Universal constructions for spaces of traffics

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ABSTRACT:

We investigate questions related to the notion of *traffics* introduced in [8] as a noncommutative probability space with numerous additional operations and equipped with the notion of *traffic independence.* We prove that any sequence of unitarily invariant random matrices that converges in noncommutative distribution converges in distribution of traffics whenever it fulfills some factorization property. We provide an explicit description of the limit which allows to recover and extend some applications (on the freeness from the transposed ensembles [12] and the freeness of infinite transitive graphs [1]). We also improve the theory of traffic spaces by considering a positivity axiom related to the notion of *state* in noncommutative probability. We construct the free product of spaces of traffics and prove that it preserves the positivity condition. This analysis leads to our main result stating that every noncommutative probability space endowed with a tracial state can be enlarged and equipped with a structure of space of traffics.

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1 Introduction

1.1 Motivations for traffics

Thanks to the fundamental work of Voiculescu [15], it is now understood that noncommutative probability is a good framework for the study of large random matrices. Here are two important considerations which sum up the role of noncommutative probability in the description of the macroscopic behavior of large random matrices:

1. A large class of families of random matrices $\mathbf{A}_N \in \mathrm{M}_N(\mathbb{C})$ *converge in noncommutative distribution* as N tends to ∞ (in the sense that the normalized trace of any polynomial in the matrices converges).

2. If two independent families of random matrices $\mathbf{A}_N$ and $\mathbf{B}_N$ converge separately in noncommutative distribution and are invariant in law when conjugating by a *unitary* matrix, then the joint noncommutative distribution of the family $\mathbf{A}_N \cup \mathbf{B}_N$ converges as well. Moreover, the joint limit can be described from the separate limits thanks to the relation of *free independence* introduced by Voiculescu.

In [8, 9, 10], it was pointed out that there are cases where other important macroscopic convergences occur in the study of large random matrices and graphs. The notion of noncommutative probability is too restrictive and should be generalized to get more information about the limit in large dimension. This is precisely the motivation to introduce the concept of *space of traffics*, which comes together with the notion of distribution of traffics and the notion of traffic independence: it is a non-commutative probability space where one can consider not only the usual operations of algebras, but also more general *n*-ary operations called *graph operations*. We will introduce those concept in details, but let us first describe the role of traffics enlightened in [8] for the description of large N asymptotics of random matrices:

1. A large class of families of random matrices $\mathbf{A}_N \in \mathrm{M}_N(\mathbb{C})$ *converge in distribution of traffics* as N tends to ∞ (in the sense that the normalized trace of any graph operation in the matrices converges).

2. If two independent families of random matrices $\mathbf{A}_N$ and $\mathbf{B}_N$ converge separately in distribution of traffics, satisfy a factorization property and are invariant in law when conjugating by a *permutation* matrix, then the joint distribution of traffics of the family $\mathbf{A}_N \cup \mathbf{B}_N$ converges as well. Moreover, the joint limit can be described from the separate limits thanks to the relation of *traffic independence* introduced in [8].

In general, asymptotic traffic independence is different than Voiculescu's notion. Nevertheless, they coincide if one family has the same limit in distribution of traffics as a family of random matrices invariant in law by conjugation by any unitary matrix. We now present our main results in the three next subsections.

1.2 Distribution of traffics of random matrices

Let us first describe how we encode new operations on space of matrices. For all $K \geqslant 0$, a *K-graph operation* is a connected graph g with K oriented and ordered edges, and two distinguished vertices (one input and one output, not necessarily distinct). The set $\mathcal{G}$ of graph operations is the set of all K-graph operations for all $n \geqslant 0$. A K-graph operation g has to be thought as an operation that accepts K objects and produces a new one.

For example, it acts on the space $\mathrm{M}_N(\mathbb{C})$ of N by N complex matrices as follows. For each K-graph operation $g \in \mathcal{G}$, we define a linear map $Z_g : \mathrm{M}_N(\mathbb{C}) \otimes \cdots \otimes \mathrm{M}_N(\mathbb{C}) \to \mathrm{M}_N(\mathbb{C})$ (or equivalently a K-linear map on $\mathrm{M}_N(\mathbb{C})^K$) in the following way. Denoting by V the vertices of g, by $(v_1, w_1), \ldots, (v_K, w_K)$ the ordered edges of g, and by $E_{k,l}$ the matrix unit $(\delta_{ik}\delta_{jl})_{i,j=1}^N \in \mathrm{M}_N(\mathbb{C})$, we set, for all $A^{(1)}, \ldots, A^{(K)} \in \mathrm{M}_N(\mathbb{C})$,

$$Z_g(A^{(1)} \otimes \cdots \otimes A^{(K)}) = \sum_{k:V \to \{1,\ldots,N\}} \left(A^{(1)}_{k(w_1),k(v_1)} \cdots A^{(K)}_{k(w_K),k(v_K)} \right) \cdot E_{k(out),k(in)}.$$

Following [13], we can think of the linear map $\mathbb{C}^N \to \mathbb{C}^N$ associated to $Z_g(A^{(1)} \otimes \cdots \otimes A^{(K)})$ as an algorithm, where we are feeding a vector into the input vertex and then operate it through the graph, each edge doing some calculation thanks to the corresponding matrix $A^{(i)}$, and each vertex acting like a logic gate, doing some compatibility checks. Those operations encode naturally the product of matrices, but also other natural operations, like the Hadamard (entry-wise) product $(A, B) \mapsto A \circ B$, the real transpose $A \mapsto A^t$ or the degree matrix $deg(A) = diag(\sum_{j=1}^{N} A_{i,j})_{i=1,\ldots,N}$.

Starting from a family $\mathbf{A} = (A_j)_{j \in J}$ of random matrices of size $N \times N$, the smallest algebra closed by the adjointness and by the action of the K-graph operations is the space of traffics generated by $\mathbf{A}_N$. The *distribution of traffics* of $\mathbf{A}_N$ is the data of the noncommutative distribution of the matrices which are in the space of traffics generated by $\mathbf{A}_N$. More concretely, it is the collection of the quantities

$$\frac{1}{N} \mathbb{E}\Big[\mathrm{Tr}\big(Z_g(A_{j_1}^{\epsilon_1} \otimes \cdots \otimes A_{j_K}^{\epsilon_K}) \big) \Big]$$

for all K-graph operations $g \in \mathcal{G}$, indices $j_1, \ldots, j_K \in J$ and labels $\epsilon_1, \cdots, \epsilon_K \in \{1, *\}$. Those quantities appear quite canonically in investigations of random matrices and have been first considered in [13]. The following theorem shows that the unitarily invariance is sufficient to deduce the convergence in distribution of traffics from the convergence in $*$-distribution.

Theorem 1.1. *For all $N \geqslant 1$, let $\mathbf{A}_N = (A_j)_{j \in J}$ be a family of random matrices in $\mathrm{M}_N(\mathbb{C})$. We assume*

1. *The unitary invariance: for all $N \geqslant 1$ and all $U \in \mathrm{M}_N(\mathbb{C})$ which is unitary, $U\mathbf{A}_N U^* := (UA_j U^*)_{j \in J}$ and $\mathbf{A}_N$ have the same law.*

2. *The convergence in $*$-distribution of $\mathbf{A}_N$: for all indices $j_1, \ldots, j_K \in J$ and labels $\epsilon_1, \cdots, \epsilon_K \in \{1, *\}$, the quantity $(1/N)\mathbb{E}[\mathrm{Tr}(A_{j_1}^{\epsilon_1} \cdots A_{j_K}^{\epsilon_K})]$ converges.*

3. *The factorization property: for all $*$-monomials $m_1, \ldots, m_k$, we have the following convergence*

$$\lim_{N \to \infty} \mathbb{E}\left[\frac{1}{N} \mathrm{Tr}\left(m_1(\mathbf{A}_N) \right) \cdots \frac{1}{N} \mathrm{Tr}\left(m_k(\mathbf{A}_N) \right) \right]$$

$$= \lim_{N \to \infty} \mathbb{E}\left[\frac{1}{N} \mathrm{Tr}\left(m_1(\mathbf{A}_N) \right) \right] \cdots \lim_{N \to \infty} \mathbb{E}\left[\frac{1}{N} \mathrm{Tr}\left(m_k(\mathbf{A}_N) \right) \right].$$

Then, $\mathbf{A}_N$ converges in distribution of traffics: for all K-graph operation $g \in \mathcal{G}$, indices $j_1, \ldots, j_K \in J$ and labels $\epsilon_1, \cdots, \epsilon_K \in \{1, *\}$, the following quantity converges

$$\frac{1}{N} \mathbb{E}\Big[\mathrm{Tr}\big(Z_g(A_{j_1}^{\epsilon_1} \otimes \ldots \otimes A_{j_K}^{\epsilon_K}) \big) \Big].$$

It has to be noticed that a similar result about the convergence of observables related to traffic distributions, for unitarily invariant random matrices, is also proved independently by Gabriel in [6]. More generally, the framework developed by Gabriel in [5, 6, 4] is related to the framework of traffic, and will certainly lead to further investigations in order to understand the precise link between both theories.

In practice, the limit of the distribution of traffic of $\mathbf{A}_N$ depends explicitly on the limit of the noncommutative $*$-distribution of $\mathbf{A}_N$. For example, a recent result of Mingo and Popa [12] tells that for all sequence of unitarily invariant random matrices $\mathbf{A}_N$ the family $\mathbf{A}_N^t$ of the transposes of $\mathbf{A}_N$ has the same noncommutative $*$-distribution as $\mathbf{A}_N$ and is asymptotically freely independent with $\mathbf{A}_N$ (under assumptions stronger than those of Theorem 1.1 that also imply the asymptotic free independence of second order). Thanks to the description of the limiting distribution of traffics of unitarily invariant matrices, we will get that for a family $\mathbf{A}_N = (A_j)_{j \in J}$ as in Theorem 1.1, $\mathbf{A}_N$, $\mathbf{A}_N^t$ and $deg(\mathbf{A}_N)$ are asymptotically free independent, as well as $\mathbf{A}_N \otimes \mathbf{A}_N := (A_j \otimes A_{j'})_{j,j' \in J}$, $deg(\mathbf{A}_N \otimes \mathbf{A}_N)$ and their transpose.

1.3 Spaces of traffics and their free product

Recall that a non commutative probability space is a pair $(\mathcal{A}, \Phi)$, where $\mathcal{A}$ is unital algebra and Φ is a trace, that is a unital linear form on $\mathcal{A}$ such that $\Phi(ab) = \Phi(ba)$ for any $a, b \in \mathcal{A}$. A *-probability space is a non commutative probability space equipped with an anti-linear involution $\cdot^*$ satisfying $(ab)^* = b^*a^*$ and such that Φ is a state, that is $\Phi(a^*a) \geqslant 0$ for any $a \in \mathcal{A}$.

The *-distribution of a family $\mathbf{a}$ of elements of $\mathcal{A}$ is the linear form $\Phi_{\mathbf{a}} : P \mapsto \Phi\big(P(\mathbf{a})\big)$ defined for non commutative polynomials in elements of $\mathbf{a}$ and their adjoint. The convergence in *-distribution of a sequence $\mathbf{a}_N$ is the pointwise convergence of $\Phi_{\mathbf{a}_N}$.

In [8], the notion of space of traffics was defined in an algebraic framework as a non-commutative probability space $(\mathcal{A}, \tau)$, with a collection of K-linear map indexed by the K-graph operations in a consistent way. It allows to consider the additional operations for matrices as the *Hadamard* (entry-wise) product, or the *real transpose* for non commutative random variables, and hopefully will lead to new probabilistic investigations in the general theory of quantum probability theory. More precisely, the set of graph operation $\mathcal{G}$ can be endowed naturally with a structure of operad, and we say that *the operad $\mathcal{G}$ acts* on a vector space $\mathcal{A}$ if to each K-graph operation $g \in \mathcal{G}$, there is a linear map

$$Z_g : \underbrace{\mathcal{A} \otimes \cdots \otimes \mathcal{A}}_{K \text{ times}} \to \mathcal{A}$$

(or equivalently a K-ary multilinear operation) subject to some requirements of compatibility (see Definition 2.2).

In Definition 2.8 of Section 2, we go further defining a *space of traffics* as a *-probability space $(\mathcal{A}, \tau)$ on which acts the graph operations $\mathcal{G}$, with two additional properties: the compatibility of the involution $\cdot^*$ with graph operations, and a positivity condition on τ which is stronger than saying that it is a state. Moreover, in Section 3, we define the free product $(*_{j \in J} \mathcal{A}_j, \star_{j \in J} \tau_j)$ of a family $(\mathcal{A}_j, \tau_j)_{j \in J}$ of algebraic spaces of traffics, in such a way that the algebras A_j seen as subspaces of traffics of $*_{j \in J} \mathcal{A}_j$ are traffic independent. The free product of spaces is compatible with the positivity condition for spaces of traffics, as the following theorem shows.

Theorem 1.2. *The free product of distributions of traffics satisfies the positivity condition for spaces of traffics, i.e. the free product of a family of spaces of traffics is well-defined as a space of traffic.*

One may be surprised by this additional positivity condition for spaces of traffics. Let us give a short explanation. The fact that the traces τ_j are states is not sufficient to ensure that $\star_{j \in J} \tau_j$ is a state as well. One has to require a bit more on τ_j to get the positivity of $\star_{j \in J} \tau_j$.

A consequence of Theorem 1.2 of conceptual importance is that for any traffic a there exists a space of traffics that contains a sequence of traffic independent variables distributed as a.

As a byproduct of the proof of Theorem 1.2, we get a new characterization of traffic independence (Theorem 3.11) which is much more similar to the usual definition of free independence. We deduce from it a simple criterion to characterize the free independence of variables assuming their traffic-independence (proving that the criterion in [8, Corollary 3.5] is actually a characterization of free independence in that context). An example is a new proof of the free independence of the spectral distributions of the free product of infinite deterministic graphs [1].

1.4 A canonical lifting from *-probability spaces to spaces of traffics

We turn now to our last result, which was the first motivation of this article and whose demonstration uses both Theorem 1.1 and Theorem 1.2. It states that the *-probability spaces of Voiculescu can be enlarged and equipped with the structure of space of traffics. Let us be more explicit. As explained, Theorem 1.1 in its full form gives a formula of the limiting distribution of traffics which involves only the limiting noncommutative distribution of the matrices. Replacing in this formula the limiting noncommutative distribution of matrices by an arbitrary distribution, we obtain a distribution of traffics which implies the following result. The difficulty consists in proving that this distribution satisfies the positivity condition.

Theorem 1.3. *Let $(\mathcal{A}, \Phi)$ be a *-probability space. There exists a space of traffics $(\mathcal{B}, \tau)$ such that $\mathcal{A} \subset \mathcal{B}$ as *-algebras and such that the trace induced by τ restricted to $\mathcal{A}$ is Φ.*

Moreover, the distribution of traffics τ is canonical in the sense that

1. *If $\mathbf{A}_N$ is a sequence of random matrices that converges in *-distribution to $\mathbf{a} \in \mathcal{A}$ as N tends to ∞ and verifies the condition of Theorem 1.1, then $\mathbf{A}_N$ converges in distribution of traffics to $\mathbf{a} \in \mathcal{B}$ as N tends to ∞.*

2. *Two families $\mathbf{a}$ and $\mathbf{b} \in \mathcal{A}$ are freely independent in $\mathcal{A}$ if and only if they are traffic independent in $\mathcal{B}$.*

Remark that, starting from an abelian non-commutative probability space $(\mathcal{A}, \Phi)$, there exists another procedure described in [8] which allows to define a space of traffics $(\mathcal{B}, \tau)$ such that $\mathcal{A} \subset \mathcal{B}$ as *-algebras and such that the state induced by τ on $\mathcal{A}$ is Φ, and where two families $\mathbf{a}$ and $\mathbf{b} \in \mathcal{A}$ are tensor independent in $\mathcal{A}$ if and only if they are traffic independent in $\mathcal{B}$. In other words, the free product of space of traffics leads to the tensor product or the free product of the probability spaces, depending on the way the *-distribution and the distribution of traffics of our random variables are linked.

The rest of the article is organized as follows. In section 2 we first recall the definition of algebraic spaces of traffics and define non-algebraic ones. Then we recall the definition of traffic independence. In Section 3 we define the free product of spaces of traffics. We state therein the new characterization of traffic independence and prove Theorem 1.2. In Section 4, we prove Theorem 1.3 on the canonical extension of *-probability spaces and Theorem 1.1 on the distribution of traffics of unitarily invariant matrices.

2   Definitions of spaces of traffics

2.1 $\mathcal{G}$-algebras

We first recall and make more precise the definition of graph operations given in the introduction.

Definition 2.1. *For all $K \geqslant 0$, a K-graph operation is a finite, connected and oriented graph with K ordered edges, and two particular vertices (one input and one output). The set of K-graph operations is denoted by $\mathcal{G}_K$, and the sequence $(\mathcal{G}_K)_{K \geqslant 0}$ is denoted by $\mathcal{G}$.*

A K-graph operation can produce a new graph operation from K different graph operations in the following way. Let us consider the *composition maps*

$$\circ : \mathcal{G}_K \times \mathcal{G}_{L_1} \times \cdots \times \mathcal{G}_{L_K} \to \mathcal{G}_{L_1 + \cdots + L_K}$$
$$(g, g_1, \ldots, g_K) \mapsto g \circ (g_1, \ldots, g_K)$$

for $K \geqslant 1$ and $L_i \geqslant 0$, which consists in replacing the i-th edge of $g \in \mathcal{G}_K$ by the L_i-graph operation g_i (which leads at the end to a $(L_1 + \cdots + L_K)$-graph operation). Let also consider the *action of the symmetric group S_K on $\mathcal{G}_K$* by defining $g \circ \sigma$ to be the K-graph operation g where the edges are reordered according to $\sigma \in S_K$ (if $e_1, \ldots, e_K$ are the ordered edges of g, $e_{\sigma^{-1}(1)}, \ldots, e_{\sigma^{-1}(K)}$ are the ordered edges in g_σ).

We introduce some important graph operations for later use:

- the constant $0 = (\cdot) \in \mathcal{G}_1$ which consists in one vertex and no edges,

- the identity $I = (\cdot \leftarrow \cdot) \in \mathcal{G}_1$ which consists in two vertices and one edge from the input to the output,

- the product $(\cdot \overset{1}{\leftarrow} \cdot \overset{2}{\leftarrow} \cdot) \in \mathcal{G}_1$ which consists in three vertices and two successive edges from the input to the output,

- the Hadamard product h, which consists in two vertices and two edges from the input to the output,

- the diagonal Δ, which consists in one vertex and one edge,

- the degree $deg = \uparrow$, which consists in two vertices, where one is the input and the output, and an edge from the input/output to the other vertex.

Endowed with those composition maps and the action of the symmetric groups, the sequence $\mathcal{G} = (\mathcal{G}_K)_{K \geqslant 0}$ is an *operad*, in the sense that it satisfies

1. the *identity* property $g \circ (I, \ldots, I) = g = I \circ g$,

2. the *associativity* property

$$g \circ \big(g_1 \circ (g_{1,1}, \ldots, g_{1,k_1}), \ldots, g_K \circ (g_{K,1}, \ldots, g_{K,k_K})\big)$$
$$= \big(g \circ (g_1, \ldots, g_K)\big) \circ (g_{1,1}, \ldots, g_{1,k_1}, \ldots, g_{K,1}, \ldots, g_{K,k_K})$$

3. the *equivariance* properties $(g \circ \pi) \circ (g_{\pi^{-1}(1)}, \ldots, g_{\pi^{-1}(K)}) = g \circ (g_1, \ldots, g_K)$; and $g \circ (g_1 \circ \sigma_1, \ldots, g_K \circ \sigma_K) = \big(g \circ (g_1, \ldots, g_K)\big) \circ (\sigma_1 \times \ldots \times \sigma_K)$.

Let us now define how a K-graph operation can produce a new element from K elements of a vector space in a linear way.

Definition 2.2. *An action of the operad $\mathcal{G} = (\mathcal{G}_K)_{K \geqslant 0}$ on a vector space $\mathcal{A}$ is the data, for all $K \geqslant 0$ and $g \in \mathcal{G}_K$, of a linear map $Z_g : \underbrace{\mathcal{A} \otimes \cdots \otimes \mathcal{A}}_{K \ times} \to \mathcal{A}$ such that*

1. $Z_I = \mathrm{Id}_\mathcal{A}$,

2. $Z_g \circ (Z_{g_1} \otimes \ldots \otimes Z_{g_K}) = Z_{g \circ (g_1, \ldots, g_K)}$,

3. $Z_g(a_1 \otimes \ldots \otimes a_K) = Z_{g \circ \sigma}(a_{\sigma^{-1}(1)} \otimes \ldots \otimes a_{\sigma^{-1}(K)})$

whenever all the objects and compositions are well-defined. By convention, for the graph 0 with a single vertex and no edge, Z_0 is a map $\mathbb{C} \to \mathcal{A}$. We denote $\mathbb{I} = Z_0(1)$ and call it the unit of $\mathcal{A}$. A vector space on which acts $\mathcal{G}$ is called a $\mathcal{G}$-algebra. A $\mathcal{G}$-subalgebra is a subvector space of a $\mathcal{G}$ algebra stable by the action of $\mathcal{G}$. A $\mathcal{G}$-morphism between two $\mathcal{G}$-algebras $\mathcal{A}$ and $\mathcal{B}$ is a linear map $f : \mathcal{A} \to \mathcal{B}$ such that $f\big(Z_g(a_1, \ldots, a_K)\big) = Z_g\big(f(a_1), \ldots, f(a_K)\big)$ for any K-graph operation g and $a_1, \ldots, a_K \in \mathcal{A}$.

Remark 2.3. The graph operation $(\cdot \overset{1}{\leftarrow} \cdot \overset{2}{\leftarrow} \cdot)$ induces a linear map $Z_{\cdot \underset{1}{\leftarrow} \underset{2}{\leftarrow} \cdot} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ which gives to $\mathcal{A}$ a structure of associative algebra over $\mathbb{C}$, with unit $\mathbb{I}$. Every $\mathcal{G}$-algebra is in particular a unital algebra. We represent graphically the element $Z_g(a_1 \otimes \ldots \otimes a_K)$ as the graph where the ordered edges are labelled by $a_1, \ldots, a_K$, and the second condition of equivariance allows to forget about the order of the edges.

Let us define also an *involution* $* : g \to g^*$ on graph operation $\mathcal{G}$, where g^* is obtained from g by reversing the orientation of its edges and interchanging the input and the output.

Definition 2.4. *A $\mathcal{G}^*$-algebra is a $\mathcal{G}$-algebra $\mathcal{A}$ endowed with an antilinear involution $* : \mathcal{A} \to \mathcal{A}$ which is compatible with the action of $\mathcal{G}$: for all K-graph operation g and $a_1, \ldots, a_K \in \mathcal{A}$, we have $Z_g(a_1 \otimes \ldots \otimes a_K)^* = Z_{g^*}(a_1^* \otimes \ldots \otimes a_K^*)$. A $\mathcal{G}^*$-subalgebra is a $\mathcal{G}$-subalgebra closed by adjointness. A $\mathcal{G}^*$-morphism between $\mathcal{A}$ and $\mathcal{B}$ is a $\mathcal{G}$-morphism $f : \mathcal{A} \to \mathcal{B}$ such that $f(a^*) = f(a)^*$ for any $a \in \mathcal{A}$.*

Remark 2.5. Recall that Δ denotes the graph operation with one vertex and one edge. Any $\mathcal{G}$-algebra $\mathcal{A}$ can be written $\mathcal{A} = \mathcal{A}_0 \oplus B$, where $\mathcal{A}_0 := \big\{\Delta(a), \ a \in \mathcal{A}\big\}$ is a commutative algebra. We call $\mathcal{A}_0$ the diagonal algebra of $\mathcal{A}$.

Example 2.6. Denote $\mathrm{M}_N(\mathbb{C})$ the algebra of N by N matrices. For any $K \geqslant 1$ and $g \in \mathcal{G}_K$ with vertex set V and ordered edges $(v_1, w_1), \ldots, (v_K, w_K)$, let us define Z_g by setting, for all $A^{(1)}, \ldots, A^{(K)} \in \mathrm{M}_N(\mathbb{C})$, the (i,j)-coefficient of $Z_g(A^{(1)} \otimes \ldots \otimes A^{(K)})$ as

$$\Big[Z_g(A^{(1)} \otimes \ldots \otimes A^{(K)})\Big]_{ij} = \sum_{\substack{k:V \to [N] \\ k(input)=j, \, k(output)=i}} A^{(1)}_{k(w_1),k(v_1)} \cdots A^{(K)}_{k(w_K),k(v_K)}.$$

This defines an action of the operad $\mathcal{G} = (\mathcal{G}_K)_{K \geqslant 0}$ on $\mathrm{M}_N(\mathbb{C})$, compatible with the usual complex transpose of matrices, and so $\mathrm{M}_N(\mathbb{C})$ is a $\mathcal{G}^*$-algebra. The product $Z_{\underset{.\leftarrow\;.\leftarrow.}{1\;\;2}}(A \otimes B)$ induced by this action coincides with the classical product of matrices, but we also have others operations like the Hadamard product $Z_h(A \otimes B) = (A_{ij} B_{ij})_{i,j=1}^N$, the projection on the diagonal $Z_\Delta(A) = (\delta_{ij} A_{ii})_{i,j=1}^N$, or the transpose $Z_{.\rightarrow.}(A) = (A_{ji})_{i,j=1}^N$. The diagonal algebra of $\mathrm{M}_N(\mathbb{C})$ defined in Remark 2.5 is the algebra of diagonal matrices.

