quantities. For example, four- and five-loop RG expansions resummed with a help of Borel-transformation-based procedures lead to the values for g_6^* differing from each other by less than 0.5% [15, 16] while the three-loop RG approximation turns out to be sufficient to provide an apparent accuracy no worse than 1.6% [15, 23]. In principle, this is not surprising since the field-theoretical RG approach proved to be highly efficient when used to estimate critical exponents, critical amplitude ratios, marginal dimensionality of the order parameter, etc. for numerous phase transition models [3, 4, 11, 19, 26, 29, 30, 33, 34].

To obtain proper numerical estimates from diverging RG expansions the resummation procedures have to be applied. Most of those being used are based upon the Borel transformation which kills the factorial growth of higher-order coefficients and paves the way to converging iteration schemes. These schemes have enabled to obtain a great number of accurate numerical results for basic models of critical phenomena. There exists, however, alternative technique turning divergent perturbative series into more convenient ones, i. e. into expansions that have smaller lower-order coefficients and much slower growing higher-order ones than those of original series. The method of pseudo-ϵ expansion invented by B. Nickel (see Ref. 19 in the paper of Le Guillou and Zinn-Justin [4]) is meant here. This approach has been shown to be very efficient numerically when used to evaluate critical exponents and other universal quantities of various 3D and 2D systems [4, 32, 35–44].

In this paper, we study renormalized effective coupling constants and universal ratios R_{2k} of the 3D scalar $\lambda\phi^4$ field theory with the help of pseudo-ϵ expansion technique. The pseudo-ϵ expansions (τ-series) for the universal values of renormalized coupling constants g_6 and g_8 will be calculated on the base of five-loop RG expansions obtained by Guida and Zinn-Justin [16]. Along with the sextic and octic coupling constants, universal critical values of ratios $R_6 = g_6/g_4^2$ and $R_8 = g_8/g_4^3$ will be found as series in τ up to τ^5 terms. The pseudo-ϵ expansions obtained will be processed by means of Padé and Padé–Borel resummation techniques as well as by direct summation when it looks reasonable. The numerical estimates for the universal ratios will be compared with the values extracted from the higher-order ϵ-expansions, from the perturbative RG expansions in three dimensions, and from the lattice calculations and some conclusions concerning the numerical power of the pseudo-ϵ expansion approach will be formulated.

So, the Hamiltonian of the model under consideration reads:

$$H = \int d^3x \left[\frac{1}{2}(m_0^2\varphi_\alpha^2 + (\nabla\varphi_\alpha)^2) + \frac{\lambda}{24}(\varphi_\alpha^2)^2\right], \tag{5}$$

where φ_α is a real scalar field, bare mass squared m_0^2 being proportional to $T - T_c^{(0)}$, $T_c^{(0)}$ – mean field transition temperature. The β-function for the model (3) has been calculated within the massive theory [3, 45] with the propagator, quartic vertex and φ^2 insertion normalized in a conventional way:

$$\begin{aligned} G_R^{-1}(0, m, g_4) &= m^2, \qquad & \frac{\partial G_R^{-1}(p, m, g_4)}{\partial p^2}\bigg|_{p^2=0} &= 1, \\ \Gamma_R(0, 0, 0, m, g) &= m^2 g_4, \qquad & \Gamma_R^{1,2}(0, 0, m, g_4) &= 1. \end{aligned} \tag{6}$$

Later, the five-loop RG series for renormalized coupling constants g_6 and g_8 of this model were obtained [16] and the six-loop pseudo-ϵ expansion for the Wilson fixed point location was reported [32]:

$$g_6 = \frac{9}{\pi} g_4^3 \Big(1 - \frac{3}{\pi} g_4 + 1.38996295 g_4^2 - 2.50173246 g_4^3 + 5.275903\ g_4^4\Big), \tag{7}$$

$$g_8 = -\frac{81}{2\pi} g_4^4 \Big(1 - \frac{65}{6\pi} g_4 + 7.77500131 g_4^2 - 18.5837685 g_4^3 + 48.16781 g_4^4\Big), \tag{8}$$

$$g_4^* = \frac{2\pi}{9}\Big(\tau + 0.4224965707\tau^2 + 0.005937107\tau^3 + 0.011983594\tau^4 - 0.04123101\tau^5 + 0.0401346\tau^6\Big). \tag{9}$$

Combining these expansions one can easily arrive to the τ-series for the values of the coupling constants g_6 and g_8 at the critical point:

$$g_6^* = \frac{8\pi^2}{81}\tau^3(1 + 0.600823045\tau + 0.104114939\tau^2 - 0.023565414\tau^3 - 0.01838783\tau^4) \tag{10}$$

$$g_8^* = -\frac{8\pi^3}{81}\tau^4(1 - 0.717421125\tau - 0.201396988\tau^2 - 0.70623903\tau^3 + 0.8824349\tau^4) \tag{11}$$

