BRAIDED C*-QUANTUM GROUPS

SUTANU ROY

ABSTRACT. We propose a general theory of braided quantum groups in the C*-algebraic framework. More precisely, we construct braided quantum groups using manageable braided multiplicative unitaries over a regular C*-quantum group. We show that braided C*-quantum groups are equivalent to C*-quantum groups with projection.

1. INTRODUCTION

Let H be a group and let p be an idempotent homomorphism H. This is equivalent to a split exact sequence of groups such that $H \cong K \ltimes G$ where $K = \ker(p)$ and $G = \mathrm{Im}(p)$. C*-quantum groups with projection is a quantum analogue of semidirect product of groups.

In a purely algebraic setting, quantum groups and Hopf algebras are (roughly) synonymous. In [10], Radford shows that Hopf algebras with projection correspond exactly to pairs consisting of a Hopf algebra A and a Hopf algebra in the monoidal category of A-Yetter-Drinfeld algebras.

The image of the projection is again a Hopf algebra A. The analogue of the kernel is a Yetter–Drinfeld algebra B over A. For instance, when $A = \mathbb{C}[\mathbb{Z}]$ then B is a A-Yetter-Drinfeld algebra if and only if B is a $\mathbb{Z}$-graded $\mathbb{Z}$-module. For two Yetter–Drinfeld algebras B_1 and B_2, the tensor product $B_1 \otimes B_2$ carries a unique multiplication for which it is again a Yetter–Drinfeld algebra; the Yetter–Drinfeld module structure is the diagonal one, which is determined by requiring the embeddings of B_1 and B_2 to be equivariant. The comultiplication on B is a homomorphism to the deformed tensor product $B \boxtimes B$, which turns B into a Hopf algebra in the monoidal category of Yetter–Drinfeld algebras.

This suggests that a braided C*-quantum group over a C*-quantum group $\mathbb{G} = (A, \Delta_A)$ should be a pair (B, Δ_B) consisting of a $\mathbb{G}$-Yetter-Drinfeld C*-algebra B and a nondegenerate *-homomorphism $\Delta_B\colon B \to \mathcal{M}(B \boxtimes B)$ respecting the $\mathbb{G}$-Yetter-Drinfeld structure. This has been studied in [7, Section 6] when A and B both are unital. In the nonunital case, we need to generalise the concept of multiplicative uniatries.

Let $\mathcal{H}$ be a separable Hilbert space. A unitary operator $\mathbb{W}\colon \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is *multiplicative* if it satisfies the *pentagon equation*

$$\mathbb{W}_{23}\mathbb{W}_{12} = \mathbb{W}_{12}\mathbb{W}_{13}\mathbb{W}_{23} \qquad \text{in } \mathcal{U}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}). \tag{1.1}$$

In [1], Baaj and Skandalis used *regularity* of multiplicative unitaries as a basic axiom to construct locally compact quantum groups in C*-algebraic framework. The notion of manageability of multiplicative unitaries, introduced by Woronowicz in [16], provides a more general approach to the C*-algebraic theory for locally compact quantum groups or, in short, C*-quantum groups (see Theorem 2.2).

2010 *Mathematics Subject Classification.* Primary 81R50, 46L89; Secondary 18D10, 46L55.

Key words and phrases. braided C*-quantum group, manageable multiplicative unitary, braided multiplicative unitary, quantum group with projection.

The author was supported by a Fields–Ontario postdoctoral fellowship.

Motivated by [1], Bücher and the author, in [2], presented a general theory of regular braided quantum groups in the C*-algebraic framework using regular braided multiplicative unitaries.

Let $\mathcal{C}$ be a braided monoidal category of separable Hilbert spaces. Thus, for any two objects $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{C}$ there is a bounded operator ${}^{\mathcal{L}_1}\times^{\mathcal{L}_2} \colon \mathcal{L}_1 \otimes \mathcal{L}_2 \to \mathcal{L}_2 \otimes \mathcal{L}_1$ that braid diagrams. Assume that ${}^{\mathcal{L}_1}\times^{\mathcal{L}_2}$ is unitary for all objects $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{C}$. For an object $\mathcal{L} \in \mathcal{C}$, a *braided multiplicative unitary* on should be unitary morphism $\mathbb{F}\colon \mathcal{L} \otimes \mathcal{L} \to \mathcal{L} \otimes \mathcal{L}$ in $\mathcal{C}$ that satisfies a braided version of (1.1) (see (3.7)).

