Quantum speed limit for noisy dynamics

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The laws of quantum physics place a limit on the speed of computation, in particular the evolution time of a system cannot be arbitrarily fast. Bounds on the speed of evolution for unitary dynamics have been long studied. A few bounds on the speed of evolution for noisy dynamics have also been obtained recently, these bounds, however, are in general not tight. In this article we present a new framework for quantum speed limit of noisy dynamics. With this framework we obtain the exact maximal angle that a noisy dynamics can achieve at any given time, this then provides tight bounds on the evolution time for noisy dynamics. The obtained bound reveals that noisy dynamics are generically different from unitary dynamics, in particular we show that the 'orthogonalization' time, which is the minimum time needed to evolve any state to its orthogonal states, is in general not applicable to noisy dynamics.

Quantum information processing may be regarded as the transformation of quantum states encoding the information to be processed or computed. The time for which the states transform dictates the speed of the quantum computation. Quantum physics imposes a limit on the transformation time. This quantum speed limit (QSL) [1] arises because the energy of the system and environment is finite and the system state may evolve according to a slow dynamics.

The transformation of quantum states may be measured by the Bures angle which characterizes the distance between two states. In a time period t, a quantum process may rotate an initial state by an angle θ. In QSL, the reverse question is asked. Given a certain angle θ, we ask what is the minimum time t needed to rotate any state by θ. The first major result of QSL was by Mandelstam and Tamm [2]. Since then, many results on unitary dynamics followed (see e.g. [3–17]), while studies on noisy dynamics are only carried out quite recently [18–26]. A few lower bounds have been obtained for the evolution time of noisy dynamics [18–20], these bounds, however, are in general not achievable. In this article we present a new framework for QSL of noisy dynamics. While previous studies mostly focus on the rotating speed of a given state under a dynamics, here we study the maximal speed of evolution that a dynamics can generate on all quantum states, which requires an optimization over all states. The obtained speed of evolution represents the limit of the quantum speed that a given dynamics can possibly induce on any quantum states, which is then a fundamental limit of the dynamics and can be used to provide bounds on the computation speed of a quantum device. While the QSL on a fixed state tells little about the ability of a dynamics to rotating the states in general, the maximal speed of evolution provides a way to gauge the dynamics.

Our framework is based on a method that gives the exact maximal rotation angle for a given dynamics, which ensures that the bound is achievable. And the bound is obtained directly from the Kraus operators of the dynamics, which allows ease of computation. The obtained bound reveals that noisy dynamics are generically different from unitary dynamics, in particular we show that the 'orthogonalization' time, a concept heavily used in QSL, is in general not applicable to noisy dynamics.

Our framework builds on a distance measure on quantum channels [31] which we describe briefly. For a $m \times m$ unitary matrix U, we denote by $e^{-i\theta_j}$ the eigenvalues of U, where $\theta_j \in (-\pi, \pi]$ for $1 \leq j \leq m$ and we call θ_j the eigen-angles of U. We define(see also[27–29]) $\| U \|_{\max} = \max_{1\leq j\leq m} |\theta_j|$, and $\| U \|_g$ as the minimum of $\| e^{i\gamma} U \|_{\max}$ over equivalent unitary operators with different global phases, i.e., $\| U \|_g = \min_{\gamma\in\mathbb{R}} \| e^{i\gamma} U \|_{\max}$. We then define

$$C_\theta(U) = \begin{cases} \| U \|_g, & if \| U \|_g \leq \frac{\pi}{2}, \\ \frac{\pi}{2}, & if \| U \|_g > \frac{\pi}{2}. \end{cases} \quad (1)$$

Essentially $C_\theta(U)$ represents the maximal angle that U can rotate a state away from itself[29, 31], i.e., $\cos[C_\theta(U)] = \min_\rho F_B(\rho, U\rho U^\dagger)$, where $F_B(\rho_1, \rho_2)$ is the fidelity between two states $F_B(\rho_1, \rho_2) = Tr\sqrt{\rho_1^{\frac{1}{2}} \rho_2 \rho_1^{\frac{1}{2}}}$.