Corresponding pseudo-ϵ expansions for the universal ratios are as follows:

$$R_6^* = 2\tau(1 - 0.244170096\tau + 0.120059430\tau^2 - 0.1075143\tau^3 + 0.1289821\tau^4). \tag{12}$$

$$R_8^* = -9\tau(1 - 1.98491084\tau + 1.76113570\tau^2 - 1.9665851\tau^3 + 2.741546\tau^4). \tag{13}$$

These τ-series will be used for evaluation of renormalized effective couplings near the Curie point.

First, let us find the numerical value of the ratio R_6 at criticality. Since the pseudo-ϵ expansion (12) has small higher-order coefficients a direct summation of this series looks quite reasonable. Within third, fourth and fifth orders in τ it gives 1.752, 1.537 and 1.795 respectively. These numbers certainly group around the estimates 1.644 and 1.649 extracted from advanced field-theoretical and lattice calculations [16, 31]. It is interesting that the value 1.537 obtained by truncation of the series (12) by the smallest term (optimal truncation [41]) differs from the estimates just mentioned by 6% only. Moreover, direct summation of τ-series for g_6^* (10) having very small higher-order coefficients gives the value $g_6^* = 1.621$ which under $g_4^* = 0.9886$ [3] results in the estimate $R_6^* = 1.659$ looking rather optimistic. This fact confirms the conclusion that the pseudo-ϵ expansion itself may be considered as some specific resummation method [32, 41–44].

Much more accurate numerical value of R_6^* can be obtained from the pseudo-ϵ expansion (12) using Padé approximants [L/M]. Padé triangle for R_6^*/τ, i. e. with the insignificant factor τ neglected is presented in Table I. Along with the numerical values given by various Padé approximants the rate of convergence of Padé estimates to the asymptotic value is shown in this Table (the lowest line, RoC). Since the diagonal and near-diagonal Padé approximants are known to possess the best approximating properties the Padé estimate of k-th order is accepted to be given by the diagonal approximant or by the average over two near-diagonal ones when corresponding diagonal approximant does not exist. As seen from Table I, the convergence of Padé estimates is well pronounced and the asymptotic value of R_6^* equals 1.6502. This number is close to the higher-order ϵ expansion estimate $R_6^* = 1.690 \pm 0.04$ [16], to the five-loop 3D RG estimate $R_6^* = 1.644 \pm 0.006$ [16] and, in particular, to the value $R_6^* = 1.649 \pm 0.002$ given by the advanced lattice calculations [31].

It is worthy to note that an account for the five-loop terms in τ-series for R_6^* and g_6^* shifts the numerical value of the universal ratio only slightly. Indeed, the Padé resummation of the four-loop τ-series for R_6^* and g_6^* result in $R_6^* = 1.642$ and $R_6^* = 1.654$, respectively [32]. This may be considered as a manifestation of the fact that the τ-series for the sextic effective coupling have a structure rather favorable from the numerical point of view. It is especially true for the pseudo-ϵ expansion (12) which, having small higher-order coefficients, is alternating what makes this series very convenient for getting numerical estimates.

For the renormalized octic coupling we have pseudo-ϵ expansions with much less favorable structure. The series for R_8^* (13) being alternating possesses rather big coefficients making a direct summation certainly useless in this case. To estimate the ratio R_8^* Padé resummation procedure should be applied. The coefficients of the τ-expansion for g_8^* (11) are considerably smaller than those of the series (13) but have irregular signs. This series may be also processed, in principle, within the technique mentioned. The numerical results thus obtained, however, turn out to be so strongly scattered that they could not be referred to as satisfactory or even meaningfull. That is why further we will concentrate on the Padé resummation of the τ-series for R_8^*.

We construct Padé approximants for the ratio R_8^*/τ neglecting, as in the case of R_6^*, the insignificant factor τ. Corresponding Padé triangle is presented in Table II. As is seen, in the case of octic coupling numerical estimates turn out to be much worse than those obtained for R_6^*. Indeed, the numbers given by Padé resummed pseudo-ϵ expansion (13) are strongly scattered. Moreover, even the estimates extracted from the highest-order available – five-loop – τ-series differ from each other considerably. On the other hand, the optimal value of R_8^*, i. e. that given by the diagonal approximant [2/2] is close to the 3D RG estimate $R_8^* = 0.857 \pm 0.086$ [16] and to the result of recent lattice calculations $R_8^* = 0.871 \pm 0.014$ [31]. It implies that use of more powerful resummation techniques may lead to acceptable results.

