STABILITY RESULTS OF A DISTRIBUTED PROBLEM INVOLVING BRESSE SYSTEM WITH HISTORY AND/OR CATTANEO LAW UNDER FULLY DIRICHLET OR MIXED BOUNDARY CONDITIONS

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Abstract. In this paper, we study the stability of a one-dimensional Bresse system with infinite memory type control and/or with heat conduction given by Cattaneo's law acting in the shear angle displacement. When the thermal effect vanishes, the system becomes elastic with memory term acting on one equation. Unlike [6], [10], and [22], we consider the interesting case of fully Dirichlet boundary conditions. Indeed, under equal speed of propagation condition, we establish the exponential stability of the system. However, in the natural physical case when the speeds of propagation are different, using a spectrum method, we show that the Bresse system is not uniformly stable. In this case, we establish a polynomial energy decay rate. Our study is valid for all other mixed boundary conditions and generalizes that of [6], [10], and [22].

1. Introduction

In this paper, we study the stability of the Bresse system with history and/or heat conduction given by Cattaneo's law. This system defined on $(0,L)\times(0,+\infty)$ takes the following form

$$
\begin{cases}
\rho_1\varphi_{tt}-k_1\left(\varphi_x+\psi+\mathrm{l}w\right)_x-\mathrm{l}k_3\left(w_x-\mathrm{l}\varphi\right)=0,\\
\rho_2\psi_{tt}-k_2\psi_{xx}+k_1\left(\varphi_x+\psi+\mathrm{l}w\right)-\displaystyle\int_0^{+\infty}g\left(s\right)\psi_{xx}\left(x,t-s\right)ds+\delta\theta_x=0,\\
\rho_1 w_{tt}-k_3\left(w_x-\mathrm{l}\varphi\right)_x+\mathrm{l}k_1\left(\varphi_x+\psi+\mathrm{l}w\right)=0,\\
\rho_3\theta_t+q_x+\delta\psi_{tx}=0,\\
\tau q_t+\beta q+\theta_x=0,
\end{cases}
\tag{1.1}
$$

with fully Dirichlet boundary conditions

$$
\begin{array}{ll}
\varphi\left(0,\cdot\right)=\varphi\left(L,\cdot\right)=\psi\left(0,\cdot\right)=\psi\left(L,\cdot\right)=0 & \text{in } \mathbb{R}_+,\\
w\left(0,\cdot\right)=w\left(L,\cdot\right)=\theta\left(0,\cdot\right)=\theta\left(L,\cdot\right)=0 & \text{in } \mathbb{R}_+,
\end{array}
\tag{1.2}
$$

or with Dirichlet-Neumann-Dirichlet-Dirichlet boundary conditions

$$
\begin{array}{ll}
\varphi\left(0,\cdot\right)=\varphi\left(L,\cdot\right)=\psi_x\left(0,\cdot\right)=\psi_x\left(L,\cdot\right)=0 & \text{in } \mathbb{R}_+,\\
w\left(0,\cdot\right)=w\left(L,\cdot\right)=\theta\left(0,\cdot\right)=\theta\left(L,\cdot\right)=0 & \text{in } \mathbb{R}_+,
\end{array}
\tag{1.3}
$$

or with Dirichlet-Neumann-Neumann-Dirichlet boundary conditions

$$
\begin{array}{ll}
\varphi\left(0,\cdot\right)=\varphi\left(L,\cdot\right)=\psi_x\left(0,\cdot\right)=\psi_x\left(L,\cdot\right)=0 & \text{in } \mathbb{R}_+,\\
w_x\left(0,\cdot\right)=w_x\left(L,\cdot\right)=\theta\left(0,\cdot\right)=\theta\left(L,\cdot\right)=0 & \text{in } \mathbb{R}_+,
\end{array}
\tag{1.4}
$$

in addition to the following initial conditions

$$
\begin{array}{l}
\varphi\left(\cdot,0\right)=\varphi_0\left(\cdot\right),\ \psi\left(\cdot,-t\right)=\psi_0\left(\cdot,t\right),\ w\left(\cdot,0\right)=w_0\left(\cdot\right),\ \theta(0,\cdot)=\theta_0,\ q(0,\cdot)=q_0,\\
\varphi_t\left(\cdot,0\right)=\varphi_1\left(\cdot\right),\ \psi_t\left(\cdot,0\right)=\psi_1\left(\cdot\right),\ w_t\left(\cdot,0\right)=w_1\left(\cdot\right)\quad \text{in } (0,L).
\end{array}
$$

The functions φ, ψ, and w model the vertical, shear angle, and longitudinal displacements of the filament. The functions θ and q model the temperature difference and the heat flux respectively. The coefficients ρ_1, ρ_2, ρ_3, k_1, k_2, k_3, l, δ, τ, β are positive constants. The integral term represents a history term with kernel g satisfying the following hypothesis:

Key words and phrases. Bresse system, history and Cattaneo law, indirect polynomial stability, non uniform stability, frequency domain approach.

(H) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-increasing differentiable function such that $\lim_{s\to 0} g(s)$ exists and there exists $c > 0$ such that

$$g'(s) \leq -cg(s).$$

Furthermore, we assume that $\widetilde{k}_2 > 0$ where $\widetilde{k}_2 := k_2 - g_0$, and $g_0 = \int_0^{+\infty} g(s)\, ds$.

Indeed, the condition $\lim_{s\to 0} g(s)$ is sufficient near zero and can replace the condition $g'(s) \geq -\widetilde{c}g(s)$ for some $\widetilde{c} > 0$ considered in [6], [10], and [22]. Therefore, in this paper, hypothesis (H) is an improved condition on the kernel function appearing in the history term.

When $\delta = 0$, decoupling occurs and the thermal effect in system (1.1) vanishes. Consequently, the study of the stability of system (1.1) is reduced to the study of the Bresse system without heat conduction but with an infinite memory type control acting only in the shear angle displacement. The Bresse system is usually considered in studying elastic structures of the arcs type (see [11]). It can be expressed by the equations of motion

$$\begin{aligned} \rho_1 \varphi_{tt} &= Q_x + \mathrm{l}N \\ \rho_2 \psi_{tt} &= M_x - Q \\ \rho_1 w_{tt} &= N_x - \mathrm{l}Q \end{aligned}$$

where

$$\begin{aligned} N &= k_3 (w_x - \mathrm{l}\varphi) \\ Q &= k_1 (\varphi_x + \psi + \mathrm{l}w) \\ M &= k_2 \psi_x - \int_0^{+\infty} g(s)\, \psi_x(x, t-s)\, ds \end{aligned}$$

are the stress strain relations for elastic behavior. Here $\rho_1 = \rho A$, $\rho_2 = \rho I$, $k_1 = k'GA$, $k_3 = EA$, $k_2 = EI$, $\mathrm{l} = R^{-1}$ where ρ is the density of the material, E is the modulus of elasticity, G is the shear modulus, k' is the shear factor, A is the cross-sectional area, I is the second moment of area of the cross-section, and R is the radius of curvature. φ, ψ, and w are the vertical, shear angle, and longitudinal displacements. The kernel g represents the memory effect acting only on the shear displacement. We note that when $R \to \infty$, then $\mathrm{l} \to 0$ and the Bresse model reduces to well-known Timoshenko beam equations.

Hago et al. in [10] showed that the Timoshenko system with history type damping is not exponentially stable under Cattaneo's law, while under Fourier's law, an exponential stability can be only attained once the speeds are equal. Moreover, no decay rate has been discussed if the speeds are different. This result has been recently improved by Fatori et al in [6], where an exponential stability is obtained with Cattaneo's law if and only if a new condition on the wave speed of propagation is verified. Otherwise, an optimal energy decay rate of type $\frac{1}{\sqrt{t}}$ is obtained. Santos et al. in [22] extend the results of [6] and [10] to the Bresse system with only one infinite memory type damping and under mixed boundary conditions. They first proved that the system is exponentially stable if and only if the three waves propagate with the same speed. Moreover, when at least two waves propagate with the same speed, a polynomial energy decay rate was established but this case has only a mathematical sense. Therefore, these results are very interesting but not complete. Indeed, from one hand, in the important physical natural case when the three waves have distinct speeds, no decay is discussed by Santos et al. On the other hand, the used techniques in all previously cited papers can not be adapted to prove the lack of exponential stability and to establish a polynomial decay rate in the interesting and difficult case of fully Dirichlet boundary conditions.

The purpose of this paper is to study the Bresse system in the presence of history type and/or heat conduction given by Cattaneo's law acting in the shear angle displacement equation and under fully Dirichlet or mixed boundary conditions. We limit our attention to the case of fully Dirichlet boundary conditions since our study can be easily adapted to the other mixed boundary conditions. Besides the mathematical case when all the speeds are equal, we treat the interesting physical case when the three waves all propagate with different speeds. In fact, from the physical interpretation, we remark that the speeds of propagation of the three waves given by $\frac{\rho_1}{k_1}$, $\frac{\rho_2}{k_2}$, and $\frac{\rho_1}{k_3}$ are all distinct. Our study is divided into two main parts. First, we study the stability of the elastic Bresse system with only one memory type damping under fully Dirichlet boundary conditions. When the speeds of the waves are all equal, we ensure the exponential decay found in [22]. On the contrary, using a spectrum method we prove the lack of uniform stability. Moreover, when only two waves propagate with the same speed, using a frequency domain approach combining with a multiplier method, we establish the energy decay rate of type $\frac{1}{t}$. Finally, in the interesting physical case, when the whole three waves propagate with different speeds, we prove an energy decay rate of type $\frac{1}{\sqrt{t}}$(see Theorem 2.20 and Theorem 2.21). In these cases, we conjecture the optimality of the energy decay rate. Next, we adapt the study of the stability of the elastic Bresse system (2.1) to the thermo-elastic Bresse system (1.1).

Last but not least, in addition to the previously cited papers, we rapidly recall some previous studies done on the Bresse system. The stability of the elastic Bresse system with different kind of damping has been studied in [13], [23], [5], [1], [17], [7], [8] and [16] . Guesmia et al. in [8] considered Bresse system with infinite memories acting in the three equations of the system. They established asymptotic stability results under some conditions on the relaxation functions regardless the speeds of propagation. Furthermore, thermal stabilization of the Bresse system has been studied in [7], [13], [16]. In [13], Liu and Rao considered the Bresse system with two thermal dissipation laws. The results of [13] are improved by Fatori and Rivera in [7] where they studied the stability of Bresse system with one distributed temperature dissipation law operating on the angle displacement equation. Recently, Najdi and Wehbe in [16] extended and improved the results of [7] when the thermal dissipation is locally distributed.

This paper is organized as follows: In Section 2.1, we prove the well-posedness of system (2.1) with fully Dirichlet boundary conditions. In Section 2.2, we prove the strong stability of the system in the lack of the compactness of the resolvent of the generator. In Section 2.3, we prove the exponential stability of the system on condition that the waves propagate with equal speeds. In Section 2.4, we prove that the Bresse system considered with fully Dirichlet boundary conditions is non-uniformly stable when the speeds of the propagation of the waves are different. More precisely, we consider the reduced Timchenko system with fully Dirichlet boundary conditions and prove that an infinite number of eigenvalues approach the imaginary axis. In Section 2.5, if the waves propagate with different speeds, we prove the polynomial stability of the system. Indeed, if only two of the waves propagate with equal speeds, we prove a faster polynomial decay rate. Finally, in Section 3, we adapt the results to the thermo-elastic Bresse system (1.1).

2. Elastic Bresse system with one memory type control

In this section, we study the stability of Bresse system with only one infinite memory damping acting in the equation about the shear angle displacement under fully Dirichlet boundary conditions (our study can be easily adapted to other mixed boundary conditions). The system is governed by the following partial differential equations:

$$
\begin{cases}
\rho_1\varphi_{tt} - k_1\left(\varphi_x + \psi + \mathrm{l}w\right)_x - \mathrm{l}k_3\left(w_x - \mathrm{l}\varphi\right) = 0, \\[2mm]
\rho_2\psi_{tt} - k_2\psi_{xx} + k_1\left(\varphi_x + \psi + \mathrm{l}w\right) + \displaystyle\int_0^{+\infty} g\left(s\right)\psi_{xx}\left(x, t-s\right)ds = 0, \\[2mm]
\rho_1 w_{tt} - k_3\left(w_x - \mathrm{l}\varphi\right)_x + \mathrm{l}k_1\left(\varphi_x + \psi + \mathrm{l}w\right) = 0,
\end{cases} \tag{2.1}
$$

with the following boundary conditions:

$$
\varphi\left(0,t\right) = \varphi\left(L,t\right) = \psi\left(0,t\right) = \psi\left(L,t\right) = w\left(0,t\right) = w\left(L,t\right) = 0, \tag{2.2}
$$

and the following initial conditions:

$$
\begin{array}{c}
\varphi\left(\cdot,0\right) = \varphi_0\left(\cdot\right), \ \psi\left(\cdot,-t\right) = \psi_0\left(\cdot,t\right), \ w\left(\cdot,0\right) = w_0\left(\cdot\right), \\
\varphi_t\left(\cdot,0\right) = \varphi_1\left(\cdot\right), \ \psi_t\left(\cdot,0\right) = \psi_1\left(\cdot\right), \ w_t\left(\cdot,0\right) = w_1\left(\cdot\right).
\end{array}
$$

Remark 2.1. *The mixed boundary conditions make the calculations easier because they do not introduce pointwise terms when we apply the multiplicative techniques. However, in the case of fully Dirichlet boundary conditions, the calculations are more complicated because the boundary terms does not vanish.*

2.1. **Well-posedness of the problem.**

In this part, using a semi-group approach, we establish well-posedness result for the system (2.1)-(2.2) under condition (H) imposed into the relaxation function. For this purpose, similar to [4] and [15], we introduce the new variable

$$
\eta\left(x,t,s\right) := \psi\left(x,t\right) - \psi\left(x,t-s\right), \quad \text{in } \left(0,L\right)\times\mathbb{R}_+\times\mathbb{R}_+. \tag{2.3}
$$

Then, system (2.1)-(2.2) becomes

$$
\begin{cases}
\rho_1\varphi_{tt}-k_1\left(\varphi_x+\psi+\mathrm{l}w\right)_x-\mathrm{l}k_3\left(w_x-\mathrm{l}\varphi\right)=0,\\[2mm]
\rho_2\psi_{tt}-\left(k_2-\displaystyle\int_0^{+\infty}g\left(s\right)ds\right)\psi_{xx}+k_1\left(\varphi_x+\psi+\mathrm{l}w\right)-\displaystyle\int_0^{+\infty}g\left(s\right)\eta_{xx}ds=0,\\[2mm]
\rho_1 w_{tt}-k_3\left(w_x-\mathrm{l}\varphi\right)_x+\mathrm{l}k_1\left(\varphi_x+\psi+\mathrm{l}w\right)=0,\\[2mm]
\eta_t+\eta_s-\psi_t=0,
\end{cases}
\tag{2.4}
$$

with the boundary conditions

$$
\begin{array}{ll}
\varphi\left(0,\cdot\right)=\varphi\left(L,\cdot\right)=\psi\left(0,\cdot\right)=\psi\left(L,\cdot\right)=w\left(0,\cdot\right)=w\left(L,\cdot\right)=0 & \text{in } \mathbb{R}_+,\\
\eta\left(0,\cdot,\cdot\right)=\eta\left(L,\cdot,\cdot\right)=0 & \text{in } \mathbb{R}_+\times\mathbb{R}_+,\\
\eta\left(\cdot,\cdot,0\right)=0 & \text{in } \left(0,L\right)\times\mathbb{R}_+,
\end{array}
\tag{2.5}
$$

and initial conditions

$$
\begin{array}{ll}
\varphi\left(\cdot,0\right)=\varphi_0\left(\cdot\right),\ \psi\left(\cdot,-t\right)=\psi_0\left(\cdot,t\right),\ w\left(\cdot,0\right)=w_0\left(\cdot\right), & \\
\varphi_t\left(\cdot,0\right)=\varphi_1\left(\cdot\right),\ \psi_t\left(\cdot,0\right)=\psi_1\left(\cdot\right),\ w_t\left(\cdot,0\right)=w_1\left(\cdot\right), & \\
\eta^0\left(\cdot,s\right):=\eta\left(\cdot,0,s\right)=\psi_0\left(\cdot,0\right)-\psi_0\left(\cdot,s\right) & \text{in } \left(0,L\right),\ s\geq 0.
\end{array}
\tag{2.6}
$$

The energy of system (2.4)-(2.5) is given by

$$
\begin{aligned}
E\left(t\right) &=\frac{1}{2}\Bigg\{\int_0^L\left(\rho_1\left|\varphi\right|^2+\rho_2\left|\psi_t\right|^2+\rho_1\left|w_t\right|^2+k_1\left|\varphi_x+\psi^2+\mathrm{l}w\right|^2+\widetilde{k}_2\left|\psi_x\right|^2\right)dx\\
&+k_3\int_0^L\left|w_x-\mathrm{l}\varphi\right|^2dx+\int_0^L\int_0^{+\infty}g\left(s\right)\left|\eta_x\right|^2dxds\Bigg\}.
\end{aligned}
\tag{2.7}
$$

Then a straightforward computation gives

$$
E'\left(t\right)=\frac{1}{2}\int_0^L\int_0^{+\infty}g'\left(s\right)\left|\eta_x\right|^2dsdx\leq 0.
\tag{2.8}
$$

Thus, the system (2.4)-(2.5) is dissipative in the sense that its energy is non increasing with respect to the time t. Now, we define the energy space $\mathcal{H}$ by

$$
\mathcal{H}=\left(H_0^1\left(0,L\right)\right)^3\times\left(L^2\left(0,L\right)\right)^3\times L_g^2\left(\mathbb{R}_+,H_0^1\right)
$$

where $L_g^2\left(\mathbb{R}_+,H_0^1\right)$ denotes the Hilbert space endowed with the inner product

$$
\left(\eta^1,\eta^2\right)_g=\int_0^L\int_0^{+\infty}g\left(s\right)\eta_x^1\left(x,s\right)\eta_x^2\left(x,s\right)dsdx.
$$