If $\theta_{\max} = \theta_1 \geq \theta_2 \geq \cdots \geq \theta_m = \theta_{\min}$ are arranged in decreasing order, then $C_\theta(U) = \frac{\theta_{\max}-\theta_{\min}}{2}$ when $\theta_{\max} - \theta_{\min} \leq \pi$[29]. A metric on unitary operators can be induced by $C_\theta(U)$ as $d(U_1, U_2) = C_\theta(U_1^\dagger U_2)$; $d(U_1, U_2)$ represents the maximal angle U_1 and U_2 can generate on the same input state, $\cos d(U_1, U_2) = \min_\rho F_B(U_1\rho U_1^\dagger, U_2\rho U_2^\dagger)$[31].

This metric can be generalized to noisy dynamics as $d(K_1, K_2) = \min_{U_{ES2}} d(U_{ES1}, U_{ES2})$, where U_{ES1} and U_{ES2} are unitary extensions of K_1 and K_2 respectively. And $d(K_1, K_2)$ can be computed from the Kraus operators of K_1 and K_2 as $d(K_1, K_2) = \arccos \max_{\|W\|\leq 1} \frac{1}{2}\lambda_{\min}(K_W + K_W^\dagger)$[31], where $\lambda_{\min}(K_W + K_W^\dagger)$ denotes the minimum eigenvalue of $K_W + K_W^\dagger$ with $K_W = \sum_{j=1}^{D}\sum_{i=1}^{D} w_{ij} F_{1i}^\dagger F_{2j}$, here F_{1i} and F_{2j}, $1 \leq i, j \leq D$, denote the Kraus operators of K_1 and K_2 respectively, w_{ij} denotes the ij-th entry of a $D \times D$ matrix W with $\|W\| \leq 1$($\|\cdot\|$ is the operator norm which is equal to the maximum singular value). $d(K_1, K_2)$ represents the maximal angle that $K_1 \otimes I_A$ and $K_2 \otimes I_A$ can generate with the same input state, $\cos d(K_1, K_2) = \min_{\rho_{SA}} F_B[K_1 \otimes I_A(\rho_{SA}), K_2 \otimes I_A(\rho_{SA})]$, where ρ_{SA} is a state of system+ancilla and I_A denotes the identity operator on the ancillary system[31]. Furthermore this distance can be efficiently calculated via semi-definite programming as $\max_{\|W\|\leq 1} \frac{1}{2}\lambda_{\min}(K_W + K_W^\dagger) =$

$$\begin{aligned} &max && \frac{1}{2}t \\ &s.t. && \begin{pmatrix} I & W^\dagger \\ W & I \end{pmatrix} \succeq 0, \\ & && K_W + K_W^\dagger - tI \succeq 0. \end{aligned} \tag{2}$$

And the dual semi-definite programming provides a way to find the optimal state[32]: $\max_{\|W\|\leq 1} \frac{1}{2}\lambda_{\min}(K_W + K_W^\dagger) =$

$$\begin{aligned} &min && \frac{1}{2}Tr(P) + \frac{1}{2}Tr(Q) \\ &s.t. && \begin{pmatrix} P & M^\dagger(\rho_S) \\ M(\rho_S) & Q \end{pmatrix} \succeq 0, \\ & && \rho_S \succeq 0, \\ & && Tr(\rho_S) = 1, \end{aligned} \tag{3}$$

where P, Q are Hermitian matrices and $M(\rho_S)$ is a $D \times D$ matrix with its ij-th entry equals to $Tr(\rho_S F_{1i}^\dagger F_{2j})$. The optimal state is any pure state ρ_{SA} with $Tr_A(\rho_{SA}) = \rho_S$, where ρ_S is obtained from the above semi-definite programming [32].

The metric can be used to obtain a saturable bound for QSL: for a dynamics $K_t(\rho) = \sum_i F_i(t)\rho F_i^\dagger(t)$, suppose it takes t units of time for the dynamics to rotate a state, possibly entangled with an ancillary system, with an angle θ. Then $\theta = \arccos F_B[\rho_{SA}, K_t \otimes I_A(\rho_{SA})] \leq d(I, K_t)$, a lower bound on the minimum time can then be obtained by this inequality where the equality can be saturated when ρ_{SA} takes the optimal input state. When ρ_{SA} is restricted to separable states, the maximal rotation speed is reduced to the case without ancillary system, which is in general slower. $d(I, K_t)$ thus provides a limit on the maximal angle that a given dynamics can generate on any state at time t.