Our setting is the following: we set $\mathcal{C}$ to be the corepresentation category of quantum codouble of a C*-quantum group $\mathbb{G}$ see [7, Proposition 3.4 & Section 5]. Then a braided multiplicative unitary $\mathbb{F}$ over $\mathbb{G}$ is a morphism in this category satisfying braided pentagon equation (see Definition 3.4). Next we recall the notion of manageability from [8].

The goal of this article is to construct braided C*-quantum groups (as outlined in the fourth paragraph above) from manageable braided multiplicative unitaries over a regular C*-quantum group $\mathbb{G}$.

Unlike nonbraided case, it is not even clear whether the set B_0 of slices $(\omega \otimes \mathrm{id}_{\mathcal{L}})\mathbb{F}$ for $\omega \in \mathbb{B}(\mathcal{L})_*$ forms an algebra. In [2, Proposition 5], it was shown that B_0 is an algebra whenever $\mathcal{C}$ is a regularly braided monoidal category: the braiding operator on $\mathcal{C}$ is regular in the sense of [2, Definition 3]. Furthemore, regularity condition on $\mathbb{F}$ ensures that $B = B_0^{-\|\cdot\|} \subset \mathbb{B}(\mathcal{L})$ is a C*-algebra and B admits a structure of a regular braided C*-quantum group see [2, Theorem 13]. Because of [2, Proposition 16] the monoidal category $\mathcal{C}$ is regularly braided.

It is shown in [11] and [8], that Radford's theorem can be generalised nicely for manageable multiplicative unitaries. Thus shows that a braided C*-quantum group (B, Δ_B) over a C*-quantum group $\mathbb{G}$ gives rise to a C*-quantum group $\mathbb{H} = (C, \Delta_C)$ with projection. As shown in [3], for the von Neumann algebraic quantum groups, the analogue of the B coincides with the algebra fixed points for the canonical coaction of $\mathbb{G}$ on $\mathbb{H}$ induced by the projection on $\mathbb{H}$. This is not the case for B. As a C*-algebra, B should be the generalised fixed point algebra. This is a special case of quantum homogeneous spaces, which is also treated by Vaes [15] that needs regularity assumptions on $\mathbb{G}$.

Therefore, it seems that the regularity of $\mathbb{G}$ turns out to be a natural assumption to construct braided C*-quantum groups from braided multiplicative unitaries.

Let us briefly outline the structure of this article. In Section 2, we recall basic necessary preliminaries. In particular, the main results on modular and manageable multiplicative unitaries, that give rise to C*-quantum groups [16], the notion of Heisenberg and anti-Heisenbegr pairs for C*-quantum groups from [6], coactions and corepresentations of C*-quantum groups, results related to Yetter-Drinfeld C*-algebras from [7]. In Section 2.5 we gather some important facts of regular C*-quantum groups. After introducing manageable braided multiplicative unitaries we state the main result (see Theorem 3.9) to construct braided C*-quantum groups in Section 3. We also construct the big C*-quantum group $\mathbb{H}$ in terms of a braided C*-quantum group (B, Δ_B) over a regular $\mathbb{G}$. In Section 4, we use the quantum version of the Landstad theorem to construct B the fixed point algebra for the action of $\mathbb{G}$ on $\mathbb{H}$ induced by the projection. Finally, in Section 5 we complete the proof of Theorem 3.9.

2. Preliminaries

All Hilbert spaces and C*-algebras (which are not explicitly multiplier algebras) are assumed to be separable. For a C*-algebra A, let $\mathcal{M}(A)$ be its multiplier algebra and let $\mathcal{U}(A)$ be the group of unitary multipliers of A. For two norm closed subsets X and Y of a C*-algebra A and $T \in \mathcal{M}(A)$, let

$$XTY := \{xTy : x \in X, y \in T\}^{\mathrm{CLS}}$$

where CLS stands for the *closed linear span*.

Let $\mathfrak{C}^*\mathfrak{alg}$ be the category of C*-algebras with nondegenerate *-homomorphisms $\varphi\colon A \to \mathcal{M}(B)$ as morphisms $A \to B$; let $\mathrm{Mor}(A, B)$ denote this set of morphisms.