The energy space $\mathcal{H}$ is endowed with the following norm

$$
\begin{aligned}
\|U\|_{\mathcal{H}}^2 &=\|(v^1,v^2,v^3,v^4,v^5,v^6,v^7)\|_{\mathcal{H}}^2\\
&=\rho_1\left\|v^4\right\|^2+\rho_2\left\|v^5\right\|^2+\rho_1\left\|v^6\right\|^2+k_1\left\|v_x^1+v^2+\mathrm{l}v^3\right\|^2+\widetilde{k}_2\left\|v_x^2\right\|^2\\
&+k_3\left\|v_x^3-\mathrm{l}v^1\right\|^2+\left\|v^7\right\|_g^2
\end{aligned}
\tag{2.9}
$$

where $\|\cdot\|$ and $\|\cdot\|_g$ denote the norms of $L^2\left(0,L\right)$ and $L_g^2\left(\mathbb{R}_+,H_0^1\right)$ respectively.
Next, we define the linear operator $\mathcal{A}$ in $\mathcal{H}$ by

$$
\begin{aligned}
D\left(\mathcal{A}\right)=\Bigg\{&U\in\mathcal{H}\mid v^1,v^3\in H^2\left(0,L\right),\ v^4,v^5,v^6\in H_0^1\left(0,L\right),\ v_s^7\in L_g^2\left(\mathbb{R}_+,H_0^1\right),\\
&v^2+\textstyle\int_0^{+\infty}g\left(s\right)v^7ds\in H^2\left(0,L\right)\cap H_0^1\left(0,L\right),\ v^7\left(x,0\right)=0\Bigg\}
\end{aligned}
$$

and

$$\mathcal{A}\begin{pmatrix} v^1 \\ v^2 \\ v^3 \\ v^4 \\ v^5 \\ v^6 \\ v^7 \end{pmatrix} = \begin{pmatrix} v^4 \\ v^5 \\ v^6 \\ \rho_1^{-1}\left(k_1\left(v_x^1+v^2+\mathrm{l}v^3\right)_x+\mathrm{l}k_3\left(v_x^3-\mathrm{l}v^1\right)\right) \\ \rho_2^{-1}\left(\widetilde{k}_2 v_{xx}^2-k_1\left(v_x^1+v^2+\mathrm{l}v^3\right)+\int_0^{+\infty} g(s)\, v_{xx}^7 ds\right) \\ \rho_1^{-1}\left(k_3\left(v_x^3-\mathrm{l}v^1\right)_x-\mathrm{l}k_1\left(v_x^1+v^2+\mathrm{l}v^3\right)\right) \\ v^5-v_s^7 \end{pmatrix} \tag{2.10}$$

for all $U=\left(v^1,v^2,v^3,v^4,v^5,v^6,v^7\right)^{\mathsf{T}} \in D(\mathcal{A})$. If $U=(\varphi,\psi,w,\varphi_t,\psi_t,w_t,\eta)^{\mathsf{T}}$ is the state of (2.4)-(2.5), then the Bresse beam system is transformed into a first order evolution equation on the Hilbert space $\mathcal{H}$:

$$\begin{cases} U_t=\mathcal{A}U, \\ U(0)=U^0 \end{cases} \tag{2.11}$$

where

$$U^0(x)=\left(\varphi_0(x),\psi_0(x,0),w_0(x),\varphi_1(x),\psi_1(x),w_1(x),\eta^0(x,\cdot)\right)^{\mathsf{T}}.$$

Remark 2.2. *It is easy to see that there exists a positive constant k_0' such that*

$$k_1\|\varphi_x+\psi+\mathrm{l}w\|^2+\widetilde{k}_2\|\psi_x\|^2+k_3\|w_x-\mathrm{l}\varphi\|^2 \leq k_0'\left(\|\varphi_x\|^2+\|\psi_x\|^2+\|w_x\|^2\right). \tag{2.12}$$

On the other hand, under hypothesis (H), *as $\widetilde{k}_2>0$, we can show by a contradiction argument the existence of a positive constant k_0 such that, for any $(\varphi,\psi,w)\in\left(H_0^1(]0,L[)\right)^3$,*

$$k_0\left(\|\varphi_x\|^2+\|\psi_x\|^2+\|w_x\|^2\right) \leq k_1\|\varphi_x+\psi+\mathrm{l}w\|^2+\widetilde{k}_2\|\psi_x\|^2+k_3\|w_x-\mathrm{l}\varphi\|^2. \tag{2.13}$$

Therefore, the norm on the energy space $\mathcal{H}$ given in (2.9) *is equivalent to the usual norm on $\mathcal{H}$.*

Proposition 2.3. *Under hypothesis* (H), *the operator $\mathcal{A}$ is m-dissipative in the energy space $\mathcal{H}$.*

Proof. For all $U\in D(\mathcal{A})$, by a straight forward calculation, we have

$$\Re\left(\langle \mathcal{A}U,U\rangle_{\mathcal{H}}\right)=\frac{1}{2}\int_0^L\int_0^{+\infty} g'(s)\left|v_x^7\right|^2 dsdx. \tag{2.14}$$

As g is non-increasing we get that $\mathcal{A}$ is dissipative. Now let $F=\left(f^1,f^2,f^3,f^4,f^5,f^6,f^7\right)^{\mathsf{T}}\in\mathcal{H}$, we prove the existence of

$$U=\left(v^1,v^2,v^3,v^4,v^5,v^6,v^7\right)^{\mathsf{T}}\in D(\mathcal{A})$$

unique solution of the equation

$$-\mathcal{A}U=F.$$

Equivalently, we have the following system

$$-v^4 = f^1, \tag{2.15}$$
$$-v^5 = f^2, \tag{2.16}$$
$$-v^6 = f^3, \tag{2.17}$$
$$-k_1\left[v_x^1+v^2+\mathrm{l}v^3\right]_x-\mathrm{l}k_3\left[v_x^3-\mathrm{l}v^1\right] = \rho_1 f^4, \tag{2.18}$$
$$-\widetilde{k}_2 v_{xx}^2+k_1\left[v_x^1+v^2+\mathrm{l}v^3\right]-\int_0^{+\infty} g(s)\, v_{xx}^7 ds = \rho_2 f^5, \tag{2.19}$$
$$-k_3\left[v_x^3-\mathrm{l}v^1\right]_x+\mathrm{l}k_1\left[v_x^1+v^2+\mathrm{l}v^3\right] = \rho_1 f^6, \tag{2.20}$$
$$v_s^7-v^5 = f^7. \tag{2.21}$$

From (2.21) and (2.16), we can determine

$$v^7(x,s)=-sf^2(x)+\int_0^s f^7(x,\tau)\,d\tau. \tag{2.22}$$

It is clear that $v^7(x,0)=0$ and $v^7_s \in L^2_g\left(\mathbb{R}_+, H^1_0\right)$. To prove that $v^7 \in L^2_g\left(\mathbb{R}_+, H^1_0\right)$ let $T, \epsilon > 0$ be arbitrary. Using hypothesis (H), we have

$$\begin{aligned}\int_\epsilon^T g(s)\left\|v_x^7\right\|^2 ds &\leq -\frac{1}{c}\int_\epsilon^T g'(s)\left\|v_x^7\right\|^2 ds \\ &\leq -\frac{g(T)}{c}\left\|v_x^7(\cdot,T)\right\|^2 + \frac{g(\epsilon)}{c}\left\|v_x^7(\cdot,\epsilon)\right\|^2 + \frac{2}{c}\int_\epsilon^T g(s)\Re\left\{\int_0^L v_x^7\overline{v_{xs}^7}dx\right\}ds.\end{aligned} \tag{2.23}$$

Using the hypothesis on g and the fact that $v^7(x,0)=0$, then as $T\to+\infty$ and $\epsilon\to 0$, we obtain from (2.23)

$$\begin{aligned}\int_0^{+\infty} g(s)\left\|v_x^7\right\|^2 ds &\leq \frac{2}{c}\int_0^{+\infty} g(s)\Re\left\{\int_0^L v_x^7\overline{v_{xs}^7}dx\right\}ds \\ &\leq \frac{1}{2}\int_0^{+\infty} g(s)\left\|v_x^7\right\|^2 ds + \frac{2}{c^2}\int_0^{+\infty} g(s)\left\|v_{sx}^7\right\|^2 ds.\end{aligned}$$

It follows that

$$\int_0^{+\infty} g(s)\left\|v_x^7\right\|^2 ds \leq \frac{4}{c^2}\int_0^{+\infty} g(s)\left\|v_{sx}^7\right\|^2 ds < +\infty.$$

Therefore $v^7 \in L^2_g\left(\mathbb{R}_+, H^1_0\right)$. Now, inserting (2.22) in (2.18)-(2.20), we get

$$\begin{cases} -k_1\left[v_x^1+v^2+\mathrm{l}v^3\right]_x - \mathrm{l}k_3\left[v_x^3-\mathrm{l}v^1\right] = \rho_1 f^4, \\ -\underline{k}_2 v_{xx}^2 + k_1\left[v_x^1+v^2+\mathrm{l}v^3\right] = \rho_2 f^5 + \displaystyle\int_0^{+\infty} g(s)\left(-sf^2(x)+\int_0^s f^7(x,\tau)\,d\tau\right)_{xx} ds, \\ -k_3\left[v_x^3-\mathrm{l}v^1\right]_x + \mathrm{l}k_1\left[v_x^1+v^2+\mathrm{l}v^3\right] = \rho_1 f^6, \end{cases} \tag{2.24}$$

where $\underline{k}_2 = k_2 - \int_0^{+\infty} e^{-s} g(s)\,ds > 0$.

Using Lax-Milgram Theorem (see [18]), we deduce that (2.24) admits a unique solution in $\left(H^1_0(0,L)\right)^3$. Thus, using (2.22) and classical regularity arguments, we conclude that $-\mathcal{A}U = F$ admits a unique solution $U \in D(\mathcal{A})$ and $0 \in \rho(\mathcal{A})$. Since $D(\mathcal{A})$ is dense in $\mathcal{H}$ then, by the resolvent identity, for small $\lambda > 0$, we have $R(\lambda I - \mathcal{A}) = \mathcal{H}$ (see Theorem 1.2.4 in [14]) and $\mathcal{A}$ is m-dissipative in $\mathcal{H}$. The proof is thus complete. ☐

Thanks to Lumer-Phillips Theorem (see [14, 18]), we deduce that $\mathcal{A}$ generates a C_0-semigroup of contraction $e^{t\mathcal{A}}$ in $\mathcal{H}$ and therefore problem (2.4)-(2.5) is well-posed. Then we have the following result:

Theorem 2.4. *Under hypothesis* (H)*, for any $U^0 \in \mathcal{H}$, problem* (2.11) *admits a unique weak solution*

$$U \in C\left(\mathbb{R}_+; \mathcal{H}\right).$$

Moreover, if $U^0 \in D(\mathcal{A})$, then

$$U \in C\left(\mathbb{R}_+; D(\mathcal{A})\right) \cap C^1\left(\mathbb{R}_+; \mathcal{H}\right).$$

2.2. **Strong stability of the system.** In this part, we use a general criteria of Arendt-Batty in [2] to show the strong stability of the C_0-semigroup $e^{t\mathcal{A}}$ associated to the Bresse system (2.4)-(2.5) in the absence of the compactness of the resolvent of $\mathcal{A}$. Our main result is the following theorem:

Theorem 2.5. *Assume that* (H) *is true. Then, the C_0-semigroup $e^{t\mathcal{A}}$ is strongly stable in $\mathcal{H}$; i.e, for all $U^0 \in \mathcal{H}$, the solution of* (2.11) *satisfies*

$$\lim_{t\to+\infty}\left\|e^{t\mathcal{A}}U^0\right\|_{\mathcal{H}} = 0.$$

For the proof of Theorem 2.5, we need the following two lemmas.

Lemma 2.6. *Under hypothesis* (H)*, we have*

$$\ker(i\lambda - \mathcal{A}) = \{0\} \quad \text{for all } \lambda \in \mathbb{R}. \tag{2.25}$$

Proof. From Proposition 2.3, we deduce that $0 \in \rho(\mathcal{A})$. We still need to show the result for $\lambda \in \mathbb{R}^*$. Suppose that there exists a real number $\lambda \neq 0$ and $U = \left(v^1, v^2, v^3, v^4, v^5, v^6, v^7\right)^\top \in D(\mathcal{A})$ such that

$$\mathcal{A}U = i\lambda U. \tag{2.26}$$

Then, we have

$$\Re\left(\langle \mathcal{A}U, U\rangle_{\mathcal{H}}\right) = \frac{1}{2}\int_0^L \int_0^{+\infty} g'(s)\left|v_x^7\right|^2 dsdx = 0.$$

Due to hypothesis (H), it follows that

$$\int_0^L \int_0^{+\infty} g(s)\left|v_x^7\right|^2 dsdx = 0.$$

This implies that

$$v^7 = 0.$$

Now equation (2.26) is equivalent to

$$\begin{aligned}
v^4 &= i\lambda v^1, && (2.27)\\
v^5 &= i\lambda v^2, && (2.28)\\
v^6 &= i\lambda v^3, && (2.29)\\
k_1\left[v_x^1 + v^2 + \mathrm{l}v^3\right]_x + \mathrm{l}k_3\left[v_x^3 - \mathrm{l}v^1\right] &= i\rho_1\lambda v^4, && (2.30)\\
\widetilde{k}_2 v_{xx}^2 - k_1\left[v_x^1 + v^2 + \mathrm{l}v^3\right] &= i\rho_2\lambda v^5, && (2.31)\\
k_3\left[v_x^3 - \mathrm{l}v^1\right]_x - \mathrm{l}k_1\left[v_x^1 + v^2 + \mathrm{l}v^3\right] &= i\rho_1\lambda v^6, && (2.32)\\
v^5 &= 0. && (2.33)
\end{aligned}$$

Using equations (2.33), (2.28), (2.31) and the fact that $v^2,\ v^5 \in H_0^1(0,L)$, we get

$$v^2 = v^5 = 0 \text{ and } v_x^1 + \mathrm{l}v^3 = 0. \tag{2.34}$$

Inserting (2.27), (2.29) and (2.34) into equations (2.30) and (2.32), we get

$$\begin{aligned}
-k_3 v_{xx}^1 + \left(\rho_1\lambda^2 - \mathrm{l}^2 k_3\right) v^1 &= 0,\\
k_3 v_{xx}^3 + \left(\rho_1\lambda^2 + \mathrm{l}^2 k_3\right) v^3 &= 0,\\
v^1(0) = v^1(L) = v^3(0) = v^3(L) &= 0.
\end{aligned}$$

By direct calculation, we deduce that $v^1 = v^3 = 0$ and therefore $U = 0$. The proof is thus complete. □

Lemma 2.7. *Under hypothesis* (H), $i\lambda - \mathcal{A}$ *is surjective for all* $\lambda \in \mathbb{R}$.

Proof. Since $0 \in \rho(\mathcal{A})$. We still need to show the result for $\lambda \in \mathbb{R}^*$. For any

$$F = \left(f^1, f^2, f^3, f^4, f^5, f^6, f^7\right)^{\mathsf{T}} \in \mathcal{H},\ \lambda \in \mathbb{R}^*,$$

we prove the existence of

$$U = \left(v^1, v^2, v^3, v^4, v^5, v^6, v^7\right)^{\mathsf{T}} \in D(\mathcal{A})$$

solution of the following equation

$$(i\lambda - \mathcal{A})\,U = F.$$

Equivalently, we have the following system

$$\begin{aligned}
i\lambda v^1 - v^4 &= f^1, && (2.35)\\
i\lambda v^2 - v^5 &= f^2, && (2.36)\\
i\lambda v^3 - v^6 &= f^3, && (2.37)\\
\rho_1 i\lambda v^4 - k_1\left[v_x^1 + v^2 + \mathrm{l}v^3\right]_x - \mathrm{l}k_3\left[v_x^3 - \mathrm{l}v^1\right] &= \rho_1 f^4, && (2.38)\\
\rho_2 i\lambda v^5 - \widetilde{k}_2 v_{xx}^2 + k_1\left[v_x^1 + v^2 + \mathrm{l}v^3\right] - \int_0^{+\infty} g(s)\, v_{xx}^7 ds &= \rho_2 f^5, && (2.39)\\
\rho_1 i\lambda v^6 - k_3\left[v_x^3 - \mathrm{l}v^1\right]_x + \mathrm{l}k_1\left[v_x^1 + v^2 + \mathrm{l}v^3\right] &= \rho_1 f^6, && (2.40)\\
i\lambda v^7 + v_s^7 - v^5 &= f^7. && (2.41)
\end{aligned}$$

From (2.41) and (2.36), we have

$$v_s^7 + i\lambda v^7 = i\lambda v^2 - f^2 + f^7.$$

It follows that

$$v^7(x,s) = \left(1 - e^{-i\lambda s}\right) v^2(x) + \frac{i}{\lambda}\left(1 - e^{-i\lambda s}\right) f^2(x) + \int_0^s e^{i\lambda(\tau - s)} f^7(x,\tau)\, d\tau. \tag{2.42}$$

From (2.35)-(2.37), we have

$$v^4 = i\lambda v^1 - f^1, \quad v^5 = i\lambda v^2 - f^2, \quad v^6 = i\lambda v^3 - f^3. \tag{2.43}$$