Unitary dynamics. For unitary dynamics $U_t = e^{-iHt}$, suppose it takes t units of time to rotate a state ρ with angle $\theta \in [0, \frac{\pi}{2}]$, then $\theta \leq d(I, U_t) = \frac{(E_{\max} - E_{\min})t}{2}$, here $E_{\max}(E_{\min})$ denote the maximum(minimum) eigenvalue of H. The minimum time needed to rotate a state away with angle θ is then bounded by $t \geq \frac{2\theta}{E_{\max} - E_{\min}}$. This recovers previous results on the quantum speed limit for unitary dynamics[7]. This bound is also known to be saturable with the input state $|\varphi\rangle = \frac{1}{\sqrt{2}}(|E_{\max}\rangle + e^{i\phi}|E_{\min}\rangle)$, which can always be rotated to an orthogonal state at time $t = \frac{\pi}{E_{\max} - E_{\min}}$.

Here $E_{\max} - E_{\min}$ can be seen as the energy scale of the system, $d(I, U_t)$ is thus proportional to the multiplication of the energy scale and time, the maximal angle that can be rotated is thus proportional to the time-energy cost of the dynamics[27–30]. For noisy dynamics as $d(I, K_t) = \min_{U_{ESt}}(I_{ES}, U_{ESt})$ where U_{ESt} is unitary extension of K_t, the maximal angle is thus proportional to the minimum time-energy cost over all unitary extensions of the noisy dynamics[27–30].

Amplitude damping. Consider the dynamics with amplitude damping $K_t(\rho) = F_{11}(t)\rho F_{11}^\dagger(t) + F_{12}(t)\rho F_{12}^\dagger(t)$, where the Kraus operators $F_{11}(t) = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{P(t)} \end{bmatrix}$, $F_{12}(t) = \begin{bmatrix} 0 & \sqrt{1 - P(t)} \\ 0 & 0 \end{bmatrix}$, here $P(t) = e^{-\gamma t}$ where γ is the decay rate. Suppose it takes t units of time for the dynamics to rotate a state ρ_{SA} with angle $\theta \in [0, \frac{\pi}{2}]$, here ρ_{SA} represents a state of the system+ancilla, then $\theta = \arccos F_B[\rho_{SA}, K_t \otimes I_A(\rho_{SA})] \leq \arccos \min_{\rho_{SA}} F_B[\rho_{SA}, K_t \otimes I_A(\rho_{SA})] = d(I, K_t)$. In the appendix we showed that $\cos d(I, K_t) = \max_{\|W\|\leq 1} \frac{1}{2}\lambda_{\min}(K_W + K_W^\dagger) = \sqrt{P(t)}$[34]. As $\theta \leq d(I, K_t)$, we have $\cos\theta \geq \cos d(I, K_t) = \sqrt{P(t)}$, which gives $t \geq \frac{2}{\gamma}\ln\sec\theta$. This provides a lower bound on the minimum time needed to rotate any state with angle θ and it is consistent with previous results[18]. We note that in this case in order to rotate a state to an orthogonal state, infinite time is needed as $\ln\sec\frac{\pi}{2} \to \infty$, this corresponds to the case where the initial state is the excited state $|1\rangle$ and it only decays completely to the ground state $|0\rangle$ with infinite amount of time.

For non-Markovian dynamics, due to the strong couplings to the environment, the decay rate $\gamma_{nM}(t)$, which is usually time-dependent, can be bigger than the decay rate in the Markovian regime[20]. In this case $P(t) = e^{-\int_0^t \gamma_{nM}(\tau)d\tau}$ where $\int_0^t \gamma_{nM}(\tau)d\tau$ is usually bigger than γt in the Markovian case, thus for the same duration the maximal angle $d(I, K_t) = \arccos\sqrt{P(t)}$ can be bigger in the non-Markovian regime than the Markovian regime. This was explored in previous study to show that non-Markovian dynamics can speed up the rotation[20]. We note that even in the non-Markovian regime as long as $\gamma_{nM}(t)$ is finite, it always needs infinite amount time for $P(t)$ to reach 0, thus it always needs infinite amount of time to achieve a $\pi/2$-rotation.