Example 2.7. Let $\mathcal{V}$ be an infinite set and let $\mathrm{M}_\mathcal{V}(\mathbb{C})$ denotes the set of complex matrices indexed by $\mathcal{V}$, $A = (A_{v,w})_{v,w \in \mathcal{V}}$ such that each row and column have a finite number of nonzero entries. For any $g \in \mathcal{G}$ and $A^{(1)}, \ldots, A^{(K)} \in \mathrm{M}_\mathcal{V}(\mathbb{C})$, we define $Z_g(A^{(1)} \otimes \ldots \otimes A^{(K)})$ by the same formula as in Example 2.6 with summation now over the maps $k : V \to \mathcal{V}$. This defines as well a structure of $\mathcal{G}^*$-algebra for $\mathrm{M}_\mathcal{V}(\mathbb{C})$. When the entries of the matrices are non negative integers, they encode the adjacency of a locally finite directed graph: the graph associated to a matrix A has $A(v, w)$ edges from a vertex $v \in V$ to a vertex $w \in V$ (see [8]).

2.2 Space of traffics

Recall the definition from [8].

Definition 2.8. *An* algebraic space of traffics *is the data of a vector space $\mathcal{A}$ with a linear functional $\Phi : \mathcal{A} \to \mathbb{C}$ such that*

- *there exists an action of $\mathcal{G}$ on $\mathcal{A}$: $\mathcal{A}$ is a $\mathcal{G}$-algebra,*

- *Φ is unital: $\Phi(\mathbb{I}) = 1$,*

- *Φ is* input-independent*: for all $g \in \mathcal{G}_n$, $\Phi \circ Z_g = \Phi \circ Z_{\Delta \circ g}$ and does not depend on the place of the input in $\Delta \circ g$.*

A homomorphism between two algebraic spaces of traffics $\mathcal{A}$ and $\mathcal{B}$ with respective linear functionals Φ and Ψ is a $\mathcal{G}$-morphism $f : \mathcal{A} \to \mathcal{B}$ such that $\Phi \circ f = \Psi$.

The condition of input-independence for Φ implies that it is a trace for the structure of associative algebra of $\mathcal{A}$ with product $(a, b) \mapsto Z_{\underset{.\leftarrow\;.\leftarrow.}{1\;\;2}}(a, b)$. Moreover, it is possible to describe completely Φ in terms of a functional defined on some graphs where the input and output are totally forgotten. For later purpose, let us define more generally a notion of n-graph monomial, where we outline $n \geqslant 0$ particular vertices, instead of two.

Definition 2.9. *A 0-graph monomial indexed by a set J (called test-graph in [8]) is a collection $t = (V, E, \gamma)$, where (V, E) is a finite, connected and oriented graph and $\gamma : E \to J$ is a labeling of the edges by indices. For any $n \geqslant 1$, a n-graph monomial indexed by J is a collection $t = (V, E, \gamma, \mathbf{v})$, where (V, E, γ) is a 0-graph monomial and $\mathbf{v} = (v_1, \ldots, v_n)$ is a n-tuple of vertices of T, considered as the outputs of t. For any $n \geqslant 0$, we set $\mathbb{C}\mathcal{G}^{(n)}\langle J \rangle$ the vector space spanned by the n-graph monomials indexed by J, whose elements are called n-graph polynomials indexed by J.*

Let us fix an algebraic space of traffics $\mathcal{A}$ with linear functional $\Phi : \mathcal{A} \to \mathbb{C}$, and consider a 0-graph monomial $t = (V, E, \gamma)$ with labels on $\mathcal{A}$. Let us list arbitrarily the edges of $E = \{e_1, \ldots, e_K\}$ and denote by g the K-graph operation (V, E) with the ordered edges $e_1, \ldots, e_K$ and choose arbitrarily for input and output a same vertex of g. Set

$$\tau(t) = \Phi\Big(Z_g\big(\gamma(e_1) \otimes \cdots \otimes \gamma(e_K) \big) \Big), \tag{2.1}$$

which does depend neither on the choice of the ordering of $e_1, \ldots, e_K$ nor on the input and output of g, thanks to the equivariance and the input-independence properties. This map extends to $\tau : \mathbb{C}\mathcal{G}^{(0)}\langle \mathcal{A} \rangle \to \mathbb{C}$ by linearity, and characterizes entirely the functional $\Phi : \mathcal{A} \to \mathbb{C}$, thanks to the relation $\Phi(a) = \tau(\circlessO_a)$.

Definition 2.10. *Let $\mathcal{A}$ be an algebraic space of traffics with linear functional $\Phi : \mathcal{A} \to \mathbb{C}$. The map $\tau : \mathbb{C}\mathcal{G}^{(0)}\langle \mathcal{A} \rangle \to \mathbb{C}$ defined above is called the* distribution of traffics *on $\mathcal{A}$. Saying that $(\mathcal{A}, \tau)$ is an algebraic space of traffics, we mean that τ denotes this functional and call Φ the associated trace on $\mathcal{A}$.*

We now define the non-algebraic spaces of traffics. Let $\mathcal{A}$ be a set with an antilinear involution $* : \mathcal{A} \to \mathcal{A}$. Let t, t' be two n-graph monomials indexed by $\mathcal{A}$. We set $t|t'$ the 0-graph monomial obtained by merging the i-th output of t and t' for any $i = 1, \ldots, n$. We extend the map $(t, t') \mapsto t|t'$ to a bilinear application $\mathbb{C}\mathcal{G}^{(n)}\langle\mathcal{A}\rangle^2 \to \mathbb{C}\mathcal{G}^{(0)}\langle\mathcal{A}\rangle$.

Moreover, given an n-graph monomial $t = (V, E, \gamma, \mathbf{v})$ we set $t^* = (V, E^*, \gamma^*, \mathbf{v})$, where E^* is obtained by reversing the orientation of the edges in E, and γ^* is given by $e \mapsto \gamma(e)^*$. We extend the map $t \mapsto t^*$ to a linear map on $\mathbb{C}\mathcal{G}^{(n)}\langle\mathcal{A}\rangle$.

Definition 2.11. *A* space of traffics *is an algebraic space of traffics $(\mathcal{A}, \tau)$ such that:*

- *$\mathcal{A}$ is a $\mathcal{G}^*$-algebra,*

- *the distribution of traffics on $\mathcal{A}$ satisfies the following* positivity condition *: for any n-graph polynomial t indexed by $\mathcal{A}$,*

$$\tau[t|t^*] \geqslant 0. \tag{2.2}$$

A homomorphism between two spaces of traffics is a $\mathcal{G}^*$-morphism which is a homomorphism of algebraic space of traffics.

Note that (2.2) for $n = 2$ is equivalent to say that the trace Φ induced by τ is a state on the $*$-algebra $\mathcal{A}$. By consequence, the product graph operation $(\cdot \overset{1}{\leftarrow} \cdot \overset{2}{\leftarrow} \cdot)$ induces a linear map $Z_{\cdot \overset{1}{\leftarrow} \cdot \overset{2}{\leftarrow} \cdot} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ which gives to $\mathcal{A}$ a structure of $*$-probability space. Hence every space of traffics is in particular a $*$-probability space. Theorem 1.3 states that the reciprocal is true.

Example 2.12. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space in the classical sense and let consider the algebra $M_N(L^{\infty-}(\Omega, \mathbb{C}))$ of matrices whose coefficient are random variables with finite moments of all orders. Endowed with the action of the operad $\mathcal{G}$ described in Example 2.6, it is a $\mathcal{G}^*$-algebra. The linear form $\Phi_N := \mathbb{E}[\mathrm{Tr}\,\cdot]/N$ equips $M_N(L^{\infty-}(\Omega, \mathbb{C}))$ with the structure of algebraic space of traffics and the distribution of traffics τ_N is given by: for any 0-graph monomial $T = (V, E, M)$ indexed by $M_N(L^{\infty-}(\Omega, \mathbb{C}))$, where $M : E \to M_N(L^{\infty-}(\Omega, \mathbb{C}))$,

$$\tau_N[T] = \mathbb{E}\Big[\frac{1}{N} \sum_{k:V \to [N]} \prod_{e=(v,w)\in E} (M(e))_{k(v),k(w)}\Big]. \tag{2.3}$$

Moreover, $(M_N(L^{\infty-}(\Omega, \mathbb{C})), \tau_N)$ is actually a space of traffics since τ_N is positive. First, for any n-graph monomial $t = (V, E, M, \mathbf{v})$, we define a random tensor $T(t) \in (\mathbb{C}^N)^{\otimes n}$ as follows. Let us denote by $\mathbf{v} = (v_1, \ldots, v_n)$ the sequence of outputs of t and by $(\xi_i)_{i=1,\ldots,N}$ the canonical basis of $\mathbb{C}^N$. Then we set,

$$T(t) = \sum_{k:V \to [N]} \left(\prod_{e=(v,w)\in E} (M(e))_{k(v),k(w)}\right) \xi_{k(v_1)} \otimes \cdots \otimes \xi_{k(v_n)}. \tag{2.4}$$

We extend the definition by linearity on n-graph polynomials Positivity is clear since one has

$$\tau_N\big[(t|t^*)(\mathbf{A}_N)\big] := \mathbb{E}\Big[\frac{1}{N} \sum_{\mathbf{i}\in[N]^n} T(t)_{\mathbf{i}}\, \overline{T(t)_{\mathbf{i}}}\Big] \geqslant 0$$

Example 2.13. Let $\mathcal{V}$ be an infinite set. A locally finite rooted graph on $\mathcal{V}$ is a pair (G, ρ) where G is a directed graph such that each vertex has a finite number of neighbors (or equivalently an element of the space $M_{\mathcal{V}}(\mathbb{C})$ of Example 2.7 with integers entries) and ρ is an element of $\mathcal{V}$. Recall briefly that the so-called weak local topology is induced by the sets of (G, ρ) such that the subgraph induced by vertices at fixed distance of the root is given [8, Section 2.7.2]. The notion of locally finite random rooted graphs refers to the Borel σ-algebra given by this topology.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{V}$ and $\rho \in \mathcal{V}$. Let $\mathbf{G}$ be a family of locally finite random rooted graphs on Ω with vertex set $\mathcal{V}$ and common root ρ. Consider the $\mathcal{G}$-subalgebra $\mathcal{A}$ of $M_{\mathcal{V}}(\mathbb{C})$ induced by the adjacency matrices of $\mathbf{G}$. In general, the linear form $\Phi_{\rho}(A) = \mathbb{E}[A(\rho, \rho)]$ is neither well defined nor input-independent.

In [8], certain situations where Φ_ρ equips $\mathcal{A}$ with the structure of algebraic space of traffics were characterized: in particular, if the degree of the vertices of the graphs $\mathbf{G}$ are uniformly bounded, then Φ_ρ is well defined and is input-independent if and only if $\mathbf{G}$ is called *unimodular*.

When Φ_ρ is well defined, then the associated map τ_ρ always satisfies the positivity condition. Indeed, for any n-graph monomial t we define a tensor $T(t) \in (\mathbb{C}^{\mathcal{V}})^{\otimes n}$ with the same formula as for matrices, but with summation over $k : V \to \mathcal{V}$ with $k(r) = \rho$, for an arbitrary vertex r of V and with $(\xi_i)_{i\in\mathcal{V}}$ the canonical basis of $\mathbb{C}^{\mathcal{V}}$. The positivity of τ follows as well since $\tau_\rho\big[t|t^*\big] := \mathbb{E}\big[\sum_{\mathbf{i}\in\mathcal{V}^n : i_1 = \rho} T(t)_{\mathbf{i}} \, \overline{T(t)_{\mathbf{i}}}\big]$ is nonnegative.

Definition 2.14. *Let $(\mathcal{A}, \tau)$ be a space of traffics, with associated trace Φ, J an arbitrary index set, and $\mathbf{a} = (a_j)_{j\in J}$ a family of elements in $\mathcal{A}$*

1. *The distribution of traffics of $\mathbf{a}$ is the linear functional $\tau_{\mathbf{a}} : \mathbb{C}\mathcal{G}^{(0)}\langle J \times \{1, *\}\rangle \to \mathbb{C}$ given by the distribution of traffics $\tau : \mathbb{C}\mathcal{G}^{(0)}\langle \mathcal{A}\rangle \to \mathbb{C}$ composed with the linear map*

$$
\begin{aligned}
\mathbb{C}\mathcal{G}^{(0)}\langle J \times \{1, *\}\rangle &\to \mathbb{C}\mathcal{G}^{(0)}\langle \mathcal{A}\rangle \\
(V, E, j \times \epsilon) &\mapsto (V, E, a_{j(\cdot)}^{\epsilon(\cdot)}),
\end{aligned}
$$

   or in other words, for all 0-graph monomial $T = (V, E, j \times \epsilon) \in \mathbb{C}\mathcal{G}^{(0)}\langle J \times \{1, *\}\rangle$, the quantity $\tau_{\mathbf{a}}(T)$ is given by $\tau(t)$, where t is the 0-graph monomial $(V, E, \gamma) \in \mathbb{C}\mathcal{G}^{(0)}\langle \mathcal{A}\rangle$ such that $\gamma(e) = a_{j(e)}^{\epsilon(e)}$.

2. *Let $(\mathcal{A}_N, \tau_N)$ a sequence of spaces of traffics, with associated trace τ_N, J an arbitrary index set, and for each $N \geqslant 1$, a family $\mathbf{a}_N = (a_j)_{j\in J}$ of elements of $\mathcal{A}_N$.*

   We say that the sequence $\mathbf{a}_N$ converges in distribution of traffics to $\mathbf{a}$ if the distribution of traffics of $\mathbf{a}_N$ converges pointwise to the distribution of traffics of $\mathbf{a}$ on $\mathbb{C}\mathcal{G}^{(0)}\langle J \times \{1, *\}\rangle$,

   or equivalently, if, for all K-graph operations $g \in \mathcal{G}$, indices $j_1, \ldots, j_K \in J$ and labels $\epsilon_1, \ldots, \epsilon_K \in \{1, *\}$, we have the following convergence

$$
\lim_{N\to\infty} \Phi_N\big[Z_g(a_{j_1}^{\epsilon_1} \otimes \ldots \otimes a_{j_K}^{\epsilon_K})\big] = \Phi\big[Z_g(a_{j_1}^{\epsilon_1} \otimes \ldots \otimes a_{j_K}^{\epsilon_K})\big].
$$

Example 2.15. The distribution of traffics of a family $\mathbf{A}_N = (A(j))_{j\in J}$ of random matrices is given, for all 0-graph monomial $T = (V, E, j \times \epsilon) \in \mathbb{C}\mathcal{G}^{(0)}\langle J \times \{1, *\}\rangle$, by

$$
\tau_{\mathbf{A}_N}[T] = \mathbb{E}\Big[\frac{1}{N} \sum_{k:V\to[N]} \prod_{e=(v,w)\in E} (A(j(e)))_{k(v),k(w)}^{\epsilon(e)}\Big].
$$

2.3 Möbius inversion and injective trace

In order to define traffic independence, we need first to define a transform of distributions of traffics. Recall that a poset is a set $\mathcal{X}$ with a partial order $\leqslant$ (see [14, Lecture 10]). If $\mathcal{X}$ is finite, then there exists a map $\mathrm{Mob}_{\mathcal{X}} : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$, called the Möbius function on $\mathcal{X}$, such that for two functions $F, G : \mathcal{X} \to \mathbb{C}$ the statement that

$$
F(x) = \sum_{x'\leqslant x} G(x'), \ \forall x \in \mathcal{X}
$$

is equivalent to

$$
G(x) = \sum_{x'\leqslant x} \mathrm{Mob}_{\mathcal{X}}(x', x) F(x'), \ \forall x \in \mathcal{X}.
$$

Hence the first formula implicitly defines the function G in terms of F.

For any set V, denote by $\mathcal{P}(V)$ the poset of partitions of V equipped with inverse refinement order, that is $\pi' \leqslant \pi$ if the blocks of π are included in blocks of π'. Let $(\mathcal{A}, \Phi)$ be a non-commutative probability space and denote by $\mathrm{N.C.}(K) \subset \mathcal{P}(\{1, \ldots, K\})$ the set of non-crossing partitions of

$\{1, \ldots, K\}$ [14, Lecture 9]. Recall that the free cumulants are the multi-linear maps κ given implicitly by

$$\Phi(a_1 \times \cdots \times a_K) = \sum_{\pi \in \text{N.C.}(K)} \prod_{\{i_1, \ldots, i_L\} \in \pi} \kappa(a_{i_1}, \ldots, a_{i_L}). \tag{2.5}$$

We introduce now a similar concept for traffics.

Let g be a 0-graph monomial in $\mathcal{G}^{(0)}\langle \mathcal{A} \rangle$, with vertex set V. For any partition $\pi \in \mathcal{P}(V)$ of V, we denote by g^π the 0-graph monomial obtained by identifying vertices in a same block of π (the edges link the associated blocks). Denote $\mathbf{1}_{\mathcal{P}(V)}$ the partition of V with singletons only.

Definition 2.16. *Let $\mathcal{A}$ be an ensemble and let $\tau : \mathbb{C}\mathcal{G}^{(0)}\langle \mathcal{A} \rangle \to \mathbb{C}$ be a linear map (for instance τ is the distribution of traffics). The linear form τ^0 on $\mathcal{G}^{(0)}\langle \mathcal{A} \rangle$, called* injective version *of τ, is implicitly given by the following formula: for any 0-graph monomial $t \in \mathcal{G}^{(0)}\langle \mathcal{A} \rangle$*

$$\tau[t] = \sum_{\pi \in \mathcal{P}(V)} \tau^0[t^\pi], \tag{2.6}$$

in such a way for each 0-graph monomial g one has

$$\tau^0[t] = \sum_{\pi \in \mathcal{P}(V)} \text{Mob}(\pi, \mathbf{1}_{\mathcal{P}(V)}) \tau[t^\pi].$$

Example 2.17. The injective version Tr^0 of the trace of 0-graph monomials in random matrices of $M_N(\mathbb{C})$ defined in (2.3) is given, for $T = (V, E, M)$ a 0-graph monomial indexed by $\text{M}_N\big(L^{\infty-}(\Omega, \mathbb{C})\big)$, by

$$\tau_N^0[T] = \mathbb{E}\Big[\frac{1}{N} \sum_{\substack{k : V \to [N] \\ \text{injective}}} \prod_{e=(v,w) \in E} (M(e))_{k(v), k(w)} \Big]. \tag{2.7}$$

2.4 Definition of traffic independence

Let J be a fixed index set and, for each $j \in J$, let $\mathcal{A}_j$ be some set. Given a family of linear maps $\tau_j : \mathbb{C}\mathcal{G}^{(0)}\langle \mathcal{A}_j \rangle \to \mathbb{C}$, $j \in J$, sending the graph with no edge to one, we shall define a linear map denoted $\star_{j \in J} \tau_j : \mathbb{C}\mathcal{G}^{(0)}\langle \bigsqcup_{j \in J} \mathcal{A}_j \rangle$ with the same property and called the free product[1] of the τ_j's. Therein, $\bigsqcup_{j \in J} \mathcal{A}_j$ has to be thought as the disjoint union of copies of $\mathcal{A}_j$, although the sets $\mathcal{A}_j$ can originally intersect (they can even be equal).

Let us consider 0-graph monomial T in $\mathbb{C}\mathcal{G}^{(0)}\langle \bigsqcup_{j \in J} \mathcal{A}_j \rangle$ and introduce the following indirect graph. We call colored components of T with respect to the families $(\mathcal{A}_j)_{j \in J}$ the maximal nontrivial connected subgraphs whose edges are labelled by elements of $\mathcal{A}_j$ for some $j \in J$ (it is an element of $\mathbb{C}\mathcal{G}^{(0)}\langle \mathcal{A}_j \rangle$). There is no ambiguity about the definition of colored components since T is labeled in $\bigsqcup_{j \in J} \mathcal{A}_j$ where $\bigsqcup$ means that we distinguish the origin of a element that can come from several $\mathcal{A}_j$'s. We call connectors of T the vertices of T belonging to at least two different colored components. The graph $\bar{T}$ defined below is called graph of colored components of T with respect to $(\mathcal{A}_j)_{j \in J}$:

- the vertices of $\bar{T}$ are the colored components of T and its connectors

- there is an edge between a colored component of $\bar{T}$ and a connector if the connector belongs to the component.

Definition 2.18. *1. For each $j \in J$, let $\mathcal{A}_j$ be a set and $\tau_j : \mathbb{C}\mathcal{G}^{(0)}\langle \mathcal{A}_j \rangle \to \mathbb{C}$ be a linear map sending the graph with no edges to one. The free product of the maps τ_j is the linear map $\star_{j \in J} \tau_j : \mathbb{C}\mathcal{G}^{(0)}\langle \bigsqcup \mathcal{A}_j \rangle \to \mathbb{C}$ whose injective version is given by: for any 0-graph monomial T,*

$$(\star_{j \in J} \tau_j)^0[T] = \mathbb{1}(\bar{T} \text{ is a tree}) \times \prod_{\substack{S \text{ colored} \\ \text{component of } T \\ \text{w.r.t. } (\mathcal{A}_j)_{j \in J}}} \tau^0[S]. \tag{2.8}$$

---

[1] The terminology *free* product should be understood as *canonical* product, and may not be confused with the terminology *free* independence

2. Let $(\mathcal{A}, \tau)$ be an algebraic space of traffics and let J be a fixed index set. For each $j \in J$, let $\mathcal{A}_j \subset \mathcal{A}$ be a $\mathcal{G}$-subalgebra. The subalgebras $(\mathcal{A}_j)_{j \in J}$ are called traffic independent whenever the restriction of τ to the $\mathcal{G}$-subalgebra induced by the $\mathcal{A}_j$ is $\star_{j \in J} \tau_j$.

3. Let $X_j, j \in J$ be subsets of $\mathcal{A}$ and let $(\mathbf{a}_j)_{j \in J}$ be a family of elements of $\mathcal{A}$. Then $(X_j)_{j \in J}$ (resp. $(\mathbf{a}_j)_{j \in J}$) are called traffic independent whenever the $\mathcal{G}$-subalgebra induced by the X_j's (resp. by the $\mathbf{a}_j$'s) are traffic independent.

4. In the context of space of traffics, we say that $(X_j)_{j \in J}$ are traffic independent whenever the $\mathcal{G}^*$-subalgebras are traffic independent.

3 The free product of spaces of traffics

3.1 Free products of algebraic spaces of traffics

The free product $*_{j \in J} \mathcal{A}_j$ of a family $(\mathcal{A}_j)_{j \in J}$ of $\mathcal{G}$-algebras will be a $\mathcal{G}$-algebra made with "graphs whose edges are labelled by elements" from the $\mathcal{A}_j$ and the free product of a family $(\mathcal{A}_j, \tau_j)_{j \in J}$ of spaces of traffics is their free product of $\mathcal{G}$-algebras $*_{j \in J} \mathcal{A}_j$ equipped with the free product $*_{j \in J} \tau_j$ of their distributions of traffics.

Let J be a fixed index set and, for each $j \in J$, $\mathcal{A}_j$ be some set. As in Section 2.4, while considering a monomial g in $\mathbb{C}\mathcal{G}^{(2)}\langle \bigsqcup_{j \in J} \mathcal{A}_j \rangle$ we mean that g is the data of a finite connected graph (V, E) with an input and an output, and that for each edge is associated an index $j \in J$ and then an element of $\mathcal{A}_j$.