Inserting (2.42) and (2.43) in (2.38)-(2.40), we get

$$\begin{cases} -\lambda^2 v^1 - k_1\rho_1^{-1}\left[v_x^1 + v^2 + \mathrm{l}v^3\right]_x - \mathrm{l}k_3\rho_1^{-1}\left[v_x^3 - \mathrm{l}v^1\right] = h^1, \\ -\lambda^2 v^2 - \breve{k}_2\rho_2^{-1}v_{xx}^2 + k_1\rho_2^{-1}\left[v_x^1 + v^2 + \mathrm{l}v^3\right] = h^2, \\ -\lambda^2 v^3 - k_3\rho_1^{-1}\left[v_x^3 - \mathrm{l}v^1\right]_x + \mathrm{l}k_1\rho_1^{-1}\left[v_x^1 + v^2 + \mathrm{l}v^3\right] = h^3, \end{cases} \tag{2.44}$$

where

$$\breve{k}_2 = k_2 - \int_0^{+\infty} e^{-i\lambda s} g(s)\,ds,$$

and

$$\begin{cases} h^1 = f^4 + i\lambda f^1, \quad h^3 = f^6 + i\lambda f^3, \\ h^2 = f^5 + i\lambda f^2 + \dfrac{i}{\lambda}\rho_2^{-1}\displaystyle\int_0^{+\infty}\left(1 - e^{-i\lambda s}\right) g(s)\,ds f_{xx}^2 + \rho_2^{-1}\int_0^{+\infty} g(s)\int_0^s e^{i\lambda(\tau - s)} f_{xx}^7 d\tau ds. \end{cases}$$

Define the operators

$$\mathcal{L}v = \begin{pmatrix} -k_1\rho_1^{-1}\left(v_x^1 + v^2 + \mathrm{l}v^3\right)_x - \mathrm{l}k_3\rho_1^{-1}\left(v_x^3 - \mathrm{l}v^1\right) \\ -\breve{k}_2\rho_2^{-1}v_{xx}^2 + k_1\rho_2^{-1}\left(v_x^1 + v^2 + \mathrm{l}v^3\right) \\ -k_3\rho_1^{-1}\left(v_x^3 - \mathrm{l}v^1\right)_x + \mathrm{l}k_1\rho_1^{-1}\left(v_x^1 + v^2 + \mathrm{l}v^3\right) \end{pmatrix}, \quad \forall\ v = \left(v^1, v^2, v^3\right)^{\mathsf{T}} \in \left(H_0^1(0,L)\right)^3.$$

Using Lax-Milgram theorem, it is easy to show that $\mathcal{L}$ is an isomorphism from $(H_0^1(0,L))^3$ onto $(H^{-1}(0,L))^3$. Let $v = \left(v^1, v^2, v^3\right)^{\mathsf{T}}$ and $h = \left(h^1, h^2, h^3\right)^{\mathsf{T}}$, then we transform system (2.44) into the following form

$$v - \lambda^2\mathcal{L}^{-1}v = \mathcal{L}^{-1}h. \tag{2.45}$$

Using the compactness embeddings from $L^2(0,L)$ into $H^{-1}(0,L)$ and from $H_0^1(0,L)$ into $L^2(0,L)$ we deduce that the operator $\mathcal{L}^{-1}$ is compact from $L^2(0,L)$ into $L^2(0,L)$. Consequently, by Fredholm alternative, proving the existence of v solution of (2.45) reduces to proving the injectivity of the operator $Id - \lambda^2\mathcal{L}^{-1}$. Indeed, if $\tilde{v} = \left(\tilde{v}^1, \tilde{v}^2, \tilde{v}^3\right)^{\mathsf{T}} \in \mathrm{Ker}(Id - \lambda^2\mathcal{L}^{-1})$, then we have $\lambda^2\tilde{v} - \mathcal{L}\tilde{v} = 0$. It follows that

$$\begin{cases} -\rho_1\lambda^2\tilde{v}^1 - k_1\left[\tilde{v}_x^1 + \tilde{v}^2 + \mathrm{l}\tilde{v}^3\right]_x - \mathrm{l}k_3\left[\tilde{v}_x^3 - \mathrm{l}\tilde{v}^1\right] = 0, \\ -\rho_2\lambda^2\tilde{v}^2 - \breve{k}_2\tilde{v}_{xx}^2 + k_1\left[\tilde{v}_x^1 + \tilde{v}^2 + \mathrm{l}\tilde{v}^3\right] = 0, \\ -\rho_1\lambda^2\tilde{v}^3 - k_3\left[\tilde{v}_x^3 - \mathrm{l}\tilde{v}^1\right]_x + Ik_1\left[\tilde{v}_x^1 + \tilde{v}^2 + \mathrm{l}\tilde{v}^3\right] = 0. \end{cases} \tag{2.46}$$

Now, it is easy to see that if $(\tilde{v}^1, \tilde{v}^2, \tilde{v}^3)$ is a solution of system (2.46) then the vector $\tilde{V}$ defined by $\tilde{V} = \left(\tilde{v}^1, \tilde{v}^2, \tilde{v}^3, i\lambda\tilde{v}^1, i\lambda\tilde{v}^2, i\lambda\tilde{v}^3, \left(1 - e^{-i\lambda s}\right)\tilde{v}^2\right)^{\mathsf{T}}$ belongs to $D(\mathcal{A})$ and we have $i\lambda\tilde{V} - \mathcal{A}\tilde{V} = 0$. Therefore, by Lemma 2.6, we get $\tilde{V} = 0$ and so $\mathrm{Ker}(Id - \lambda^2\mathcal{L}^{-1}) = \{0\}$. Thanks to Fredholm alternative, the equation (2.45) admits a unique solution $v = (v^1, v^2, v^3) \in \left(H_0^1(0,L)\right)^3$. Thus, using (2.42), (2.43) and a classical regularity arguments, we conclude that $(i\lambda - \mathcal{A})U = F$ admits a unique solution $U \in D(\mathcal{A})$. The proof is thus complete. □

Proof of Theorem 2.5. Following a general criteria of Arendt-Batty in [2], the C_0-semigroup $e^{t\mathcal{A}}$ of contractions is strongly stable if $\mathcal{A}$ has no pure imaginary eigenvalues and $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable. By Lemma 2.6, the operator $\mathcal{A}$ has no pure imaginary eigenvalues and by Lemma 2.7, $\mathrm{R}(i\lambda - \mathcal{A}) = \mathcal{H}$ for all $\lambda \in \mathbb{R}$. Therefore, the closed graph theorem of Banach implies that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$. The proof is thus complete.

Remark 2.8. *Using a multiplier method, Santos et al. in* [22] *established the strong stability of Bresse system with only one infinite memory damping. Their study is only available for one dimensional case. In Theorem 2.5, our approach can be generalized to multi-dimensional spaces. In addition, our condition (H) on the relaxation function g is weaker than that used in* [22].

Remark 2.9. *We mention* [20] *for a direct approach of the strong stability of Kirchhoff plates in the absence of compactness of the resolvent.*

2.3. **Exponential stability in the case $\frac{k_1}{\rho_1}=\frac{k_2}{\rho_2}$ and $k_1=k_3$.** In this part, under hypothesis (H), we prove the exponential stability of the Bresse system (2.1)-(2.2) provided that

$$\frac{k_1}{\rho_1}=\frac{k_2}{\rho_2} \quad \text{and} \quad k_1=k_3. \tag{2.47}$$

Our main result in this part is the following stability estimate:

Theorem 2.10. *Assume that* (2.47) *is satisfied. Under hypothesis* (H)*, the C_0-semigroup $e^{t\mathcal{A}}$ is exponentially stable, i.e, there exist constants $M\geq 1$, and $\epsilon>0$ independent of U^0 such that*

$$\left\|e^{t\mathcal{A}}U^0\right\|_{\mathcal{H}}\leq Me^{-\epsilon t}\left\|U^0\right\|_{\mathcal{H}},\quad t\geq 0.$$

According to [9] and [19], we have to check if the following conditions hold,

$$i\mathbb{R}\subseteq\rho\left(\mathcal{A}\right)\qquad \text{(H1)},$$

and

$$\sup_{\lambda\in\mathbb{R}}\left\|\left(i\lambda Id-\mathcal{A}\right)^{-1}\right\|_{\mathcal{L}(\mathcal{H})}=O\left(1\right)\qquad \text{(H2)}.$$

Condition $i\mathbb{R}\subseteq\rho\left(\mathcal{A}\right)$ is already proved in Lemma 2.6 and Lemma 2.7. We will prove condition (H2) by a contradiction argument. Suppose that there exist a sequence of real numbers $\left(\lambda_n\right)_n$, with $|\lambda_n|\to+\infty$, and a sequence of vectors

$$U_n=\left(v_n^1,v_n^2,v_n^3,v_n^4,v_n^5,v_n^6,v_n^7\right)^{\mathsf{T}}\in D\left(\mathcal{A}\right)\quad\text{with}\quad \|U_n\|_{\mathcal{H}}=1 \tag{2.48}$$

such that

$$i\lambda_nU_n-\mathcal{A}U_n=\left(f_n^1,f_n^2,f_n^3,f_n^4,f_n^5,f_n^6,f_n^7\right)^{\mathsf{T}}\to 0\ \text{ in }\mathcal{H}. \tag{2.49}$$

That we detail as

$$i\lambda_nv_n^1-v_n^4 = h_n^1, \tag{2.50}$$

$$i\lambda_nv_n^2-v_n^5 = h_n^2, \tag{2.51}$$

$$i\lambda_nv_n^3-v_n^6 = h_n^3, \tag{2.52}$$

$$\rho_1\lambda_n^2v_n^1+k_1\left[\left(v_n^1\right)_x+v_n^2+\mathrm{l}v_n^3\right]_x+\mathrm{l}k_3\left[\left(v_n^3\right)_x-\mathrm{l}v_n^1\right] = h_n^4, \tag{2.53}$$

$$\rho_2\lambda_n^2v_n^2+\widetilde{k}_2\left(v_n^2\right)_{xx}-k_1\left[\left(v_n^1\right)_x+v_n^2+\mathrm{l}v_n^3\right]+\int_0^{+\infty}g\left(s\right)\left(v_n^7\right)_{xx}ds = h_n^5, \tag{2.54}$$

$$\rho_1\lambda_n^2v_n^3+k_3\left[\left(v_n^3\right)_x-\mathrm{l}v_n^1\right]_x-\mathrm{l}k_1\left[\left(v_n^1\right)_x+v_n^2+\mathrm{l}v_n^3\right] = h_n^6, \tag{2.55}$$

$$i\lambda_nv_n^7+\left(v_n^7\right)_s-i\lambda_nv_n^2 = h_n^7, \tag{2.56}$$

where

$$\begin{cases} h_n^1=f_n^1,\ h_n^2=f_n^2,\ h_n^3=f_n^3,\ h_n^7=f_n^7-f_n^2,\\ \\ h_n^4=-\rho_1\left(f_n^4+i\lambda_nf_n^1\right),\ h_n^5=-\rho_2\left(f_n^5+i\lambda_nf_n^2\right),\ h_n^6=-\rho_1\left(f_n^6+i\lambda_nf_n^3\right).\end{cases}$$

In the following we will check the condition (H2) by finding a contradiction with (2.48) such as $\|U_n\|_{\mathcal{H}}=o(1)$. For clarity, we divide the proof into several lemmas. From now on, for simplicity, we drop the index n.

Lemma 2.11. *Assume that hypothesis* (H) *is verified. Then we have*

$$\int_0^L\int_0^{+\infty}g\left(s\right)\left|v_x^7\right|^2dsdx=o\left(1\right). \tag{2.57}$$

Proof. Taking the inner product of (2.49) with U in $\mathcal{H}$. Then, using (2.14) and the fact that U is uniformly bounded in $\mathcal{H}$, we get

$$\frac{1}{2}\int_0^L\int_0^{+\infty}g'\left(s\right)\left|v_x^7\right|^2dsdx=\Re\left(\mathcal{A}U,U\right)_{\mathcal{H}}=-\Re\left(i\lambda U-\mathcal{A}U,U\right)_{\mathcal{H}}=o\left(1\right). \tag{2.58}$$

Using condition (H) into (2.58), we obtain the desired asymptotic equation (2.57). Thus the proof is complete. □

Lemma 2.12. *Assume that hypothesis* (H) *is verified. Then we have*

$$\int_0^L\left|v_x^2\right|dx=o\left(1\right)\ \text{ and }\ \int_0^L\left|\lambda v^2\right|^2dx=o\left(1\right). \tag{2.59}$$

Proof. Multiplying (2.56) by $\overline{v^2}$ in $L_g^2\left(\mathbb{R}_+, H_0^1\right)$. Then, using the fact that $\left\|v^2\right\|_g^2 = g_0\left\|v_x^2\right\|^2$, v_x^2 is uniformly bounded in $L^2(0,L)$, f^2 converges to zero in $H_0^1(0,L)$ and f^7 converges to zero in $L_g^2\left(\mathbb{R}_+, H_0^1\right)$, we get

$$g^0\int_0^L \left|v_x^2\right|^2 dx = \int_0^L\int_0^{+\infty} g\left(s\right) v_x^7\overline{v_x^2}dsdx + \frac{1}{i\lambda}\int_0^L\int_0^{+\infty} g\left(s\right) v_{xs}^7\overline{v_x^2}dsdx + \frac{o(1)}{\lambda}. \tag{2.60}$$

Using by parts integration, condition (H) and the fact that $v^7\left(x,0\right)=0$, we get

$$\frac{1}{i\lambda}\int_0^L\int_0^{+\infty} g\left(s\right) v_{xs}^7\overline{v_x^2}dsdx = -\frac{1}{i\lambda}\int_0^L\int_0^{+\infty} g'\left(s\right) v_x^7\overline{v_x^2}dsdx.$$

Then, applying Holder's inequality in $L^2(0,L)$ and $L^2(0,+\infty)$ and using (2.58) and the fact that v_x^2 is uniformly bounded in $L^2(0,L)$ and $\lim\limits_{s\to 0}\sqrt{g\left(s\right)}$ exists, it follows that

$$\left|\frac{1}{\lambda}\int_0^L\int_0^{+\infty} g\left(s\right) v_{xs}^7\overline{v_x^2}dsdx\right| \leq \frac{\lim\limits_{s\to0}\sqrt{g\left(s\right)}}{|\lambda|}\left(\int_0^L\int_0^{+\infty} -g'\left(s\right)\left|v_x^7\right|^2 dsdx\right)^{1/2}\left\|v_x^2\right\| = \frac{o(1)}{\lambda}. \tag{2.61}$$

Using Lemma 2.11 and the fact that v_x^2 is uniformly bounded in $L^2(0,L)$, we get

$$\left|\int_0^L\int_0^{+\infty} g\left(s\right) v_x^7\overline{v_x^2}dsdx\right| = o\left(1\right). \tag{2.62}$$

Using equation (2.61) and (2.62) in equation (2.60), we get the first asymptotic estimate of (2.59). Now, multiplying (2.54) by $\overline{v^2}$ in $L^2(0,L)$. Then, using the fact that v^2 is uniformly bounded in $L^2(0,L)$, f^2 converges to zero in $H_0^1(0,L)$ and f^5 converges to zero in $L^2\left(0,L\right)$, we get

$$\begin{aligned}\rho_2\lambda^2\int_0^L\left|v^2\right|^2 dx &= \widetilde{k}_2\int_0^L\left|v_x^2\right|^2 dx + \int_0^L\int_0^{+\infty} g\left(s\right) v^7\overline{v^2}dsdx\\ &\quad + k_1\int_0^L\left(v_x^1 + v^2 + \mathrm{l}v^3\right)\overline{v^2}dx + o\left(1\right).\end{aligned} \tag{2.63}$$

Using Lemma 2.11, the first estimation of (2.59) and the fact that v_x^1 is uniformly bounded in $L^2(0,L)$, $\left\|v^2\right\| = o(1)$, v^2, v^3 converge to zero in $L^2\left(0,L\right)$ in equation (2.63) we obtain the second asymptotic estimate of (2.59). Thus the proof is complete. □

Lemma 2.13. *Assume that hypothesis* (H) *is verified. If* $\|U\|_{\mathcal{H}} = o\left(1\right)$*, on* $(\alpha,\beta)\subset(0,L)$ *then* $\|U\|_{\mathcal{H}} = o\left(1\right)$ *on* $(0,L)$*.*

Proof. From Lemma 2.12, we have $\left\|v_x^2\right\| = o\left(1\right)$ and $\left\|\lambda v^2\right\| = o\left(1\right)$. Therefore, we only need to prove the same results for v^1 and v^3. Let $\phi\in H_0^1\left(0,L\right)$ be a given function. We proceed the proof in two steps.