Dynamics with dephasing noises. Let $K_t(\rho) = F_{11}(t)\rho F_{11}^\dagger(t) + F_{12}(t)\rho F_{12}^\dagger(t)$ as the dynamics with dephasing noises, with the Kraus

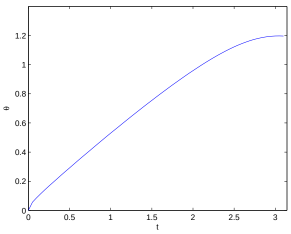

FIG. 1: The maximal angle that can be rotated at different time t with dephasing noises, with $\gamma = 0.1, \omega = 1$.

operators $F_{11}(t) = \sqrt{\frac{1+P(t)}{2}}\begin{bmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2}\end{bmatrix}$, $F_{12}(t) = \sqrt{\frac{1-P(t)}{2}}\begin{bmatrix} e^{-i\omega t/2} & 0 \\ 0 & -e^{i\omega t/2}\end{bmatrix}$, here $P(t) = e^{-\gamma t}$ where γ is the dephasing rate. Again suppose it takes t units of time for the dynamics to rotate a state ρ_{SA} with angle $\theta \in [0, \frac{\pi}{2}]$, then $\theta \leq d(I, K_t)$. In the appendix[34] we showed that $\cos d(I, K_t) = \sqrt{\frac{1+P(t)\cos(\omega t)}{2}}$. Then from $\cos\theta \geq \cos d(I, K_t)$,we obtain the minimum time needed to rotate a state with angle θ, which is plotted in Fig.1.

In this case as long as $\gamma > 0$, $\cos d(I, K_t) = \sqrt{\frac{1+e^{-\gamma t}\cos(\omega t)}{2}} > 0$, thus $d(I, K_t) < \pi/2$, i.e., the dynamics can not rotate any state to its orthogonal states. This is a much stronger statement than previous study in [18] where it stated that only when $\frac{\omega}{\gamma} > r_{crit} \approx 2.6$ the dynamics can not rotate any state to its orthogonal states. Such difference arises as the previous bound is obtained from the integration of the quantum Fisher metric along the path $\rho_t = K_t \otimes I_A(\rho_{SA})$. Such path is fixed by the dynamics and usually not the geodesic between the initial state and the final state. As a result the integration of the quantum Fisher metric along the path is in general bigger than the actual distance between the initial state and the final state, which in turn leads to a looser bound and inaccurate classifications for noisy dynamics. The bound obtained in [19] for dephasing dynamics is also not tight, which resulted a finite orthogonalization time. As a contrast the bound obtained here is tight and can be saturated with the input state $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$, ancillary system is not needed to saturate the bound in this case.

Generic noisy dynamics. We show that generically noisy dynamics can not rotate any state to its orthogonal states. First we show that for a given dynamics $K_t(\rho) = \sum_{i=1}^D F_i(t)\rho F_i^\dagger(t)$, if $I \in \{F_1(t), F_2(t), \cdots, F_D(t)\}$, then K_t can not rotate any state to its orthogonal states, i.e., if identity matrix belongs to the span space of the Kraus operators then $d(I, K_t)$ is always smaller than $\frac{\pi}{2}$. As if $I \in \{F_1(t), F_2(t), \cdots, F_D(t)\}$, then there exists w_{1i} such that $I = \sum_{i=1}^D w_{1i}F_i(t)$. Now let $\alpha = 1/\sqrt{\sum_{i=1}^D |w_{1i}|^2} > 0$, then $\alpha I = \sum_{i=1}^D w'_{1i}F_i(t)$ where $w'_{1i} = \alpha w_{1i}$. Define W' as a $D \times D$ matrix with the entries of the first row equal to w'_{1i} and other entries equal to 0. It is obvious that $\|W'\| = 1$, thus

$$\begin{aligned}\cos d(I, K_t) &= \max_{\|W\|\leq 1} \frac{1}{2}\lambda_{\min}(K_W + K_W^\dagger) \\ &\geq \frac{1}{2}\lambda_{\min}(K_{W'} + K_{W'}^\dagger) \\ &= \frac{1}{2}\lambda_{\min}[\sum_{i=1}^D w'_{1i}F_i(t) + (\sum_{i=1}^D w'_{1i}F_i(t))^\dagger] \\ &= \alpha > 0,\end{aligned} \tag{4}$$

so $d(I, K_t) \leq \arccos\alpha < \pi/2$, i.e., the dynamics can not rotate any state to its orthogonal states.