Definition 3.1. *For all family of $\mathcal{G}$-algebras $(\mathcal{A}_j)_{j \in J}$, we denote by $*_{j \in J} \mathcal{A}_j$ the $\mathcal{G}$-algebra $\mathbb{C}\mathcal{G}^{(2)}\langle \bigsqcup_{j \in J} \mathcal{A}_j \rangle$, quotiented by the space generated by the following relations:*

$$Z_g(\cdot \xleftarrow{Z_{g_1}(a_1 \otimes \cdots \otimes a_k)} \cdot \otimes \cdot \xleftarrow{a_{k+1}} \cdot \otimes \ldots \otimes \cdot \xleftarrow{a_n} \cdot) = Z_g(Z_{g_1}(\cdot \xleftarrow{a_1} \cdot \otimes \cdots \otimes \cdot \xleftarrow{a_k} \cdot) \otimes \cdot \xleftarrow{a_{k+1}} \cdot \otimes \ldots \otimes \cdot \xleftarrow{a_n} \cdot)$$

whenever $a_1, \ldots, a_k$ are in a same algebra $\mathcal{A}_j$; which allows to consider the $\mathcal{G}$-algebra homomorphisms $V_j : \mathcal{A}_j \to *_{j \in J} \mathcal{A}_j$ given by the image of $a \mapsto (\cdot \xleftarrow{a} \cdot)$ by the quotient map.

The $\mathcal{G}$-algebra $*_{j \in J} \mathcal{A}_j$ is the free product of the $\mathcal{G}$-algebras in the following sense.

Proposition 3.2. *Let $\mathcal{B}$ be a $\mathcal{G}$-algebra, and $f_j : \mathcal{A}_j \to \mathcal{B}$ a family of $\mathcal{G}$-morphism. There exists a unique $\mathcal{G}$-morphism $*_{j \in J} f_j : *_{j \in J} \mathcal{A}_j \to \mathcal{B}$ such that $f_j = (*_{j \in J} f_j) \circ V_j$ for all $j \in J$. As a consequence, the maps V_j are injective.*

Proof. The existence is given by the following definition of $*_{j \in J} f_j$ on $*_{j \in J} \mathcal{A}_j$:

$$*_{j \in J} f_j(Z_g(\cdot \xleftarrow{a_1} \cdot \otimes \ldots \otimes \cdot \xleftarrow{a_n} \cdot)) = Z_g(f_{j(1)}(a_1) \otimes \ldots \otimes f_{j(n)}(a_n))$$

whenever $a_1 \in \mathcal{A}_{j(1)}, \ldots, a_n \in \mathcal{A}_{j(n)}$; which obviously respects the relation defining $*_{j \in J} \mathcal{A}_j$.

The uniqueness follows from the fact that $*_{j \in J} f_j$ is uniquely determined on $\bigcup_j V_j(\mathcal{A}_j)$ (indeed, $*_{j \in J} f_j(a)$ must be equal to $f_j(b)$ whenever $a = V_j(b)$) and that $\bigcup_j V_j(\mathcal{A}_j)$ generates $*_{j \in J} \mathcal{A}_j$ as a $\mathcal{G}$-algebra. $\square$

Proposition 3.3. *Let $(\mathcal{A}_j, \tau_j)_{j \in J}$ be a family of algebraic spaces of traffics. The free product of distributions of traffics $\star_{j \in J} \tau_j : \mathbb{C}\mathcal{G}^{(0)}\langle \bigsqcup_{j \in J} \mathcal{A}_j \rangle \to \mathbb{C}$ of Definition 2.18 respects the quotient structure of $*_{j \in J} \mathcal{A}_j$, and consequently yields an algebraic space of traffics $(*_{j \in J} \mathcal{A}_j, \star_{j \in J} \tau_j)$. Furthermore, we have $\tau_i = (\star_{j \in J} \tau_j) \circ V_i$, where V_i is the canonical injective algebra homomorphism from $\mathcal{A}_i$ to $*_{j \in J} \mathcal{A}_j$.*

Proof. We first need to prove that we have: for any graph operations g, g_1,

$$\star_{j \in J} \tau_j \left(Z_g(\cdot \xleftarrow{Z_{g_1}(a_1 \otimes \cdots \otimes a_k)} \cdot \otimes \cdot \xleftarrow{a_{k+1}} \cdot \otimes \ldots \otimes \cdot \xleftarrow{a_n} \cdot) \right)$$

$$= \star_{j \in J} \tau_j \left(Z_g(Z_{g_1}(\cdot \xleftarrow{a_1} \cdot \otimes \cdots \otimes \cdot \xleftarrow{a_k} \cdot) \otimes \cdot \xleftarrow{a_{k+1}} \cdot \otimes \ldots \otimes \cdot \xleftarrow{a_n} \cdot) \right).$$

Let us prove the corresponding properties at the level of the injective trace.

Lemma 3.4. *Let π be a partition of the vertices V of g. We denote by V_1 the vertices in g_1 different from the output or the input of g_1. We have*

$$(\star_{j\in J}\tau_j)^0\left(Z_g(\cdot\overset{Z_{g_1}(a_1\otimes\cdots\otimes a_k)}{\longleftarrow}\cdot\otimes\cdot\overset{a_{k+1}}{\longleftarrow}\cdot\otimes\ldots\otimes\cdot\overset{a_n}{\longleftarrow}\cdot)^\pi\right)$$
$$=\sum_{\substack{\sigma\in\mathcal{P}(V\cup V_1)\\\sigma\backslash V_1=\pi}}(\star_{j\in J}\tau_j)^0\left(Z_g(Z_{g_1}(\cdot\overset{a_1}{\longleftarrow}\cdot\otimes\cdots\otimes\cdot\overset{a_k}{\longleftarrow}\cdot)\otimes\cdot\overset{a_{k+1}}{\longleftarrow}\cdot\otimes\ldots\otimes\cdot\overset{a_n}{\longleftarrow}\cdot)^\sigma\right).$$

Proof. Because the colored component containing $Z_{g_1}(a_1\otimes\cdots\otimes a_k)$ has the same edges on both sides on the equation, and because $(\star_{j\in J}\tau_j)^0$ factorizes on colored component, it suffices to prove the lemma when only one color (let say j_0) is involved. In this case, $(\star_{j\in J}\tau_j)^0=\tau_{j_0}^0$, and we can compute in $(\mathcal{A}_{j_0},\tau_{j_0})$. Below, we denote by $\mathrm{Mob}(\,\cdot\,,\,\cdot\,)$ the Möbis function on $\mathcal{P}(V)$.

$$\tau_{j_0}^0\left(Z_g(\cdot\overset{Z_{g_1}(a_1\otimes\cdots\otimes a_k)}{\longleftarrow}\cdot\otimes\cdot\overset{a_{k+1}}{\longleftarrow}\cdot\otimes\ldots\otimes\cdot\overset{a_n}{\longleftarrow}\cdot)^\pi\right)$$

$$=\sum_{\pi\leqslant\pi'\in\mathcal{P}(V)}\mathrm{Mob}(\pi,\pi')\tau_{j_0}\left(Z_g(\cdot\overset{Z_{g_1}(a_1\otimes\cdots\otimes a_k)}{\longleftarrow}\cdot\otimes\cdot\overset{a_{k+1}}{\longleftarrow}\cdot\otimes\ldots\otimes\cdot\overset{a_n}{\longleftarrow}\cdot)^{\pi'}\right)$$

$$=\sum_{\pi\leqslant\pi'\in\mathcal{P}(V)}\mathrm{Mob}(\pi,\pi')\tau_{j_0}\left(Z_g(Z_{g_1}(\cdot\overset{a_1}{\longleftarrow}\cdot\otimes\cdots\otimes\cdot\overset{a_k}{\longleftarrow}\cdot)\otimes\cdot\overset{a_{k+1}}{\longleftarrow}\cdot\otimes\ldots\otimes\cdot\overset{a_n}{\longleftarrow}\cdot)^{(\pi'\cup 0_{V_2})}\right)$$

$$=\sum_{\pi\leqslant\pi'\in\mathcal{P}(V)}\sum_{(\pi'\cup 0_{V_2})\leqslant\sigma\in\mathcal{P}(V\cup V_1)}\mathrm{Mob}(\pi,\pi')\tau_{j_0}^0\left(Z_g(Z_{g_1}(\cdot\overset{a_1}{\longleftarrow}\cdot\otimes\cdots\otimes\cdot\overset{a_k}{\longleftarrow}\cdot)\otimes\cdot\overset{a_{k+1}}{\longleftarrow}\cdot\otimes\ldots\otimes\cdot\overset{a_n}{\longleftarrow}\cdot)^\sigma\right)$$

$$=\sum_{\sigma\in\mathcal{P}(V\cup V_1)}\left(\sum_{\pi\leqslant\pi'\leqslant\sigma\backslash V_1\in\mathcal{P}(V)}\mathrm{Mob}(\pi,\pi')\right)\tau_{j_0}^0\left(Z_g(Z_{g_1}(\cdot\overset{a_1}{\longleftarrow}\cdot\otimes\cdots\otimes\cdot\overset{a_k}{\longleftarrow}\cdot)\otimes\cdot\overset{a_{k+1}}{\longleftarrow}\cdot\otimes\ldots\otimes\cdot\overset{a_n}{\longleftarrow}\cdot)^\sigma\right)$$

$$=\sum_{\sigma\in\mathcal{P}(V\cup V_1)}\delta_{\pi,\sigma\backslash V_1}\tau_{j_0}^0\left(Z_g(Z_{g_1}(\cdot\overset{a_1}{\longleftarrow}\cdot\otimes\cdots\otimes\cdot\overset{a_k}{\longleftarrow}\cdot)\otimes\cdot\overset{a_{k+1}}{\longleftarrow}\cdot\otimes\ldots\otimes\cdot\overset{a_n}{\longleftarrow}\cdot)^\sigma\right)$$

$$\sum_{\substack{\sigma\in\mathcal{P}(V\cup V_1)\\\sigma\backslash V_1=\pi}}(\tau_{j_0})^0\left(Z_g(Z_{g_1}(\cdot\overset{a_1}{\longleftarrow}\cdot\otimes\cdots\otimes\cdot\overset{a_k}{\longleftarrow}\cdot)\otimes\cdot\overset{a_{k+1}}{\longleftarrow}\cdot\otimes\ldots\otimes\cdot\overset{a_n}{\longleftarrow}\cdot)^\sigma\right).$$

$\square$

Now we can conclude, since

$$\star_{j\in J}\tau_j\left(Z_g(\cdot\overset{Z_{g_1}(a_1\otimes\cdots\otimes a_k)}{\longleftarrow}\cdot,\cdot\otimes\cdot\overset{a_{k+1}}{\longleftarrow}\cdot\otimes\ldots\otimes\cdot\overset{a_n}{\longleftarrow}\cdot)\right)$$

$$=\sum_{\pi\in V}(\star_{j\in J}\tau_j)^0\left(Z_g(\cdot\overset{Z_{g_1}(a_1\otimes\cdots\otimes a_k)}{\longleftarrow}\cdot,\cdot\otimes\cdot\overset{a_{k+1}}{\longleftarrow}\cdot\otimes\ldots\otimes\cdot\overset{a_n}{\longleftarrow}\cdot)^\pi\right)$$

$$=\sum_{\substack{\pi\in V\ \sigma\in\mathcal{P}(V\cup V_1)\\\sigma\backslash V_1=\pi}}(\star_{j\in J}\tau_j)^0\left(Z_g(Z_{g_1}(\cdot\overset{a_1}{\longleftarrow}\cdot\otimes\cdots\otimes\cdot\overset{a_k}{\longleftarrow}\cdot)\otimes\cdot\overset{a_{k+1}}{\longleftarrow}\cdot\otimes\ldots\otimes\cdot\overset{a_n}{\longleftarrow}\cdot)^\sigma\right)$$

$$=\sum_{\sigma\in\mathcal{P}(V\cup V_1)}(\star_{j\in J}\tau_j)^0\left(Z_g(Z_{g_1}(\cdot\overset{a_1}{\longleftarrow}\cdot\otimes\cdots\otimes\cdot\overset{a_k}{\longleftarrow}\cdot)\otimes\cdot\overset{a_{k+1}}{\longleftarrow}\cdot\otimes\ldots\otimes\cdot\overset{a_n}{\longleftarrow}\cdot)^\sigma\right)$$

$$=\star_{j\in J}\tau_j\left(Z_g(Z_{g_1}(\cdot\overset{a_1}{\longleftarrow}\cdot\otimes\cdots\otimes\cdot\overset{a_k}{\longleftarrow}\cdot)\otimes\cdot\overset{a_{k+1}}{\longleftarrow}\cdot\otimes\ldots\otimes\cdot\overset{a_n}{\longleftarrow}\cdot)\right).$$

$\square$

3.2 A new characterization of traffic independence

Let $(\mathcal{A}_j,\tau_j)_{j\in J}$ be spaces of traffics and $(*_{j\in J}\mathcal{A}_j,*_{j\in J}\tau_j)$ their algebraic free product. in order to finish the construction of the free product of spaces of traffics, it remains to prove Theorem 1.2,

that is the positivity of the free product $*_{j \in J} \tau_j$ of positive distributions of traffics τ_j. We will reason as for the construction of the free product of *-probability spaces [14, Lecture 6] using a structure result (3.20) for $(*_{j \in J} \mathcal{A}_j, *_{j \in J} \tau_j)$. Before that, we shall first state in Proposition 3.11 a characterization of traffic independence whose statement is closer to the more familiar free-independence than Definition 2.18.

Definition 3.5. *A* bigraph *is a finite, connected and bipartite graph g, endowed with a bipartition of its vertices into two sets $V_{in}(g)$ and $V_{co}(g)$, whose elements we call* inputs *and* connectors.

For all $L, n \geqslant 0$, a (L, n)-bigraph operation is the data of a bigraph with exactly L ordered inputs, together with an ordering of its edges around each input, and the data of an ordered subset $V_{out}(g)$ consisting in n elements of the connectors $V_{co}(g)$ that we call output, and such that all connectors which are not an output have a degree larger than 2. For all integer $(L, n) \geqslant 0$ and tuple $\mathbf{d} = (d_1, \ldots, d_L) \in (\mathbb{N}^*)^L$, we denote by $\mathcal{G}^{(n)}_{L,\mathbf{d}}$ (if $L \neq 0$ and by $\mathcal{G}^{(n)}_0$ otherwise) the set of (L, n)-bigraph operations such that the k-th inputs have a degree d_k.

A (L, n)-bigraph operation with degrees $d_1, \ldots, d_L$ is to be thought as an operation that accepts L objects with types $d_1, \ldots, d_L$, and produces a new object of type n. In particular, a (L, n)-bigraph operation can produce a new n-graph monomial from L different graphs monomials in the following way. See figure 1
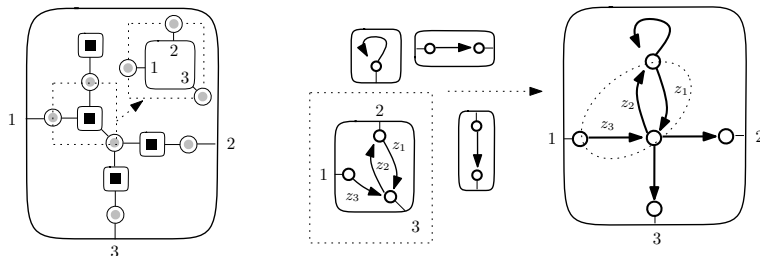

Figure 1: Left: a bigraph with four inputs (squares), five connectors (circles) and three outputs (with links exiting the box). The ordering of adjacent connectors is noticed for an input. Substituing in an obvious way the inputs of the bigraph by graph monomials one get the rightmost 3-graph monomial.

Let us consider L graph monomials $t_1, \ldots, t_L$ on some set of labels $\mathcal{A}$, with respective number of outputs given by $\mathbf{d} \in (\mathbb{N}^*)^L$ (that is $t_\ell \in \mathbb{C}\mathcal{G}^{(d_\ell)}\langle \mathcal{A} \rangle$), and a (L, n)-bigraph operation $g \in \mathcal{G}^{(n)}_{L,\mathbf{d}}$. Replacing the ℓ-th input of g and its adjacent ordered edges $(e_1, \ldots, e_{d_\ell})$ by the graph of t_ℓ, identifying for each $k \in [L]$, the connector attached to e_k with the k-th output of t_ℓ, yields a connected graph. We denote by $T_g(t_1 \otimes \ldots \otimes t_L) \in \mathbb{C}\mathcal{G}^{(n)}\langle \mathcal{A} \rangle$ the n-graph monomial whose labelling is induced by those of $t_1, \ldots, t_L$, and with outputs given by the outputs of g. We then define by linear extension

$$T_g : \mathbb{C}\mathcal{G}^{(d_1)}\langle \mathcal{A} \rangle \otimes \cdots \otimes \mathbb{C}\mathcal{G}^{(d_L)}\langle \mathcal{A} \rangle \longrightarrow \mathbb{C}\mathcal{G}^{(n)}\langle \mathcal{A} \rangle$$
$$t_1 \otimes \ldots \otimes t_L \longmapsto T_g(t_1 \otimes \ldots \otimes t_L).$$

Remark 3.6. One can show that the set of bigraph operations defines an operad with a compatible action on n-graph polynomials. It acts on the tensors of order n in a slight generalization of Example 2.6 of [7] We do not use this fact here.

Definition 3.7. *Let J be an index set and $(\mathcal{A}_j)_{j \in J}$ be a family ensembles, and let $g \in \mathcal{G}^{(n)}_{L,\mathbf{d}}$ be a bigraph operation with $\mathbf{d} = (d_1, \ldots, d_L)$. A sequence $t_1 \in \mathbb{C}\mathcal{G}^{(d_1)}\langle \mathcal{A}_{j_1} \rangle, \ldots, t_L \in \mathbb{C}\mathcal{G}^{(d_L)}\langle \mathcal{A}_{j_L} \rangle$ of graph polynomials is g-alternated if for all $p, q \in [L]$ such that the p-th and the q-th inputs are neighbors of a same connector, then $j_p \neq j_q$.*

If $t_1, \ldots, t_L$ are graph monomials alternated along $g \in \mathcal{G}^{(0)}_{L,\mathbf{d}}$, then $T_g(t_1 \otimes \ldots \otimes t_L)$ is a graph monomial with graph of colored components g, and its colored components are $t_1, \ldots, t_L$, (considered as graphs with no outputs).

For any $n \geqslant 1$, $\pi \in \mathcal{P}_n$ and any n-graph monomial g made of a finite graph with outputs $(v_1, \ldots, v_n)$, let us denote by g^π the quotient graph obtained by identifying vertices $(v_1, \ldots, v_n)$ according to π, with outputs given by the images of $(v_1, \ldots, v_n)$ by the quotient map, so that edges of g^π can be identified with the one of g. This defines a linear map $\Delta_\pi : \mathbb{C}\mathcal{G}^{(n)}\langle \mathcal{A} \rangle \to \mathbb{C}\mathcal{G}^{(n)}\langle \mathcal{A} \rangle$ such that $\Delta_\pi(g) = g^\pi$ for n-graph monomials g. Denote respectively by 1_n and 0_n the partition of n made of n singletons and of 1 single block.

Definition 3.8. *Let $\phi : \mathbb{C}\mathcal{G}^{(1)}\langle \mathcal{A} \rangle \to \mathbb{C}$ be a linear form. A graph polynomial $t \in \mathbb{C}\mathcal{G}^{(n)}\langle \mathcal{A} \rangle$ is called* reduced *with respect to ϕ, if $n \geqslant 2$ and for any $\pi \in \mathcal{P}(n) \backslash \{1_n\}$, $\Delta_\pi(t) = 0$ or $n = 1$ and $\phi(t) = 0$.*

Example 3.9. For any $t \in \mathbb{C}G^{(n)}\langle \mathcal{A} \rangle$, one has $\Delta_{1_n}(t) = t$. If $n = 2$, then $\Delta_{0_2}(t) = \Delta(t)$, where we recall that Δ is the graph operation with one vertex and one edge (and so t is reduced if and only if $\Delta(t) = 0$.)

Example 3.10. Let $\mathbf{A}_N$ be a family of matrices of size N by N and let g be a n-graph polynomial. Recall that we defined in Example 2.12 a tensor $Z_g(\mathbf{A}_N) = (B_\mathbf{i})_{\mathbf{i} \in [N]^n}$ of order n. Then $T_g(\mathbf{A}_N)$ is reduced if and only if $B_\mathbf{i} = 0$ as soon as two indices of $\mathbf{i}$ are equal. In particular for $n = 2$, a matrix is reduced whenever its diagonal is null.

Proposition 3.11. *Let $(\mathcal{A}, \tau)$ be a space of traffics. Denote by $\phi : \mathbb{C}\mathcal{G}^{(1)}\langle \mathcal{A} \rangle \to \mathbb{C}$ the linear map given by $\phi(g) = \tau(\tilde{g})$ where $\tilde{g} \in \mathbb{C}\mathcal{G}^{(0)}\langle \mathcal{A} \rangle$ is obtained by forgetting the position of the output of g. Say that a graph polynomial is reduced when it is reduced with respect to ϕ. Then, the $\mathcal{G}$-subalgebras $(\mathcal{A}_j)_{j \in J}$ are traffic independent if and only if for any bigraph $g \in \mathcal{G}_{L,\mathbf{d}}^{(0)}$ and any g-alternated sequence $(t_1, \ldots, t_L)$ of reduced graph polynomials in $\mathbb{C}\mathcal{G}\langle \mathcal{A} \rangle$, one has $\tau[T_g(t_1 \otimes \ldots \otimes t_L)] = 0$.*

This characterization shows that traffic independence is stronger than free independence in the following situation, which has to be compared with [8, Corollary 3.5] and will be satisfied in Section 4.

Lemma 3.12. *Let $(\mathcal{A}, \tau)$ be a space of traffics. Denote Φ the associated trace on $\mathcal{A}$ and $\eta(a) = \Phi(\Delta(a^*)\Delta(a)) - |\Phi(a)|^2 = \Phi(a^* \circ a) - |\Phi(a)|^2$. If for any $a \in \mathcal{A}$, $\eta(a) = 0$ then any family of $\mathcal{G}$-subalgebras that is traffic independent is free independent in the *-probability space $(\mathcal{A}, \Phi)$. Similarly, for any subalgebra $\mathcal{B}$ of $\mathcal{A}$, if $\eta(a) = 0$ for all $a \in \mathcal{B}$, then the free independence of families in $(\mathcal{B}, \Phi_{|\mathcal{B}})$ is a consequence of traffic independence in $(\mathcal{A}, \tau)$.*

Proof. The two statement are proved in a similar way, and we only prove the first one. Since the trace defined on $\mathcal{A}$ is a state, the assumption implies, for every $a \in \mathcal{A}$, that $\Delta(a)$ has the same *-distribution as $\Phi(a)\mathbb{I}$. Let $(\mathcal{A}_j)_{j \in J}$ be traffic-independent $\mathcal{G}$-subalgebras and let $a_1, \ldots, a_n \in \mathcal{A}$, such that for any $k \in [n]$, $\Phi(a_k) = 0$ and $a_k \in \mathcal{A}_{j_k}$, with $j_k \neq j_{k+1}$, whenever $k < n$. Then,

$$\Phi((a_1 - \Delta(a_1)) \ldots (a_n - \Delta(a_n))) = \Phi((a_1 - \Phi(a_1)) \ldots (a_n - \Phi(a_n))) = \Phi(a_1 \ldots a_n).$$

Let g be the bigraph with no outputs, n inputs and $n-1$ connectors whose graph is a segment, with inputs vertices (alternating with the connectors) labeled consecutively from one side to the other, from 1 to n. Then one has

$$\Phi((a_1 - \Delta(a_1)) \ldots (a_n - \Delta(a_n))) = \tau\Big(T_g((a_1 - \Delta(a_1)) \otimes \ldots \otimes (a_n - \Delta(a_n)))\Big),$$

and $(a_1 - \Delta(a_1)) \otimes \ldots \otimes (a_n - \Delta(a_n))$ is a g-alternated reduced tensor, so that by Proposition 3.11 we get $\Phi(a_1 \ldots a_n) = 0$ as desired. $\square$

Remark 3.13. Recall Example 2.13 of the $\mathcal{G}$-algebra $\mathcal{A}$ of locally finite rooted graphs on a set of vertices $\mathcal{V}$. It is a classical fact that an element A of $\mathcal{A}$ which is both deterministic and unimodular is vertex-transitive (there exists automorphisms exchanging each pair of vertices). This property implies that the diagonal $\Delta(A) = (A(v,v)\mathbb{1}_{v=w})_{v,w \in \mathcal{V}}$ of A is constant, and so one can apply the lemma. This gives a new proof of the free independence of the spectral distributions of the free product of infinite deterministic graphs of [1], thanks to [8, Proposition 7.2].

3.3 Proof of Proposition 3.11

We start by stating two preliminary lemmas.