(1) Multiplying equation (2.53) by $2\phi\overline{v_x^1}$ in $L^2(0,L)$ and use Dirichlet boundary conditions to get

$$\begin{aligned}&-\rho_1\int_0^L\phi'\left|\lambda v^1\right|^2dx - k_1\int_0^L\phi'\left|v_x^1\right|^2dx\\ &+2\Re\left\{k_1\int_0^L\phi v_x^2\overline{v_x^1}dx + \mathrm{l}\left(k_1+k_3\right)\int_0^L\phi v_x^3\overline{v_x^1}dx - \mathrm{l}^2k_3\int_0^L\phi v^1\overline{v_x^1}dx\right\}\\ &= -2\rho_1\Re\left\{\int_0^L\phi f^4\overline{v_x^1}dx - i\int_0^L\left(\phi' f^1 + \phi f_x^1\right)\lambda\overline{v^1}dx\right\}.\end{aligned} \tag{2.64}$$

From (2.48) and (2.50)-(2.52), we remark that

$$\left\|v^1\right\| = O\left(\frac{1}{\lambda}\right),\ \left\|v^2\right\| = O\left(\frac{1}{\lambda}\right),\ \left\|v^3\right\| = O\left(\frac{1}{\lambda}\right). \tag{2.65}$$

Then, using equation (2.65), Lemma 2.12 and the facts that v_x^1, λv^1 are uniformly bounded in $L^2(0,L)$, f^1 converges to zero in $H_0^1(0,L)$, f^4 converges to zero in $L^2(0,L)$ in (2.64), we get

$$-\rho_1\int_0^L\phi'\left|\lambda v^1\right|^2dx - k_1\int_0^L\phi'\left|v_x^1\right|^2dx + 2\mathrm{l}\left(k_1+k_3\right)\Re\left\{\int_0^L\phi v_x^3\overline{v_x^1}dx\right\} = o\left(1\right). \tag{2.66}$$

Similarly, multiplying equation (2.55) by $2\phi\overline{v_x^3}$ in $L^2(0,L)$, we get

$$-\rho_1\int_0^L \phi'\left|\lambda v^3\right|^2 dx - k_3\int_0^L \phi'\left|v_x^3\right|^2 dx - 2\mathrm{l}\left(k_1+k_3\right)\Re\left\{\int_0^L \phi v_x^1\overline{v_x^3}dx\right\} = o\left(1\right). \tag{2.67}$$

Adding (2.66) and (2.67), we get

$$\rho_1\int_0^L \phi'\left(\left|\lambda v^1\right|^2+\left|\lambda v^3\right|^2\right)dx + k_1\int_0^L \phi'\left|v_x^1\right|^2 dx + k_3\int_0^L \phi'\left|v_x^3\right|^2 dx = o\left(1\right). \tag{2.68}$$

(2) Let $\epsilon>0$ such that $\alpha+\epsilon<\beta$ and define the cut-off function ς_1 in $C^1\left([0,L]\right)$ by

$$0\le\varsigma_1\le 1,\ \varsigma_1=1 \text{ on } [0,\alpha] \text{ and } \varsigma_1=0 \text{ on } [\alpha+\epsilon,L].$$

Take $\phi=x\varsigma_1$ in (2.68) and use the fact that $\|U\|_{\mathcal{H}}=o\left(1\right)$ on (α,β), we get

$$\rho_1\int_0^\alpha \left|\lambda v^1\right|^2 dx + \rho_1\int_0^\alpha \left|\lambda v^3\right|^2 dx + k_1\int_0^\alpha \left|v_x^1\right|^2 dx + k_3\int_0^\alpha \left|v_x^3\right|^2 dx = o\left(1\right). \tag{2.69}$$

Using Lemmas 2.11 and 2.12, in (2.69), we get

$$\|U\|_{\mathcal{H}}=o\left(1\right) \text{ on } (0,\alpha).$$

Similarly, by symmetry, we can prove that $\|U\|_{\mathcal{H}}=o\left(1\right)$ on (β,L) and therefore

$$\|U\|_{\mathcal{H}}=o\left(1\right) \text{ on } (0,L).$$

Thus the proof is complete. □

In the sequel, let $0<\alpha<\beta<L$ and consider the function $\varsigma\in C^1\left([0,L]\right)$ such that $0\le\varsigma\le 1$, $\varsigma=1$ on $[\alpha+\epsilon,\beta-\epsilon]\subset[0,L]$ and $\varsigma=0$ on $[0,\alpha]\cup[\beta,L]$. Our aim is to prove that $\|U\|_{\mathcal{H}}=o\left(1\right)$ on $[\alpha,\beta]$ and so by Lemma 2.13, we get $\|U\|_{\mathcal{H}}=o\left(1\right)$ on $(0,L)$ contradicting (2.48).

Lemma 2.14. *Suppose that hypothesis* (H) *and* (2.47) *are satisfied. Then we have*

$$\int_0^L \varsigma\left|v_x^1\right|^2 dx = o\left(1\right), \quad \textit{and} \quad \int_0^L \varsigma\left|\lambda v^1\right|^2 dx = o\left(1\right). \tag{2.70}$$

Proof. We show the first estimation of (2.70). We proceed in two main steps.

(1) Our first aim is to show that

$$\begin{aligned} &\frac{k_1}{k_2}\int_0^L \varsigma\left|v_x^1\right|^2 dx + \left(\frac{\rho_2}{k_2}-\frac{\rho_1}{k_1}\right)\Re\left\{\int_0^L \lambda^2 v_x^2\varsigma\overline{v^1}dx\right\} \\ &+\Re\left\{\frac{\rho_1\lambda^2}{k_1k_2}\int_0^L\left(g^0v_x^2-\int_0^{+\infty}g\left(s\right)v_x^7ds\right)\varsigma\overline{v^1}dx\right\} = o\left(1\right). \end{aligned} \tag{2.71}$$

Multiplying (2.54) by $\varsigma\overline{v_x^1}$ in $L^2(0,L)$ and using by parts integration. Then, using Lemmas 2.11, 2.12 and the facts that v_x^1, λv^1 are uniformly bounded in $L^2(0,L)$, f^2 converges to zero in $H_0^1(0,L)$, f^5 converges to zero in $L^2(0,L)$, we get

$$\begin{aligned} &k_1\int_0^L \varsigma\left|v_x^1\right|^2 dx + \rho_2\lambda^2\int_0^L \varsigma v_x^2\overline{v^1}dx \\ &+\int_0^L\left(\widetilde{k}_2v_x^2+\int_0^{+\infty}g\left(s\right)v_x^7ds\right)\varsigma\overline{v_{xx}^1}dx = o\left(1\right). \end{aligned} \tag{2.72}$$

Furthermore, multiplying (2.53) by $\frac{\varsigma}{k_1}\left(\widetilde{k}_2\overline{v_x^2}+\int_0^{+\infty}g\left(s\right)\overline{v_x^7}ds\right)$ in $L^2(0,L)$ and using by parts integration. Then, using Lemmas 2.11, 2.12 and the facts that v_x^3, λv^1 are uniformly bounded in $L^2(0,L)$, f^1 converges to zero in $H_0^1(0,L)$, f^4 converges to zero in $L^2(0,L)$, we get

$$\begin{aligned} &\frac{\rho_1\lambda^2}{k_1}\int_0^L\left(\widetilde{k}_2\overline{v_x^2}+\int_0^{+\infty}g\left(s\right)\overline{v_x^7}ds\right)\varsigma v^1dx + \int_0^L\left(\widetilde{k}_2\overline{v_x^2}+\int_0^{+\infty}g\left(s\right)\overline{v_x^7}ds\right)\varsigma v_{xx}^1dx \\ &+\frac{i\rho_1}{k_1}\int_0^L\left(\lambda\int_0^{+\infty}g\left(s\right)\overline{v_x^7}ds\right)\varsigma f^1dx = o\left(1\right). \end{aligned} \tag{2.73}$$

Subtracting (2.72) from (2.73) and take the real part of the resulting equation, we get

$$
\begin{aligned}
&k_1\int_0^L \varsigma\left|v_x^1\right|^2 dx-\Re\left\{\frac{\rho_1\lambda^2}{k_1}\int_0^L\left(\widetilde{k}_2\overline{v_x^2}+\int_0^{+\infty}g(s)\,\overline{v_x^7}ds\right)\varsigma v^1dx\right\}\\
&+\Re\left\{\rho_2\lambda^2\int_0^L\varsigma v_x^2\overline{v^1}dx-\frac{i\rho_1}{k_1}\int_0^L\left(\lambda\int_0^{+\infty}g(s)\,\overline{v_x^7}ds\right)\varsigma f^1dx\right\}=o(1)\,.
\end{aligned}\tag{2.74}
$$

From (2.56), we have

$$
\lambda v_x^7-iv_{xs}^7-\lambda v_x^2=-ih_x^7,\quad\text{in}\quad L_g^2\left(\mathbb{R}_+,L^2\right).\tag{2.75}
$$

Multiplying (2.75) by $\varsigma\overline{f^1}$ in $L_g^2\left(\mathbb{R}_+,L^2\right)$ and using by parts integration. Then, using hypothesis (H), Lemmas 2.11, 2.12 and the facts that f^1, f^2 converge to zero in $H_0^1(0,L)$, f^7 converges to zero in $L_g^2(\mathbb{R}_+;H_0^1(0,L))$, we get

$$
\begin{aligned}
&\int_0^L\left(\lambda\int_0^{+\infty}g(s)\,v_x^7ds\right)\varsigma\overline{f^1}dx=-i\int_0^L\left(\int_0^{+\infty}g'(s)\,v_x^7ds\right)\varsigma\overline{f^1}dx\\
&-g^0\int_0^L\lambda v^2\left(\varsigma'\overline{f^1}+\varsigma\overline{f_x^1}\right)dx+ig^0\int_0^L f_x^2\varsigma\overline{f^1}dx-i\int_0^L\int_0^{+\infty}g(s)f_x^7\varsigma\overline{f^1}dx=o(1)\,.
\end{aligned}\tag{2.76}
$$

Finally, inserting (2.76) in (2.74) and using the fact that $\tilde{k}_2=k_2-g^0$, we get (2.71).

(2) Our next aim is to prove

$$
k_1\int_0^L\varsigma\left|v_x^1\right|^2dx+\lambda^2k_2\left(\frac{\rho_2}{k_2}-\frac{\rho_1}{k_1}\right)\int_0^L\varsigma v_x^2\overline{v^1}dx=o(1)\,.\tag{2.77}
$$

Multiplying (2.75) by $\frac{\rho_1}{k_1}\lambda\varsigma\overline{v^1}$ in $L_g^2\left(\mathbb{R}_+,L^2\right)$ and using by parts integration. Then, using hypothesis (H), Lemma 2.11, and the facts that λv^1 is uniformly bounded in $L^2(0,L)$, f^2 converges to zero in $H_0^1(0,L)$, f^7 converges to zero in $L_g^2(\mathbb{R}_+;H_0^1(0,L))$, we get

$$
\begin{aligned}
&\frac{\rho_1\lambda^2}{k_1k_2}\int_0^L\int_0^{+\infty}g(s)\,v_x^7\varsigma\overline{v^1}dsdx=-i\frac{\rho_1}{k_1k_2}\int_0^L\int_0^{+\infty}g'(s)\,v_x^7\varsigma\lambda\overline{v^1}dsdx\\
&+\frac{\rho_1\lambda^2}{k_1k_2}g^0\int_0^L v_x^2\varsigma\overline{v^1}dx-i\frac{\rho_1}{k_1k_2}\int_0^L\int_0^{+\infty}g(s)\,(f_x^7-f_x^2)\varsigma\lambda\overline{v^1}dsdx=o(1)\,.
\end{aligned}\tag{2.78}
$$

Adding (2.71) and (2.78), we deduce (2.77).

(3) Finally, using condition (2.47) in (2.77), we get the first estimation of (2.70). Moreover, multiplying (2.53) by $\varsigma\overline{v^1}$ in $L^2(0,L)$, using (2.48), (2.49), (2.65), and the first estimation of (2.70), we can easily prove that

$$
\int_0^L\varsigma\left|\lambda v^1\right|^2dx=o(1)\,.
$$

Thus the proof is complete. □

Lemma 2.15. *Assume that hypothesis* (H) *and* (2.47) *are satisfied. Then*

$$
\int_0^L\varsigma\left|v_x^3\right|^2dx=o(1)\quad\text{and}\quad\int_0^L\varsigma\left|\lambda v^3\right|^2dx=o(1)\,.\tag{2.79}
$$

Proof. Multiplying (2.53) by $\varsigma\overline{v_x^3}$ in $L^2(0,L)$ and using by parts integration. Then, using Lemmas 2.12, 2.14 and the facts that f^1 converges to zero in $H_0^1(0,L)$, f^4 converges to zero in $L^2(0,L)$, we get

$$
\rho_1\int_0^L\lambda^2v^1\varsigma\overline{v_x^3}dx+1(k_1+k_3)\int_0^L\varsigma\left|v_x^3\right|^2dx-k_1\int_0^L v_x^1\varsigma\overline{v_{xx}^3}dx=o(1)\,.\tag{2.80}
$$

Moreover, multiplying (2.55) by $\varsigma\overline{v_x^1}$ in $L^2(0,L)$ and using by parts integration. Then, using Lemmas 2.12, 2.14 and the facts that λv^3 is uniformly bounded in $L^2(0,L)$, f^3 converges to zero in $H_0^1(0,L)$, f^6 converges to zero in $L^2(0,L)$, we get

$$
-\rho_1\int_0^L\lambda^2v_x^3\varsigma\overline{v^1}dx+k_3\int_0^L\overline{v_x^1}\varsigma v_{xx}^3dx=o(1)\,.\tag{2.81}
$$

Take the real part of the sum of (2.80) and (2.81). Then, using the fact that $k_1 = k_3$, we get

$$\int_0^L \varsigma \left|v_x^3\right|^2 dx = o(1). \tag{2.82}$$

Next, if we multiplying (2.55) by $\varsigma\overline{v^3}$ in $L^2(0,L)$, then from (2.65), (2.82) and Lemma 2.14, we deduce that

$$\rho_1 \int_0^L \varsigma \left|\lambda v^3\right|^2 dx = o(1).$$

Thus the proof is complete. □

Proof of Theorem 2.10 Using Lemma 2.11, Lemma 2.12, Lemma 2.14, and Lemma 2.15, we get $\|U\|_{\mathcal{H}} = o(1)$ on $[\alpha+\epsilon, \beta-\epsilon]$. Hence, by Lemma 2.13, we get $\|U\|_{\mathcal{H}} = o(1)$ on $[0, L]$ which contradicts (2.48). Therefore, (H2) holds and so, by [9] and [19], we deduce the exponential stability of the system (2.4)-(2.5) propagating with equal speeds.

Remark 2.16. *It is easy to see that our technique used for the proof of the exponential stability of the Bresse system under fully Dirichlet boundary conditions is also valid under mixed boundary conditions.*

2.4. **Lack of exponential stability with different speed.** In this part, our goal is to show that the elastic Bresse system (2.4)-(2.5) with fully Dirichlet boundary conditions is not exponentially stable if the speeds of propagation of the waves are different. In particular, we consider the case when $\mathrm{l} \to 0$; i.e, when (2.4)-(2.5) reduces to the Timoshenko system (2.83)-(2.84) with $\frac{\rho_1}{k_1} \neq \frac{\rho_2}{k_2}$. In fact, when the speeds of propagation are different, if mixed Dirichlet-Neumann boundary conditions are considered in system (2.4) instead of fully Dirichlet boundary conditions, then we can easily show that the system is not exponentially decaying. Indeed, similar to [1], [16], [6], [10], and [22], the idea is to find a sequence of $(\lambda_n)_n \subseteq \mathbb{R}$ with $|\lambda_n| \longrightarrow +\infty$ and a sequence of vectors $(U_n)_n \subseteq D(\mathcal{A})$ with $\|U_n\|_{\mathcal{H}} = 1$ such that $\|(i\lambda_n Id - \mathcal{A})U_n\|_{\mathcal{H}} \longrightarrow 0$. In the case of Dirichlet-Neumann-Neumann boundary condition, this approach worked well due to the fact that all eigenmodes are separable, i.e., the system operator can be decomposed to a block-diagonal form according to the frequency when the state variables are expanded into Fourier series. However, in the case of fully Dirichlet boundary conditions, this approach has no success in the literature to our knowledge and the problem is still be open. Consequently, in this section, we use another approach based on the behavior of the spectrum to prove the lack of exponential stability of the system mainly in the case when $\mathrm{l} \to 0$. For simplicity, in this section, we take $L = 1$ so (2.4)-(2.5) reduces to the following Timoshenko system:

$$\begin{cases} \rho_1 \varphi_{tt} - k_1 (\varphi_x + \psi)_x = 0, \\ \rho_2 \psi_{tt} - \widetilde{k}_2 \psi_{xx} + k_1 (\varphi_x + \psi) - \int_0^{+\infty} g(s)\, \eta_{xx} ds = 0, \\ \eta_t + \eta_s - \psi_t = 0, \end{cases} \tag{2.83}$$

with the initial conditions

$$\begin{array}{l} \varphi(\cdot,0) = \varphi_0(\cdot),\ \psi(\cdot,-t) = \psi_0(\cdot,t), \\ \varphi_t(\cdot,0) = \varphi_1(\cdot),\ \psi_t(\cdot,0) = \psi_1(\cdot), \\ \eta^0(\cdot,s) := \eta(\cdot,0,s) = \psi_0(\cdot,0) - \psi_0(\cdot,s) \quad \text{in } (0,1),\ s \geq 0, \end{array}$$

and fully Dirichlet boundary conditions

$$\begin{array}{ll} \varphi(0,\cdot) = \varphi(1,\cdot) = \psi(0,\cdot) = \psi(1,\cdot) = 0 & \text{in } \mathbb{R}_+, \\ \eta(0,\cdot,\cdot) = \eta(1,\cdot,\cdot) = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}_+, \\ \eta(\cdot,\cdot,0) = 0 & \text{in } (0,1) \times \mathbb{R}_+. \end{array} \tag{2.84}$$

In this case, the energy space $\mathcal{H}$ reduces to

$$\mathcal{H}_1 = \left(H_0^1(0,1)\right)^2 \times \left(L^2(0,1)\right)^2 \times L_g^2\left(\mathbb{R}_+, H_0^1\right)$$

and the generator $\mathcal{A}$ becomes the operator $\mathcal{A}_1$ defined by

$$D(\mathcal{A}_1) = \left\{ U = (v^1, v^2, v^3, v^4, v^5)^\mathsf{T} \in \mathcal{H}_1 \mid v^1 \in H^2(0,1),\ v^3, v^4 \in H_0^1(0,1),\ v_s^5 \in L_g^2\left(\mathbb{R}_+, H_0^1\right), \right.$$
$$\left. v^2 + \int_0^{+\infty} g(s)\, v^5 ds \in H^2(0,1) \cap H_0^1(0,1),\ v^5(x,0) = 0 \right\}$$

and

$$\mathcal{A}_1 U = \begin{pmatrix} v^3 \\ v^4 \\ \rho_1^{-1} k_1 \left(v_x^1 + v^2\right)_x \\ \rho_2^{-1}\left(\widetilde{k}_2 v_{xx}^2 - k_1\left(v_x^1 + v^2\right) + \int_0^{+\infty} g\left(s\right) v_{xx}^5 ds\right) \\ v^2 - v_s^5 \end{pmatrix} \tag{2.85}$$

for all $U = \left(v^1, v^2, v^3, v^4, v^5\right)^{\mathsf{T}} \in D\left(\mathcal{A}_1\right).$

Throughout this part, in addition to hypothesis (H), we assume that

$$\frac{\rho_1}{k_1} \neq \frac{\rho_2}{k_2} \text{ and } \left|g''\left(s\right)\right| \leq c_2 g\left(s\right) \text{ for some } c_2 > 0. \tag{H$'$}$$

Theorem 2.17. *Under hypothesis* (H) *and (H$'$), system* (2.83)-(2.84) *is not uniformly stable in the energy space* $\mathcal{H}_1$.