Let $\mathcal{H}$ be a Hilbert space. A *representation* of a C*-algebra A is a nondegenerate *-homomorphism $A \to \mathbb{B}(\mathcal{H})$. Since $\mathbb{B}(\mathcal{H}) = \mathcal{M}(\mathbb{K}(\mathcal{H}))$ and the nondegeneracy conditions $A\mathbb{K}(\mathcal{H}) = \mathbb{K}(\mathcal{H})$ and $A\mathcal{H} = \mathcal{H}$ are equivalent, this is the same as a morphism from A to $\mathbb{K}(\mathcal{H})$.

We write Σ for the tensor flip $\mathcal{H} \otimes \mathcal{K} \to \mathcal{K} \otimes \mathcal{H}$, $x \otimes y \mapsto y \otimes x$, for two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. We write σ for the tensor flip isomorphism $A \otimes B \to B \otimes A$ for two C*-algebras A and B.

2.1. **Multiplicative unitaries and quantum groups.** Let $\mathcal{H}$ be a Hilbert space. A multiplicative unitary $\mathbb{W} \in \mathcal{U}(\mathcal{H}\otimes\mathcal{H})$ is *manageable* if there are a strictly positive operator Q on $\mathcal{H}$ and a unitary $\widetilde{\mathbb{W}} \in \mathcal{U}(\overline{\mathcal{H}} \otimes \mathcal{H})$ with $\mathbb{W}^*(Q \otimes Q)\mathbb{W} = Q \otimes Q$ and

$$\big(x \otimes u \mid \mathbb{W} \mid z \otimes y\big) = \big(\overline{z} \otimes Qu \mid \widetilde{\mathbb{W}} \mid \overline{x} \otimes Q^{-1}y\big) \tag{2.1}$$

for all $x, z \in \mathcal{H}$, $u \in \mathcal{D}(Q)$ and $y \in \mathcal{D}(Q^{-1})$ (see [16, Definition 1.2]). Here $\overline{\mathcal{H}}$ is the conjugate Hilbert space, and an operator is *strictly positive* if it is positive and self-adjoint with trivial kernel. The condition $\mathbb{W}^*(Q \otimes Q)\mathbb{W} = Q \otimes Q$ means that the unitary $\mathbb{W}$ commutes with the unbounded operator $Q \otimes Q$.

Theorem 2.2 ([14,16])**.** *Let $\mathcal{H}$ be a separable Hilbert space and $\mathbb{W} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$ a manageable multiplicative unitary. Let*

$$A := \{(\omega \otimes \mathrm{id}_{\mathcal{H}})\mathbb{W} : \omega \in \mathbb{B}(\mathcal{H})_*\}^{\mathrm{CLS}}, \tag{2.3}$$

$$\hat{A} := \{(\mathrm{id}_{\mathcal{H}} \otimes \omega)\mathbb{W} : \omega \in \mathbb{B}(\mathcal{H})_*\}^{\mathrm{CLS}}. \tag{2.4}$$

(1) *A and $\hat{A}$ are separable, nondegenerate* C*-*subalgebras of* $\mathbb{B}(\mathcal{H})$.
(2) $\mathbb{W} \in \mathcal{U}(\hat{A} \otimes A) \subseteq \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$. *We write* W^A *for* $\mathbb{W}$ *viewed as a unitary multiplier of* $\hat{A} \otimes A$ *and call it* reduced bicharacter*.*
(3) *The map $\Delta_A(a) := \mathbb{W}(a \otimes 1_{\mathcal{H}})\mathbb{W}^*$ defines a unique morphism $A \to A \otimes A$ satisfying*

$$(\mathrm{id}_{\hat{A}} \otimes \Delta_A)\mathrm{W}^A = \mathrm{W}^A_{12}\mathrm{W}^A_{13} \qquad \text{in } \mathcal{U}(\hat{A} \otimes A \otimes A). \tag{2.5}$$

Moreover, Δ_A is coassociative*:*

$$(\Delta_A \otimes \mathrm{id}_A)\Delta_A = (\mathrm{id}_A \otimes \Delta_A)\Delta_A, \tag{2.6}$$

and satisfies the cancellation conditions*:*

$$\Delta_A(A)(1_A \otimes A) = A \otimes A = (A \otimes 1_A)\Delta_A(A). \tag{2.7}$$

(4) *There is a unique ultraweakly continuous, linear anti-automorphism* R_A *of A with*

$$\Delta_A \mathrm{R}_A = \sigma(\mathrm{R}_A \otimes \mathrm{R}_A)\Delta_A, \tag{2.8}$$

where $\sigma(x \otimes y) = y \otimes x$. It satisfies $\mathrm{R}_A^2 = \mathrm{id}_A$.