Lemma 3.14. *Let m a graph monomial with output set $\mathcal{O}$. For each partition π of $\mathcal{O}$, denote by m^π the graph operation obtained by identifying the outputs of m that belong to a same block of π. Let us denote by Mob the Möbius function for the poset of partitions of $\mathcal{O}$ and $0_{\mathcal{O}}$ the partition of $\mathcal{O}$ made of singletons. Then, $p(m) = \sum_{\pi \in \mathcal{P}(\mathcal{O})} \mathrm{Mob}(0_{\mathcal{O}}, \pi) m^\pi$ is a reduced graph polynomial, and every reduced graph polynomial m satisfies $m = p(m)$.*

Proof. For any $\nu \in \mathcal{P}(\mathcal{O})$,

$$\Delta_\nu \left(\sum_{\pi \in \mathcal{P}(\mathcal{O})} \mathrm{Mob}(0_{\mathcal{O}}, \pi) m^\pi \right) = \sum_{\mu \in \mathcal{P}(\mathcal{O})} \left(\sum_{\pi \in \mathcal{P}(\mathcal{O}): \pi \vee \nu = \mu} \mathrm{Mob}(0_{\mathcal{O}}, \pi) \right) m^\mu$$

Now, for any $\mu \in \mathcal{P}(\mathcal{O})$,

$$\sum_{\pi \in \mathcal{P}(\mathcal{O}): \pi \vee \nu = \mu} \mathrm{Mob}(0_{\mathcal{O}}, \pi) = \sum_{\pi \leqslant \mu} \sum_{\pi \vee \nu \leqslant \sigma \leqslant \mu} \mathrm{Mob}(\sigma, \mu) \mathrm{Mob}(0_{\mathcal{O}}, \pi)$$

$$= \sum_{\nu \leqslant \sigma \leqslant \mu} \mathrm{Mob}(\sigma, \mu) \left(\sum_{\pi \leqslant \sigma} \mathrm{Mob}(0_{\mathcal{O}}, \pi) \right)$$

$$= \sum_{\nu \leqslant \sigma \leqslant \mu} \mathrm{Mob}(\sigma, \mu) \delta_{\sigma, 0_{\mathcal{O}}} = \delta_{\nu, 0_{\mathcal{O}}} \mathcal{M}(0_{\mathcal{O}}, \mu).$$

$\square$

Lemma 3.15. *For any linear from ϕ on $\mathcal{G}^{(1)}\langle \bigsqcup_{j \in J} \mathcal{A}_j \rangle$ sending the graph with no edges to one, and calling* reduced *graph polynomials reduced with respect to ϕ, one has*

$$\mathcal{G}^{(n)}\langle \bigsqcup_{j \in J} \mathcal{A}_j \rangle = \mathbb{C}\mathbb{I} \oplus \bigoplus_{\substack{g \in \mathcal{G}^{(n)}_{L, \mathbf{d}} \\ \gamma: \{1, \ldots, n\} \to J}} \underbrace{\mathrm{span}\left\{ T_g(\mathbf{t}) \,\Big|\, \mathbf{t} = (t_\ell)_{\ell=1}^L \, g\text{--alternated, reduced, } t_\ell \in \mathcal{G}^{(n)}\langle \mathcal{A}_{\gamma(\ell)} \rangle \right\}}_{:= \mathcal{W}_{g, \gamma}},$$

Proof. Let us denote by $\mathcal{E}$ the vector space spanned by the right hand side. For any integers $k \geqslant 1, s \geqslant 0$, let us consider the vector space $\mathcal{E}_k^s$ spanned by the family of graphs polynomials $T_g(\mathbf{t})$, where $g \in \mathcal{G}^{(n)}_{L, \mathbf{d}}$ has less than k vertices and $\mathbf{t} = t_1 \otimes \ldots \otimes t_L$ is such that the number of $k \in [L]$ with t_k reduced is greater than $\max\{0, L - s\}$. Let us set $\mathcal{E}_k = \mathrm{Span}(\mathcal{E}_k^s)_{s \geqslant 0}$ and prove by induction that for any $k \geqslant 0$, $\mathcal{E}_k \subset \mathcal{E}$, which shall conclude the proof. To begin with, note that $\mathcal{E}_1 = \mathbb{C}\mathbb{I} \subset \mathcal{E}$. Let us assume the claim for $k \in \mathbb{N}$ and prove by induction on $s \geqslant 0$ that $\mathcal{E}_{k+1}^s \subset \mathcal{E}$. First,

$$\mathcal{E}_{k+1}^0 \subset \bigcup_{\substack{g \in \mathcal{G}^{(n)}_d \\ m \text{ a } g\text{-alternated sequence} \\ \text{of graph monomials}}} \{T_g(\mathbf{t}) : \mathbf{t} \in \Lambda_{\mathbf{m}} \text{ and } \mathbf{t} \text{ is reduced}\} \subset \mathcal{E}.$$

Let us assume that $\mathcal{E}_{k+1}^s \subset \mathcal{E}$ and consider g a bigraph with $k+1$ connectors and $\mathbf{t} = t_1 \otimes \ldots \otimes t_L \in \Lambda_{\mathbf{m}}$ a g-alternated tensor with $\max\{L - s - 1, 0\}$ reduced components. Let us assume that $t_1 \in \mathbb{C}\mathcal{G}^{(d_1)}\langle \mathcal{A}_j \rangle$ is not reduced, for some $j \in J, d_1 \geqslant 1$. If $d_1 = 1$, then $T_g \left((t_1 - \tau_j(t_1)) \otimes t_2 \ldots \otimes t_L \right) \in \mathcal{E}_{k+1}^s$ and $\tau_j(t_1) T_g (1 \otimes t_2 \ldots \otimes t_L) \in \mathcal{E}_k$, so that $T_g(\mathbf{t}) \in \mathcal{E}$. If $d_1 \geqslant 2$, according to Lemma 3.14, we can write $t_1 = r + \sum_{i=1}^m x_i$, where $r \in \mathbb{C}\mathcal{G}^{(d_1)} \mathcal{A}_j$ is a reduced graph polynomial and $x_1, \ldots, x_m \in \mathbb{C}\mathcal{G}^{(d_1)}\langle \mathcal{A}_j \rangle$ are graph monomials having two ouputs equal to the same vertex. Then, for any $i \in [m]$, $T_g (x_i \otimes t_2 \ldots \otimes t_L) \in \mathcal{E}_k$ and $T_g(r \otimes t_2 \otimes \ldots \otimes t_L) \in \mathcal{E}_{k+1}^s$, so that $T_g(\mathbf{t}) \in \mathcal{E}$. $\square$

To prove Proposition 3.11 it is then sufficient to prove that if $(\mathcal{A}_j)_{j \in J}$ are traffic independent in $(\mathcal{A}, \tau)$ then for each bi-graph operation $g \in \mathcal{G}^{(2)}_{L, \mathbf{d}}$ and each g-alternated sequence $\mathbf{t}$ of reduced graph polynomials one has $\tau[T_g(\mathbf{t})] = 0$. Indeed, it implies that this property is true for the free product $*_{j \in J} \tau_j$ and so the reciprocal assertion follows from Lemma 3.15 since it implies that τ

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coincides with $*_{j\in J}\tau_j$. The formal difficulty is that we shall use Definition 2.16 of the injective trace in order to prove that $\tau[T_g(\mathbf{t})]$ vanishes. Formula (2.6) is only valid for graph monomials as the summation involves the vertex set of the graph, but h is not a monomial because of *reduceness* of $\mathbf{t}$.

We fix from now a sequence $\mathbf{m} = (m_1, \ldots, m_L)$ of graph monomials with respective vertex sets $V_1, \ldots, V_L$, and define

$$\Lambda_{\mathbf{m}} = \text{Span} \left\{ m_1^{\pi_1} \otimes \ldots \otimes m_L^{\pi_L} \mid \forall k \in [L], \pi_k \in \mathcal{P}(V_k) \right\}.$$

We claim that it suffices to prove that $\tau[h] = 0$ for any $h = T_g(\mathbf{t})$, where g is a bigraph operation and $\mathbf{t}$ is g-alternated, reduced and belongs to $\Lambda_{\mathbf{m}}$. Indeed, let $\mathbf{t} = (t_1, \ldots, t_L)$ be an arbitrary sequence of g-alternated, reduced graph polynomials and denote $\mathbf{t} = \sum_i \alpha_i \mathbf{x}_i$ where the $\mathbf{x}_i$, are the sequences of graph monomials of $\mathbf{t}$. Let p be the projection of Lemma 3.14 and denote $p(\mathbf{t}) = \big(p(t_1), \ldots, p(t_L)\big)$. Then $t = p(t) = \sum_i \alpha_i p(\mathbf{x}_i)$ where $p(\mathbf{x}_i) \in \Lambda_{\mathbf{m}_i}$ is reduced for each i.

The interest in fixing the monomial $\mathbf{m}$ is that each *monomial* $\mathbf{x} \in \Lambda_{\mathbf{m}}$ satisfies that $T_g(\mathbf{x}) = T_g(\mathbf{m})^{\nu_{g,\mathbf{m}}(\mathbf{x})}$ for a unique partition $\nu_{g,\mathbf{m}}(\mathbf{x})$ of the set V of vertices of $T_g(\mathbf{m})$. Denoting $\nu = \nu_{g,\mathbf{m}}(\mathbf{x})$, Formula (2.6) yields $\tau[T_g(\mathbf{x})] = \tau[T_g(\mathbf{m})^{\nu}] = \sum_{\pi \geqslant \nu \in \mathcal{P}(V)} \tau^0[T_g(\mathbf{m})^{\pi}]$, where we recall that $\pi \geqslant \nu$ means that π refines the identifications made by ν. We then define the linear form defined for *monomials* by

$$\alpha_\pi : \mathbf{x} \mapsto \mathbb{1}(\pi \geqslant \nu_{g,\mathbf{m}}(\mathbf{x}))$$

on $\Lambda_{\mathbf{m}}$. By linearity of τ^0, for any $\mathbf{t}$ reduced and g-alternated graph polynomial we get

$$\tau[h] = \tau[T_g(\mathbf{t})] = \sum_{\pi \in \mathcal{P}(V)} \tau^0\big[\alpha_\pi(\mathbf{t}).T_g(\mathbf{m})^{\pi}\big].$$

Moreover, one can write $T_g(\mathbf{m})^{\pi} = T_{G_\pi}\big(F_\pi\big)$, where G_π is the bigraph of colored components of $T_g(\mathbf{m})^{\pi}$ with an arbitrary choice of ordering of the inputs and of edges around inputs, and F_π is the sequence of colored component of $T_g(\mathbf{m})^{\pi}$ (with ordering fixed by the previous choice). Both G_π and F_π depend implicitly on g and $\mathbf{m}$.

Remark 3.16. Making this operation $g \mapsto G_\pi$ can increase the number of connectors, so that G_π is not a quotient of g. However, it cannot increase the number of colored component: the set $V_{in}(G_\pi)$ of inputs of G_π is a quotient of $V_{in}(g)$. The mapping $p_\pi : V_{in}(g) \rightarrow V_{in}(G_\pi)$ induced by the quotient by π, of $T_g(\mathbf{m})$ respects the bipartition, is 1-Lipschitz for the graph distances, and is surjective.

We set $F_\pi(\mathbf{t}) = \alpha_\pi(\mathbf{t}).T_g(\mathbf{m})^{\pi}$ and obtain

$$\tau[h] = \sum_{\pi \in \mathcal{P}(V)} \tau^0\Big[T_{G_\pi}\big(F_\pi(\mathbf{t})\big)\Big]. \tag{3.1}$$

Recall now that traffic-independence means that for any bi-graph G and any G-alternated sequence of graph monomials $\mathbf{x} = (x_\ell)_{\ell=1}^L$, one has

$$\tau^0[T_G(\mathbf{x})] = \mathbb{1}(G \text{ is a tree }) \times \prod_{\ell=1}^L \tau^0[x_\ell] \tag{3.2}$$

where x_ℓ is considered has a 0-graph monomial. This equality is then valid for $\tau^0[T_G(\mathbf{t})]$ when $\mathbf{t}$ is a G-alternated sequence of graph polynomial. We need the following Lemma whose proof is postponed to the end of the proof.

Lemma 3.17. *Let $g \in \mathcal{G}_{L,\mathbf{d}}^{(0)}$ and let $\mathbf{m}$ be a g-alternated sequence of graph monomials. Let $\mathbf{t} \in \Lambda_{\mathbf{m}}$ be a sequence of reduced graph polynomials and π a partition of the vertex set V of $T_g(\mathbf{m})$. With G_π and α_π defined as above, if G_π is a tree then, $G_\pi = g$ or $\alpha_\pi(\mathbf{t}) = 0$.*

Assuming for the moment Lemma 3.17, we deduce from (3.1) that

$$\tau[h] = \mathbb{1}(g \text{ is a tree }) \times \sum_{\substack{\pi \in \mathcal{P}(V) \\ \text{s.t. } G_\pi = g}} \tau^0\Big[T_g\big(F_\pi(\mathbf{t})\big)\Big]$$

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which is zero if g is not a tree. From now, we assume that g is a tree. Note that the partitions π such that $G_\pi = g$ are those given by first considering a sequence $(\pi_1, \ldots, \pi_L) \in \prod_{\ell=1}^L \mathcal{P}(V_k)$ of partitions of the vertex sets of the monomials of $\mathbf{m}$ such that $\pi_{|V_k} = 0_{V_k}$ for all $k \in [L]$ (i.e. does not identifies outputs of the t_k's), and forming a smallest partition $\bar{\pi}$ of V. Moreover, for such $\pi = \bar{\pi}$ one has $F_\pi = (m_1^{\pi_1}, \ldots, m_L^{\pi_L})$ and that the linear map α_π factorizes $\alpha_\pi(\mathbf{t}) = \prod_\ell \alpha_{\pi_\ell}(t_\ell)$. By (3.2) we can therefore rewrite

$$\tau[h] \quad = \quad \sum_{\substack{(\pi_\ell)_\ell \in \prod_{\ell=1}^L \mathcal{P}(V_\ell) \\ \text{s.t. } \pi_{\ell|\partial V_\ell} = 0_{\partial V_\ell}, \, \forall \ell \in [L]}} \prod_{\ell=1}^L \tau^0[\alpha_{\pi_\ell}(t_\ell) \times m_\ell^{\pi_\ell}],$$

where in the r.h.s. we see $m_\ell^{\pi_\ell}$ as a 0-graph monomial. By definition of $\alpha_\pi(\mathbf{t})$ and since the graphs t_ℓ are reduced we get

$$\tau[h] = \prod_{\ell=1}^L \sum_{\pi_\ell \in \mathcal{P}(V_\ell)} \tau^0[t_\ell^{\pi_\ell}] = \prod_{\ell=1}^L \tau[t_\ell],$$

where t_ℓ is also seen as a 0-graph monomial. Since g a tree it possesses a leaf for which reduceness condition implies $\tau[t_\ell] = 0$. Hence we get $\tau[h] = 0$ as desired.

The rest of this section is devoted to the proof of Lemma 3.17

Lemma 3.18. *Let $g \in \mathcal{G}_{L,\mathbf{d}}^{(0)}$ and let $\mathbf{m} = (m_1, \ldots, m_L)$ be a g-alternated sequence of graph monomials. Let $\mathbf{t} \in \Lambda_\mathbf{m}$ be a sequence of reduced graph polynomials and π a partition of the vertex set V of $T_g(\mathbf{m})$. Assume G_π is a tree and there exists ω a simple path on g visiting exactly $R \geqslant 3$ inputs of g whose source and destination are identified in G_π. More precisely, denote the inputs that ω visits in consecutive order $v_{i_1}, \ldots, v_{i_R}$, with $i_1, \ldots, i_R \in [L]$ (pairwise distinct by simplicity of ω). Recall that p_π is the distance map on the inputs of g induced by π and assume $p_\pi(v_{i_1}) = p_\pi(v_{i_R})$. Then $\alpha_\pi(\mathbf{t}) = 0$ and we can allow $v_{i_1} = v_{i_R}$ without changing this conclusion.*

Proof. As G_π is a tree it has two leaves and, since $\mathbf{m}$ is g-alternated, there exists $1 < r < R$ such that when ω enters and exit neighboring connectors c^- and c^+ of v_r that are identified. More precisely, let $\pi_{-,+}$ be the finest partition of the outputs of t_{i_r} including $\{c^-, c^+\}$. Then, $\alpha_\pi(\mathbf{t}) = \alpha_\pi(t_1 \otimes \ldots \otimes \Delta_{\pi_{-,+}} t_{i_r} \otimes \ldots \otimes t_L)$. But $\Delta_{\pi_{-,+}} t_{i_r} = 0$ since t_{i_r} is reduced. $\qquad \square$

Proof of Lemma 3.17. Assume that g is not a tree. Since G_π is a tree, there exist two distinct inputs $\underline{v}, \overline{v}$ of g with $p_\pi(\underline{v}) = p_\pi(\overline{v})$, so as $\mathbf{t}$ is g-alternated, there exists a path ω in g going through at least three inputs satisfying the condition of Lemma 3.18, hence $\alpha_\pi(\mathbf{t}) = 0$. $\qquad \square$

Remark 3.19. The conclusion of Lemma 3.18 remains valid when we relax the condition that $\mathbf{t}$ is reduced and only assume that $\Delta_{\pi_{-,+}} t_{i_r} = 0$ at each input v_{i_r} with $1 < r < R$. Moreover if exactly one graph polynomial t_{i_r} does not satisfy $\Delta_{\pi_{-,+}} t_{i_r} = 0$ then π must identify the entering and exiting output of t_{i_r} in order for $\alpha_\pi(\mathbf{t})$ not to vanish.

3.4 Proof of Theorem 1.2

For each $j \in J$ let τ_j be a distribution of traffics. It remains to prove that the free product $\tau := \star_{j \in J} \tau_j$ is also a distribution of traffics, showing that it satisfies the positivity condition (2.2). Therefor, we reason as in [14, Chapter 6] where is stated a structural result for the free product of unital algebras with identification of units [14, Formula (6.2)].

Let us consider for $n \geqslant 1$ a bigraph $g \in \mathcal{G}_{L,\mathbf{d}}^{(n)}$ and a g-alternated sequence $\mathbf{m} = (m_1, \ldots, m_L)$ of graph monomials such that for any $k \in [L]$, $m_k \in \mathbb{C}\mathcal{G}^{(d_k)}\langle \mathcal{A}_{\gamma(k)} \rangle$, where $\gamma(k) \in J$ and $d_k \in \{1, 2, \ldots\}$ for traffic independent $\mathcal{G}$-subalgebras $\mathcal{A}_j$, $j \in J$. Let us denote by $Aut_{g,\mathbf{m}}$ the set of automorphisms σ of the bigraph g i.e. the set of maps from the vertex set of g to itself preserving

- the adjacency, the bipartition and the set of outputs of g,

- the coloring of g given by $\mathbf{m}$, i.e. $\gamma \circ \sigma = \gamma$ on the inputs. It does not necessarily respect the ordering of the edges around inputs.

Every $\sigma \in Aut_{g,\mathbf{m}}$ and $\mathbf{t} \in \Lambda_{\mathbf{m}}$ induces a new g-alternated sequence of graph polynomials $\mathbf{t}_\sigma = t_{1,\sigma} \otimes \cdots \otimes t_{L,\sigma}$: we define $t_{i,\sigma}$ to be $t_{\sigma(i)}$ with a reordering of labels of inputs and ordering of neighbor connectors in such a way that $T_g(\mathbf{t}) = T_g(\mathbf{t}_\sigma)$. We have the property $(\mathbf{t}_{\sigma_1})_{\sigma_2} = \mathbf{t}_{\sigma_2\sigma_1}$ for all $\sigma_1, \sigma_2 \in Aut_{g,\mathbf{m}}$.

Lemma 3.20. *Let us fix $n \geqslant 1$, g be a bigraph in $\mathcal{G}_{L,\mathbf{d}}^{(n)}$ and $\mathbf{m}$ be a g-alternated sequence of graph monomials. Let g' be a bigraph in $\mathcal{G}_{L',\mathbf{d}'}^{(n)}$ and $\mathbf{m}'$ be a g-alternated sequence of graph monomials. Let $\mathbf{t} \in \Lambda_{\mathbf{m}}$ and $\mathbf{t}' \in \Lambda_{\mathbf{m}'}$ be reduced.*

1. *If $\tau[T_g(\mathbf{t})|T_{g'}(\mathbf{t}')] \neq 0$, then g is a tree, and $T_{g'}(\mathbf{t}') = T_g(\mathbf{t}'')$ for some reduced graph polynomials $\mathbf{t}'' \in \Lambda_{\mathbf{m}''}$ and $\mathbf{m}''$ some g-alternated sequence of graph monomials which have the same coloring as $\mathbf{m}$. In particular the spaces $W_{g,\gamma}$ of Lemma 3.15 are orthogonal.*

2. *Assume that g is a tree and that $\mathbf{m}$ and $\mathbf{m}'$ have the same coloring. Then we have*

$$\tau[T_g(\mathbf{t})|T_g(\mathbf{t}')] = \sum_{\sigma \in Aut_{g,\mathbf{m}}} \tau[t_1|t'_{1,\sigma}] \cdots \tau[t_L|t'_{L,\sigma}].$$

Assuming this lemma for the moment, let us deduce Theorem 1.2. By the same reasoning as in the previous section, it suffices to prove that $\tau[h|h^*] \geqslant 0$ for each finite combination $h = \sum_i \beta_i T_{g_i}(\mathbf{t}^i)$ for bigraphs g_i and sequences of reduced polynomials $\mathbf{t}^i \in \Lambda_{\mathbf{m}^i}$ where the $\mathbf{m}^i$'s are fixed sequences of g_i-alternated monomials. Moreover the previous lemma allows to restrict our consideration to the case where all g_i are equal to one particular tree g and all $\mathbf{m}^i$ have the same coloring (after a modification of the $\mathbf{t}^i$s and $\mathbf{m}^i$s if necessary). In this case, we denote by $Aut_{g,\mathbf{m}}$ the sets $Aut_{g_i,\mathbf{m}^i}$ (which are all equal), and we have

$$\begin{aligned}
\tau\Big[\sum_i \beta_i T_g(\mathbf{t}^i)\Big|\sum_j \bar{\beta}_j T_g(\mathbf{t}^j)^*\Big] &= \sum_{ij} \beta_i \bar{\beta}_j \tau\big[T_g(\mathbf{t}^i)|T_g(\mathbf{t}^{j^*})\big] \\
&= \frac{1}{\sharp Aut_{g,\mathbf{m}}} \sum_{\substack{ij \\ \sigma \in Aut_{g,\mathbf{m}}}} \beta_i \bar{\beta}_j \tau\big[T_g(\mathbf{t}_\sigma^i)|T_g(\mathbf{t}_\sigma^{j^*})\big], \\
&= \frac{1}{\sharp Aut_{g,\mathbf{m}}} \sum_{\substack{ij \\ \sigma,\sigma' \in Aut_{g,\mathbf{m}}}} \beta_i \bar{\beta}_j \tau\big[t_{1,\sigma}^i|t_{1,\sigma'\circ\sigma}^{j^*}\big] \cdots \tau\big[t_{L,\sigma}^i|t_{L,\sigma'\circ\sigma}^{j^*}\big], \\
&= \frac{1}{\sharp Aut_{g,\mathbf{m}}} \sum_{\substack{ij \\ \sigma,\sigma' \in Aut_{g,\mathbf{m}}}} \beta_i \bar{\beta}_j \tau\big[t_{1,\sigma}^i|t_{1,\sigma'}^{j^*}\big] \cdots \tau\big[t_{L,\sigma}^i|t_{L,\sigma'}^{j^*}\big].
\end{aligned}$$

We can now see that the r.h.s. is nonnegative. First, the matrices $\big(\tau[t_{\ell,\sigma}^i|t_{\ell,\sigma'}^{j^*}]\big)_{i,j,\sigma,\sigma'}$, $\ell = 1, \dots, L$ are positive definite since τ is positive on each $\mathcal{G}$-subalgebra $\mathcal{A}_j$. Moreover, their entry wise product (also called Schur product) $\big(\tau[t_{1,\sigma}^i|t_{1,\sigma'}^{j^*}] \cdots \tau[t_{L,\sigma}^i|t_{L,\sigma'}^{j^*}]\big)_{i,j,\sigma,\sigma'}$ is also positive ([14, Lemma 6.11]). This yields by consequence the positivity of the free product.

Proof of Lemma 3.20. We will prove this lemma by induction on the number of inputs g. If this number is 0, this means that g consists in a single connector (and the sequence of outputs of g is constant and equal to this single connector) and then $\tau\big[T_g(\mathbf{t})|T_{g'}(\mathbf{t})\big]$ is proportional to $\tau\big[T_{\tilde{g}'}(\mathbf{t})\big]$ where $\tilde{g}'$ is the bigraph with no outputs obtained by identifying the outputs of g'. Hence, by Proposition 3.11, $\tau\big[T_g(\mathbf{t})|T_{g'}(\mathbf{t})\big]$ is zero if g' as one input or more and so the lemma is true.