For the proof of Theorem 2.17, we aim to show that an infinite number of eigenvalues of $\mathcal{A}_1$ approach the imaginary axis which prevents the Timoshenko system (2.83)-(2.84) from being exponentially stable. First we determine the characteristic equation satisfied by the eigenvalues of $\mathcal{A}_1$. For this aim, Let $\lambda \in \mathbb{C}$ be an eigenvalue of $\mathcal{A}_1$ and let $U = \left(v^1, v^2, v^3, v^4, v^5\right)^{\mathsf{T}} \in D(\mathcal{A}_1)$ be an associated eigenvector such that $\|U\|_{\mathcal{H}_1} = 1$. Then

$$v^3 = \lambda v^1, \tag{2.86}$$

$$v^4 = \lambda v^2, \tag{2.87}$$

$$k_1\left(v_x^1 + v^2\right)_x = \rho_1 \lambda v^3, \tag{2.88}$$

$$\widetilde{k}_2 v_{xx}^2 - k_1\left(v_x^1 + v^2\right) + \int_0^{+\infty} g\left(s\right) v_{xx}^5 ds = \rho_2 \lambda v^4, \tag{2.89}$$

$$v^4 - v_s^5 = \lambda v^5. \tag{2.90}$$

From (2.90) and (2.87), we have

$$v_s^5 + \lambda v^5 = \lambda v^2.$$

Integrating this equation and using the fact that $v^5\left(x,0\right) = 0$, we get

$$v^5 = v^2\left(1 - e^{-\lambda s}\right). \tag{2.91}$$

Inserting (2.91), (2.86)-(2.87) in (2.88)-(2.89), we get

$$\frac{k_1}{\rho_1}\left(v_x^1 + v^2\right)_x = \lambda^2 v^1, \tag{2.92}$$

$$\frac{\underline{k_2}}{\rho_2} v_{xx}^2 - \frac{k_1}{\rho_2}\left(v_x^1 + v^2\right) = \lambda^2 v^2, \tag{2.93}$$

where $\underline{k_2} = k_2 - \int_0^{+\infty} g\left(s\right) e^{-\lambda s} ds$. Equivalently, we have

$$\begin{cases} v_{xxxx}^2 - \left(\dfrac{\rho_2}{\underline{k_2}} + \dfrac{\rho_1}{k_1}\right)\lambda^2 v_{xx}^2 + \dfrac{\rho_1 \rho_2}{k_1 \underline{k_2}} \lambda^2 \left(\lambda^2 + \dfrac{k_1}{\rho_2}\right) v^2 = 0, \\[2ex] v^2\left(\zeta\right) = 0,\ v_{xxx}^2\left(\zeta\right) - \dfrac{\rho_2}{\underline{k_2}} \lambda^2 v_x^2\left(\zeta\right) = 0,\ \zeta = 0, 1. \end{cases} \tag{2.94}$$

The solution of (2.94) is given by

$$v^2\left(x\right) = \sum_{j=1}^{4} c_j e^{r_j x},$$

where $c_j \in \mathbb{C}$ for all $1 \leq j \leq 4$ and

$$\begin{cases} r_1 = \lambda \sqrt{\dfrac{\left(\frac{\rho_2}{\underline{k}_2} + \frac{\rho_1}{k_1}\right) + \sqrt{\left(\frac{\rho_2}{\underline{k}_2} - \frac{\rho_1}{k_1}\right)^2 - \frac{4\rho_1}{\underline{k}_2 \lambda^2}}}{2}}, \ r_2 = -r_1, \\ r_3 = \lambda \sqrt{\dfrac{\left(\frac{\rho_2}{\underline{k}_2} + \frac{\rho_1}{k_1}\right) - \sqrt{\left(\frac{\rho_2}{\underline{k}_2} - \frac{\rho_1}{k_1}\right)^2 - \frac{4\rho_1}{\underline{k}_2 \lambda^2}}}{2}}, \ r_4 = -r_3. \end{cases}$$

The boundary conditions in (2.94) can be expressed by

$$MC = 0$$

where

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ e^{r_1} & e^{-r_1} & e^{r_3} & e^{-r_3} \\ f(r_1) & -f(r_1) & f(r_3) & -f(r_3) \\ f(r_1)e^{r_1} & -f(r_1)e^{-r_1} & f(r_3)e^{r_3} & -f(r_3)e^{-r_3} \end{pmatrix}, \ C = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix},$$

and $f(r) = r^3 - \frac{\rho_2}{\underline{k}_2} r\lambda^2$. For shortness, denote by $f(r_1) = f_1$ and $f(r_3) = f_3$. Then

$$\begin{cases} f_1 = \dfrac{r_1\lambda^2}{2}\left[\left(\dfrac{\rho_1}{k_1} - \dfrac{\rho_2}{\underline{k}_2}\right) + \sqrt{\left(\dfrac{\rho_2}{\underline{k}_2} - \dfrac{\rho_1}{k_1}\right)^2 - \dfrac{4\rho_1}{\underline{k}_2\lambda^2}}\right], \\ f_3 = \dfrac{r_3\lambda^2}{2}\left[\left(\dfrac{\rho_1}{k_1} - \dfrac{\rho_2}{\underline{k}_2}\right) - \sqrt{\left(\dfrac{\rho_2}{\underline{k}_2} - \dfrac{\rho_1}{k_1}\right)^2 - \dfrac{4\rho_1}{\underline{k}_2\lambda^2}}\right], \end{cases} \tag{2.95}$$

and

$$\begin{cases} (f_1 + f_3)^2 = \lambda^6 \dfrac{\rho_1}{k_1}\left(\dfrac{\rho_1}{k_1} - \dfrac{\rho_2}{\underline{k}_2}\right)^2 - \lambda^4 \dfrac{\rho_1}{\underline{k}_2}\left(\dfrac{3\rho_1}{k_1} - \dfrac{\rho_2}{\underline{k}_2}\right) + 2\lambda^4 \dfrac{\rho_1}{\underline{k}_2}\sqrt{\dfrac{\rho_1\rho_2}{k_1\underline{k}_2}\left(1 + \dfrac{k_1}{\rho_2\lambda^2}\right)}, \\ (f_1 - f_3)^2 = \lambda^6 \dfrac{\rho_1}{k_1}\left(\dfrac{\rho_1}{k_1} - \dfrac{\rho_2}{\underline{k}_2}\right)^2 - \lambda^4 \dfrac{\rho_1}{\underline{k}_2}\left(\dfrac{3\rho_1}{k_1} - \dfrac{\rho_2}{\underline{k}_2}\right) - 2\lambda^4 \dfrac{\rho_1}{\underline{k}_2}\sqrt{\dfrac{\rho_1\rho_2}{k_1\underline{k}_2}\left(1 + \dfrac{k_1}{\rho_2\lambda^2}\right)}. \end{cases} \tag{2.96}$$

Therefore, using (2.95) and (2.96), we get

$$\begin{aligned} det(M) &= -2(f_1 - f_3)^2 \cosh(r_1 + r_3) + 2(f_1 + f_3)^2 \cosh(r_1 - r_3) - 8 f_1 f_3 \\ &= -4\lambda^6 \frac{\rho_1}{k_1}\left(\frac{\rho_1}{k_1} - \frac{\rho_2}{\underline{k}_2}\right)^2 \sinh(r_1)\sinh(r_3) + 4\lambda^4 \frac{\rho_1}{\underline{k}_2}\left(\frac{3\rho_1}{k_1} - \frac{\rho_2}{\underline{k}_2}\right)\sinh(r_1)\sinh(r_3) \\ &\quad -8\lambda^4 \frac{\rho_1}{\underline{k}_2}\sqrt{\frac{\rho_1\rho_2}{k_1\underline{k}_2}\left(1 + \frac{k_1}{\rho_2\lambda^2}\right)}\cosh(r_1)\cosh(r_3) - 8\lambda^4 \frac{\rho_1}{\underline{k}_2}\sqrt{\frac{\rho_1\rho_2}{k_1\underline{k}_2}\left(1 + \frac{k_1}{\rho_2\lambda^2}\right)}. \end{aligned}$$

Equation (2.94) admits a non trivial solution if and only if $det(M) = 0$; i.e, if and only if the eigenvalues of $\mathcal{A}_1$ are roots of the function F defined by:

$$\begin{aligned} F(\lambda) = \ & \frac{\lambda^6}{k_1}\left(\frac{\rho_1}{k_1} - \frac{\rho_2}{\underline{k}_2}\right)^2 \sinh(r_1)\sinh(r_3) - \frac{\lambda^4}{\underline{k}_2}\left(\frac{3\rho_1}{k_1} - \frac{\rho_2}{\underline{k}_2}\right)\sinh(r_1)\sinh(r_3) \\ & + \frac{2\lambda^4}{\underline{k}_2}\sqrt{\frac{\rho_1\rho_2}{k_1\underline{k}_2}\left(1 + \frac{k_1}{\rho_2\lambda^2}\right)}\cosh(r_1)\cosh(r_3) + \frac{2\lambda^4}{\underline{k}_2}\sqrt{\frac{\rho_1\rho_2}{k_1\underline{k}_2}\left(1 + \frac{k_1}{\rho_2\lambda^2}\right)}. \end{aligned} \tag{2.97}$$

Lemma 2.18. *Let $\lambda \in \mathbb{C}$ be an eigenvalue of $\mathcal{A}_1$. Then $\Re(\lambda)$ is bounded.*

Proof. Multiplying (2.92) and (2.93) by $-\rho_1\overline{\varphi}$, and $-\rho_2\overline{\psi}$ respectively, and integrating their sum, we get

$$\rho_1 \|\lambda\varphi\|^2 + \rho_2 \|\lambda\psi\|^2 + k_1 \|\varphi_x + \psi\|^2 + k_2 \|\psi_x\|^2 - \|\psi_x\|^2 \int_0^{+\infty} g(s)\, e^{-\lambda s} ds = 0.$$

Since $\|U\|_{\mathcal{H}_1} = 1$ then $\rho_1 \|\lambda\varphi\|^2 + \rho_2 \|\lambda\psi\|^2 + k_1 \|\varphi_x + \psi\|^2 + k_2 \|\psi_x\|^2$ and $\|\psi_x\|^2$ are bounded. Therefore

$$\int_0^{+\infty} g(s)\, e^{-\lambda s} ds < +\infty. \tag{2.98}$$

Hence,

$$\lim_{s \to +\infty} g(s)\, e^{-\Re(\lambda) s} = 0.$$

Since $\mathcal{A}_1$ is dissipative in $\mathcal{H}_1$ then $\Re(\lambda) \leq 0$ and consequentially there exists constant $a > 0$ such that

$$-a \leq \Re(\lambda) < 0$$

and hence the proof is complete. □

Proposition 2.19. *Assume that hypothesis* (H) *and (H′) are satisfied. Then, there exist* $n_0, n_0' \in \mathbb{N}$ *sufficiently large such that*

$$\sigma(\mathcal{A}_1) \supset \tilde{\sigma_0} \cup \tilde{\sigma_1}, \tag{2.99}$$

where $\tilde{\sigma_0} \cup \tilde{\sigma_1}$ *is the set of eigenvalues of the operator* $\mathcal{A}_1$ *such that*

$$\tilde{\sigma_1} = \left\{ \tilde{\lambda}_j^{(0)}, \tilde{\lambda}_j^{(1)} \right\}_{j \in J}, \quad \tilde{\sigma_0} = \left\{ \lambda_n^{(0)},\ \lambda_{n'}^{(1)} \right\}_{\substack{n,\ n' \in \mathbb{Z} \\ |n| \geq n_0,\ |n'| \geq n_0'}}, \quad \tilde{\sigma_0} \cap \tilde{\sigma_1} = \emptyset, \tag{2.100}$$

where J *is a finite set. Moreover,* $\lambda_n^{(0)}$ *and* $\lambda_n^{(1)}$ *are simple and satisfies the following asymptotic behavior*

$$\lambda_n^{(0)} = in\pi \sqrt{\frac{k_2}{\rho_2}} - \frac{g(0)}{2k_2} + o(1), \quad \forall\ |n| \geq n_0 \tag{2.101}$$

and

$$\lambda_{n'}^{(1)} = in'\pi \sqrt{\frac{k_1}{\rho_1}} + o(1), \quad \forall\ |n'| \geq n_0'. \tag{2.102}$$

Proof. The proof is divided into three steps. Step 1 and Step 2 furnish an asymptotic development of the characteristic equation for large λ. Step 3 gives a limited development of the large eigenvalues λ.

Step 1. In this step, we prove the following asymptotic behavior estimate

$$\frac{1}{\underline{k}_2} = \frac{1}{k_2} + \frac{g(0)}{k_2^2 \lambda} + O\left(\frac{1}{\lambda^2}\right). \tag{2.103}$$

Indeed, integration by parts yields

$$\underline{k}_2 = k_2 - \int_0^{+\infty} g(s)\, e^{-\lambda s} ds = k_2 - \frac{g(0)}{\lambda} - \frac{g'(0)}{\lambda^2} - \frac{1}{\lambda^2} \int_0^{+\infty} g''(s)\, e^{-\lambda s} ds. \tag{2.104}$$

From hypothesis (H′), since $|g''(s)| \leq c_2 g(s)$, then

$$\left| \int_0^{+\infty} g''(s)\, e^{-\lambda s} ds \right| \leq c_2 \int_0^{+\infty} g(s)\, e^{\Re(\lambda) s} ds. \tag{2.105}$$

on the other hand, since

$$\int_0^1 \int_0^{+\infty} g(s)\, |\eta_x|^2\, ds\ dx < +\infty,$$

then, from (2.91) and (2.105), we get

$$\int_0^{+\infty} g''(s)\, e^{-\lambda s} ds = O(1). \tag{2.106}$$

Finally, (2.104) and (2.106) yield (2.103).

Step 2. In this step, we furnish an asymptotic development of the function $F(\lambda)$ for large λ. Assume that $\frac{\rho_1}{k_1} \neq \frac{\rho_2}{k_2}$, then we have

$$
\left\{
\begin{aligned}
r_1 &= \lambda \sqrt{\frac{\frac{\rho_2}{k_2} + \frac{\rho_1}{k_1} + \left|\frac{\rho_2}{k_2} - \frac{\rho_1}{k_1}\right|}{2}} + \frac{\rho_2 g(0)}{2\sqrt{2}k_2^2} \frac{1 + sign\left(\frac{\rho_2}{k_2} - \frac{\rho_1}{k_1}\right)}{\sqrt{\frac{\rho_2}{k_2} + \frac{\rho_1}{k_1} + \left|\frac{\rho_2}{k_2} - \frac{\rho_1}{k_1}\right|}} + O\left(\frac{1}{\lambda}\right), \\
r_3 &= \lambda \sqrt{\frac{\frac{\rho_2}{k_2} + \frac{\rho_1}{k_1} - \left|\frac{\rho_2}{k_2} - \frac{\rho_1}{k_1}\right|}{2}} + \frac{\rho_2 g(0)}{2\sqrt{2}k_2^2} \frac{1 - sign\left(\frac{\rho_2}{k_2} - \frac{\rho_1}{k_1}\right)}{\sqrt{\frac{\rho_2}{k_2} + \frac{\rho_1}{k_1} - \left|\frac{\rho_2}{k_2} - \frac{\rho_1}{k_1}\right|}} + O\left(\frac{1}{\lambda}\right),
\end{aligned}
\right. \tag{2.107}
$$

where $sign\left(\frac{\rho_2}{k_2} - \frac{\rho_1}{k_1}\right) = \frac{\left|\frac{\rho_2}{k_2} - \frac{\rho_1}{k_1}\right|}{\frac{\rho_2}{k_2} - \frac{\rho_1}{k_1}}$. If $sign\left(\frac{\rho_2}{k_2} - \frac{\rho_1}{k_1}\right) = 1$, then (2.107) is equivalent to

$$
\left\{
\begin{aligned}
r_1 &= \lambda \sqrt{\frac{\rho_2}{k_2}} + \frac{g(0)}{2k_2}\sqrt{\frac{\rho_2}{k_2}} + O\left(\frac{1}{\lambda}\right), \\
r_3 &= \lambda \sqrt{\frac{\rho_1}{k_1}} + O\left(\frac{1}{\lambda}\right).
\end{aligned}
\right. \tag{2.108}
$$

If $sign\left(\frac{\rho_2}{k_2} - \frac{\rho_1}{k_1}\right) = -1$, then (2.107) is equivalent to

$$
\left\{
\begin{aligned}
r_1 &= \lambda \sqrt{\frac{\rho_1}{k_1}} + O\left(\frac{1}{\lambda}\right). \\
r_3 &= \lambda \sqrt{\frac{\rho_2}{k_2}} + \frac{g(0)}{2k_2}\sqrt{\frac{\rho_2}{k_2}} + O\left(\frac{1}{\lambda}\right).
\end{aligned}
\right. \tag{2.109}
$$

In the sequel, we suppose that (2.108) holds since the analysis follows similarly. Now, inserting (2.108) in (2.97) and using Lemma 2.18, we get

$$
F(\lambda) = \frac{\lambda^6}{k_1}\left(\frac{\rho_1}{k_1} - \frac{\rho_2}{k_2}\right)^2 \sinh(r_1)\sinh(r_3) + O(\lambda^5). \tag{2.110}
$$