For example with the dephasing noises, we have $F_{11}(t) = \sqrt{\frac{1+P(t)}{2}}\begin{bmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2}\end{bmatrix}$, $F_{12}(t) = \sqrt{\frac{1-P(t)}{2}}\begin{bmatrix} e^{-i\omega t/2} & 0 \\ 0 & -e^{i\omega t/2}\end{bmatrix}$, with $P(t) = e^{-\gamma t}$ where γ is the dephasing rate. In this case $I = \sqrt{\frac{2}{1+P(t)}}\cos(\omega t/2)F_{11}(t) + i\sqrt{\frac{2}{1-P(t)}}\sin(\omega t/2)F_{12}(t)$, then

$$\begin{aligned}\alpha &= \frac{1}{\sqrt{\frac{2}{1+P(t)}\cos^2(\omega t/2) + \frac{2}{1-P(t)}\sin^2(\omega t/2)}} \\ &= \frac{\sqrt{1-P^2(t)}}{\sqrt{2-2P(t)\cos(\omega t)}},\end{aligned} \tag{5}$$

which is positive for any $P(t) < 1$. Thus at the presence of dephasing noises, $d(I, K_t) \leq \arccos\alpha < \pi/2$.

This fact can also be easily seen from the equivalent representations of the Kraus operators: when $I \in \{F_1(t), F_2(t), \cdots, F_D(t)\}$ then there exists an equivalent representation of Kraus operators such that αI is one of them, then the fidelity between the initial state and final state will be at least α, thus such dynamics can not rotate any state to its orthogonal state. Our bound not only captures this fact, but can also provide tighter bound by exploring different choices of W. For example with the dephasing noises, the choice of $W = \begin{bmatrix} \frac{\sqrt{1+P(t)}\cos(\omega t/2)}{\sqrt{1+P(t)\cos(\omega t)}} & \frac{i\sqrt{1-P(t)}\sin(\omega t/2)}{\sqrt{1+P(t)\cos(\omega t)}} \\ 0 & 0 \end{bmatrix}$ leads to the tight bound. It is also easy to see that if the span of the Kraus operators contains any matrix M, such that $\lambda_{\min}(M + M^\dagger) > 0$, the above argument still holds thus the dynamics can not rotate any state to its orthogonal states. For example in the case of the amplitude damping, the span of the Kraus operators contains $M = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{P(t)}\end{bmatrix}$, which satisfies the condition as $\lambda_{\min}(M + M^\dagger) = 2\sqrt{P(t)} > 0$ except at the infinite time point $P(\infty) = 0$.

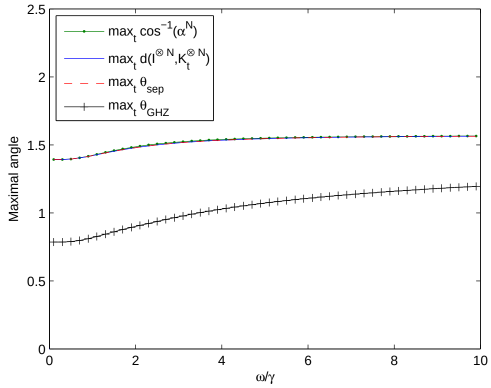

FIG. 2: The maximal angle that can be rotated by the composite dynamics with dephasing noises, and its lower and upper bounds at different values of $\frac{\omega}{\gamma}$. The curves are obtained by finding the optimal time t that gives the maximal angle for the separable state, $d(I^{\otimes^N}, K_t^{\otimes N})$ and $\arccos(\alpha^N)$ with $N = 5$. It can be seen that $\max_t \theta_{sep}$ and $\max_t \arccos(\alpha^N)$ provides pretty tight bounds for $\max_t d(I^{\otimes^N}, K_t^{\otimes N})$. The maximal angle that can be achieved with the GHZ state is also plotted for comparison.

One immediate implication is that all dynamics with full rank, i.e., with the Kraus operators span the whole space(or equivalently the number of linearly independent Kraus operators $d = n^2$ where n denotes the dimension of the system), can not rotate any state to its orthogonal states. Such dynamics is generic among all completely positive trace preserving maps, thus generically noisy dynamics can not rotate any state to its orthogonal states.