The hypotheses is that the number of colored components of $T_g(\mathbf{t})$ is larger than 1 and that the lemma is true for all inferior numbers of colored components of $T_g(\mathbf{t})$. Let us assume that $\tau[T_g(\mathbf{t})|T_{g'}(\mathbf{t}')] \neq 0$. Remark that we have $T_g(\mathbf{m})|T_{g'}(\mathbf{m}') = T_{g|g'}(\mathbf{m} \otimes \mathbf{m}')$ for the bigraph $g|g'$ with no outputs and $L + L'$ inputs which consists in collapsing the outputs of g with those of g'. Then, denoting by V the vertex set of $T_{g|g'}(\mathbf{m} \otimes \mathbf{m}')$, for all $\pi \in \mathcal{P}(V)$ we define the linear map α_π, the bigraph G_π and the sequence of monomials F_π as in the previous section, namely

- $T_{g|g'}(\mathbf{m} \otimes \mathbf{m}')^\pi = T_{G_\pi}(F_\pi)$ where G_π is the bigraph of colored components of $T_{g|g'}(\mathbf{m} \otimes \mathbf{m}')$,

- for each sequence of monomials $\mathbf{x} \in \Lambda_m, \mathbf{x}' \in \Lambda_{m'}$, one has $\alpha_\pi(\mathbf{x}, \mathbf{x}') = \mathbb{1}(\pi \geqslant \nu)$ where $T_{g|g'}(\mathbf{x}, \mathbf{x}') = T_{g|g'}(\mathbf{m}, \mathbf{m}')^\nu$.

Denoting $F_\pi(\mathbf{t}, \mathbf{t}') = \alpha_\pi(\mathbf{t}, \mathbf{t}') \times T_{G_\pi}(F_\pi)$ we get

$$\tau\big[T_g(\mathbf{t})|T_{g'}(\mathbf{t}')\big] = \sum_{\pi \in \mathcal{P}(V)} \tau^0\Big[T_{G_\pi}\big(F_\pi(\mathbf{t}, \mathbf{t}')\big)\Big],$$

and by definition of traffic independence the terms in the sum are possibly nonzero only if G_π is a tree.

We shall use the following Lemma.

Lemma 3.21. *With notations as above, let $\pi \in \mathcal{P}(V)$ such that G_π is a tree and $\alpha_\pi(\mathbf{t}, \mathbf{t}') \neq 0$. Then*

1. *g and g' are trees.*

2. *π respects the decomposition of $T_g(\mathbf{m})$ and $T_{g'}(\mathbf{m}')$ into colored components, in the sense that the image by π of two vertices that belong to different colored components of $T_g(\mathbf{m})$ (resp. $T_{g'}(\mathbf{m}')$) belong to different colored components of $T_{g|g'}(\mathbf{m} \otimes \mathbf{m}')^\pi$.*

3. *two different connectors of g (resp. g') are not identified by π in $T_{g|g'}(\mathbf{m} \otimes \mathbf{m}')^\pi$.*

Proof. By Lemma 3.18 and Remark 3.19, if G_π is a tree and there is a simple path ω on g (resp. g') visiting exactly in $V_{in}(g)$ (resp. $V_{in}(g')$) the vertices $v_{i_1}, \ldots, v_{i_l}$, in consecutive order, with $l \geqslant 3$ and $i_1, \ldots, i_l \in [L]$, such that $p_\pi(v_{i_1})$ and $p_\pi(v_{i_l})$ belong to a same colored component of $T_{g|g'}(\mathbf{m} \otimes \mathbf{m}')^\pi$, then $\alpha_\pi(\mathbf{t}, \mathbf{t}') = 0$. We can allow $v_{i_1} = v_{i_l}$ without changing the conclusion. Indeed, since the path ω is in g the graph polynomials corresponding to the inputs it visits are reduced.

If g is not a tree, then there exists again a simple loop ω in g from a colored component to itself which visit another colored component, which then satisfies the condition of Lemma 3.18, and so $\alpha_\pi(\mathbf{t}, \mathbf{t}') = 0$.

Now, let us take two vertices $\underline{v}$ and $\overline{v}$ in different colored components of $T_g(\mathbf{m})$. If their images by π belong to a same colored component, there exists a path ω in g from $\underline{v}$ to $\overline{v}$ going through at least three inputs, satisfying the condition of Lemma 3.18, so that $\alpha_\pi(\mathbf{t}, \mathbf{t}') = 0$.

Finally, let us take two different connectors $\underline{c}$ and $\overline{c}$ which are identified by π. Here again there exists a path ω in g from $\underline{c}$ to $\overline{c}$ going through at least three inputs, satisfying the condition of Lemma 3.18, yielding the same conclusion.

We can deduce the same properties for g' by interchanging g and g'. $\qquad\square$

We then can assume that g and g' are trees. Moreover, we know from Proposition 3.11 that $\tau\big[T_g(\mathbf{t})|T_{g'}(\mathbf{t}')\big]$ vanishes if $(\mathbf{m} \otimes \mathbf{m}')$ is $(g|g')$-alternated and reduced. Hence we can assume that there is a k in $\{1, \ldots, n\}$ such that the color of one particular neighbor input v next to the k-th output of $T_g(\mathbf{m})$ is the same that the color of some neighbor input v' next to the k-th output of $T_{g'}(\mathbf{m}')$. Without loss of generality we assume that v and v' are neighbors of the first output of $T_g(\mathbf{m})$ and $T_{g'}(\mathbf{m}')$, corresponding to the graph monomials m_1 and m_1' respectively.

We denote by $c_1, \ldots, c_m$ the connectors around v in T_g and by $s_1, \ldots, s_m$ the connected components of T_g when v is removed, in such a way that c_i belongs to s_i for each $i = 1, \ldots, m$, with c_i considered as an additional output. Some of these bigraphs have a single output (which is the corresponding c_i) and we assume that those bigraphs are $c_1, \ldots, c_p$ for $0 \leqslant p \leqslant m$. Similarly we define c_i', s_i', $i = 1, \ldots, m'$ and p' by considering $T_{g'}$ instead of T_g. Moreover, given $\sigma \in \Sigma(p)$ a permutation of $\{1, \ldots, p'\}$, we denote by $t_{1,\sigma}'$ the graph polynomial obtained from t_1' by permuting the outputs attached to the connectors $c_1, \ldots, c_p$ according to σ.

Lemma 3.22. *With the above notations, up to a reordering of the s_i' for $i > p'$ and a reordering of its outputs different from c_i,*

$$\tau\big[T_g(\mathbf{t})|T_g(\mathbf{t}')\big] = \mathbb{1}(p = p')\mathbb{1}(m = m') \sum_{\sigma \in \Sigma(p)} \tau[t_1|t_{1,\sigma}] \times \prod_{i=1}^{p} \tau\big[s_i|s_{\sigma(i)}'\big] \times \prod_{i=p+1}^{m} \tau\big[s_i|s_i'\big].$$

Assuming momentarily this Lemma, let us finish the proof. Applying the induction hypothesis, we get that if $T_{g'}(\mathbf{t}')$ cannot be written $T_g(\mathbf{t}'')$ for some reduced graph polynomials $\mathbf{t}'' \in \Lambda_{\mathbf{m}''}$ and $\mathbf{m}''$ some g-alternated sequence of graph monomials which has the same coloring as $\mathbf{m}$, then we would have a vanishing expression. For the second part of the lemma, let us assume that $g = g'$ and that $\mathbf{m}'$ has the same coloring as $\mathbf{m}$. Remark that an element of $Aut_{g,\mathbf{m}}$ is nothing else than a bijection from $\{c_1, \ldots, c_m\}$ to itself, and an automorphism of each s_i. Using the induction hypothesis on each s_i, we see that a non-vanishing term is such that s_i and $s'_{i,\sigma}$ for $i \leqslant p$, resp. s_i and s'_i for $i > p$, are of the same type of tree of colored component, and there exists an automorphism for each of these couples, which allows to define a global automorphism $\sigma \in Aut_{g,\mathbf{m}}$. Hence by recurrence we get as expected

$$\tau[T_g(\mathbf{t})|T_{g'}(\mathbf{t}')] = \sum_{\sigma \in Aut_{g,\mathbf{t}}} \tau[t_1|t'_{1,\sigma}] \cdots \tau[t_L|t'_{L,\sigma}].$$

$\square$

Proof of Lemma 3.22. Let $\pi \in \mathcal{P}(V)$ such that $\alpha_\pi(\mathbf{t}, \mathbf{t}' \neq 0)$. Assume that π identifies a vertex of v_1 of s_i with a vertex of v_2 of $s'_{i'}$ for $i > p$ and $i' > p'$.

Consider a simple path in $T_{g|g'}(\mathbf{m} \otimes \mathbf{m}')$ from (the colored component of) v_1 to (the one of) v_2, consisting in a sub-path in T_g from v_1 to c_1, going through t_1 and t'_1, and finishing with a subpath from c_2 to v_2. By Lemma 3.18 and the last sentence of Remark 3.19 we get that π identifies c_i and $c'_{i'}$. By Lemma 3.21, π does not identify c_i and $c'_{i'}$ with an other connector of $T_{g|g'}$, and so up to a reordering we can assume $i' = i$.

Assume now that an output o of s_i is not attached to any output of s'_i in $T_{g|g'}$. Consider a simple path from (the colored component of) v_1 to (the one of) v_2, consisting in a sub-path in T_g from v_1 to o (which then does not visit c_i), continuing with a sub-path in T'_g going v_2. While entering in T'_g the path goes through a subgraph s'_j for $j \neq i$ and goes through t'_1. Necessarily π must identify a vertex of s_i with a vertex of s'_j (since otherwise $\alpha_\pi(\mathbf{t}, \mathbf{t}' \neq 0)$ by Lemma 3.18 and the last sentence of Remark 3.19). But by the previous paragraph this implies that c_i and c'_j are identified. This is absurd since c'_i and c'_j cannot be identified by π. Hence, since the argument of this paragraph remains true by exchanging the roles of s_i and s'_i, the outputs of s_i and s'_i are in correspondence and up to a reordering we can assume that the k-th output of s_i is attached with the k-th output of s'_i.

Moreover, since different connectors of T_g (resp. $T_{g'}$) cannot be identified by π, there exists an injective partial function σ_π from $\{1, \ldots, p\}$ to $\{1, \ldots, p'\}$ such that for each $i \leqslant p, i' \leqslant p'$, π identifies c_i and $c'_{i'}$ if and only if $i \in Dom(\sigma)$ and $\sigma(i) = i'$. With the same argument as a the beginning of the proof, vertices of s_i for $i \leqslant p$ can only be identified with vertices $s'_{i'}$ for $i' \leqslant p'$ with $i \in Dom(\sigma)$ and $\sigma(i) = i'$.

At last, a direct use of Lemma 3.18 implies that a vertex of t_1 (resp. t'_1) cannot be identified with a vertex of any $c'_{i'}$ for any $i' = 1, \ldots, m'$ (resp. c_i for any $i = 1, \ldots, m$). With a small abuse of notations, for each partial functions $\sigma : \{1, \ldots, p\} \to \{1, \ldots, p'\}$ denote by $t'_{1,\sigma}$ the graph polynomial t'_1 where outputs are ordered in such a way in $t_1|t'_{1,\sigma}$ the outputs c_i and $c'_{i'}$ for $i < p$ and $i' < p'$ are identified if $\sigma(i) = i'$, and are not identified if $i \notin Dom(\sigma)$ of $i' \notin Im(\sigma)$. The conclusion so far is that

$$\sum_{\pi \, s.t. \, \sigma_\pi = \sigma} \tau^0\Big[T_{G_\pi}\big(F_\pi(\mathbf{t}, \mathbf{t}')\big)\Big] = \mathbb{1}(m - p = m' - p')\tau\big[t_1|t'_{1,\sigma}\big] \times \prod_{i=1,\ldots,m-p} \tau\big[s_{i+p}|s'_{i+p'}\big]$$

$$\times \prod_{\substack{i \leqslant p \\ c_i \in Dom(\sigma)}} \tau\big[s_i|s'_{\sigma(i)}\big] \prod_{\substack{i \leqslant p \\ i \notin Dom(\sigma)}} \tau\big[s_i|\mathbf{1}\big] \prod_{\substack{i' \leqslant p' \\ i' \notin Im(\sigma)}} \tau\big[\mathbf{1}|s'_{i'}\big],$$

where $\mathbf{1}$ stands for the graph with no edges. But $\tau[s_i|\mathbf{1}] = \tau[\mathbf{1}|s'_{i'}] = 0$, thanks to Proposition 3.11. Hence this sum is zero if $p \neq p'$ and otherwise this is equivalent to consider σ as a permutation. This yields the Lemma and conclude the proof.

$\square$

4 Canonical extension of non-commutative spaces into traffic spaces

This section is dedicated to the proof of Theorem 1.3. We define, for all *-probability space, a space of traffics $(\mathcal{B}, \tau)$ such that $\mathcal{A} \subset \mathcal{B}$ as $*$-algebras and such that the trace induced by τ restricted to $\mathcal{A}$ is Φ. The first step is to give this construction at the algebraic level, which is the aim of Section 4.1, where we also prove the second item of Theorem 1.3. Then we prove in Section 4.2 a version of Theorem 1.1 which yields the first item of Theorem 1.3. In Section 4.3, we prove the positivity of the distribution of traffics we introduce now.

4.1 Definition and properties

Definition 4.1. *Let $\mathcal{A}$ be an algebra. We denote by $\mathcal{G}(\mathcal{A})$ the $\mathcal{G}$-algebra $\mathbb{C}\mathcal{G}^{(2)}\langle \mathcal{A} \rangle$, quotiented by the following relations: for all $g \in \mathcal{G}_n$, $a_1, \ldots, a_n \in \mathcal{A}$ and P non-commutative polynomial in n variables, we have*

$$Z_g(\cdot \overset{P(a_1,\ldots,a_k)}{\longleftarrow} \cdot \otimes \cdot \overset{a_{k+1}}{\leftarrow} \cdot \otimes \ldots \otimes \cdot \overset{a_n}{\leftarrow} \cdot) = Z_g(P(\cdot \overset{a_1}{\leftarrow} \cdot, \ldots, \cdot \overset{a_k}{\leftarrow} \cdot) \otimes \cdot \overset{a_{k+1}}{\leftarrow} \cdot \otimes \ldots \otimes \cdot \overset{a_n}{\leftarrow} \cdot) \quad (4.1)$$

which allows to consider the algebra homomorphism $V : \mathcal{A} \to \mathcal{G}(\mathcal{A})$ given by $a \mapsto (\cdot \overset{a}{\leftarrow} \cdot)$.

The algebra $\mathcal{G}(\mathcal{A})$ is the free $\mathcal{G}$-algebra generated by the algebra $\mathcal{A}$ in the following sense.

Proposition 4.2. *Let $\mathcal{B}$ be a $\mathcal{G}$-algebra and $f : \mathcal{A} \to \mathcal{B}$ a algebra homomorphism. There exists a unique $\mathcal{G}$-algebra homomorphism $f' : \mathcal{G}(\mathcal{A}) \to \mathcal{B}$ such that $f = f' \circ V$. As a consequence, the algebra homomorphism $V : \mathcal{A} \to \mathcal{G}(\mathcal{A})$ is injective.*

Proof. The existence is given by the following definition of f' on $\mathcal{G}(\mathcal{A})$:

$$f'(Z_g(\cdot \overset{a_1}{\longleftarrow} \cdot \otimes \ldots \otimes \cdot \overset{a_n}{\leftarrow} \cdot)) = Z_g(f(a_1) \otimes \ldots \otimes f(a_n))$$

for all $a_1, \ldots, a_n \in \mathcal{A}$; which obviously respects the relation defining $*_{j \in J} \mathcal{A}_j$.

The uniqueness follows from the fact that f' is uniquely determined on $V(\mathcal{A})$ (indeed, $f'(a)$ must be equal to $f(b)$ whenever $a = V(b)$) and that $V(\mathcal{A})$ generates $\mathcal{G}(\mathcal{A})$ as a $\mathcal{G}$-algebra. $\square$

For example, the free $\mathcal{G}$-algebra generated by the variables $\mathbf{x} = (x_i)_i \in J$ and $\mathbf{x}^* = (x_i^*)_i \in J$ is the $\mathcal{G}$-algebra $\mathbb{C}\mathcal{G}\langle \mathbf{x}, \mathbf{x}^* \rangle$ of graphs whose edges are labelled by $\mathbf{x}$ and $\mathbf{x}^*$.

For all 0-graph monomial $T = (G, \gamma, \mathbf{v})$ indexed by $\mathcal{A}$, we say that T is a cactus whenever each edge belongs exactly to one cycle of T. See figure 2. Equivalently, for all vertices v_1 and v_2 of G, the two following equivalent statement are true:

- the minimum number of edges whose removal disconnect v_1 and v_2 is exactly 2;

- the maximum number of edge-disjoint paths from v_1 to v_2 is exactly 2.
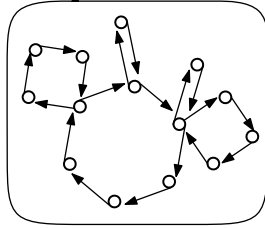
Figure 2: A cactus whose cycle are oriented.

Definition 4.3. *For all non-commutative probability space $(\mathcal{A}, \Phi)$, we define the linear functional $\tau_\Phi : \mathbb{C}\mathcal{G}^{(0)}\langle \mathcal{A} \rangle \to \mathbb{C}$ by it injective trace $\tau_\Phi^0 : \mathbb{C}\mathcal{G}^{(0)}\langle \mathcal{A} \rangle \to \mathbb{C}$ given by:*

1. $\tau_\Phi^0[C] = \kappa(a_1, \ldots, a_n)$ *if C is an oriented cycle with edges labelled by $a_1, \ldots, a_n$, where κ denotes the free cumulants defined in* (2.5).

2. $\tau_\Phi^0[T] = \prod_{C \in T} \tau_\Phi^0[C]$, *if T is a cactus with oriented cycles, where the product is over all cycles of T.*

3. $\tau_\Phi^0[T] = 0$, *otherwise.*

Proposition 4.4. *The linear form $\Psi : \mathbb{C}\mathcal{G}^{(2)}\langle \mathcal{A} \rangle \to \mathbb{C}$ given by $\Psi(t) = \tau_\Phi(\tilde{\Delta}(t))$, where $\tilde{\Delta}(t)$ is $\Delta(t)$ where the input and output are forgotten, is invariant under the relations (4.1) defining $\mathcal{G}(A)$, and consequently yields to an algebraic space of traffics $(\mathcal{G}(A), \tau_\Phi)$. Furthermore, we have $\Phi = \Psi \circ V$, where φ is the canonical injective algebra homomorphism from $\mathcal{A}$ to $\mathcal{G}(A)$.*

Remark 4.5. Before proving the Proposition, let us underline the motivation to introduce the distribution of traffics of Definition 4.3: it gives a parallel between the relation moments-free cumulants of formula (2.5) and the relation trace-injective trace of (2.6). Let t be the graph consisting in a simple cycle labeled $a_1 \times \cdots \times a_K$ along the orientation. One has

$$\Phi(a_1 \times \cdots \times a_K) = \tau[t] = \sum_{\pi \in \mathcal{P}(V)} \tau[t^\pi] = \sum_{\substack{\pi \in \mathcal{P}(V) \\ \text{s.t. } t^\pi \text{ cactus}}} \prod_{c \text{ cycle of } t^\pi} \tau^0[c].$$

It can be seen that the partitions π of the set of vertices for which t^π is a cactus are the Kreweras dual of the non crossing partitions ν of the edges of the cycle, see figure 3. The cycles of the cactus correspond to the blocks of ν, so that getting (2.5) from the above r.h.s. is a matter of change of variables.
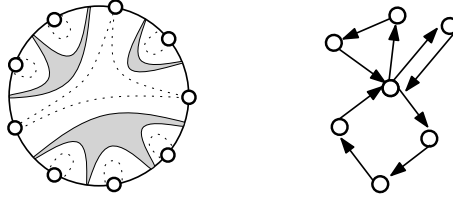

Figure 3: Left: A cycle of length nine, a non crossing partition ν of its edges (grey) and the Kreweras complement π (dotted) of ν. Right: the quotient of the cycle by π.

Proof. Proving that Ψ is invariant under the relations (4.1) is equivalent to the prove the following: for all 0-graph monomial g, with a slight abuse of notation, denoting $Z_g(\cdot \overset{a_1}{\leftarrow} \cdot \otimes \ldots \otimes \cdot \overset{a_n}{\leftarrow} \cdot)$ the element of $\mathbb{C}\mathcal{G}^{(0)}\langle \mathcal{A} \rangle$ obtained by replacing the corresponding edges of g by the a_i's, one has

1. $\tau_\Phi(Z_g(\cdot \overset{a_1 + \alpha a_2}{\leftarrow} \cdot \otimes \cdot \overset{a_3}{\leftarrow} \cdot \otimes \ldots \otimes \cdot \overset{a_n}{\leftarrow} \cdot)) = \tau_\Phi(Z_g(\cdot \overset{a_1}{\leftarrow} \cdot \otimes \cdot \overset{a_3}{\leftarrow} \cdot \otimes \ldots \otimes \cdot \overset{a_n}{\leftarrow} \cdot)) + \alpha \tau_\Phi(Z_g(\cdot \overset{a_2}{\leftarrow} \cdot \otimes \cdot \overset{a_3}{\leftarrow} \cdot \otimes \ldots \otimes \cdot \overset{a_n}{\leftarrow} \cdot))$,

2. $\tau_\Phi(Z_g(\cdot \overset{1_\mathcal{A}}{\leftarrow} \cdot \otimes \cdot \overset{a_1}{\leftarrow} \cdot \otimes \ldots \otimes \cdot \overset{a_n}{\leftarrow} \cdot)) = \tau_\Phi(Z_g(\cdot \otimes \cdot \overset{a_1}{\leftarrow} \cdot \otimes \ldots \otimes \cdot \overset{a_n}{\leftarrow} \cdot))$,

3. $\tau_\Phi(Z_g(\cdot \overset{a_1 a_2}{\leftarrow} \cdot \otimes \cdot \overset{a_3}{\leftarrow} \cdot \otimes \ldots \otimes \cdot \overset{a_n}{\leftarrow} \cdot)) = \tau_\Phi(Z_{g^+}(\cdot \overset{a_1}{\leftarrow} \cdot \otimes \cdot \overset{a_2}{\leftarrow} \cdot \otimes \ldots \otimes \cdot \overset{a_n}{\leftarrow} \cdot))$, where g^+ is the graph g where the first edge is replaced by two consecutive edges.

The first property is an immediate consequence of the linearity of the cumulants. Let us prove the others properties at the level of the injective trace.

Lemma 4.6. *Let $a_1, \ldots, a_n \in (\mathcal{A}, \Phi)$ and π be a partition of the vertices V of g. We denote by v_0 the new vertex in g^+. We have*

1. $\tau_\Phi^0(Z_g(\cdot \overset{1_\mathcal{A}}{\leftarrow} \cdot \otimes \cdot \overset{a_3}{\leftarrow} \cdot \otimes \ldots \otimes \cdot \overset{a_n}{\leftarrow} \cdot)) = \tau_\Phi^0(Z_g(\cdot \otimes \cdot \overset{a_3}{\leftarrow} \cdot \otimes \ldots \otimes \cdot \overset{a_n}{\leftarrow} \cdot))$ *if the goal and the source of $\cdot \overset{1_\mathcal{A}}{\leftarrow} \cdot$ are identified in g, and $\tau_\Phi^0(Z_g(\cdot \overset{1_\mathcal{A}}{\leftarrow} \cdot \otimes \cdot \overset{a_3}{\leftarrow} \cdot \otimes \ldots \otimes \cdot \overset{a_n}{\leftarrow} \cdot)) = 0$ if not;*

2. $\tau_\Phi^0(Z_g(\cdot \overset{a_1 a_2}{\leftarrow} \cdot \otimes \ldots \otimes \cdot \overset{a_n}{\leftarrow} \cdot)^\pi) = \sum_{\substack{\sigma \in \mathcal{P}(V \cup \{v_0\}) \\ \sigma \backslash \{v_0\} = \pi}} \tau_\Phi^0(Z_{g^+}(\cdot \overset{a_1}{\leftarrow} \cdot \otimes \cdot \overset{a_2}{\leftarrow} \cdot \otimes \ldots \cdot \overset{a_n}{\leftarrow} \cdot)^\sigma)$.