A C*-*quantum group* $\mathbb{G}$ is a pair (C, Δ_C) consisting of a C*-algebra C and an element $\Delta_C \in \mathrm{Mor}(C, C \otimes C)$ constructed from a modular or managebale multiplicative unitary $\mathbb{W}$. Then we say $\mathbb{G} = (C, \Delta_C)$ is generated by $\mathbb{W}$. We do not need Haar weights.

The *dual* multiplicative unitary is $\widehat{\mathbb{W}} := \Sigma\mathbb{W}^*\Sigma \in \mathcal{U}(\mathcal{H}\otimes\mathcal{H})$, where $\Sigma(x\otimes y) = y\otimes x$. It is modular or manageable if $\mathbb{W}$ is. The C^*-quantum group $\widehat{\mathbb{G}} = (\hat{A}, \hat{\Delta}_A)$ generated by $\widehat{\mathbb{W}}$ is the *dual* of $\mathbb{G}$. Its comultiplication is characterised by

$$(\hat{\Delta}_A \otimes \mathrm{id}_A)\mathrm{W}^A = \mathrm{W}^A_{23}\mathrm{W}^A_{13} \qquad \text{in } \mathcal{U}(\hat{A}\otimes\hat{A}\otimes A). \tag{2.9}$$

2.2. **Heisenberg pairs.** Let $\mathbb{G} = (A, \Delta_A)$ be a C^*-quantum group. Let $\widehat{\mathbb{G}} = (\hat{A}, \hat{\Delta}_A)$ be its dual, and $\mathrm{W}^A \in \mathcal{U}(\hat{A}\otimes A)$ be the reduced bicharacter.

A pair of representations $(\pi, \hat{\pi})$ of A and $\hat{A}$ on a Hilbert space $\mathcal{H}$ is a $\mathbb{G}$-*Heisenberg pair* if and only if

$$\mathrm{W}^A_{\hat{\pi}3}\mathrm{W}^A_{1\pi} = \mathrm{W}^A_{1\pi}\mathrm{W}^A_{13}\mathrm{W}^A_{\hat{\pi}3} \qquad \text{in } \mathcal{U}(\hat{A}\otimes\mathbb{K}(\mathcal{H})\otimes A). \tag{2.10}$$

Here $\mathrm{W}^A_{1\pi} := ((\mathrm{id}_{\hat{A}}\otimes\pi)\mathrm{W}^A)_{12}$ and $\mathrm{W}^A_{\hat{\pi}3} := ((\hat{\pi}\otimes\mathrm{id}_A)\mathrm{W}^A)_{23}$. Theorem 2.2 ensures the existance of a faithful $\mathbb{G}$-Heisenberg pairs and [12, Proposition 3.2] shows that any $\mathbb{G}$-Heisenberg pair is faithful.

Similarly, a pair of representations $(\rho, \hat{\rho})$ of A and $\hat{A}$ on $\mathcal{H}$ is a $\mathbb{G}$-*anti-Heisenberg pair* on $\mathcal{H}$ if and only if

$$\mathrm{W}^A_{1\rho}\mathrm{W}^A_{\hat{\rho}3} = \mathrm{W}^A_{\hat{\rho}3}\mathrm{W}^A_{13}\mathrm{W}^A_{1\rho} \qquad \text{in } \mathcal{U}(\hat{A}\otimes\mathbb{K}(\mathcal{H})\otimes A). \tag{2.11}$$