Composite systems. Given a noisy dynamics $K_t(\rho) = \sum_{j=1}^{D} F_j(t)\rho F_j^\dagger(t)$, suppose there are N such dynamics, which we denote as $K_t^{\otimes^N}$, acts independently on a composite systems. In the supplemental material[34] we showed that if $I \in \{F_1(t), F_2(t), \cdots, F_D(t)\}$, then $d(I^{\otimes^N}, K_t^{\otimes N}) \leq \arccos(\alpha^N) < \pi/2$, here α is the same α used in Eq.(4), thus in this case any state of the composite system can not be rotated to its orthogonal states.

For the dephasing case, by substituting the value of α in Eq.(5), one can immediate get an upper bound for $d(I^{\otimes^N}, K_t^{\otimes N})$. One can also get a lower bound for $d(I^{\otimes^N}, K_t^{\otimes N})$ by taking the input state as the separable state $|+\cdots+\rangle$, where $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, it is easy to get the rotated angle on this state which is $\theta_{sep} = \arccos(\beta^N)$ with $\beta = \sqrt{\frac{1+e^{-\gamma t}\cos(\omega t)}{2}}$. We thus get $\arccos(\beta^N) \leq d(I^{\otimes^N}, K_t^{\otimes N}) \leq \arccos(\alpha^N)$. If we maximize over t, then $\max_t \arccos(\beta^N) \leq \max_t d(I^{\otimes^N}, K_t^{\otimes N}) \leq \max_t \arccos(\alpha^N)$, which bounds the maximal angle that can be rotated on composited systems. In Fig.2, we plotted these bounds and the exact maximal angle for the composite systems with dephasing noises which shows that these bounds are quite tight.

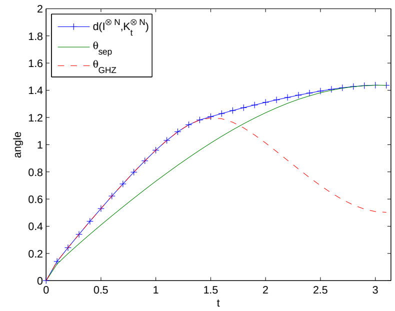

FIG. 3: Rotation angle of the composite dynamics with dephasing noises on GHZ state and the separable state, compared with the maximal angle $d(I^{\otimes^N}, K_t^{\otimes N})$, with $\gamma = 0.1, \omega = 1, N = 2$.

For composite system the GHZ state, $(|0\cdots0\rangle + |1\cdots1\rangle)/\sqrt{2}$, is usually used as a benchmark for the QSL[18, 19]. The rotation angle on the GHZ state can be explicitly computed as $\cos\theta_{GHZ} = \sqrt{\frac{1+e^{-N\gamma t}\cos(N\omega t)}{2}}$. It can be seen from Fig.3 that while for small t, i.e., when the noise effect is still small, the GHZ state achieves the maximal speed of evolution, for big t GHZ state is no longer the optimal state that achieves the maximal angle, in fact GHZ state can be even worse than the separable state. This is clearly seen in Fig.4, where we plotted the entanglement for the optimal state that saturates $d(I^{\otimes^2}, K_t^{\otimes 2})$. It clearly shows that the maximally entangled state is only optimal when t is smaller than a threshold, when t is bigger than the threshold, the optimal state that achieves maximal rotated angle gradually changes from the maximally entangled state to separable states. From Fig.2 we can also see that the maximal angle on the GHZ state is far below the maximal angle on the separable state. This is because the maximal angle on the GHZ state actually does not change with N(it only shortens the optimal time achieving the maximal angle by N times), i.e., $\max_t \theta_{GHZ} = \max_t \arccos\beta$ with $\beta = \sqrt{\frac{1+e^{-\gamma t}\cos(\omega t)}{2}}$, while $\max_t \theta_{sep} = \max_t \arccos(\beta^N)$ increases with N. If we take the rotated angle as the degenerate effect under the noisy dynamics, this also means that while the GHZ state deteriorates fast under the dephasing noises at short time, in the long time the entanglement in the GHZ state mitigates the maximal degeneration.