This implies the proposition, since it gives

$$\tau_\Phi(Z_g(\cdot \overset{1_\mathcal{A}}{\leftarrow} \cdot \otimes \cdot \overset{a_1}{\leftarrow} \cdot \otimes \ldots \otimes \cdot \overset{a_n}{\leftarrow} \cdot)) = \sum_{\pi \in \mathcal{P}(V)} \tau_\Phi^0(Z_g(\cdot \overset{1_\mathcal{A}}{\leftarrow} \cdot \otimes \cdot \overset{a_1}{\leftarrow} \cdot \otimes \ldots \otimes \cdot \overset{a_n}{\leftarrow} \cdot)^\pi)$$

$$= \sum_{\substack{\pi \in \mathcal{P}(V) \\ \text{the goal and the source of } \cdot \overset{1_\mathcal{A}}{\leftarrow} \cdot \\ \text{are identified in } g}} \tau_\Phi^0(Z_g(\cdot \otimes \cdot \overset{a_1}{\leftarrow} \cdot \otimes \ldots \otimes \cdot \overset{a_n}{\leftarrow} \cdot)^\pi)$$

$$= \sum_{\pi \in \mathcal{P}(V)} \tau_\Phi^0(Z_g(\cdot \otimes \cdot \overset{a_1}{\leftarrow} \cdot \otimes \ldots \otimes \cdot \overset{a_n}{\leftarrow} \cdot)^\pi)$$

$$= \tau_\Phi(Z_g(\cdot \otimes \cdot \overset{a_1}{\leftarrow} \cdot \otimes \ldots \otimes \cdot \overset{a_n}{\leftarrow} \cdot)),$$

and

$$\tau_\Phi(Z_g(\cdot \overset{a_1 a_2}{\leftarrow} \cdot \otimes \ldots \otimes \cdot \overset{a_n}{\leftarrow} \cdot)) = \sum_{\pi \in \mathcal{P}(V)} \tau_\Phi^0(Z_g(\cdot \overset{a_1 a_2}{\leftarrow} \cdot \otimes \ldots \otimes \cdot \overset{a_n}{\leftarrow} \cdot)^\pi)$$

$$= \sum_{\pi \in \mathcal{P}(V)} \sum_{\substack{\sigma \in \mathcal{P}(V \cup \{v_0\}) \\ \sigma \backslash \{v_0\} = \pi}} \tau_\Phi^0(Z_{g^+}(\cdot \overset{a_1}{\leftarrow} \cdot \otimes \cdot \overset{a_2}{\leftarrow} \cdot \otimes \ldots \cdot \overset{a_n}{\leftarrow} \cdot)^\sigma)$$

$$= \sum_{\sigma \in \mathcal{P}(V)} \tau_\Phi^0(Z_{g^+}(\cdot \overset{a_1}{\leftarrow} \cdot \otimes \cdot \overset{a_2}{\leftarrow} \cdot \otimes \ldots \cdot \overset{a_n}{\leftarrow} \cdot)^\sigma)$$

$$= \tau_\Phi(Z_{g^+}(\cdot \overset{a_1}{\leftarrow} \cdot \otimes \cdot \overset{a_2}{\leftarrow} \cdot \otimes \ldots \cdot \overset{a_n}{\leftarrow} \cdot)).$$

Now, let $a \in \mathcal{A}$. We can write

$$\Psi(V(a)) = \Psi(\cdot \overset{a}{\leftarrow} \cdot) = \tau_\Phi(\circlearrowleft_a) = \tau_\Phi^0(\circlearrowleft_a) = \kappa(a) = \Phi(a)$$

which finishes the proof of the proposition. $\qquad\square$

Proof of Lemma 4.6. The first item follows from the fact that a cumulant involving $1_\mathcal{A}$ is equal to 0, except $\kappa(1_\mathcal{A}) = 1$ (see [14, Proposition 11.15]). As a consequence, for any cactus $Z_g(\cdot \overset{1_\mathcal{A}}{\leftarrow} \cdot \otimes \cdot \overset{a_3}{\leftarrow} \cdot \otimes \ldots \otimes \cdot \overset{a_n}{\leftarrow} \cdot)$ on which τ_Φ^0 is not vanishing, we can just remove the little cycle $\circlearrowleft_{1_\mathcal{A}}$ of the cactus and it yields exactly $Z_g(\cdot \otimes \cdot \overset{a_3}{\leftarrow} \cdot \otimes \ldots \otimes \cdot \overset{a_n}{\leftarrow} \cdot)$.

Let us prove the second item. We denote $Z_g(\cdot \overset{a_1 a_2}{\leftarrow} \cdot \otimes \ldots \otimes \cdot \overset{a_n}{\leftarrow} \cdot)$ by T and $Z_{g^+}(\cdot \overset{a_1}{\leftarrow} \cdot \otimes \cdot \overset{a_2}{\leftarrow} \cdot \otimes \ldots \cdot \overset{a_n}{\leftarrow} \cdot)$ by T_+. If T^π is not a cactus, then the two side of the equation are equal to zero. Assume that T^π is a cactus. We denote by c the cycle of $\cdot \overset{a_1 a_2}{\leftarrow} \cdot$ in T^π and $a_1 a_2, b_2, \ldots, b_{k-1}$ the elements of the cycle c starting at $a_1 a_2$.

Let us consider a partition $\sigma \in \mathcal{P}(V \cup \{v_0\})$ such that T_+^σ is a cactus and $\pi = \sigma \backslash \{v_0\}$. Then, we have two cases:

1. v_0 is of degree 2 (this occurs for only one partition σ given by $\pi \cup \{\{v_0\}\}$). Denoting by c^+ the cycle of T_+^σ which contains v_0, we have $c^+ = (a_2, b_2, \ldots, b_{k-1}, a_1)$. The cycles of T^π and T_+^σ different from c and c^+ are the same, and by consequence

$$\tau_\Phi^0[T^\pi]/k(a_1 a_2, b_2, \ldots, b_{k-1}) = \tau_\Phi^0[T_+^\sigma]/k(a_2, b_2, \ldots, b_{k-1}, a_1).$$

2. v_0 is of degree > 2. We denote by c_1 the cycle of $\cdot \overset{a_2}{\leftarrow} \cdot$ in T_+^σ, c_2 the cycle of $\cdot \overset{a_1}{\leftarrow} \cdot$ in T_+^σ (of course, c_1 and c_2 are not equal, because if it is the case, T^π would be disconnected, which is not possible). The cycles of T^π different from c are exactly the cycles of T_+^σ different from c_1 or c_2. We have $c_1 = (a_2, b_2, \ldots, b_l)$ and $c_2 = (b_{l+1}, \ldots, b_k, a_1)$ with l the place of the vertex which is identified with v_0 in T_+^σ. By definition, we have

$$\tau_\Phi^0[T^\pi]/k(a_1 a_2, b_2, \ldots, b_{k-1}) = \tau_\Phi^0[T_+^\sigma]/(k(a_2, b_2, \ldots, b_l) \cdot k(b_{l+1}, \ldots, b_k, a_1)).$$

Conversely, for each vertex v_1 in the cycle c, we are in the above situation for $\sigma = \pi_{|v_0 \simeq v_1}$.

Finally, using [14, Theorem 11.12] for computing $k(a_1a_2, b_2, \ldots, b_{k-1})$, we can compute

$$
\begin{aligned}
\tau_\Phi^0[T^\pi] =& \tau_\Phi^0[T^\pi]/k(a_1a_2, b_2, \ldots, b_{k-1}) \cdot k(a_1a_2, b_2, \ldots, b_{k-1})\\
=& \tau_\Phi^0[T^\pi]/k(a_1a_2, b_2, \ldots, b_{k-1})\\
& \cdot \left(k(a_2, b_2, \ldots, b_{k-1}, a_1) + \sum_{1 \leqslant l \leqslant k} k(a_2, b_2, \ldots, b_l) \cdot k(b_{l+1}, \ldots, b_k, a_1) \right)\\
=& \tau_\Phi^0[T_+^{\pi \cup \{\{v_0\}\}}] + \sum_{\substack{\sigma \in \mathcal{P}(V \cup \{v_0\}) \backslash \{\pi \cup \{\{v_0\}\}\}\\ \sigma \backslash \{v_0\} = \pi}} \tau_\Phi^0[T_+^\sigma]\\
=& \sum_{\substack{\sigma \in \mathcal{P}(V \cup \{v_0\})\\ \sigma \backslash \{v_0\} = \pi}} \tau_\Phi^0(T_+^\sigma).
\end{aligned}
$$

$\square$

Corollary 4.7. *The algebras* $\mathbb{C}\langle \cdot \overset{a}{\leftarrow} \cdot : a \in \mathcal{A}\rangle$, $\mathbb{C}\langle \cdot \overset{a}{\to} \cdot : a \in \mathcal{A}\rangle$ *and* $\mathbb{C}\langle \overset{\uparrow a}{\cdot} : a \in \mathcal{A}\rangle$ *are free in the sense of Voiculescu in* $(\mathbb{C}\mathcal{G}^{(2)}\langle\mathcal{A}\rangle, \tau_\Phi)$, *or equivalently the algebras* $\mathbb{C}\langle a : a \in \mathcal{A}\rangle$, $\mathbb{C}\langle {}^t a : a \in \mathcal{A}\rangle$ *and* $\mathbb{C}\langle deg(a) : a \in \mathcal{A}\rangle$ *are free in the sense of Voiculescu in* $(\mathcal{G}(\mathcal{A}), \tau_\Phi)$.

Proof. We first prove that $\mathbb{C}\langle \cdot \overset{a}{\leftarrow} \cdot : a \in \mathcal{A}\rangle$ and $\mathbb{C}\langle \cdot \overset{a}{\to} \cdot : a \in \mathcal{A}\rangle$ are free. Let us consider $2n$ elements $c_1, \ldots, c_{2n}$ alternatively in $\mathbb{C}\langle \cdot \overset{a}{\leftarrow} \cdot : a \in \mathcal{A}\rangle$ and $\mathbb{C}\langle \cdot \overset{a}{\to} \cdot : a \in \mathcal{A}\rangle$ such that $\tau_\Phi(c_1) = \ldots = \tau_\Phi(c_{2n}) = 0$. We want to prove that $\tau_\Phi(\Delta(c_1 \ldots c_{2n})) = 0$. Using Proposition 4.4 in order to regroup consecutive edges which are oriented in the same direction, we can assume that the $c_i's$ are written as $\cdot \overset{a_i}{\leftarrow} \cdot$ with $a_i \in \mathcal{A}$ such that $\Phi(a_i) = 0$, and c_i and c_{i+1} not oriented in the same direction. Consider now a partition π such that $\tau_\Phi^0(\Delta(c_1 \ldots c_{2n})^\pi) \neq 0$. Then, take a leaf of the oriented cactus $\Delta(c_1 \ldots c_{2n})^\pi$. This leaf is a cycle of only one edge, because if not, the cycle cannot be oriented, since two consecutive edges in $\Delta(c_1 \ldots c_{2n})$ are not oriented in the same way. This produces a term $\tau_\Phi^0(\Delta(c_i)) = 0$ in the product $\tau_\Phi^0(\Delta(c_1 \ldots c_{2n})^\pi)$, which leads at the end to a vanishing contribution. Finally, $\tau_\Phi(c_1 \ldots c_{2n}) = 0$ and we have the freeness wanted.

Now, let us prove that $\mathbb{C}\langle \overset{\uparrow a}{\cdot} : a \in \mathcal{A}\rangle$ is free from $\mathbb{C}\langle \cdot \overset{a}{\leftarrow} \cdot, \cdot \overset{a}{\to} \cdot : a \in \mathcal{A}\rangle$. By the same argument as above, we can consider that we have a cycle $\Delta(c_1 \ldots c_n)$ which consists in an alternating sequence of $c_i's$ written as $\cdot \overset{a_i}{\leftarrow} \cdot$ with $a_i \in \mathcal{A}$ such that $\Phi(a_i) = 0$, $\cdot \overset{a_i}{\to} \cdot$ with $a_i \in \mathcal{A}$ such that $\Phi(a_i) = 0$, and $c_i \in \mathbb{C}\langle \overset{\uparrow a}{\cdot} : a \in \mathcal{A}\rangle$ such that $\tau_\Phi(c_i) = 0$. We want to prove that $\tau_\Phi(\Delta(c_1 \ldots c_n)) = 0$. If there is no term $c_i \in \mathbb{C}\langle \overset{\uparrow a}{\cdot} : a \in \mathcal{A}\rangle$, we are in the case of the previous paragraph. Let us assume that there exists at least one such term, say c_1. By linearity, we can consider that the term $c_1 \in \mathbb{C}\langle \overset{\uparrow a}{\cdot} : a \in \mathcal{A}\rangle$ is written as $\overset{\uparrow b_1}{\cdot} \ldots \overset{\uparrow b_k}{\cdot} - \tau_\Phi(\overset{\uparrow b_1}{\cdot} \ldots \overset{\uparrow b_k}{\cdot})$, where $\overset{\uparrow b_1}{\cdot} \ldots \overset{\uparrow b_k}{\cdot}$ is some vertex input/output from which start k edges labelled by $b_1, \ldots, b_k \in \mathcal{A}$. Let us prove that $\tau_\Phi(\Delta((\overset{\uparrow b_1}{\cdot} \ldots \overset{\uparrow b_k}{\cdot})c_2 \ldots c_n))$ and $\tau_\Phi(\overset{\uparrow b_1}{\cdot} \ldots \overset{\uparrow b_k}{\cdot})\tau_\Phi(\Delta(c_2 \ldots c_n))$ are equal, which implies by linearity that $\tau_\Phi(\Delta(c_1 \ldots c_n)) = 0$. Decomposing into injective trace, we are left to prove that for all partition π of the vertices of $\Delta((\overset{\uparrow b_1}{\cdot} \ldots \overset{\uparrow b_k}{\cdot})c_2 \ldots c_n)$ which do not respect the blocks $(\overset{\uparrow b_1}{\cdot} \ldots \overset{\uparrow b_k}{\cdot})$ and $\Delta(c_2 \ldots c_n)$, $\tau_\Phi^0(\Delta((\overset{\uparrow b_1}{\cdot} \ldots \overset{\uparrow b_k}{\cdot})c_2 \ldots c_n)^\pi) = 0$. The same argument as previous paragraph works again. If one of the vertex of $(\overset{\uparrow b_1}{\cdot} \ldots \overset{\uparrow b_k}{\cdot})$ is identified by π with one of the vertex of $\Delta(c_2 \ldots c_n)$, and $\Delta((\overset{\uparrow b_1}{\cdot} \ldots \overset{\uparrow b_k}{\cdot})c_2 \ldots c_n)^\pi$ is a cactus there exists a cycle not oriented or a leaf labelled by one a_i, which leads to a vanishing contribution. $\square$

We can now prove the second item of Theorem 1.3.

Proposition 4.8. *Let* $(\mathcal{A}, \Phi)$ *be a non-commutative probability space* $(\mathcal{A}, \Phi)$. *We define* $(\mathcal{G}(\mathcal{A}), \tau_\Phi)$ *as in Proposition 4.4. Two families* $\mathbf{a}$ *and* $\mathbf{b} \in \mathcal{A}$ *are freely independent in* $\mathcal{A}$ *if and only if they are traffic independent in* $(\mathcal{G}(\mathcal{A}), \tau_\Phi)$.

Proof. Let $\mathbf{a}$ and $\mathbf{b} \in \mathcal{A}$ be freely independent in $\mathcal{A}$. The mixed cumulants of $\mathbf{a}$ and $\mathbf{b}$ vanish (see [14, Theorem 1.16]). Definition 4.3 of τ_Φ^0 implies in particular that, for all 0-graph monomial T indexed by $\mathcal{A}$, $\tau_\Phi^0(T) = 0$ whenever the graph of color component of T is not a tree and is equal to the product of τ_Φ^0 applied on each color component in the other case. In other words, $\mathbf{a}$ and $\mathbf{b}$ are traffic independent in $(\mathcal{G}(\mathcal{A}), \tau_\Phi)$.

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Now, let $\mathbf{a}$ and $\mathbf{b} \in \mathcal{A}$ be traffic independent in $(\mathcal{G}(\mathcal{A}), \tau_\Phi)$. We denote by Ψ the trace on $\mathcal{G}(\mathcal{A})$ associated to τ_Φ. Lemma 3.12 says that, in order to prove that $\mathbf{a}$ and $\mathbf{b}$ are freely independent, it suffices to prove that $\Psi(\Delta(a^*)\Delta(a)) - |\Psi(a)|^2 = 0$ for our variables in $\mathbf{a}$ and $\mathbf{b}$. We compute

$$\Psi(\Delta(a^*)\Delta(a)) - |\Psi(a)|^2 = \tau_\Phi\left(a \,\mathrel{\reflectbox{G}\mkern-2mu\circlearrowleft}\, a^*\right) - |\tau_\Phi(\circlearrowleft_a)|^2$$
$$= \tau_\Phi^0\left(a \,\mathrel{\reflectbox{G}\mkern-2mu\circlearrowleft}\, a^*\right) - |\tau_\Phi^0(\circlearrowleft_a)|^2 = \Phi(a)\Phi(a^*) - |\Phi(a)|^2 = 0,$$

which allows us to conclude. $\qquad\square$

4.2 Proof of the convergence of random matrices (Theorem 1.1)

The purpose of this section is to prove Theorem 1.1 in the following more precise form.

Theorem 4.9. *For all $N \geqslant 1$, let $\mathbf{X}_N = (X_j)_{j \in J}$ be a family of random matrices in $\mathrm{M}_N(\mathbb{C})$ satisfying the hypothesises of Theorem 1.1. Let $\mathbf{x} = (x_j)_{j \in J}$ be a family in some non-commutative probability space $(\mathcal{A}, \Phi)$ which is the limit in $*$-distribution of $\mathbf{X}_N$. Let consider the algebraic space of traffic $(\mathcal{G}(\mathcal{A}), \tau_\Phi) \supset (\mathcal{A}, \Phi)$ given by Proposition 4.4. Then $\mathbf{X}_N$ converges in distribution of traffics to $\mathbf{x}$.*

In other words, for all 0-graph polynomials t in $\mathbb{C}\mathcal{G}^{(0)}\langle J \times \{1, *\}\rangle$, we have

$$\tau_{\mathbf{X}_N}[t] \xrightarrow[N\to\infty]{} \tau_{\mathbf{x}}[t],$$

where $\tau_{\mathbf{x}}[t]$ is given by Definition 4.3 .

Let us first derive some consequences of this theorem. The first one is obtained as an application of Corollary 4.7, and generalize a recent result of Mingo and Popa [12].

Corollary 4.10. *For all family of random matrices $\mathbf{X}_N$ satisfying the previous theorem, the family $\mathbf{X}_N$, the family of the transposes ${}^t\mathbf{X}_N$ and the family of the degrees $\deg(\mathbf{X}_N)$ are asymptotically free .*

Proposition 4.11. *Let $\mathbf{X}_N = (X_j)_{j \in J}, \mathbf{Y}_M = (Y_k)_{k \in K}$ be independent unitarily invariant families of random matrices of size N and M respectively. Assume that $\mathbf{X}_N$ and $\mathbf{Y}_M$ converge in $*$-distribution as N, M goes to infinity. Then $\mathbf{X}_N \otimes \mathbf{Y}_N = (X_j \otimes Y_k)_{j \in J, k \in K}$, seen as an element of $\left(\mathrm{M}_{NM}(\mathbb{C}), \mathbb{E}[\frac{1}{NM}\mathrm{Tr}]\right)$, converges in distribution of traffics. Moreover, in the set of 0-graph monomials T such that there exists a cycle visiting each edge once, the limiting distribution of $\mathbf{X}_N \otimes \mathbf{Y}_N$ has the form of the distribution in Definition 4.3. In particular the conclusion of Corollary 4.10 for this family of matrices holds true, namely, $\mathbf{X}_N \otimes \mathbf{Y}_N$ is asymptotically free from $\mathbf{X}_N^t \otimes \mathbf{Y}_N^t$.*

Proof. We index the entries of a matrix $X \otimes Y \in \mathrm{M}_{NM}(\mathbb{C})$ by pairs of indices $\mathbf{i} = (i, i'), \mathbf{j} = (j, j') \in [N] \times [M]$ with the convention that the entry $(X \otimes Y)_{\mathbf{i},\mathbf{j}}$ is $X_{i,j}Y_{i',j'}$. Let $T = (V, E, j \times \epsilon \in \mathbb{C}\mathcal{G}^{(0)}\langle J \times K \times \{1, *\}\rangle$. Then one has

$$\tau^0_{\mathbf{X}_N \otimes \mathbf{Y}_M}[T]$$
$$= \frac{1}{NM} \sum_{\substack{\phi: V \to [N] \times [M] \\ \text{injective}}} \mathbb{E}\Big[\prod_{(v,w)\in E} \big(X^{\epsilon(e)}_{j(e)} \otimes Y^{\epsilon(e)}_{k(e)}\big)_{\phi(v),\phi(w)}\Big]$$
$$= \frac{1}{NM} \sum_{\substack{\phi_1: V \to [N], \ \phi_2: V \to [M] \\ \text{s.t.}\forall v,w\in V, \forall i \neq j \in \{1,2\} \\ \phi_i(v)=\phi_j(w) \Rightarrow \phi_j(v)\neq\phi_j(w)}} \mathbb{E}\Big[\prod_{(v,w)\in E} \big(X^{\epsilon(e)}_{j(e)}\big)_{\phi_1(v),\phi_1(w)}\Big] \times \mathbb{E}\Big[\prod_{(v,w)\in E} \big(Y^{\epsilon(e)}_{k(e)}\big)_{\phi_2(v),\phi_2(w)}\Big]$$

Denote by Λ_T the set of pairs $(\pi_1, \pi_2) \in \mathcal{P}(V)^2$ such that if two elements belong to a same block of π_i then they belong to different blocks of π_j, $i \neq j \in \{1, 2\}$. Denote also by $\ker\phi_i$ the partition

of V such that $v \sim_{\ker\phi_i} w$ if and only if $\phi_i(v) = \phi_i(w)$. Then we get

$$
\begin{aligned}
\tau^0_{\mathbf{X}_N \otimes \mathbf{Y}_M}[T] &= \frac{1}{NM} \sum_{(\pi_1,\pi_2)\in\Lambda_T} \sum_{\substack{\phi_1:V\to[N] \\ \text{s.t. } \ker\phi_1=\phi_1}} \mathbb{E}\Big[\prod_{(v,w)\in E} \big(X^{\epsilon(e)}_{j(e)}\big)_{\phi_1(v),\phi_1(w)} \Big] \\
&\quad \times \sum_{\substack{\phi_2:V\to[M] \\ \text{s.t. } \ker\phi_2=\phi_2}} \mathbb{E}\Big[\prod_{(v,w)\in E} \big(Y^{\epsilon(e)}_{k(e)}\big)_{\phi_2(v),\phi_2(w)} \Big] \\
&= \sum_{(\pi_1,\pi_2)\in\Lambda_T} \tau^0_N\big[T^{\pi_1}(\mathbf{X}_N)\big] \times \tau^0_N\big[T^{\pi_2}(\mathbf{Y}_N)\big]. \qquad (4.2)
\end{aligned}
$$

By Theorem 1.1 we get that $\mathbf{X}_N \otimes \mathbf{Y}_M$ converges in distribution of traffics. Moreover, the partitions π_1, π_2 which contribute in the limit are those such that T^{π_1} and T^{π_2} are cacti with oriented cycles. Recall that cacti are characterized by the fact that each edge belong exactly to one cycle. But for a graph T' and partition π' of its vertices, the number of cycles an edge of T' belongs to can only increase in the quotient graph $(T')^{\pi'}$.

Hence we deduce that $\tau^0_{\mathbf{X}_N \otimes \mathbf{Y}_M}[T]$ does not vanishes at infinity only if each edge of T belongs at most to one cycle. In particular, if there is a cycle visiting each edge of T once, then T must be a cactus.

Assume from now on, that T is a cactus and let $\pi \in \mathcal{P}(v)$ such that T^π is a cactus. Then denoting by π_c the restriction of π on a cycle c of T, π is the smallest partition that contains the blocks of the π_c for any cycle c of T (otherwise there will exist an edge belonging to more than a cycle of T^π).

Moreover, given a pair $(\pi_1, \pi_2) \in \mathcal{P}(V)^2$ such that T^{π_1} and T^{π_2} are cacti, one has $(\pi_1, \pi_2) \in \Lambda_T$ if and only if for each cycle c of T the partitions $\pi_{1,c}, \pi_{2,c}$ restricted to c are such that $(\pi_{1,c}, \pi_{2,c}) \in \Lambda_c$. Since $\tau^0\big[T^{\pi_1}(\mathbf{X}_N)\big]$ and $\tau^0\big[T^{\pi_2}(\mathbf{Y}_N)\big]$ are asymptotically multiplicative with respect to the cycles of T^{π_1}, T^{π_2}, we get

$$
\tau^0_{\mathbf{X}_N \otimes \mathbf{Y}_M}[T] = \prod_{c \text{ cycle of} T} \sum_{(\pi_1,\pi_2)\in\Lambda_c} \tau^0_N\big[c^{\pi_1}(\mathbf{X}_N)\big] \times \tau^0_N\big[c^{\pi_2}(\mathbf{Y}_N)\big] + o(1). \qquad (4.3)
$$

This proves the first part of the result. For the asymptotic freeness of the ensemble $\mathbf{X}_N \otimes \mathbf{Y}_N$ with its transpose it suffices to remark that the *-distribution of an ensemble depends only on the distribution of traffics of this ensemble restricted to 0-graph monomials such that there is a cycle visiting each edge once. $\qquad \square$

In the following three paragraphs, we respectively review some results about the free cumulants, some results about the Weingarten function, and the links between those two objects in large dimension.