One of such techniques effectively suppressing divergence of the series being resummed is the Padé–Borel machinery. The results of resummation of pseudo-ϵ expansion (13) with the help of this method are presented in Table III. As is seen, use of Padé–Borel technique makes numerical estimates much less scattered and markedly accelerates the iteration procedure. Moreover, the asymptotic value 0.890 the iterations result in only slightly differs from the high-precision estimates mentioned above. This shows that the pseudo-ϵ expansion technique remains workable when employed to estimate the universal value of the higher-order coupling constants.

At the same time, the question arises: what is the origin of the poor approximating properties of the τ-series for g_8^* and R_8^*? Of course, pronounced divergence of these pseudo-ϵ expansions may be thought of as a main source of such a misfortune. There exists, however, an extra moment making the situation rather unfavorable. The point is that the series (11), (13) have some specific feature. Namely, their first terms are negative and big in modulo while the estimates these series imply turn out to be an order of magnitude smaller and have an opposite sign. It means that numerical values resulting from (11), (13) are nothing but small differences of big numbers what drastically lowers an approximating power of these τ-series.

So, we have calculated pseudo-ϵ expansions for the universal values of renormalized coupling constants g_6, g_8 and of the ratios R_6, R_8 for 3D Euclidean scalar $\lambda\phi^4$ field theory. Numerical estimates for R_6^* and R_8^* have been found using Padé and Padé–Borel resummation techniques. The pseudo-ϵ expansion machinery has been shown to lead to high-precision value of R_6^* which is in very good agreement with the numbers obtained by means of other methods including lattice calculations. For the octic coupling this technique has been found to be less efficient. However the numbers extracted from τ-series for R_8^* by means of the Padé–Borel resummation turn out to be rather close to high-precision lattice and field-theoretical estimates. These results confirm a conclusion that the pseudo-ϵ expansion approach may be referred to as a specific resummation method converting divergent RG series into the expansions very convenient from the numerical point of view.

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TABLE I: Padé table for pseudo-ϵ expansion of the ratio R_6^*. Padé approximants [L/M] are derived for R_6^*/τ, i. e. with factor τ omitted. The lowest line (RoC) shows the rate of convergence of Padé estimates to the asymptotic value. Here the Padé estimate of k-th order is that given by diagonal approximant or by the average over two near-diagonal ones when corresponding diagonal approximant does not exist. The values of R_6^* resulting from the 3D RG analysis [16] and recent lattice calculations [31] are equal to 1.644 ± 0.006 and 1.649 ± 0.002 respectively.

$M \setminus L$	0	1	2	3	4
0	2	1.5117	1.7518	1.5368	1.7947
1	1.6075	1.6726	1.6383	1.6540	
2	1.6896	1.6465	1.6502		
3	1.6036	1.6504			
4	1.7135				
RoC	2	1.5596	1.6726	1.6424	1.6502

TABLE II: Padé triangle for pseudo-ϵ expansion of the ratio R_8^*. Padé approximants [L/M] are constructed for R_8^*/τ, i. e. with factor τ omitted. The lowest line (RoC) demonstrates the rate of convergence of Padé estimates where Padé estimate of k-th order is that given by diagonal approximant or by the average over two near-diagonal ones if diagonal approximant does not exist.

$M \setminus L$	0	1	2	3	4
0	-9	8.864	-6.986	10.713	-13.961
1	-3.015	0.466	1.376	0.407	
2	-1.743	1.910	0.879		
3	-1.131	0.095			
4	-0.831				
RoC	-9	2.925	0.466	1.643	0.879

TABLE III: The values of universal ratio R_8 obtained by means of Padé–Borel resummation of the series (13). Padé approximants [L/M] are employed for analytical continuation of the expansion of Borel transform of R_8^*/τ in powers of τ. The lowest line (RoC) demonstrates the rate of convergence of Padé–Borel estimates to the asymptotic value. The estimate of k-th order is that found using corresponding diagonal approximant or the average over two values given by approximants [M/M−1] [M−1/M] when the diagonal approximant does not exist. The values of R_8^* resulting from 3D RG analysis [16] and extracted from advanced lattice calculations [31] are equal to 0.857 ± 0.086 and 0.871 ± 0.014 respectively.

$M \setminus L$	1	2	3	4	5
0	−9	8.8642	−6.9860	10.7132	−13.9607
1	−3.6473	1.1000	0.8854	0.8906	
2	−2.4658	0.9041	0.8905		
3	−2.0048	0.8916			
4	−1.7788				
RoC	−9	2.608	1.1000	0.8948	0.8905