In summary we presented a new framework to calculate the exact maximal rotation angle for any noisy dynamics in a given evolution time, which provided tight bounds for QSL of noisy dynamics. With this we showed that the commonly used concept in QSL, the 'orthogonalization' time, is in general not applicable to noisy dynamics. It is also shown that although maximally entangled states, such as the GHZ state, evolves faster at short time, they are in general not the optimal states that lead to the

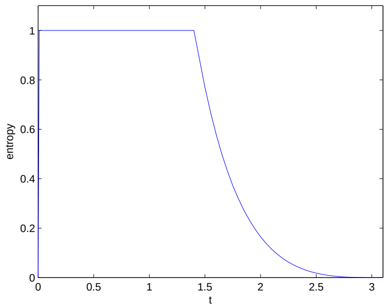

FIG. 4: The entanglement of the optimal input state which achieves the maximal rotated angle under the composite dynamics with dephasing noises($\gamma = 0.1, \omega = 1, N = 2$).

maximal rotation angle under the noisy dynamics.

Our work has implications to quantum computing in which the state transformation time bounds the speed of computation, and to quantum memory in which the amount of state degradation is bounded by the storage time.

Appendix

In the appendix, we compute $d(I, K_t)$ for dynamics with amplitude damping and dephasing noises respectively and provide the detailed derivation for the upper bound on the maximal angle on composite systems.

Amplitude damping

Let $K_t(\rho) = F_{11}(t)\rho F_{11}^\dagger(t) + F_{12}(t)\rho F_{12}^\dagger(t)$, where the Kraus operators $F_{11}(t) = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{P(t)} \end{bmatrix}$, $F_{12}(t) = \begin{bmatrix} 0 & \sqrt{1-P(t)} \\ 0 & 0 \end{bmatrix}$, here $P(t) = e^{-\gamma t}$ where γ is the decay rate. As $\cos d(I, K_t) = \max_{\|W\|\leq 1} \frac{1}{2}\lambda_{\min}(K_W + K_W^\dagger)$, here $K_W = \sum_{ij} w_{ij} F_{0i}^\dagger F_{1j}$, with $F_{01} = I$, and $F_{02} = \mathbf{0}$ as the Kraus operators for the identity operator(where we added a zero operator), w_{ij} is the ij-th entry of 2×2 matrix W with $\|W\| \leq 1$. Thus $K_W + K_W^\dagger = \begin{bmatrix} a & c \\ c^* & b \end{bmatrix}$, with $a = 2Re(w_{11})$, $b = 2Re(w_{11})\sqrt{P(t)}$, $c = w_{12}\sqrt{1-P(t)}$, here $Re(w_{11})$ denotes the real part of w_{11}. The minimum eigenvalue of $K_W + K_W^\dagger$ is given by $\lambda_{\min}(K_W + K_W^\dagger) = \frac{a+b-\sqrt{(a-b)^2+4|c|^2}}{2}$. It is obvious that to maximize the minimum eigenvalue, c should be 0, i.e., $w_{12} = 0$, then $\lambda_{\min}(K_W + K_W^\dagger) = b = 2Re(w_{11})\sqrt{P(t)}$ which achieves the maximum value when $w_{11} = 1$. Thus $\cos d(I, K_t) = \max_{\|W\|\leq 1} \frac{1}{2}\lambda_{\min}(K_W + K_W^\dagger) = \sqrt{P(t)}$.

Dephasing noises

Let $K_t(\rho) = F_{11}(t)\rho F_{11}^\dagger(t) + F_{12}(t)\rho F_{12}^\dagger(t)$ as the dynamics with dephasing noises, with the Kraus operators $F_{11}(t) = \sqrt{\frac{1+P(t)}{2}} \begin{bmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{bmatrix}$, $F_{12}(t) = \sqrt{\frac{1-P(t)}{2}} \begin{bmatrix} e^{-i\omega t/2} & 0 \\ 0 & -e^{i\omega t/2} \end{bmatrix}$, here $P(t) = e^{-\gamma t}$ where γ is the dephasing rate. Again we have $\cos d(I, K_t) = \max_{\|W\|\leq 1} \frac{1}{2}\lambda_{\min}(K_W + K_W^\dagger)$, where $K_W = \sum_{ij} w_{ij} F_{0i}^\dagger F_{1j}$, with $F_{01} = I$, and $F_{02} = \mathbf{0}$ as the Kraus operators for the identity operator, w_{ij} is the ij-th entry of 2×2 matrix W with $\|W\| \leq 1$. We have $K_W + K_W^\dagger = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ with $a = 2Re[\sqrt{\frac{1+P(t)}{2}} w_{11} e^{-i\omega t/2} + \sqrt{\frac{1-P(t)}{2}} w_{12} e^{-i\omega t/2}]$, $b = 2Re[\sqrt{\frac{1+P(t)}{2}} w_{11} e^{i\omega t/2} - \sqrt{\frac{1-P(t)}{2}} w_{12} e^{i\omega t/2}]$. Since