The Weingarten function. To prove Theorem 4.9, we have to integrate polynomials against the $\mathrm{U}(N)$-Haar measure. Expressions for these integrals appeared in [16] and were first proven in [3] and given in terms of a function on symmetric group called the Weingarten function. We recall here its definition and some of its properties. For any $n \in \mathbb{N}^*$ and any permutation $\sigma \in \mathcal{S}_n$, let us set

$$
\Omega_{n,N}(\sigma) = N^{\#\sigma},
$$

where $\#\sigma$ is the number of cycles of σ. When n is fixed and $N \to \infty$, $N^{-n}\Omega_{n,N} \to \delta_{\mathrm{Id}_n}$. For any pair of functions $f, g : \mathcal{S}_n \to \mathbb{C}$ and $\pi \in \mathcal{S}_n$, let us define the convolution product

$$
f \star g(\sigma) = \sum_{\pi \leqslant \sigma} f(\pi)g(\pi^{-1}\sigma),
$$

Hence, for N large enough, $\Omega_{n,N}$ is invertible in the algebra of function on $\mathcal{S}_n$ endowed with convolution as a product. We denote by $\mathrm{Wg}_{n,N}$ the unique function on $\mathcal{S}_n$ such that

$$
\mathrm{Wg}_{n,N} * \Omega_{n,N} = \Omega_{n,N} * \mathrm{Wg}_{n,N} = \delta_{\mathrm{Id}_n}.
$$

Then, [3, Corollary 2.4] says that, for any indices $i_1, i_1', j_1, j_1' \ldots, i_n, i_n', j_n, j_n' \in \{1, \ldots, N\}$ and $U = \big(U(i,j)\big)_{i,j=1,\ldots,N}$ a Haar distributed random matrix on $\mathrm{U}(N)$,

$$\mathbb{E}[U(i_1, j_1) \ldots U(i_n, j_n)\overline{U}(i_1', j_1') \ldots \overline{U}(i_n', j_n')] = \sum_{\substack{\alpha, \beta \in \mathcal{S}_n \\ i_{\alpha(k)} = i_k', j_{\beta(k)} = j_k'}} \mathrm{Wg}_{n,N}(\alpha\beta^{-1}). \qquad (4.4)$$

Free cumulants and the Möbius function μ. As explained in [2], it is equivalent to consider lattices of non-crossing partitions or sets of permutations endowed with an appropriate distance. For our purposes, it is more suitable to define the free cumulants using sets of permutations. Let us endow $\mathcal{S}_n$ with the metric d, by setting for any $\alpha, \beta \in \mathcal{S}_n$,

$$d(\alpha, \beta) = n - \#(\beta\alpha^{-1}),$$

where $\#(\beta\alpha^{-1})$ is the number of cycles of $\beta\alpha^{-1}$. We endow the set $\mathcal{S}_n$ with the partial order given by the relation $\sigma_1 \preceq \sigma_2$ if $d(\mathrm{Id}_n, \sigma_1) + d(\sigma_1, \sigma_2) = d(\mathrm{Id}_n, \sigma_2)$, or similarly if σ_1 is on a geodesic between Id_n and σ_2.

Given a state $\Phi : \mathbb{C}\langle x_j, x_j^* \rangle_{j \in J} \to \mathbb{C}$, we define the free cumulants $(\kappa_n)_{n \in \mathbb{N}}$ recursively on $\mathbb{C}\langle x_j, x_j^* \rangle_{j \in J}$ by the system of equations

$$\Phi(y_1 \cdots y_n) = \sum_{\sigma \preceq (1 \cdots n)} \prod_{\substack{(c_1 \ldots c_k) \\ \text{cycle of } \sigma}} \kappa(y_{c_1}, \ldots, y_{c_k}), \qquad \forall y_1, \ldots, y_n \in \mathbb{C}\langle x_j, x_j^* \rangle_{j \in J}. \qquad (4.5)$$

Let us fix $y_1, \ldots, y_n \in \mathbb{C}\langle x_j, x_j^* \rangle_{j \in J}$ and denote by respectively ϕ and k the functions from $\mathcal{S}_n$ to $\mathbb{C}$ given by

$$\phi(\alpha) = \prod_{\substack{(c_1 \ldots c_k) \\ \text{cycle of } \sigma}} \Phi(y_{c_1} \ldots y_{c_k}) \quad \text{and} \quad k(\alpha) = \prod_{\substack{(c_1 \ldots c_k) \\ \text{cycle of } \sigma}} \kappa(y_{c_1}, \ldots, y_{c_k}),$$

which are such that $\phi((1 \cdots n)) = \sum_{\pi \preceq (1 \cdots n)} k(\pi)$. In fact, we have more generally the relation

$$\phi(\alpha) = \sum_{\pi \preceq \sigma} k(\pi).$$

Note that $\phi = k \star \zeta$, where ζ is identically equal to one. The identically one function ζ is invertible for the convolution $\star$ (see [2]), and its inverse μ is called Möbius function. It allows us to express the free cumulants in terms of the trace:

$$k = \phi \star \mu. \qquad (4.6)$$

Asymptotics of the Weingarten function. One can observe that, for any pair of functions $f, g : \mathcal{S}_n \to \mathbb{C}$ and $\pi \in \mathcal{S}_n$,

$$\sum_{\pi \in \mathcal{S}_n} N^{d(\mathrm{Id}_n, \sigma) - d(\mathrm{Id}_n, \pi) - d(\pi, \sigma)} f(\pi)g(\pi^{-1}\sigma) = f \star g(\sigma) + o(1).$$

Defining the convolution $\star_N$ as

$$f \star_N g = N^n \Omega_{n,N}^{-1}((N^{-n}\Omega_{n,N}f) * (N^{-n}\Omega_{n,N}g)) = \sum_{\pi \in \mathcal{S}_n} N^{d(\mathrm{Id}_n, \sigma) - d(\mathrm{Id}_n, \pi) - d(\pi, \sigma)} f(\pi)g(\pi^{-1}\sigma),$$

it follows that $\star$ is the limit of $\star_N$. Because $\mathrm{Wg}_{n,N}$ is the inverse of $\Omega_{n,N}$ for the convolution $*$, we have $(N^{2n}\Omega_{n,N}^{-1}\mathrm{Wg}_{n,N}) \star_N \zeta = N^{-n}\Omega_{n,N}$, from which we deduce that $(N^{2n}\Omega_{n,N}^{-1}\mathrm{Wg}_{n,N}) \star \zeta = \delta_{\mathrm{Id}_n} + o(1)$, or similarly that

$$N^{2n}\Omega_{n,N}^{-1}\mathrm{Wg}_{n,N} = \mu + o(1).$$

More generally, if $f, f_N : \mathcal{S}_n \to \mathbb{C}$ are such that $f_N = f + o(1)$, then

$$N^n \Omega_{n,N}^{-1}((\Omega_{n,N}f_N) * \mathrm{Wg}_{n,N}) = (f_N) \star_N (\mathrm{Wg}_{n,N}) = f \star \mu + o(1). \qquad (4.7)$$

Proof of Theorem 4.9. We can now prove Theorem 1.1. Let $\mathbf{X}_N = (X_j)_{j\in J}$ a family of unitary invariant random matrices which converges in *-distribution, as N goes to infinity, to $\mathbf{x} = (x_j)_{j\in J}$ family of some noncommutative probability space $(\mathcal{A}, \Phi)$. We extend $(\mathcal{A}, \Phi)$ to a space of traffics $(\mathcal{G}(\mathcal{A}), \tau_\Phi)$. We fix a $*$-test graph $T = (V, E, j \times \epsilon) \in \mathbb{C}\mathcal{G}^{(0)}\langle J \times \{1, *\}\rangle$ and prove that $\tau_{\mathbf{X}_N}[T]$ converges to $\tau_{\mathbf{X}}[T]$ as $N \to \infty$. By taking the real and the imaginary parts, we can assume that the matrices of $\mathbf{X}_N$ are Hermitian and so assume $\varepsilon(e) = 1$ for any $e \in E$.

We consider a random unitary matrix U, distributed according to the Haar distribution, and independent of $\mathbf{X}_N$. By assumption $\mathbf{Z}_N := U\mathbf{X}_N U^* \in M_N(\mathbb{C})$ has the same distribution as $\mathbf{X}_N$. We denote respectively by $\underline{e}$ and $\overline{e}$ the origin vertex and the goal vertex of e. Then

$$\tau_{\mathbf{X}_N}[T] = \frac{1}{N}\sum_{\phi:V\to[N]}\mathbb{E}\left[\prod_{e\in E}Z_{\gamma(e)}\big(\phi(\underline{e}), \phi(\overline{e})\big)\right]$$

$$= \frac{1}{N}\sum_{\substack{\phi:V\to[N]\\ \varphi,\varphi':E\to[N]}}\mathbb{E}\left[\prod_{e\in E}U\big(\phi(\underline{e}),\varphi(e)\big)\overline{U}\big(\phi(\overline{e}),\varphi'(e)\big)\right]\mathbb{E}\left[\prod_{e\in E}X_{\gamma(e)}\big(\varphi(e),\varphi'(e)\big)\right].$$

In the integration formula (4.4), the number n of occurrence of each term $U(i,j)$ is the cardinality of E and the sum over permutations of $\{1,\ldots,n\}$ is replaced by a sum over the set $\mathcal{S}_E$ of permutations of the edge set E. By identifying E with the set of integers $\{1,\ldots,|E|\}$, we consider that $\mathrm{Wg}_{n,N}$ is defined on $\mathcal{S}_E$ instead of $\mathcal{S}_n$. Then, one has

$$\tau_{\mathbf{X}_N}[T] = \frac{1}{N}\sum_{\alpha,\beta\in\mathcal{S}_E}\mathrm{Wg}_{n,N}(\alpha\beta^{-1})\sum_{\substack{\phi:V\to\{1,\ldots,N\}\\ \varphi,\varphi':E\to\{1,\ldots,N\}\\ \phi(\underline{\alpha(e)})=\phi(\overline{e}),\varphi(\beta(e))=\varphi'(e)}}\mathbb{E}\left[\prod_{e\in E}X_{\gamma(e)}\big(\varphi(e),\varphi'(e)\big)\right].$$

For any permutation $\alpha \in \mathcal{S}_E$, let $\pi(\alpha)$ be the smallest partition of V such that, for all $e \in E$, $\overline{e}$ is in the same block with $\underline{\alpha(e)}$. Summing over ϕ in the previous expression yields

$$\tau_{\mathbf{X}_N}[T] = \sum_{\alpha,\beta\in\mathcal{S}_E}N^{\#\pi(\alpha)-1}\mathrm{Wg}_{n,N}(\alpha\beta^{-1})\sum_{\substack{\varphi,\varphi':E\to\{1,\ldots,N\}\\ \varphi(\beta(e))=\varphi'(e)}}\mathbb{E}\left[\prod_{e\in E}X_{\gamma(e)}\big(\varphi(e),\varphi'(e)\big)\right]$$

$$= \sum_{\alpha,\beta\in\mathcal{S}_E}N^{\#\pi(\alpha)-1}\mathrm{Wg}_{n,N}(\alpha\beta^{-1})\mathbb{E}\left[\prod_{\substack{(e_1\ldots e_k)\\ \text{cycle of }\beta}}\mathrm{Tr}(X_{\gamma(e_1)}X_{\gamma(e_2)}\ldots X_{\gamma(e_k)})\right].$$

To conclude we will need the following

Lemma 4.12. *i) For any permutation $\alpha \in \mathcal{S}_E$, $\#\pi(\alpha) + \#\alpha \leqslant \#E + 1$ and the equality implies that the graph of $T^{\pi(\alpha)}$ is an oriented cactus.*
ii) The map

$$\pi : \{\alpha : \#\pi(\alpha) + \#\alpha = \#E + 1\} \longrightarrow \{\pi : \text{the graph of } T^\pi \text{ is an oriented cactus}\}$$

is a bijection whose inverse γ is given, for all $\pi \in \mathcal{P}(V)$ such that T^π is an oriented cactus, by the permutation $\gamma(\pi)$ whose cycles are the biconnected components of T^π.

Proof Lemma 4.12. i) Let $\alpha \in \mathcal{S}_E$. Let us define a connected graph G_α whose vertices are the cycles of α all together with the blocks of $\pi(\alpha)$, and whose edges are defined are follow. There is an edge between a cycle c of α and a block b of $\pi(\alpha)$ if and only if there is an edge e of T such that $e \in c$ and $\overline{e} \in b$. This way, the edges of G_α are in bijective correspondence with the edges of T. Therefore, $\#\pi(\alpha) + \#\alpha \leqslant \#E + 1$ with equality if and only G_α is a tree.

In fact, each cycle of α yields a cycle in $T^{\pi(\alpha)}$, and in the case where G_α is a tree, there exist no others cycles in G_α. By consequence, the biconnected component of $T^{\pi(\alpha)}$ are exactly the cycles of α, and $T^{\pi(\alpha)}$ is therefore an oriented cactus.

ii) $\pi \circ \gamma$ and $\gamma \circ \pi$ are the identity functions: π is one-to-one and its inverse is γ. $\qquad\square$

For all $\alpha \in \mathcal{S}_E$, set

$$\phi_N(\alpha) = N^{-\#\alpha} \mathbb{E}\left[\prod_{\substack{(e_1...e_k) \\ \text{cycle of } \sigma}} \text{Tr}(X_{\gamma(e_1)} X_{\gamma(e_2)} \dots X_{\gamma(e_k)})\right]$$

and

$$\phi(\alpha) = \prod_{\substack{(e_1...e_k) \\ \text{cycle of } \sigma}} \Phi(x_{\gamma(e_1)} x_{\gamma(e_2)} \dots x_{\gamma(e_k)})$$

in such a way that that $\phi_N = \phi + o(1)$. Let us fix $\alpha \in \mathcal{S}_E$. On one hand we have

$$N^{\#\pi(\alpha)+\#\alpha-\#E-1} = \mathbb{1}_{\#\pi(\alpha)+\#\alpha=\#E+1} + o(1).$$

On the other hand, according to (4.7), we have

$$\sum_{\beta \in \mathcal{S}_E} N^{\#E-\#\alpha} \text{Wg}_{n,N}(\alpha\beta^{-1}) \mathbb{E}\left[\prod_{\substack{(e_1...e_k) \\ \text{cycle of } \beta}} \text{Tr}(X_{\gamma(e_1)} X_{\gamma(e_2)} \dots X_{\gamma(e_k)})\right] = ((\phi_N) \star_N \text{Wg}_{n,N})(\alpha)$$

$$= (\phi \star \mu)(\alpha) + o(1).$$

It follows that

$$\tau_{\mathbf{X}_N}(T) = \sum_{\substack{\alpha \in \mathcal{S}_E \\ \#\pi(\alpha)+\#\alpha=\#E+1}} (\phi \star \mu)(\alpha) + o(1).$$

From (4.6), we know that $(\phi \star \mu)(\alpha) = k(\alpha) = \prod_{\substack{(e_1...e_k) \\ \text{cycle of } \alpha}} \kappa(x_{\gamma(e_1)}, \dots, x_{\gamma(e_k)})$. Thanks to Lemma 4.12, we can now write

$$\tau_{\mathbf{X}_N}(T) = \sum_{\substack{\pi \in \mathcal{P}(V) \\ T^\pi \text{ is an oriented cactus}}} \prod_{\substack{(e_1...e_k) \\ \text{cycle of } \gamma(\pi)}} \kappa(x_{\gamma(e_1)}, \dots, x_{\gamma(e_k)}) + o(1)$$

$$= \sum_{\substack{\pi \in \mathcal{P}(V) \\ T^\pi \text{ is an oriented cactus}}} \prod_{\substack{(e_1...e_k) \\ \text{cycle of } T^\pi}} \kappa(x_{\gamma(e_1)}, \dots, x_{\gamma(e_k)}) + o(1)$$

In order to pursue the computation, let t be the 0-graph monomial $(V, E, \lambda(e)) \in \mathbb{C}\mathcal{G}^{(0)}\langle\mathcal{G}(\mathcal{A})\rangle$ such that $\lambda(e) = x_{\gamma(e)}$. By Definition 4.3, we get

$$\tau_{\mathbf{X}_N}(T) = \sum_{\pi \in \mathcal{P}(V)} \tau_\Phi^0[t^\pi] + o(1)$$

$$= \tau_\Phi[t] + o(1)$$

$$= \tau_{\mathbf{x}}[T] + o(1)$$

so that $\tau_{\mathbf{X}_N}(T)$ converges towards the expected limit. $\qquad\square$

4.3 Proof of Theorem 1.3

Let $(\mathcal{A}, \Phi)$ be a non-commutative probability space $(\mathcal{A}, \Phi)$. We define $(\mathcal{G}(\mathcal{A}), \tau_\Phi)$ as in Proposition 4.4, in such a way that $(\mathcal{A}, \Phi) \subset (\mathcal{G}(\mathcal{A}), \Psi)$ if Ψ denote the trace induced by τ_Φ. To prove the full statement of Theorem 1.3, it remains to prove that τ_Φ satisfies the positivity condition (2.2) and the two following items:

- If $\mathbf{A}_N$ is a sequence of random matrices that converges in *-distribution to $\mathbf{a} \in \mathcal{A}$ as N tends to ∞ and verifies the condition of Theorem 1.1, then $\mathbf{A}_N$ converges in distribution of traffics to $\mathbf{a} \in \mathcal{G}(\mathcal{A})$ as N tends to ∞ (already proved in Theorem 4.9).

- Two families **a** and **b** $\in \mathcal{A}$ are freely independent in $\mathcal{A}$ if and only if they are traffic independent in $G(\mathcal{A})$.

In other words, it remains to prove Theorem 4.13 and Proposition 4.8 below.

Theorem 4.13. *For all non-commutative probability space $(\mathcal{A}, \Phi)$, the linear functional $\tau_\Phi :$ $\mathbb{C}\mathcal{G}^{(0)}\langle \mathcal{A} \rangle \to \mathbb{C}$ given by Definition 4.3 satisfies the positivity condition* (2.2).

Proof. In the four steps of the proof, we will prove successively that $\tau_\Phi\big[t|t^*\big] \geqslant 0$ for all $t = \sum_{i=1}^{L} \alpha_i t_i$ a n-graph polynomial such that

1. the t_i are 2-graph monomials without cycles and the leaves are outputs, that is chains of edges with possibly different orientations;

2. the t_i are trees whose leaves are the outputs;

3. the t_i are such that $t_i|t_i^*$ have no cutting edges (see definition below);

4. the t_i are n-graph monomials.

In the different steps, we will use those two direct corollaries of Menger's theorem.

Theorem 4.14 (Menger's theorem [11])**.** *Let T be a graph and v_1 and v_2 two distinct vertices. Then the minimum number of edges whose removal disconnect v_1 and v_2 is equal to the maximum number of edge-disjoint paths from v_1 to v_2 (i.e. sharing no edges out of the v_k's).*

A cutting edge of a graph T is an edge whose removal disconnects T. A graph T is two edge connected (t.e.c.) if it has no cutting edge.

Corollary 4.15. *Let T be a graph which is t.e.c. and two distinct vertices v_1 and v_2. Then, there exists two edge-disjoint simple paths between v_1 and v_2.*

Corollary 4.16. *Let T be a graph such that there exist two distinct vertices v_1 and v_2, and three edge-disjoint simple paths γ_1, γ_2 and γ_3 between v_1 and v_2. Then, T is not a cactus.*

Step 1 Proposition 4.4 shows the positivity if all the t_i's consist in chains of edges all oriented in the same direction. Indeed, we can write $t_i = \cdot \overset{a_i}{\leftarrow} \cdot$ for all i (or $t_i = \cdot \overset{a_i}{\to} \cdot$ for all i) and so, we get

$$\tau_\Phi\big[t|t^*\big] = \tau_\Phi\Big[\sum_{i,j=1}^{L} \alpha_i \bar{\alpha}_j t_i t_j^* \Big] = \Phi\Big(\sum_{i,j=1}^{L} \alpha_i \bar{\alpha}_j a_i a_j^* \Big) \geqslant 0.$$

We deduce that the trace Ψ induced by τ_Φ is positive on the algebras $\mathbb{C}\langle \cdot \overset{a}{\leftarrow} \cdot : a \in \mathcal{A} \rangle$ and $\mathbb{C}\langle \cdot \overset{a}{\to} \cdot : a \in \mathcal{A} \rangle$. From Corollary 4.7, we also know that Ψ is also positive on the mixed algebra $(\mathbb{C}\langle \cdot \overset{a}{\leftarrow} \cdot, \cdot \overset{a}{\to} \cdot : a \in \mathcal{A} \rangle, \tau_\Phi)$ (the free product of positive trace is positive [14, Lecture 6]). Finally, if the t_i's consist in chains of edges indexed by element of $\mathcal{A}$, we know that

$$\tau_\Phi\big[t|t^*\big] = \Psi\Big[\sum_{i,j=1}^{L} \alpha_i \bar{\alpha}_j t_i t_j^* \Big] \geqslant 0.$$

Step 2 Assume that the t_i's are trees whose leaves are the outputs. Let us prove by induction on the number D of all edges of the t_i's that we have $\tau_\Phi\big[t|t^*\big] \geqslant 0$.

If the number of edges of the t_i's is 0, we have $\tau_\Phi\big[t|t^*\big] = \sum_{i,j} \alpha_i \alpha_j^* \geqslant 0$. We suppose that $D \geqslant 1$ and that this result is true whenever the number of edges of the t_i's is less than $D - 1$.

We can remove one edge in the following way. Let us choose one leaf v of one of the $t_i's$ which has at least one edge. It is an output and for each tree t_i we denote by $v^{(i)}$ the first node (or distinct leaf if there is no node) of the tree of t_i encountered by starting from this output v, and by $t^{(i)}$ the branch of t_i between this output v and $v^{(i)}$. Of course, $v^{(i)}$ can be equal to v and $t^{(i)}$ can be trivial, but there is at least one of the $t^{(i)}$'s which is not trivial. Denote by $\breve{t}_i$ the n-graph obtained from t_i after discarding the $t^{(i)}$'s, and whose output v is replaced by $v^{(i)}$. We claim that

$$\tau_\Phi\big[t|t^*\big] = \tau_\Phi\big[t|t^*\big] \times \tau_\Phi\big[\breve{t}_i|\breve{t}_j^*\big].$$

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Firstly, we can identify the pairs $v^{(i)}$ and $v^{(j)}$ in the computation of the left hand-side. Indeed, we write $\tau_\Phi\big[t_i|t_j^*\big] = \sum_\pi \tau_\Phi^0\big[(t_i|t_j^*)^\pi\big]$, and consider a term in the sum for which π does not identify $v^{(i)}$ and $v^{(j)}$. Because $\check{t}_i|\check{t}_j^*$ is t.e.c., there exists two disjoints paths between $v^{(i)}$ and $v^{(j)}$. But because $t^{(i)}|t^{(j)*}$ contains a third distinct path, by Corollary 4.16 π cannot be a cactus if it does not identify $v^{(i)}$ and $v^{(j)}$ and so $\tau_\Phi^0\big[(t_i|t_j^*)^\pi\big]$ is zero.

Consider a term in the sum $\sum_\pi \tau_\Phi^0\big[(t_i|t_j^*)^\pi\big]$ for which π identifies the pairs $v^{(i)}, v^{(j)}$. Assume that a vertex v_1 of $\check{t}_i|\check{t}_j^*$ is identified with a vertex v_2 which is not in $\check{t}_i|\check{t}_j^*$. Assume that π does not identify $v^{(i)}$ with v_1 and v_2. Because $\check{t}_i|\check{t}_j^*$ is t.e.c. there exists two distinct paths between v_1 and $v^{(i)}$ out of $\tau_\Phi\big[t^{(i)}|t^{(j)*}\big]$. But there exists also a path between v_2 and $v^{(i)}$ in $t^{(i)}|t^{(j)*}$. By Corollary 4.16, we get that $(t_i|t_j^*)^\pi$ is not a cactus and so $\tau_\Phi^0\big[(t_i|t_j^*)^\pi\big]$ is zero.