Step 3. In this step, we perform a limited development of of the large eigenvalues of the operator $\mathcal{A}_1$. Let λ be a large eigenvalue of $\mathcal{A}_1$, then from (2.110), λ is large root of the following asymptotic equation

$$
h(\lambda) = h_0(\lambda) + O\left(\frac{1}{\lambda}\right) = 0, \tag{2.111}
$$

where $h_0(\lambda) = \sinh(r_1)\sinh(r_3)$. Now, we prove that

$$
h_0(\lambda) = 0 \;\text{ if and only if }\; r_1 = in\pi \;\text{ and } r_3 = in'\pi, \quad n, n' \in \mathbb{Z}.
$$

Indeed, Suppose that

$$
r_1 = in\pi, \;\text{ and } r_3 \neq in'\pi, \quad n, n' \in \mathbb{Z}.
$$

Then

$$
M = \begin{pmatrix}
1 & 1 & 1 & 1 \\
(-1)^n & (-1)^n & e^{r_3} & e^{-r_3} \\
f_1 & -f_1 & f_3 & -f_3 \\
(-1)^n f_1 & -(-1)^n f_1 & f_3 e^{r_3} & -f_3 e^{-r_3}
\end{pmatrix}.
$$

Using Gaussian elimination, M is equivalent to the following matrix, denoted by

$$
\widetilde{M} = \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 0 & e^{r_3} - (-1)^n & e^{-r_3} - (-1)^n \\
f_1 & -f_1 & f_3 & -f_3 \\
0 & 0 & f_3\left(e^{r_3} - (-1)^n\right) & -f_3\left(e^{-r_3} - (-1)^n\right)
\end{pmatrix}. \tag{2.112}
$$

Hence,

$$
\left\{
\begin{aligned}
\left(e^{r_3} - (-1)^n\right)c_3 + \left(e^{-r_3} - (-1)^n\right)c_4 &= 0 \\
f_3\left(e^{r_3} - (-1)^n\right)c_3 - f_3\left(e^{-r_3} - (-1)^n\right)c_4 &= 0.
\end{aligned}
\right.
$$

From (2.95), we can check that $f_1 \neq 0$ and $f_3 \neq 0$ for λ large enough. Since $r_3 \neq in'\pi$ for all $n' \in \mathbb{Z}$, then

$$c_3 = c_4 = 0.$$

From (2.112), we have

$$\begin{cases} c_1 + c_2 = 0 \\ f_1 c_1 - f_1 c_2 = 0. \end{cases}$$

Since $f_1 \neq 0$, we get

$$c_1 = c_2 = 0 \quad \text{and} \quad v^2 = 0$$

which is a contradiction with $\|U\|_{\mathcal{H}_1} = 1$. Similarly if

$$r_1 \neq in\pi \text{ and } r_3 = in'\pi, \quad n, n' \in \mathbb{Z}$$

we get $v^2 = 0$. We conclude that

$$h_0(\lambda) = 0 \quad \Leftrightarrow \quad r_1 = in\pi \text{ and } r_3 = in'\pi, \quad n, n' \in \mathbb{Z}.$$

Then from asymptotic equation (2.108), the large roots of h_0 satisfy the following asymptotic equations

$$\mu_n^{(0)} = in\pi\sqrt{\frac{k_2}{\rho_2}} - \frac{g(0)}{2k_2} + O\left(\frac{1}{n}\right), \quad \forall\, |n| \geq n_0 \tag{2.113}$$

and

$$\mu_{n'}^{(1)} = in'\pi\sqrt{\frac{k_1}{\rho_1}} + O\left(\frac{1}{n}\right), \quad \forall\, |n'| \geq n_0'. \tag{2.114}$$

Next, with the help of Rouché's Theorem and using the asymptotic equation (2.111), it is easy to see that the large roots of h, $\lambda_n^{(0)}$ and $\lambda_n^{(1)}$, are closed to those of h_0. Thus the proof is complete.

□

Proof of Theorem 2.17 From Proposition 2.19, the operator $\mathcal{A}_1$ has two branches of eigenvalues, the energy corresponding to the first branch $\lambda_n^{(0)}$ decays exponentially and the energy corresponding to the second branch of eigenvalues $\lambda_{n'}^{(1)}$ has no exponential decaying. Therefore the total energy of the Timoshenko system (2.83)-(2.84) has no exponential decaying when $\frac{\rho_1}{k_1} \neq \frac{\rho_2}{k_2}$. The proof is thus complete.

2.5. **Polynomial stability in the general case.** In this part, we prove that the system (2.4)-(2.5) is polynomially stable if (2.47) is not satisfied. We prove the following Theorems.

Theorem 2.20. *Under hypothesis* (H), *if*

$$\frac{\rho_1}{k_1} \neq \frac{\rho_2}{k_2} \quad \text{and} \quad k_1 \neq k_3, \tag{2.115}$$

then there exists $c > 0$ *such that for every* $U^0 \in D(\mathcal{A})$, *we have*

$$E(t) \leq \frac{c}{\sqrt{t}} \left\|U^0\right\|^2_{D(\mathcal{A})}, \qquad t > 0. \tag{2.116}$$

Theorem 2.21. *Under hypothesis* (H), *if*

$$\frac{\rho_1}{k_1} \neq \frac{\rho_2}{k_2} \quad \text{and} \quad k_1 = k_3, \tag{2.117}$$

then there exists $c > 0$ *such that for every* $U^0 \in D(\mathcal{A})$, *we have*

$$E(t) \leq \frac{c}{t} \left\|U^0\right\|^2_{D(\mathcal{A})}, \qquad t > 0. \tag{2.118}$$

Since $i\mathbb{R} \subseteq \rho(\mathcal{A})$, then for the proof of Theorem 2.20 and Theorem 2.21, according to [3] (see also [12]), we still need to prove that

$$\sup_{\lambda \in \mathbb{R}} \left\|(i\lambda Id - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} = O\left(|\lambda|^l\right) \qquad \text{(H3)},$$

where $l = 4$ if condition (2.115) holds and $l = 2$ if condition (2.117) holds. By a contradiction argument, suppose there exist a sequence of real numbers $(\lambda_n)_n$, with $\lambda_n \to +\infty$, and a sequence of vectors

$$U_n = \left(v_n^1, v_n^2, v_n^3, v_n^4, v_n^5, v_n^6, v_n^7\right)^{\mathsf{T}} \in D(\mathcal{A}) \quad \text{with} \quad \|U_n\|_{\mathcal{H}} = 1 \tag{2.119}$$

such that

$$\lambda_n^l \left(i\lambda_n U_n - \mathcal{A} U_n\right) = \left(f_n^1, f_n^2, f_n^3, f_n^4, f_n^5, f_n^6, f_n^7\right)^{\mathsf{T}} \to 0 \ \text{ in } \mathcal{H}. \tag{2.120}$$

Equivalently, we have

$$i\lambda_n v_n^1 - v_n^4 = h_n^1, \tag{2.121}$$

$$i\lambda_n v_n^2 - v_n^5 = h_n^2, \tag{2.122}$$

$$i\lambda_n v_n^3 - v_n^6 = h_n^3, \tag{2.123}$$

$$\rho_1\lambda_n^2 v_n^1 + k_1\left[\left(v_n^1\right)_x + v_n^2 + \mathrm{l}v_n^3\right]_x + \mathrm{l}k_3\left[\left(v_n^3\right)_x - \mathrm{l}v_n^1\right] = h_n^4, \tag{2.124}$$

$$\rho_2\lambda_n^2 v_n^2 + \widetilde{k}_2\left(v_n^2\right)_{xx} - k_1\left[\left(v_n^1\right)_x + v_n^2 + \mathrm{l}v_n^3\right] + \int_0^{+\infty} g(s)\left(v_n^7\right)_{xx} ds = h_n^5, \tag{2.125}$$

$$\rho_1\lambda_n^2 v_n^3 + k_3\left[\left(v_n^3\right)_x - \mathrm{l}v_n^1\right]_x - \mathrm{l}k_1\left[\left(v_n^1\right)_x + v_n^2 + \mathrm{l}v_n^3\right] = h_n^6, \tag{2.126}$$

$$i\lambda_n v_n^7 + \left(v_n^7\right)_s - i\lambda_n v_n^2 = h_n^7, \tag{2.127}$$

where

$$\begin{cases} \lambda_n^l h_n^1 = f_n^1,\ \lambda_n^l h_n^2 = f_n^2,\ \lambda_n^l h_n^3 = f_n^3, \lambda_n^l h_n^7 = f_n^7 - f_n^2, \\ \lambda_n^l h_n^4 = -\rho_1\left(f_n^4 + i\lambda_n f_n^1\right),\ \lambda_n^l h_n^5 = -\rho_2\left(f_n^5 + i\lambda_n f_n^2\right),\ \lambda_n^l h_n^6 = -\rho_1\left(f_n^6 + i\lambda_n f_n^3\right). \end{cases}$$

In the following we will check the condition (H3) by finding a contradiction with (2.119) such as $\|U_n\|_{\mathcal{H}} = o(1)$. For clarity, we divide the proof into several lemmas. From now on, for simplicity, we drop the index n. From (2.121)-(2.123), we remark that

$$\left\|v^1\right\| = O\left(\frac{1}{\lambda}\right),\ \left\|v^2\right\| = O\left(\frac{1}{\lambda}\right),\ \left\|v^3\right\| = O\left(\frac{1}{\lambda}\right). \tag{2.128}$$

Therefore, from (2.124)-(2.126), we remark that

$$\left\|v_{xx}^1\right\| = O(\lambda),\ \left\|v_{xx}^2 + \int_0^{+\infty} g(s)\, v_{xx}^7 ds\right\| = O(\lambda),\ \left\|v_{xx}^3\right\| = O(\lambda). \tag{2.129}$$

Lemma 2.22. *Let $l \geq 0$. Under hypothesis* (H)*, we have*

$$\int_0^L \int_0^{+\infty} g(s)\left|v_x^7\right|^2 ds dx = o\left(\frac{1}{\lambda^l}\right). \tag{2.130}$$

Proof. Taking the inner product of (2.120) with U in $\mathcal{H}$. Then, using (2.14) and the fact that U is uniformly bounded in $\mathcal{H}$, we get

$$\frac{1}{2}\int_0^L \int_0^{+\infty} g'(s)\left|v_x^7\right|^2 ds dx = \Re\left(\langle \mathcal{A}U, U\rangle_{\mathcal{H}}\right) = -\Re\left(\langle i\lambda U - \mathcal{A}U, U\rangle_{\mathcal{H}}\right) = o\left(\frac{1}{\lambda^l}\right). \tag{2.131}$$

Using condition (H) in (2.131), we get

$$\int_0^L \int_0^{+\infty} g(s)\left|v_x^7\right|^2 ds dx = o\left(\frac{1}{\lambda^l}\right).$$

Thus the proof is complete. □

Lemma 2.23. *Let $l \geq 0$. Under hypothesis* (H)*, we have*

$$\int_0^L \left|v_x^2\right|^2 dx = o\left(\frac{1}{\lambda^{\frac{l}{2}}}\right). \tag{2.132}$$

Proof. Multiplying (2.127) by $\overline{v^2}$ in $L_g^2\left(\mathbb{R}_+, H_0^1\right)$. Then, using the fact that $\left\|v^2\right\|_g^2 = g^0\left\|v_x^2\right\|^2$, v_x^2 is uniformly bounded in $L^2(0,L)$, f^2 converges to zero in $H_0^1(0,L)$ and f^7 converges to zero in $L_g^2\left(\mathbb{R}_+, H_0^1\right)$, we get

$$g^0\lambda\int_0^L \left|v_x^2\right|^2 dx = \lambda\int_0^L\int_0^{+\infty} g(s)\, v_x^7\overline{v_x^2} ds dx - i\int_0^L\int_0^{+\infty} g(s)\, v_{xs}^7\overline{v_x^2} ds dx + o\left(\frac{1}{\lambda^l}\right). \tag{2.133}$$

From equation (2.119) and Lemma 2.22, we get

$$\lambda\int_0^L\int_0^{+\infty} g(s)\, v_x^7\overline{v_x^2} ds dx = o\left(\frac{1}{\lambda^{\frac{l}{2}-1}}\right). \tag{2.134}$$

Applying by parts integration, Holder's inequality in $L^2(0,L)$ and $L^2(0,+\infty)$. Then, using (2.131), the fact that v_x^2 is uniformly bounded in $L^2(0,L)$ and $\lim\limits_{s\to 0}\sqrt{g(s)}$ exists, we get

$$\left|\int_0^L\int_0^{+\infty} g(s)\, v_{xs}^7\overline{v_x^2} ds dx\right| \leq \lim_{s\to 0}\sqrt{g(s)}\left(\int_0^L\int_0^{+\infty} -g'(s)\left|v_x^7\right|^2 ds dx\right)^{1/2}\left\|v_x^2\right\| = o\left(\frac{1}{\lambda^{\frac{l}{2}}}\right). \tag{2.135}$$

Inserting (2.134) and (2.135) into (2.133), we deduce the estimation of (2.132). Thus the proof is complete. $\square$

Lemma 2.24. *Let $l \geq 0$. Under hypothesis* (H)*, we have*

$$\int_0^L \left|v_x^2\right|^2 dx = o\left(\frac{1}{\lambda^l}\right). \tag{2.136}$$

Proof. Let $l_N = \frac{l}{2}\sum_{k=0}^{N}\frac{1}{2^k}$. Since $\lim_{N\to+\infty} l_N = l$, it is enough to prove by induction on $N \in \mathbb{N}$ that

$$\int_0^L \left|v_x^2\right|^2 dx = o\left(\frac{1}{\lambda^{l_N}}\right). \tag{2.137}$$

When $N = 0$, estimation (2.137) holds by Lemma 2.23. Suppose that (2.137) holds for $N-1$; i.e,

$$\lambda^{l_{N-1}}\int_0^L \left|v_x^2\right|^2 dx = o\left(1\right). \tag{2.138}$$

Multiplying (2.127) by $\lambda^{l_N}\overline{v^2}$ in $L_g^2\left(\mathbb{R}_+, H_0^1\right)$. Then using the fact that $\left\|v^2\right\|_g^2 = g^0\left\|v_x^2\right\|^2$, v_x^2 is uniformly bounded in $L^2(0,L)$, f^2 converges to zero in $H_0^1\left(0,L\right)$ and f^7 converges to zero in $L_g^2\left(\mathbb{R}_+, H_0^1\right)$, we get

$$\begin{aligned}\lambda^{l_N} g^0\int_0^L \left|v_x^2\right|^2 dx &= \int_0^L\int_0^{+\infty} g\left(s\right)\lambda^{\frac{l}{2}} v_x^7 \lambda^{l_N-\frac{l}{2}}\overline{v_x^2}dsdx \\ &-\tfrac{i}{\lambda}\int_0^L\int_0^{+\infty} g\left(s\right)\lambda^{\frac{l}{2}} v_{xs}^7 \lambda^{l_N-\frac{l}{2}}\overline{v_x^2}dsdx + o\left(\frac{1}{\lambda^{l+1-l_N}}\right).\end{aligned} \tag{2.139}$$

Using the fact that $l_N - \frac{l}{2} = \frac{l_{N-1}}{2}$, Lemma 2.22, (2.135) and (2.138), we get

$$\lambda^{l_N}\int_0^L \left|v_x^2\right|^2 dx = o\left(1\right).$$

Thus the proof is complete. $\square$

Lemma 2.25. *Let $2 \leq l \leq 4$. Under hypothesis* (H)*, we have*

$$\int_0^L \varsigma\left|v_x^1\right|^2 dx = o\left(\frac{1}{\lambda^{\frac{l}{2}-1}}\right), \tag{2.140}$$

where ς is the cut-off function defined in Section 2.3.