$$
\begin{aligned}
\lambda_{\min}(K_W + K_W^\dagger) \leq & \frac{1}{2} Tr(K_W + K_W^\dagger) \\
= & 2Re[\sqrt{\frac{1+P(t)}{2}} w_{11} \cos(\omega t/2) - i\sqrt{\frac{1-P(t)}{2}} w_{12} \sin(\omega t/2)] \\
\leq & 2[|\sqrt{\frac{1+P(t)}{2}} w_{11} \cos(\omega t/2)| + |\sqrt{\frac{1-P(t)}{2}} w_{12} \sin(\omega t/2)| \\
\leq & 2\sqrt{\frac{1+P(t)}{2} \cos^2(\omega t/2) + \frac{1-P(t)}{2} \sin^2(\omega t/2)} \times \sqrt{|w_{11}|^2 + |w_{12}|^2} \\
\leq & 2\sqrt{\frac{1+P(t)\cos(\omega t)}{2}},
\end{aligned}
$$

where the last inequality we used the fact $|w_{11}|^2 + |w_{12}|^2 \leq 1$ for any W with $\|W\| \leq 1$, the second to the last inequality is from the Cauchy-Schwarz inequality. It is also easy to check that the equality is saturated when

$w_{11}=\frac{\sqrt{1+P(t)}\cos(\omega t/2)}{\sqrt{1+P(t)\cos(\omega t)}}, w_{12}=\frac{i\sqrt{1-P(t)}\sin(\omega t/2)}{\sqrt{1+P(t)\cos(\omega t)}}$. Thus

$$\cos d(I,K_t)=\sqrt{\frac{1+P(t)\cos(\omega t)}{2}}. \tag{6}$$

Composite system

Given a noisy dynamics $K_t(\rho)=\sum_{j=1}^{D}F_j(t)\rho F_j^\dagger(t)$, suppose there are N such dynamics, which we denote as $K_t^{\otimes^N}$, acts independently on a composite systems. We show that if $I\in\{F_1(t),F_2(t),\cdots,F_D(t)\}$, then such composite system also can not achieve the $\pi/2$ rotation. As in the main text if $I\in\{F_1(t),F_2(t),\cdots,F_D(t)\}$, then there exists w_{1i} such that $I=\sum_{i=1}w_{1i}F_i(t)$. Let $\alpha=1/\sqrt{\sum_{i=1}^{D}|w_{1i}|^2}>0$, then $\alpha I=\sum_{i=1}w'_{1i}F_i(t)$ with $w'_{1i}=\alpha w_{1i}$ and define W' as a $D\times D$ matrix with the entries of the first row equal to w'_{1i} and other entries equal to 0. It is obvious that $\|W'\|=1$. Now one representation of the Kraus operators for $K_t^{\otimes^N}$ can be written as $\tilde{F}_{i_1,i_2,\cdots,i_N}(t)=F_{i_1}(t)\otimes F_{i_2}(t)\otimes\cdots\otimes F_{i_N}(t)$, let $\tilde{W}=W'^{\otimes N}$, then $K_{\tilde{W}}^{\otimes N}=(K_{W'})^{\otimes^N}=\alpha^N I^{\otimes^N}$. Thus

$$\begin{aligned}\cos d(I^{\otimes^N},K_t^{\otimes N})&=\max_{\|W\|\le 1}\frac{1}{2}\lambda_{\min}(K_W^{\otimes N}+(K_W^{\otimes N})^\dagger)\\&\ge\frac{1}{2}\lambda_{\min}(K_{\tilde{W}}^{\otimes N}+(K_{\tilde{W}}^{\otimes N})^\dagger)\\&=\lambda_{\min}(\alpha^N I^{\otimes^N})\\&=\alpha^N>0,\end{aligned} \tag{7}$$

thus $d(I^{\otimes^N},K_t^{\otimes N})\le\arccos(\alpha^N)<\pi/2$, which shows that in this case any state of the composite system can not be rotated to its orthogonal states.

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