Hence, to determine which vertices of $\check{t}_i|\check{t}_j^*$ are identified with some vertices of $t^{(i)}|t^{(j)*}$, one can first determine which vertices of $\check{t}_i|\check{t}_j^*$ are identifies with $v^{(i)} = v^{(j)}$ and which vertices of $t^{(i)}|t^{(j)*}$ are identified with this vertex. Hence the sum over π partition of the set of vertices of $t_i|t_j^*$ can be reduced to a sum over π_1 partition of the set of vertices of $\check{t}_i|\check{t}_j^*$ and a sum over π_2 partition of the set of vertices of the graph $t^{(i)}|t^{(j)*}$. Moreover, by definition of τ_Φ, for two 0-graph monomials T_1 and T_2, if T is obtained by considering the disjoint union of T_1 and T_2 and merging one of their vertices, one has $\tau_\Phi^0[T] = \tau_\Phi^0[T_1] \times \tau_\Phi^0[T_2]$. Hence, the contribution of $\check{t}_i|\check{t}_j^*$ factorizes in $\tau_\Phi\big[\check{t}_i|\check{t}_j^*\big]$ and the contribution of $t^{(i)}|t^{(j)*}$ factorizes in $\tau_\Phi\big[t^{(i)}|t^{(j)*}\big]$, and we get the expected result.

From Step 1, we know that $A = \big(\tau_\Phi\big[t^{(i)}|t^{(j)*}\big]\big)_{i,j}$ is nonnegative. By induction hypothesis, we know that $B = \big(\tau_\Phi\big[\check{t}_i|\check{t}_j^*\big]\big)_{i,j}$ is also nonnegative. We obtain as desired that the Hadamard product of A and B is nonnegative ([14, Lemma 6.11]) and in particular, for all α_i, we have

$$\sum_{i,j} \alpha_i \bar{\alpha}_j \tau_\Phi\big[t_i|t_j^*\big] \geqslant 0.$$

Step 3   Let us prove that, for all t_i such that $t_i|t_i^*$ have no cutting edges, we have $\tau_\Phi\big[t|t^*\big] \geqslant 0$.

For a graph T, let call t.e.c. components the maximal subgraphs of T with no cutting edges. The tree of t.e.c. of T is the graph whose vertices are the t.e.c. components of T and whose edges are the cutting edges of T. First of all, our condition is equivalent to the condition that, for each t_i, any leaf of the tree of the t.e.c. components of t_i is a component containing an output. Here again, we can proceed by induction. Let D be the total number of t.e.c. components of the t_i's which do not consists in a single vertex.

If $D = 0$, we are in the case of the previous step. Let us assume that $D > 0$ and that the result is true up to the case $D - 1$. We can remove one t.e.c. in the following way. Let us choose a t.e.c. component $t^{(k)}$ which is not a single vertex of a certain n-graph monomial t_k, for some k in $\{1, \ldots, L\}$. We consider $t^{(k)}$ as a multi *-graph monomial, where the outputs are the vertices which are attached to cutting edges. Let $\check{t}_k$ be the n-graph monomial obtained from t_k by replacing the component $t^{(k)}$ by one single vertex. We define also for $i \neq k$ the *-graph monomial $t^{(i)}$ to be the trivial leaf and set $\check{t}_i = t_i$. We claim that

$$\tau_\Phi\big[T(t_i, t_j^*)\big] = \tau_\Phi\big[T(\check{t}_i, \check{t}_j^*)\big] \times \tau_\Phi\big[t^{(i)}\big] \times \tau_\Phi\big[t^{(j)*}\big]$$

(of course, this equality is nontrivial only if we consider $i = k$ or $j = k$).

Firstly, the outputs of $t^{(i)}$ can be identified. Indeed, consider v_1, v_2 two distinct ouputs of $t^{(i)}$. Writing $\tau_\Phi\big[t_i|t_j^*\big] = \sum_\pi \tau_\Phi^0\big[(t_i|t_j^*)^\pi\big]$, consider a term in the sum for which π does not identify v_1 and v_2. Since $t^{(i)}$ is t.e.c. there exist two distinct simple paths γ_1 and γ_2 between v_1 and v_2. Consider a path from v_2 to v_1 that does not visit $t^{(i)}$ in $t_i|t_j^*$. Such a path exists as v_1 and v_2 belong to two subtrees of t_i that are attached to outputs of t_i, themselves being attached to the connected graph t_j^*. The quotient by π yields three distinct paths γ between v_1 and v_2 in $(t_i|t_j^*)^\pi$ which implies that $(t_i|t_j^*)^\pi$ is not a cactus by Corollary 4.16. Hence, by definition of τ_Φ, $\tau_\Phi^0\big[(t_i|t_j^*)^\pi\big]$ is zero. Thus, when we write $\tau_\Phi\big[(t_i|t_j^*)\big] = \sum_\pi \tau_\Phi^0\big[(t_i|t_j^*)^\pi\big]$ we can restrict the sum over the partition π that identify v_1 and v_2, therefore, we can replace t_i by the graph $\tilde{t}_i$ where we have identify v_1 and v_2. Hence we have $\tau_\Phi\big[t_i|t_j^*\big] = \tau_\Phi\big[\tilde{t}_i|\tilde{t}_j^*\big]$.

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Let us write $\tau_\Phi\big[\tilde{t}_i|\tilde{t}_j^*\big] = \sum_\pi \tau_\Phi^0\big[(\tilde{t}_i|\tilde{t}_j^*)^\pi\big]$. Let π be as in the sum. Assume that a vertex v_1 of $t^{(i)}$ is identified by π with a vertex v_2 which is not in $t^{(i)}$. Assume that π does not identify $w^{(i)}$ with v_1 and v_2. Since $t^{(i)}$ is t.e.c. there exist two distinct paths between v_1 and $w^{(i)}$ in $t^{(i)}$. But $\breve{t}_i$ is connected and there exists a third path between v_2 and $w^{(i)}$. As usual this implies that $(\tilde{t}_i|\tilde{t}_j^*)^\pi$ is not a cactus and so $\tau_\Phi^0\big[(\tilde{t}_i|\tilde{t}_j^*)^\pi\big]$ is zero.

Hence, to determine which vertices of $t^{(i)}$ are identified with some vertices out of $t^{(i)}$, one can first determine which vertices of $t^{(i)}$ are identifies with $w^{(i)}$ and which vertices out of $t^{(i)}$ are identified with this vertex. Thus the sum over π partition of the set of vertices of $\tilde{t}_i|\tilde{t}_j^*$ can be reduced to a sum over π_1 partition of the set of vertices of $t^{(i)}$ and a sum over π_2 partition of the set of vertices of the graph with $t^{(i)}$ removed. Moreover, by definition of τ_Φ, for two * test graphs T_1 and T_2, if T is obtained by considering the disjoint union of T_1 and T_2 and merging one of their vertices, one has $\tau_\Phi^0[T] = \tau_\Phi^0[T_1] \times \tau_\Phi^0[T_2]$. Hence, the contribution of $T(t_i, t_j^*)$ factorizes in $\tau_\Phi\big[T(\breve{t}_i, t_j^*)\big]$ and the contribution of $t^{(i)}$ factorizes in $\tau_\Phi\big[t^{(i)}\big]$. We can do the same factorization for the n-graph monomial t_j^*, and we get the expected result.

Now, setting $\beta_i = \alpha_i \tau\big[t^{(i)}\big]$, we have

$$\tau_\Phi\big[T(t, t^*)\big] = \sum_{i,j} \beta_i \bar{\beta}_j \tau_\Phi\big[T(\breve{t}_i, \breve{t}_j^*)\big]$$

which is nonnegative thanks to the induction hypothesis.

Step 4 We are not able to prove the positivity in general case so we prove it in an indirect way using the positivity of the free product proves in Theorem 1.2. To bypass this difficultly, we define an auxiliary distribution of traffic $\tilde{\tau}$ which is defined to be equal to τ_Φ on the 0-graph monomials without cutting edges and equal to 0 on the 0-graph monomials with cutting edges. For t_i's some n-graph monomials and $t = \sum_i \alpha_i t_i$, we have

$$\tilde{\tau}\big[t|t^*\big] = \sum_{i,j} \alpha_i \bar{\alpha}_j \tilde{\tau}\big[t_i|t_j^*\big] = \sum_{\substack{i,j \\ t_i|t_i^*, t_j|t_j^* \\ \text{without cutting edges}}} \alpha_i \bar{\alpha}_j \tilde{\tau}\big[t_i|t_j^*\big] = \sum_{\substack{i,j \\ t_i|t_i^*, t_j|t_j^* \\ \text{without cutting edges}}} \alpha_i \bar{\alpha}_j \tau_\Phi\big[t_i|t_j^*\big] \geqslant 0$$

using the result of the previous step.

Therefore, $\tilde{\tau} : \mathbb{C}\mathcal{G}^{(0)}\langle \mathcal{A}\rangle \to \mathbb{C}$ is positive.

Let us consider the Haar unitary traffic distribution $\tau_u : \mathbb{C}\mathcal{G}^{(0)}\langle u, u^*\rangle \to \mathbb{C}$, already mentionned in [8], and which is the (positive) limit of a random matrix distributed according to the Haar measure on the unitary group $U(N)$, and which is well-defined thanks to Theorem 4.9. We do not need the precise form of τ_u. Let us just says the following: u is unitary as a limit of unitaries, which means that $u = u^*$, and Theorem 4.9 implies that τ_u is in the form of Definition 4.3, that is to say supported on oriented cacti (see [8] for a precise formula). The traffic free product $(\tilde{\tau} \star \tau_u) : \mathbb{C}\mathcal{G}^{(0)}\langle \mathcal{A} \cup \{u, u^*\}\rangle \to \mathbb{C}$ is satisfies the positivity condition thanks to Theorem 1.2.

For any *-graph test T in $\mathbb{C}\mathcal{G}^{(0)}\langle \mathcal{A}\rangle$, we define uTu^* as the *-graph test in $\mathbb{C}\mathcal{G}^{(0)}\langle \mathcal{A} \cup \{u, u^*\}\rangle$ obtained from T by replacing each edge $\xrightarrow{a}$ by $\xrightarrow{u}\xrightarrow{a}\xrightarrow{u^*}$. We claim that

$$\tau_\Phi\big[T\big] = (\tilde{\tau} \star \tau_u)\big[uTu^*\big], \tag{4.8}$$

which implies of course the positivity of τ_Φ because $(\tilde{\tau} \star \tau_u)$ is positive as a traffic independent product of positive distribution.

By definition, there is a natural correspondence between the vertices of uTu^*, and $V \sqcup V_1 \sqcup V_2$, where V are the vertices of T and V_1 and V_2 are two copies of the edges E of T. Indeed, each edge of T adds two vertices in uTu^* (one at the beginning and one at the end), and we can denote by V_1 the vertices which appear at the beginning of an edge of T, and by V_2 the vertices which appear at the end of an edge of T. Moreover, there is a natural correspondence between the edges of uTu^*, and $E \sqcup E_1 \sqcup E_2$. Indeed, each edge of T adds two edges in uTu^* (one at the beginning and one at the end), and we can denote by E_1 the edges with appears at the beginning of an edge of T, and by E_2 the edges which appears at the end of an edge of T.

Let us denote by σ the partition which is composed of $|V|$ blocks, each one containing one vertex $v \in V$ and all its adjacent vertices of $V_1 \sqcup V_2$. For any partition $\pi \in \mathcal{P}(V \sqcup V_1 \sqcup V_2)$ which is greater than σ, let us set $\pi_{|V} \in \mathcal{P}(V)$ the restriction of the partition π to the set V.

Les us fix $T \in \mathbb{C}\mathcal{G}^{(0)}\langle \mathcal{A} \rangle$ and write

$$(\tilde{\tau} \star \tau_u)\big[uTu^*\big] = \sum_{\pi' \in \mathcal{P}(V \sqcup V_1 \sqcup V_2)} (\tilde{\tau} \star \tau_u)^0((uTu^*)^{\pi'})$$

$$= \sum_{\pi \in \mathcal{P}(V)} \sum_{\substack{\pi' \in \mathcal{P}(V \sqcup V_1 \sqcup V_2) \\ (\pi' \vee \sigma)_{|V} = \pi}} (\tilde{\tau} \star \tau_u)^0((uTu^*)^{\pi'}).$$

We claim that

Lemma 4.17.   *1. For all $\pi' \in \mathcal{P}(V \sqcup V_1 \sqcup V_2)$ such that $(uTu^*)^{\pi'}$ is not an oriented cactus, we have $(\tilde{\tau} \star \tau_u)^0((uTu^*)^{\pi'}) = 0$.*

2. For all $\pi \in \mathcal{P}(V)$, we have

$$\sum_{\substack{\pi' \in \mathcal{P}(V \sqcup V_1 \sqcup V_2) \\ (\pi' \vee \sigma)_{|V} = \pi}} (\tilde{\tau} \star \tau_u)^0((uTu^*)^{\pi'}) = \tau_\Phi^0\big[T^\pi\big].$$

Proof.   1. If $(uTu^*)^{\pi'}$ is not a cactus, either $(uTu^*)^{\pi'}$ has a cut edge, or $(uTu^*)^{\pi'}$ has three edge-disjoint simple paths γ_1, γ_2 and γ_3 between two distinct vertices v_1 and v_2. First of all, assume that $(uTu^*)^{\pi'}$ has a cut edge indexed by u or u^*, then it is also a cut edge in its colored component when $(uTu^*)^{\pi'}$ is decomposed according to the traffic freeness, and so $(\tilde{\tau} \star \tau_u)^0((uTu^*)^{\pi'})$ vanishes because of the vanishing of the injective traffic distribution of u. Let us assume now that $(uTu^*)^{\pi'}$ has a cut edge indexed by an element of $\mathcal{A}$, then it cuts $(uTu^*)^{\pi'}$ into two component, each one containing an odd number of $\{u, u^*\}$. Because of the traffic independence condition, it implies that there exists in the product an injective trace which contains an odd numbers of $\{u, u^*\}$, which is by consequence equal to 0.

Let us assume now that $(uTu^*)^{\pi'}$ has no cut edge but has three edge-disjoint simple paths γ_1, γ_2 and γ_3 between two distinct vertices v_1 and v_2. If v_1 and v_2 are in a same colored component of $(uTu^*)^{\pi'}$, we can assume that the edge-disjoint simple paths γ_1, γ_2 and γ_3 are also in this colored component, erasing each excursion which go outside of this component, and consequently, $(\tilde{\tau} \star \tau_u)^0((uTu^*)^{\pi'})$ vanishes because of the vanishing of the injective traffic distribution of $\tilde{\tau}$ and of τ_u on 2-edge connected graph which are not cactus. If v_1 and v_2 are not in a same colored component of $(uTu^*)^{\pi'}$, we can replace v_2 by the vertex v_2' in the colored component of v_1 that each simple path from v_1 to v_2 has to visit, due to the tree condition. We can also replace γ_1, γ_2 and γ_3 by edge-disjoint simple paths γ_1', γ_2' and γ_3' which are also in this colored component, erasing each excursion which go outside of this component, and stopping at the first visit of v_2'. We see at the end that there exists three edge-disjoint simple paths γ_1', γ_2' and γ_3' between two distinct vertices v_1' and v_2' inside a colored component of $(uTu^*)^{\pi'}$ and consequently, $(\tilde{\tau} \star \tau_u)^0((uTu^*)^{\pi'})$ vanishes because of the vanishing of the injective traffic distribution of $\tilde{\tau}$ and of τ_u on 2-edge connected graph which are not cactus.

Finally, let us assume that $(uTu^*)^{\pi'}$ is a cactus, but not oriented. Then, there exists two consecutive edges in the same cycle of $(uTu^*)^{\pi'}$ which are not oriented. Then $(\tilde{\tau} \star \tau_u)^0((uTu^*)^{\pi'})$ vanishes because of the vanishing of the injective traffic distribution of $\tilde{\tau}$ and of τ_u on cactus which are not oriented.

2. First of all, let us prove that, if T^π is not a cactus, then

$$\sum_{\substack{\pi' \in \mathcal{P}(V \sqcup V_1 \sqcup V_2) \\ (\pi' \vee \sigma)_{|V} = \pi}} (\tilde{\tau} \star \tau_u)^0((uTu^*)^{\pi'}) = 0(= \tau_\Phi^0\big[T^\pi\big]). \tag{4.9}$$

Assume that T^π is not a cactus. Either T^π has a cut edge, or T^π has three edge-disjoint simple paths γ_1, γ_2 and γ_3 between two distinct vertices v_1 and v_2. If $e \in E$ a cut edge of T^π,

for all $\pi' \in \mathcal{P}(V \sqcup V_1 \sqcup V_2)$ such that $(\pi' \vee \sigma)_{|V} = \pi$, e seen as an edge of $(uTu^*)^{\pi'}$ is also a cut edge, which means that $(\tilde\tau \star \tau_u)^0((uTu^*)^{\pi'}) = 0$ thanks to the first item. In the case where there exist three edge-disjoint simple paths γ_1, γ_2 and γ_3 between two distinct vertices v_1 and v_2 of T^π, it leads to three simple paths γ_1', γ_2' and γ_3' in $(uTu^*)^{\pi'}$ between two distinct vertices v_1 and v_2 of T^π which does not share any edges in E (they can share edges in E_1 or E_2). Of course, because v_1 and v_2 are distinct in T^π, they are not in the same colored component of $(uTu^*)^{\pi'}$ when decomposed according to the traffic independence condition into a tree of colored component. It means that the γ_1', γ_2' and γ_3' go trough all colored component of $(uTu^*)^{\pi'}$ between v_1 and v_2, and in particular, there is a component of $(uTu^*)^{\pi'}$ colored by elements of $\mathcal{A}$ with three edge-disjoint simple paths β_1, β_2 and β_3 between two distinct vertices, and because of the first item, it leads also to $(\tilde\tau \star \tau_u)^0((uTu^*)^{\pi'}) = 0$. Finally, we have (4.9).

Thus we can assume that T^π is a cactus. For all $\pi' \in \mathcal{P}(V \sqcup V_1 \sqcup V_2)$ such that $(\pi' \vee \sigma)_{|V} = \pi$, the graph T^π is the graph $(uTu^*)^{\pi'}$ where all edges labelled by u or u^* are removed. So, if one of the cycle of T^π is unoriented, it comes from an unoriented cycle of $(uTu^*)^{\pi'}$, which means that $(\tilde\tau \star \tau_u)^0((uTu^*)^{\pi'}) = 0$ because of the first item. By consequence, we can also assume that T^π is an oriented cactus.

Let us consider π' such that $(uTu^*)^{\pi'}$ is an oriented cactus. The computation of $(\pi' \vee \sigma)$ consists in contracting all edges labelled by u and u^*, or equivalently contracting every colored component indexed by u and u^* in one vertex. Because $(uTu^*)^{\pi'}$ is a cactus, this contraction doesn't change the cycles of $(uTu^*)^{\pi'}$ which are indexed by elements of $\mathcal{A}$. In other words, the cycles indexed by elements of $\mathcal{A}$ of $(uTu^*)^{\pi'}$ and $(uTu^*)^{\pi' \vee \sigma}$ are the same. But the cycles indexed by elements of $\mathcal{A}$ of $(uTu^*)^{\pi' \vee \sigma}$ are exactly those of $T^{(\pi' \vee \sigma)_{|V}}$. Finally, if $(\pi' \vee \sigma)_{|V} = \pi$, the cycles of $(uTu^*)^{\pi'}$ indexed by elements of $\mathcal{A}$ are those of T^π, and we have

$$(\tilde\tau \star \tau_u)^0((uTu^*)^{\pi'}) = \tilde\tau^0(T^\pi) \cdot \prod_{\substack{\text{component } c \text{ of } (uTu^*)^{\pi'} \\ \text{indexed by } u,u^*}} \tau_u^0(c) = \tau_\Phi^0(T^\pi) \cdot \prod_{\substack{\text{component } c \text{ of } (uTu^*)^{\pi'} \\ \text{indexed by } u,u^*}} \tau_u^0(c).$$

We are left to prove that

$$\sum_{\pi' \in S_\pi} \prod_{\substack{\text{component } c \text{ of } (uTu^*)^{\pi'} \\ \text{indexed by } u,u^*}} \tau_u^0(c) = 1,$$

where $S_\pi = \{\pi' \in \mathcal{P}(V \sqcup V_1 \sqcup V_2) : (\pi' \vee \sigma)_{|V} = \pi, (\tilde\tau \star \tau_u)^0((uTu^*)^{\pi'}) \neq 0\}$. Here is the good news: there exists a lower bound of S_π, that we will denote by Π, and which will allows us (with forthcoming justifications) to write:

$$\sum_{\pi' \in S_\pi} \prod_{\substack{\text{component } c \text{ of } (uTu^*)^{\pi'} \\ \text{indexed by } u,u^*}} \tau_u^0(c) = \prod_{\substack{\text{component } c \text{ of } (uTu^*)^{\Pi} \\ \text{indexed by } u,u^*}} \tau_u(c) = 1.$$

The partition $\Pi \in \mathcal{P}(V \sqcup V_1 \sqcup V_2)$ is the lower partition such that if e and e' are two consecutive edges of a cycle of T^π (oriented $\xrightarrow{e} \xrightarrow{e'}$), then the source of e' and the goal of e viewed as edges of uTu^* are in the same block. Remark that Π is not necessary a cactus, and consequently, not necessary in S_π. However, for all $\pi' \in S_\pi$, we have $\Pi \preceq \pi'$. Indeed, because we proved that the cycles of $(uTu^*)^{\pi'}$ indexed by $\mathcal{A}$ are those of T^π, the source of e' and the goal of e must be identified in π'.

Π consists in the cycles of T^π linked by some nontrivial components labelled by u and u^*. Of course, an identification of two vertices in two different colored component labelled by u and u^* would modify the traffic independence condition, and as a consequence, we know that every $\pi' \in S_\pi$ is obtained by a collection of separate identification in each u-colored component of Π which transform it into an oriented cactus. Adding all the other vanishing terms (the identifications which do not lead to a cactus), we see that

$$\sum_{\pi' \in S_\pi} \prod_{\substack{\text{component } c \text{ of } (uTu^*)^{\pi'} \\ \text{indexed by } u,u^*}} \tau_u^0(c) = \prod_{\substack{\text{component } c \text{ of } (uTu^*)^{\Pi} \\ \text{indexed by } u,u^*}} \sum_{c \preceq c'} \tau_u^0(c') = \prod_{\substack{\text{component } c \text{ of } (uTu^*)^{\Pi} \\ \text{indexed by } u,u^*}} \tau_u(c),$$

(where the sum $c \preceq c'$ is the sum over $c^{\pi'}$ with π' an identification of the vertices of c).

It suffices to conclude to prove that $\tau_u(c) = 1$ for every u-colored component c of $(uTu^*)^{\Pi}$. This fact comes from the particular structure of c: it is a graph whose edges are of the form $\xrightarrow{u}\xrightarrow{u^*}$. Indeed, each u^* in c comes from some local structure $\xrightarrow{u}\xrightarrow{e}\xrightarrow{u^*}$ of uTu^*, and if we consider the consecutive edge e' of e in the cycle of T^τ, we know that the source of e' and the goal of e are identified in $(uTu^*)^{\Pi}$, which means also that the source of u^* in $\xrightarrow{u}\xrightarrow{e}\xrightarrow{u^*}$ is identified with the goal of u in $\xrightarrow{u}\xrightarrow{e'}\xrightarrow{u^*}$, and leads to a local structure $\xrightarrow{u}\xrightarrow{u^*}$ in c (with no other identifications for the vertex in the middle of $\xrightarrow{u}\xrightarrow{u^*}$). Similarly, each u in c can be seen in a local structure $\xrightarrow{u}\xrightarrow{u^*}$ in c (with no other identifications for the vertex in the middle of $\xrightarrow{u}\xrightarrow{u^*}$).

Finally, every u-colored component c of $(uTu^*)^{\Pi}$ is composed of a graph whose edges are $\xrightarrow{u}\xrightarrow{u^*}$, and it is of public notoriety that it implies that $\tau_u(c) = 1$ (use once again Proposition 4.4 to replace each occurence of $\xrightarrow{u}\xrightarrow{u^*}$ by $\xrightarrow{u^*u}=\xrightarrow{1}$, and finally by $\cdot$, which leads to $\tau_u(c) = \tau_u(\cdot) = 1$). $\qquad\square$

This lemma allows us to conclude the proof, since

$$(\tilde{\tau} \star \tau_u)\big[uTu^*\big] = \sum_{\pi' \in \mathcal{P}(V \sqcup V_1 \sqcup V_2)} (\tilde{\tau} \star \tau_u)^0((uTu^*)^{\pi'})$$

$$= \sum_{\substack{\pi \in \mathcal{P}(V) \\ (\pi' \vee \sigma)_{|V} = \pi}} \sum_{\pi' \in \mathcal{P}(V \sqcup V_1 \sqcup V_2)} (\tilde{\tau} \star \tau_u)^0((uTu^*)^{\pi'}) = \sum_{\pi \in \mathcal{P}(V)} \tau_\Phi^0\big[T^\pi\big] = \tau_\Phi\big[T\big].$$

$\qquad\square$

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