Proof. Since $l \geq 2$, from Lemma 2.22 and Lemma 2.24 we have

$$\int_0^L \left|v_x^2\right|^2 = o\left(\frac{1}{\lambda^2}\right) \quad\text{and}\quad \int_0^L\int_0^{+\infty} g\left(s\right)\left|v_x^7\right|^2 dsdx = o\left(\frac{1}{\lambda^2}\right). \tag{2.141}$$

Multiplying (2.125) by $\varsigma\overline{v_x^1}$ in $L^2\left(0,L\right)$. Then, using (2.128) , (2.141), and the fact that v_x^1 is uniformly bounded in $L^2(0,L)$, f^2 converges to zero in $H_0^1\left(0,L\right)$, f^5 converges to zero in $L^2\left(0,L\right)$, we get we get

$$\int_0^L \varsigma\left|v_x^1\right|^2 dx = o\left(1\right). \tag{2.142}$$

Next, multiplying (2.125) by $\lambda^{\frac{l}{2}-1}\varsigma\overline{v_x^1}$ in $L^2\left(0,L\right)$. Then using the fact that v_x^1 is uniformly bounded in $L^2(0,L)$, f^2 converges to zero in $H_0^1\left(0,L\right)$ and f^5 converges to zero in $L^2\left(0,L\right)$, we get

$$\begin{aligned}&-k_1\lambda^{\frac{l}{2}-1}\int_0^L \varsigma\left|v_x^1\right|^2 dx - \rho_2\int_0^L \varsigma\lambda^{\frac{l}{2}} v_x^2\lambda\overline{v^1}dx - \rho_2\int_0^L \varsigma'\lambda^{\frac{l}{2}} v^2\lambda\overline{v^1}dx \\ &-\int_0^L \lambda^{\frac{l}{2}}\left(\widetilde{k}_2 v_x^2 + \int_0^{+\infty} g\left(s\right)v_x^7 ds\right)\varsigma\lambda^{-1}\overline{v_{xx}^1}dx - \int_0^L \lambda^{\frac{l}{2}-1}\left(\widetilde{k}_2 v_x^2 + \int_0^{+\infty} g\left(s\right)v_x^7 ds\right)\varsigma'\overline{v_x^1}dx \\ &-k_1\int_0^L \lambda^{\frac{l}{2}-1}\left(v^2 + \mathrm{l}v^3\right)\varsigma\overline{v_x^1}dx = o\left(\frac{1}{\lambda^{\frac{l}{2}}}\right).\end{aligned}$$

Since $l \leq 4$, due to (2.128), we have

$$\lambda^{\frac{l}{2}-1}\int_0^L \left(v^2 + \mathrm{l}v^3\right)dx = O(1).$$

From (2.119), (2.120), (2.128), (2.142), Lemma 2.22, and Lemma 2.24 we obtain (2.140). Thus the proof is complete. □

Lemma 2.26. *Let* $2 \leq l \leq 4$. *Under hypothesis* (H), *we have*

$$\int_0^L \varsigma \left|v_x^1\right|^2 dx = o\left(\frac{1}{\lambda^{l-2}}\right), \quad \text{and} \quad \int_0^L \varsigma \left|v^1\right|^2 dx = o\left(\frac{1}{\lambda^{l}}\right). \tag{2.143}$$

Proof. Let $l_N = \frac{l-2}{2}\sum_{k=0}^{N}\frac{1}{2^k}$. Since $\lim_{N\to+\infty}\frac{l-2}{2}\sum_{k=0}^{N}\frac{1}{2^k} = l-2$, we prove by induction on $N \in \mathbb{N}$ that

$$\int_0^L \varsigma \left|v_x^1\right|^2 dx = o\left(\frac{1}{\lambda^{l_N}}\right). \tag{2.144}$$

If $N = 0$, estimation (2.144) holds by Lemma 2.25. Suppose that

$$\lambda^{l_{N-1}}\int_0^L \varsigma \left|v_x^1\right|^2 dx = o(1). \tag{2.145}$$

Multiplying (2.124) by $\lambda^{l_{N-1}}\varsigma\overline{v^1}$ in $L^2(0,L)$. Then, using the fact that v^1 is uniformly bounded in $L^2(0,L)$, f^1 converges to zero in $H_0^1(0,L)$ and f^4 converges to zero in $L^2(0,L)$, we get

$$\rho_1\lambda^{l_{N-1}+2}\int_0^L \varsigma\left|v^1\right|^2 dx - k_1\lambda^{l_{N-1}}\int_0^L \varsigma\left|v_x^1\right|^2 dx + k_1\int_0^L \lambda^{\frac{l_{N-1}}{2}}\left(\varsigma v_x^2 - \varsigma' v_x^1\right)\lambda^{\frac{l_{N-1}}{2}}\overline{v^1}dx$$

$$-\mathrm{l}\left(k_1+k_3\right)\int_0^L \lambda^{\frac{l_{N-1}}{2}} v^3 \lambda^{\frac{l_{N-1}}{2}}\left(\varsigma v^1\right)_x dx - \mathrm{l}k_3\lambda^{l_{N-1}}\int_0^L \varsigma\left|v^1\right|^2 dx = o\left(\frac{1}{\lambda^{l-l_{N-1}}}\right).$$

As $\frac{l_{N-1}}{2} \leq 1$, from (2.128), (2.145) and Lemma 2.24, we obtain

$$\lambda^{l_{N-1}+2}\int_0^L \varsigma\left|v^1\right|^2 dx = o(1). \tag{2.146}$$

On the other hand, using (2.146) and Lemma 2.24, we get from (2.124) that

$$\lambda^{-1+\frac{l_{N-1}}{2}}\int_0^L \varsigma\left|v_{xx}^1\right|^2 dx = O(1). \tag{2.147}$$

Multiplying (2.125) by $\lambda^{l_N}\varsigma\overline{v_x^1}$ in $L^2(0,L)$. Then, using the fact that $l_N = \frac{l-2}{2} + \frac{l_{N-1}}{2}$, v_x^1 is uniformly bounded in $L^2(0,L)$, f^2 converges to zero in $H_0^1(0,L)$ and f^5 converges to zero in $L^2(0,L)$, we get

$$-\rho_2\int_0^L \lambda^{\frac{l}{2}} v_x^2 \lambda^{1+\frac{l_{N-1}}{2}}\varsigma\overline{v^1}dx - \rho_2\int_0^L \lambda^{\frac{l}{2}} v^2 \lambda^{1+\frac{l_{N-1}}{2}}\varsigma'\overline{v^1}dx - k_1\lambda^{l_N}\int_0^L \varsigma\left|v_x^1\right|^2 dx$$

$$-\int_0^L \lambda^{\frac{l}{2}}\left(\widetilde{k}_2 v_x^2 + \int_0^{+\infty} g(s)\, v_x^7 ds\right)\lambda^{-1+\frac{\ell_{N-1}}{2}}\left(\varsigma\overline{v_{xx}^1} + \varsigma'\overline{v_x^1}\right)dx$$

$$-k_1\int_0^L \lambda^{\frac{l}{2}-1}\left(v^2 + l v^3\right)\lambda^{\frac{l_{N-1}}{2}}\varsigma\overline{v_x^1}dx = o\left(\frac{1}{\lambda^{l-l_N}}\right).$$

Using the fact that $2 \leq l \leq 4$, (2.120), (2.128), (2.145), (2.146), (2.147), Lemma 2.22, and Lemma 2.24, we get (2.144). Therefore,

$$\int_0^L \varsigma\left|v_x^1\right|^2 dx = o\left(\frac{1}{\lambda^{l-2}}\right). \tag{2.148}$$

Finally, multiplying (2.124) by $\lambda^{l-2}\varsigma\overline{v^1}$ in $L^2(0,L)$. Then, using (2.120), (2.128), (2.148), and Lemma 2.24, we get the second estimation of (2.143). Thus the proof is complete. □

Lemma 2.27. *Let* $2 \leq l \leq 4$. *Under hypothesis* (H), *we have*

$$\mathrm{l}(k_1+k_3)\int_0^L \varsigma\left|v_x^3\right|^2 dx + (k_3-k_1)\Re\left\{\int_0^L \lambda\varsigma v_x^1\lambda^{-1}\overline{v_{xx}^3}dx\right\} = o(1). \tag{2.149}$$

Proof. Multiplying (2.124) by $\varsigma\overline{v_x^3}$ in $L^2(0,L)$. Then, using (2.128), Lemma 2.24, Lemma 2.26, v_x^3 is uniformly bounded in $L^2(0,L)$, f^1 converges to zero in $H_0^1(0,L)$ and f^4 converges to zero in $L^2(0,L)$, we get

$$\rho_1\int_0^L \lambda^2 v^1\varsigma\overline{v_x^3}dx + 1(k_1+k_3)\int_0^L \varsigma\left|v_x^3\right|^2 dx - k_1\int_0^L \lambda\varsigma v_x^1\lambda^{-1}\overline{v_{xx}^3} = o(1). \tag{2.150}$$

Multiplying (2.126) by $\varsigma\overline{v_x^1}$ in $L^2(0,L)$. Then, using (2.128), Lemma 2.26, v_x^1 is uniformly bounded in $L^2(0,L)$, f^3 converges to zero in $H_0^1(0,L)$ and f^6 converges to zero in $L^2(0,L)$, we get

$$-\rho_1\int_0^L \lambda^2\overline{v^1}\varsigma v_x^3dx + k_3\int_0^L \lambda\varsigma\overline{v_x^1}\lambda^{-1}v_{xx}^3dx = o(1). \tag{2.151}$$

Adding (2.150) and (2.151), then take the real part of the resulting equation, we get (2.149). Thus the proof is complete. □

Proof of Theorem 2.20 If (2.115) hold, take $\ell = 4$ in Lemma 2.22, Lemma 2.24, and Lemma 2.26, we get

$$\int_0^L\int_0^{+\infty} g(s)\left|v_x^7\right|^2 dsdx = o\left(\frac{1}{\lambda^4}\right), \quad \int_0^L \left|v_x^2\right|^2 dx = o\left(\frac{1}{\lambda^4}\right). \tag{2.152}$$

and

$$\int_0^L \varsigma\left|v_x^1\right|^2 dx = o\left(\frac{1}{\lambda^2}\right), \quad \int_0^L \varsigma\left|v^1\right|^2 dx = o\left(\frac{1}{\lambda^4}\right). \tag{2.153}$$

Using (2.153) and (2.129), we get

$$\int_0^L \lambda\varsigma v_x^1\lambda^{-1}\overline{v_{xx}^3}dx = o(1). \tag{2.154}$$

From Lemma 2.27 and (2.154), we get

$$\int_0^L \varsigma\left|v_x^3\right|^2 dx = o(1). \tag{2.155}$$

Next, multiplying (2.126) by $\varsigma\overline{v^3}$ in $L^2(0,L)$. Then, using (2.152), (2.153) , (2.155), v^3 is uniformly bounded in $L^2(0,L)$, f^3 converges to zero in $H_0^1(0,L)$ and f^6 converges to zero in $L^2(0,L)$, we get

$$\int_0^L \varsigma\left|v^3\right|^2 dx = o\left(\frac{1}{\lambda^2}\right). \tag{2.156}$$

Finally, using (2.152), (2.153), (2.155) and (2.156), we get $\|U\|_{\mathcal{H}} = o(1)$, over $(\alpha+\epsilon, \beta-\epsilon)$. Then by applying Lemma 2.13, we deduce $\|U\|_{\mathcal{H}} = o(1)$, over $(0,L)$ which contradicts (2.119). This implies that

$$\sup_{\lambda\in\mathbb{R}}\left\|(i\lambda Id - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} = O\left(\lambda^4\right).$$

The result follows from [3].

Proof of Theorem 2.21 If (2.117) hold, take $\ell = 2$ in Lemma 2.27, we get directly

$$\int_0^L \varsigma\left|v_x^3\right|^2 dx = o(1). \tag{2.157}$$

Moreover, from Lemma 2.22, Lemma 2.24, and Lemma 2.26, we get

$$\int_0^L\int_0^{+\infty} g(s)\left|v^7\right|^2 dsdx = o\left(\frac{1}{\lambda^2}\right), \quad \int_0^L \left|v_x^2\right|^2 dx = o\left(\frac{1}{\lambda^2}\right). \tag{2.158}$$

and

$$\int_0^L \varsigma\left|v_x^1\right|^2 dx = o(1), \quad \int_0^L \varsigma\left|v^1\right|^2 dx = o\left(\frac{1}{\lambda^2}\right). \tag{2.159}$$

Next, multiplying (2.126) by $\varsigma\overline{v^3}$ in $L^2(0,L)$. Then, using (2.157), (2.158), (2.159) , v^3 is uniformly bounded in $L^2(0,L)$, f^3 converges to zero in $H_0^1(0,L)$ and f^6 converges to zero in $L^2(0,L)$, we get

$$\int_0^L \varsigma\left|v^3\right|^2 dx = o\left(\frac{1}{\lambda^2}\right). \tag{2.160}$$

Finally, using (2.157), (2.158), (2.159), and (2.160), we get $\|U\|_{\mathcal{H}} = o(1)$, over $(\alpha+\epsilon, \beta-\epsilon)$. Then by applying Lemma 2.13, we deduce $\|U\|_{\mathcal{H}} = o(1)$, over $(0, L)$ which contradicts (2.119). This implies that

$$\sup_{\lambda\in\mathbb{R}} \left\|(i\lambda Id - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} = O\left(\lambda^2\right).$$

The result follows from [3].

3. Thermo-elastic Bresse system with history and Cattaneo law

We can adapt similar analysis done in Section 2 to study the stability of the thermo-elastic Bresse system (1.1) with various boundary conditions given by (1.2), (1.3) or (1.4). In this section, we consider system (1.1) with fully Dirichlet boundary conditions given by (1.2) since the analysis of the stability of system (1.1) with the other boundary conditions follows easily.

After introducing the new variable

$$\eta(x,t,s) := \psi(x,t) - \psi(x,t-s), \quad \text{in} \quad (0,L)\times\mathbb{R}_+\times\mathbb{R}_+,$$

our system (1.1) takes the form

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x+\psi+\mathrm{l}w)_x - \mathrm{l}k_3(w_x - \mathrm{l}\varphi) = 0, \\ \rho_2\psi_{tt} - \left(k_2 - \int_0^{+\infty} g(s)\,ds\right)\psi_{xx} + k_1(\varphi_x+\psi+\mathrm{l}w) - \int_0^{+\infty} g(s)\,\eta_{xx}ds + \delta\theta_x = 0, \\ \rho_1 w_{tt} - k_3(w_x - \mathrm{l}\varphi)_x + \mathrm{l}k_1(\varphi_x+\psi+\mathrm{l}w) = 0, \\ \eta_t + \eta_s - \psi_t = 0, \\ \rho_3\theta_t + q_x + \delta\psi_{tx} = 0, \\ \tau q_t + \beta q + \theta_x = 0, \end{cases} \tag{3.1}$$

with the initial conditions

$$\begin{array}{ll} \varphi(\cdot,0) = \varphi_0(\cdot),\ \psi(\cdot,-t) = \psi_0(\cdot,t),\ w(\cdot,0) = w_0(\cdot), & \\ \varphi_t(\cdot,0) = \varphi_1(\cdot),\ \psi_t(\cdot,0) = \psi_1(\cdot),\ w_t(\cdot,0) = w_1(\cdot), & \\ \theta(\cdot,0) = \theta_0(\cdot),\ q(\cdot,0) = q_0(\cdot), & \\ \eta^0(\cdot,s) := \eta(\cdot,0,s) = \psi_0(\cdot,0) - \psi_0(\cdot,s), & \text{in } (0,L),\ s\geq 0, \end{array}$$

and fully Dirichlet boundary conditions

$$\begin{array}{ll} \varphi(0,\cdot) = \varphi(L,\cdot) = \psi(0,\cdot) = \psi(L,\cdot) = 0 & \text{in } \mathbb{R}_+, \\ w(0,\cdot) = w(L,\cdot) = \theta(0,\cdot) = \theta(L,\cdot) = 0 & \text{in } \mathbb{R}_+, \\ \eta(0,\cdot,\cdot) = \eta(L,\cdot,\cdot) = 0 & \text{in } \mathbb{R}_+\times\mathbb{R}_+, \\ \eta(\cdot,\cdot,0) = 0 & \text{in } (0,L)\times\mathbb{R}_+. \end{array}$$

We consider the energy space

$$\mathcal{H} = \left(H_0^1(0,L)\right)^3 \times \left(L^2(0,L)\right)^3 \times L_g^2\left(\mathbb{R}_+, H_0^1\right) \times \left(L^2(0,L)\right)^2,$$

equipped with the norm

$$\begin{aligned} \|U\|_{\mathcal{H}}^2 &= \|\left(v^1,v^2,v^3,v^4,v^5,v^6,v^7,v^8,v^9\right)\|_{\mathcal{H}}^2 \\ &= \rho_1\left\|v^4\right\|^2 + \rho_2\left\|v^5\right\|^2 + \rho_1\left\|v^6\right\|^2 + k_1\left\|v_x^1 + v^2 + \mathrm{l}v^3\right\|^2 + \widetilde{k}_2\left\|v_x^2\right\|^2 \\ &+ k_3\left\|v_x^3 - \mathrm{l}v^1\right\|^2 + \left\|v^7\right\|_g^2 + \rho_3\left\|v^8\right\|^2 + \tau\left\|v^9\right\|^2. \end{aligned}$$

Consider the linear unbounded operator $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ defined by

$$\begin{aligned} D(\mathcal{A}) = \Big\{ & U \in \mathcal{H} \mid v^1, v^3 \in H^2(0,L),\ v^4, v^5, v^6 \in H_0^1(0,L), \\ & v_s^7 \in L_g^2\left(\mathbb{R}_+, H_0^1\right),\ v^8 \in H_0^1(0,L),\ v^9 \in H^1(0,L), \\ & v^2 + \textstyle\int_0^{+\infty} g(s)\,v^7 ds \in H^2(0,L)\cap H_0^1(0,L),\ v^7(x,0) = 0\Big\} \end{aligned}$$

and

$$\mathcal{A}\begin{pmatrix} v^1\\ v^2\\ v^3\\ v^4\\ v^5\\ v^6\\ v^7\\ v^8\\ v^9 \end{pmatrix} = \begin{pmatrix} v^4\\ v^5\\ v^6\\ \rho_1^{-1}\left(k_1\left(v^1_x+v^2+\mathrm{l}v^3\right)_x+\mathrm{l}k_3\left(v^3_x-\mathrm{l}v^1\right)\right)\\ \rho_2^{-1}\left(\widetilde{k}_2 v^2_{xx}-k_1\left(v^1_x+v^2+\mathrm{l}v^3\right)+\int_0^{+\infty} g(s)\, v^7_{xx}ds-\delta v^8_x\right)\\ \rho_1^{-1}\left(k_3\left(v^3_x-\mathrm{l}v^1\right)_x-\mathrm{l}k_1\left(v^1_x+v^2+\mathrm{l}v^3\right)\right)\\ v^5-v^7_s\\ \rho_3^{-1}\left(-\delta v^5_x-v^9_x\right)\\ \tau^{-1}\left(-v^8_x-\beta v^9\right) \end{pmatrix},$$

for all $U=\left(v^1,v^2,v^3,v^4,v^5,v^6,v^7,v^8,v^9\right)^{\mathsf{T}}\in D(\mathcal{A})$.
Then system (1.1) is equivalent to the Cauchy problem

$$\left\{\begin{array}{c} U_t=\mathcal{A}U,\\ U(x,0)=U^0(x), \end{array}\right. \tag{3.2}$$

where

$$U=\left(\varphi,\psi,w,\varphi_t,\psi_t,w_t,\eta,\theta,q\right)^{\mathsf{T}}$$

and

$$U^0(x)=\left(\varphi_0(x),\psi_0(x,0),w_0(x),\varphi_1(x),\psi_1(x),w_1(x),\eta^0(x,.),\theta_0(x),q_0(x)\right)^{\mathsf{T}}.$$

Note that $D(\mathcal{A})$ is dense in $\mathcal{H}$ and that for all $U\in D(\mathcal{A})$, we have

$$\langle \mathcal{A}U,U\rangle_{\mathcal{H}}=\frac{1}{2}\int_0^L\int_0^{+\infty} g'(s)\left|v^7_x\right|^2 dsdx-\beta\int_0^L\left|v^9\right|^2dx. \tag{3.3}$$

Consequently, under hypothesis (H), the system becomes dissipative. We can easily adapt the proofs in Subsection 2.1 and Subsection 2.2 to prove the well-posedness and the strong stability of system (3.1). Furthermore, similar to [21], we define the following stability number

$$\chi_0=\left(\tau-\frac{\rho_1}{\rho_3 k_1}\right)\left(\rho_2-\frac{k_2\rho_1}{k_1}\right)-\frac{\tau\rho_1\delta^2}{\rho_3 k_1}.$$

Theorem 3.1. *Under hypothesis* (H), *if*

$$\chi_0=0 \quad and \quad k_1=k_3, \tag{3.4}$$

then system (3.1) *with fully Dirichlet boundary conditions is exponentially stable.*

Proof. Similar to Theorem 2.10, we have to check conditions (H1) and (H2). We will prove condition (H2) by a contradiction argument. Suppose that there exists a sequence of real numbers $(\lambda_n)_n$, with $|\lambda_n|\to+\infty$, and a sequence of vectors

$$U_n=\left(v^1_n,v^2_n,v^3_n,v^4_n,v^5_n,v^6_n,v^7_n,v^8_n,v^9_n\right)^{\mathsf{T}}\in D(\mathcal{A}) \quad \text{with} \quad \|U_n\|_{\mathcal{H}}=1 \tag{3.5}$$

such that

$$i\lambda_n U_n-\mathcal{A}U_n=\left(f^1_n,f^2_n,f^3_n,f^4_n,f^5_n,f^6_n,f^7_n,f^8_n,f^9_n\right)^{\mathsf{T}}\to 0 \ \text{ in } \mathcal{H}. \tag{3.6}$$

Equivalently, we have

$$i\lambda_n v^1_n-v^4_n = h^1_n, \tag{3.7}$$

$$i\lambda_n v^2_n-v^5_n = h^2_n, \tag{3.8}$$

$$i\lambda_n v^3_n-v^6_n = h^3_n, \tag{3.9}$$

$$\rho_1\lambda_n^2 v^1_n+k_1\left[\left(v^1_n\right)_x+v^2_n+\mathrm{l}v^3_n\right]_x+\mathrm{l}k_3\left[\left(v^3_n\right)_x-\mathrm{l}v^1_n\right] = h^4_n, \tag{3.10}$$

$$\rho_2\lambda_n^2 v^2_n+\widetilde{k}_2\left(v^2_n\right)_{xx}-k_1\left[\left(v^1_n\right)_x+v^2_n+\mathrm{l}v^3_n\right]+\int_0^{+\infty} g(s)\left(v^7_n\right)_{xx}ds-\delta\left(v^8_n\right)_x = h^5_n, \tag{3.11}$$

$$\rho_1\lambda_n^2 v^3_n+k_3\left[\left(v^3_n\right)_x-\mathrm{l}v^1_n\right]_x-\mathrm{l}k_1\left[\left(v^1_n\right)_x+v^2_n+\mathrm{l}v^3_n\right] = h^6_n, \tag{3.12}$$

$$i\lambda_n v^7_n+\left(v^7_n\right)_s-i\lambda_n v^2_n = h^7_n, \tag{3.13}$$

$$i\rho_3\lambda_n v^8_n+i\delta\lambda_n\left(v^2_n\right)_x+\left(v^9_n\right)_x = h^8_n \tag{3.14}$$

$$i\tau\lambda_n v^9_n+\beta v^9_n+\left(v^8_n\right)_x = h^9_n, \tag{3.15}$$

where
$$\begin{cases} h_n^1 = f_n^1,\ h_n^2 = f_n^2,\ h_n^3 = f_n^3, \\ h_n^4 = -\rho_1\left(f_n^4 + i\lambda_n f_n^1\right),\ h_n^5 = -\rho_2\left(f_n^5 + i\lambda_n f_n^2\right),\ h_n^6 = -\rho_1\left(f_n^6 + i\lambda_n f_n^3\right), \\ h_n^7 = f_n^7 - f_n^2,\ h_n^8 = \rho_3 f_n^8 + \delta\left(f_n^2\right)_x,\ h_n^9 = \tau f_n^9. \end{cases}$$

In the sequel, for shortness, we drop the index n. Taking the inner product of (3.6) with U in $\mathcal{H}$. Then, using (3.3), hypothesis (H) and the fact that U is uniformly bounded in $\mathcal{H}$, we get

$$\int_0^L\int_0^{+\infty} g(s)\left|v_x^7\right|^2 dsdx = o(1) \quad\text{and}\quad \int_0^L \left|v^9\right|^2 dx = o(1). \tag{3.16}$$

Similar to Lemma 2.12, multiplying (3.13) by $\overline{v^2}$ in $L_g^2\left(\mathbb{R}_+, H_0^1\right)$. Then, using (3.5) and (3.16), we get

$$\int_0^L \left|v_x^2\right|^2 dx = o(1). \tag{3.17}$$

Multiplying (3.14) and (3.15) by $\overline{v^8}$ and $\overline{v^9}$ respectively in $L^2(0,L)$. Then, using (3.6), (3.16) and (3.17), we get

$$\int_0^L \left|v^8\right|^2 dx = o(1). \tag{3.18}$$

Multiplying (3.11) by $\overline{v^2}$ in $L^2(0,L)$. Then, using (3.6) and (3.16)- (3.18), we get

$$\int_0^L \left|\lambda v^2\right|^2 dx = o(1). \tag{3.19}$$

Multiplying (3.10) and (3.11) by $\frac{\varsigma}{k_1}\left(\widetilde{k}_2\overline{v_x^2} + \int_0^{+\infty} g(s)\,\overline{v_x^7}ds\right)$ and $\varsigma\overline{v_x^1}$ respectively in $L^2(0,L)$. Then, take the real part of the resulting equation, using (3.6) and (3.16)-(3.18), we get

$$k_1\int_0^L \varsigma\left|v_x^1\right|^2 dx + \lambda^2\left(\rho_2 - \frac{\rho_1 k_2}{k_1}\right)\Re\left\{\int_0^L \varsigma v_x^2\overline{v^1}dx\right\} + \delta\Re\left\{\int_0^L v_x^8\overline{v_x^1}dx\right\} = o(1), \tag{3.20}$$

where ς is the cut-off function defined in Subsection 2.3. Multiplying (3.10), (3.14) , and (3.15) by $\frac{\rho_3\tau}{\rho_1}\varsigma\overline{v^8}$, $i\tau\lambda\varsigma\overline{v^1}$, and $\varsigma\overline{v_x^1}$ respectively in $L^2(0,L)$. Then, take the real part of the resulting equation, using (3.6) and (3.16)-(3.18), we get

$$\lambda^2\Re\left\{\int_0^L \varsigma v_x^2\overline{v^1}dx\right\} = -\frac{\rho_3 k_1}{\rho_1\delta\tau}\left(\tau - \frac{\rho_1}{\rho_3 k_1}\right)\Re\left\{\int_0^L \varsigma v_x^8\overline{v^1}dx\right\} + o(1). \tag{3.21}$$

Inserting (3.21) in (3.20), we get

$$k_1\int_0^L \varsigma\left|v_x^1\right|^2 dx - \frac{\rho_3 k_1}{\rho_1\delta\tau}\chi_0\Re\left\{\int_0^L v_x^8\overline{v_x^1}dx\right\} = o(1).$$

Using the fact that $\chi_0 = 0$, we get

$$\int_0^L \varsigma\left|v_x^1\right|^2 dx = o(1). \tag{3.22}$$

Multiplying (3.10) by $\varsigma\overline{v^1}$ in $L^2(0,L)$. Then, using (3.6), (3.17) and (3.22), we get

$$\int_0^L \varsigma\left|\lambda v^1\right|^2 dx = o(1). \tag{3.23}$$

Multiplying (3.10) and (3.12) by $\varsigma\overline{v_x^3}$ and $\varsigma\overline{v_x^1}$ respectively in $L^2(0,L)$. Then, take the real part of the resulting equation, using (3.6), (3.16)-(3.17) and (3.22), we get

$$\mathrm{l}\left(k_1 + k_3\right)\int_0^L \varsigma\left|v_x^3\right|^2 dx + \left(k_3 - k_1\right)\Re\left\{\int_0^L v_x^1\varsigma\overline{v_{xx}^3}dx\right\} = o(1).$$

Using the fact that $k_1 = k_3$, we get

$$\int_0^L \varsigma\left|v_x^3\right|^2 dx = o(1). \tag{3.24}$$

Moreover, multiplying (3.12) by $\varsigma\overline{v^3}$ in $L^2(0,L)$. Then, using (3.6), (3.17) and (3.22)-(3.24), we get

$$\int_0^L \varsigma\left|\lambda v^3\right|^2 dx = o(1). \tag{3.25}$$

Finally, using (3.16)-(3.19) and (3.22)-(3.25), we can proceed similar to the proof of Theorem 2.10 to get the result of Theorem 3.1. □

Note that when $\tau = 0$, Cattaneo's law turns into Fourier law. In this case, condition (3.4) becomes equivalent to (2.47). However, if $\chi_0 \neq 0$ we can adapt the proof of Theorem 2.20 and Theorem 2.21 to show the following Theorems:

Theorem 3.2. *Under hypothesis* (H)*, if*

$$\chi_0 \neq 0 \quad \text{and} \quad k_1 \neq k_3, \tag{3.26}$$

then system (3.1) *with fully Dirichlet boundary conditions is polynomially stable with an energy rate of decay* $\dfrac{1}{\sqrt{t}}$*, i.e, there exists* $c>0$ *such that for every* $U^0 \in D(\mathcal{A})$*, we have*

$$E(t) \leq \frac{c}{\sqrt{t}}\left\|U^0\right\|^2_{D(\mathcal{A})}, \qquad t>0. \tag{3.27}$$

Theorem 3.3. *Under hypothesis* (H)*, if*

$$\chi_0 \neq 0 \quad \text{and} \quad k_1 = k_3, \tag{3.28}$$

then system (3.1) *with fully Dirichlet boundary conditions is polynomially stable with an energy rate of decay* $\dfrac{1}{t}$*, i.e, there exists* $c>0$ *such that for every* $U^0 \in D(\mathcal{A})$*, we have*

$$E(t) \leq \frac{c}{t}\left\|U^0\right\|^2_{D(\mathcal{A})}, \qquad t>0. \tag{3.29}$$

Similar to Theorem 2.20 and Theorem 2.21, we have to check (H3) where $l=4$ if condition (3.26) holds and $l=2$ if condition (3.28) holds. We will prove condition (H3) by a contradiction argument, suppose there exists a sequence of real numbers $(\lambda_n)_n$, with $\lambda_n \to +\infty$, and a sequence of vectors

$$U_n = \left(v_n^1, v_n^2, v_n^3, v_n^4, v_n^5, v_n^6, v_n^7, v_n^8, v_n^9\right)^\mathsf{T} \in D(\mathcal{A}) \quad \text{with} \quad \|U_n\|_{\mathcal{H}} = 1 \tag{3.30}$$

such that

$$\lambda_n^l\left(i\lambda_n U_n - \mathcal{A}U_n\right) = \left(f_n^1, f_n^2, f_n^3, f_n^4, f_n^5, f_n^6, f_n^7, f_n^8, f_n^9\right)^\mathsf{T} \to 0 \ \text{ in } \mathcal{H}; \tag{3.31}$$

Equivalently, we have

$$i\lambda_n v_n^1 - v_n^4 = h_n^1, \tag{3.32}$$

$$i\lambda_n v_n^2 - v_n^5 = h_n^2, \tag{3.33}$$

$$i\lambda_n v_n^3 - v_n^6 = h_n^3, \tag{3.34}$$

$$\rho_1\lambda_n^2 v_n^1 + k_1\left[\left(v_n^1\right)_x + v_n^2 + \mathrm{l}v_n^3\right]_x + \mathrm{l}k_3\left[\left(v_n^3\right)_x - \mathrm{l}v_n^1\right] = h_n^4, \tag{3.35}$$

$$\rho_2\lambda_n^2 v_n^2 + \widetilde{k}_2\left(v_n^2\right)_{xx} - k_1\left[\left(v_n^1\right)_x + v_n^2 + \mathrm{l}v_n^3\right] + \int_0^{+\infty} g(s)\left(v_n^7\right)_{xx} ds - \delta\left(v_n^8\right)_x = h_n^5, \tag{3.36}$$

$$\rho_1\lambda_n^2 v_n^3 + k_3\left[\left(v_n^3\right)_x - \mathrm{l}v_n^1\right]_x - \mathrm{l}k_1\left[\left(v_n^1\right)_x + v_n^2 + \mathrm{l}v_n^3\right] = h_n^6, \tag{3.37}$$

$$i\lambda_n v_n^7 + \left(v_n^7\right)_s - i\lambda_n v_n^2 = h_n^7, \tag{3.38}$$

$$i\rho_3\lambda_n v_n^8 + i\delta\lambda_n\left(v_n^2\right)_x + \left(v_n^9\right)_x = h_n^8 \tag{3.39}$$

$$i\tau\lambda_n v_n^9 + \beta v_n^9 + \left(v_n^8\right)_x = h_n^9. \tag{3.40}$$

where

$$\begin{cases} \lambda_n^l h_n^1 = f_n^1,\ \lambda_n^l h_n^2 = f_n^2,\ \lambda_n^l h_n^3 = f_n^3, \\[2mm] \lambda_n^l h_n^4 = -\rho_1\left(f_n^4 + i\lambda_n f_n^1\right),\ \lambda_n^l h_n^5 = -\rho_2\left(f_n^5 + i\lambda_n f_n^2\right),\ \lambda_n^l h_n^6 = -\rho_1\left(f_n^6 + i\lambda_n f_n^3\right), \\[2mm] \lambda_n^l h_n^7 = f_n^7 - f_n^2,\ \lambda_n^l h_n^8 = \rho_3 f_n^8 + \delta\left(f_n^2\right)_x,\ \lambda_n^l h_n^9 = \tau f_n^9. \end{cases}$$

In the sequel, for shortness, we drop the index n. Taking the inner product of (3.31) with U in $\mathcal{H}$. Then, using (3.3), hypothesis (H) and the fact that U is uniformly bounded in $\mathcal{H}$, we get

$$\int_0^L\int_0^{+\infty} g(s)\left|v_x^7\right|^2 dsdx = o\left(\frac{1}{\lambda^l}\right) \quad \text{and} \quad \int_0^L \left|v^9\right|^2 dx = o\left(\frac{1}{\lambda^l}\right). \tag{3.41}$$

Similar to Lemma 2.24, multiplying (3.38) by $\overline{v^2}$ in $L_g^2\left(\mathbb{R}_+, H_0^1\right)$. Then, using (3.31) and (3.41), we get

$$\int_0^L \left|v_x^2\right|^2 dx = o\left(\frac{1}{\lambda^l}\right). \tag{3.42}$$

From (3.40) and (3.41), we get

$$\int_0^L \left|v_x^8\right|^2 dx = o\left(\frac{1}{\lambda^{l-2}}\right). \tag{3.43}$$

Similar to Lemma 2.25 and using (3.43), we can prove that for $2 \leq l \leq 4$, we have

$$\int_0^L \varsigma\left|v_x^1\right|^2 dx = o\left(\frac{1}{\lambda^{\frac{l}{2}-1}}\right),$$

where ς is the cut-off function defined in Subsection 2.3. Consequently, for $2 \leq l \leq 4$, we can adapt the proof of Lemma 2.26 to show that

$$\int_0^L \varsigma\left|v_x^1\right|^2 dx = o\left(\frac{1}{\lambda^{l-2}}\right) \quad \text{and} \quad \int_0^L \varsigma\left|v^1\right|^2 dx = o\left(\frac{1}{\lambda^l}\right). \tag{3.44}$$

Finally, multiplying (3.35) and (3.37) by $\varsigma\overline{v_x^3}$ and $\varsigma\overline{v_x^1}$ respectively in $L^2(0,L)$. Then, take the real part of the resulting equation, using (3.31), (3.41)-(3.42) and (3.44), we get

$$1(k_1+k_3)\int_0^L \varsigma\left|v_x^3\right|^2 dx + (k_3-k_1)\Re\left\{\int_0^L \lambda\varsigma v_x^1\lambda^{-1}\overline{v_{xx}^3}dx\right\} = o(1). \tag{3.45}$$

Proof of Theorem 3.2 If (3.26) hold, we remark that $l=4$ is the optimal value we can choose to get

$$\int_0^L \lambda\varsigma v_x^1\lambda^{-1}\overline{v_{xx}^3}dx = o(1). \tag{3.46}$$

Therefore, from (3.45) and (3.46), we get

$$\int_0^L \varsigma\left|v_x^3\right|^2 dx = o(1). \tag{3.47}$$

Proceeding similar to the proof of Theorem 2.20, we get the result of Theorem 3.2.

Proof of Theorem 3.2 If (3.28) hold, then (3.45) yields directly (3.47). In this case, we choose $l=2$ as the optimal value of $2 \leq l \leq 4$. Proceeding similar to the proof of Theorem 2.21, we get the result of Theorem 3.3.

Remark 3.4. *Following Theorem 4.1 in* [16] *the energy of the Bresse system with fully Dirichlet or mixed boundary conditions decays as* $\frac{1}{\sqrt{t}}$ *if only one thermal dissipation given by Fourier law is considered and* $\frac{1}{\sqrt[3]{t}}$ *if only one thermal dissipation given by Cattaneo law is considered.*

4. Conclusion and open questions

Bresse system (1.1) with dissipative thermal effect given by Cattaneo's law and history type control is expected to decay faster than system (2.1) without heat conduction. Nevertheless, Theorem 2.20 and Theorem 3.2 show that the heat dissipation does not affect the rate of energy decay. Consequently, the optimality of the polynomial decay rate of system (1.1) and the influence of the Cattaneo law on the stability of the system remain an open problem.

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