Let $\overline{\mathcal{H}}$ be the conjugate Hilbert space to the Hilbert space $\mathcal{H}$. The *transpose* of an operator $x \in \mathbb{B}(\mathcal{H})$ is the operator $x^{\mathsf{T}} \in \mathbb{B}(\overline{\mathcal{H}})$ defined by $x^{\mathsf{T}}(\overline{\xi}) := \overline{x^*\xi}$ for all $\xi \in \mathcal{H}$. The transposition is a linear, involutive anti-automorphism $\mathbb{B}(\mathcal{H}) \to \mathbb{B}(\overline{\mathcal{H}})$. Let R_A and $\hat{\mathrm{R}}_A$ be the unitary antipodes of $\mathbb{G}$ and $\hat{\mathbb{G}}$, respectively. A pair of representations $(\pi, \hat{\pi})$ of A and $\hat{A}$ on $\mathcal{H}$ is a $\mathbb{G}$-Heisenberg pair if and only if the the pair of representations $(\rho, \hat{\rho})$ of A and $\hat{A}$ on $\overline{\mathcal{H}}$, defined by

$$\rho(a) := (\mathrm{R}_A(a))^{\mathsf{T}}, \qquad \hat{\rho}(\hat{a}) := (\hat{\mathrm{R}}_A(\hat{a}))^{\mathsf{T}}, \tag{2.12}$$

is a $\mathbb{G}$-anti-Heisenberg pair on $\overline{\mathcal{H}}$ (see [6, Lemma 3.4]). This shows that the set of $\mathbb{G}$-Heisenberg pairs and $\mathbb{G}$-anti-Heisenberg pairs are in bijective correspondance.

2.3. **Corepresentations.**

Definition 2.13. A (right) *corepresentation* of $\mathbb{G}$ on a Hilbert space $\mathcal{L}$ is a unitary $\mathrm{U} \in \mathcal{U}(\mathbb{K}(\mathcal{L})\otimes A)$ with

$$(\mathrm{id}_{\mathbb{K}(\mathcal{L})}\otimes\Delta_A)\mathrm{U} = \mathrm{U}_{12}\mathrm{U}_{13} \qquad \text{in } \mathcal{U}(\mathbb{K}(\mathcal{L})\otimes A\otimes A). \tag{2.14}$$

Let $\mathrm{U}^1 \in \mathcal{U}(\mathbb{K}(\mathcal{L}_1)\otimes A)$ and $\mathrm{U}^2 \in \mathcal{U}(\mathbb{K}(\mathcal{L}_2)\otimes A)$ be corepresentations of $\mathbb{G}$. An element $t \in \mathbb{B}(\mathcal{L}_1, \mathcal{L}_2)$ is called an *intertwiner* if $(t\otimes 1_A)\mathrm{U}^1 = \mathrm{U}^2(t\otimes 1_A)$. The set of all intertwiners between U^1 and U^2 is denoted $\mathrm{Hom}(\mathrm{U}^1, \mathrm{U}^2)$. This gives corepresentations a structure of W^*-category (see [14, Sections 3.1–2]).

The *tensor product* of two corepresentations $\mathrm{U}^{\mathcal{L}_1}$ and $\mathrm{U}^{\mathcal{L}_2}$ is defined by

$$\mathrm{U}^1 \mathbin{\bigcirc\!\!\!\!\top}\, \mathrm{U}^2 := \mathrm{U}^1_{13}\mathrm{U}^2_{23} \qquad \text{in } \mathcal{U}(\mathbb{K}(\mathcal{L}_1\otimes\mathcal{L}_2)\otimes A). \tag{2.15}$$

Routine computations show the following: $\mathrm{U}^1 \mathbin{\bigcirc\!\!\!\!\top}\, \mathrm{U}^2$ is a corepresentation; $\mathbin{\bigcirc\!\!\!\!\top}$ is associative; and the trivial 1-dimensional representation is a tensor unit. Thus corepresentations form a monoidal W^*-category, which we denote by $\mathfrak{Corep}(\mathbb{G})$; see [14, Section 3.3] for more details.

2.4. **Coactions.**

Definition 2.16. A *continuous* (*right*) *coaction* of $\mathbb{G}$ on a C^*-algebra C is a morphism $\gamma\colon C \to C\otimes A$ with the following properties:

(1) γ is injective;

(2) γ is a comodule structure, that is,

$$(\mathrm{id}_C\otimes\Delta_A)\gamma = (\gamma\otimes\mathrm{id}_A)\gamma; \tag{2.17}$$

(3) γ satisfies the *Podleś condition*:

$$\gamma(C)(1_C \otimes A) = C \otimes A. \tag{2.18}$$

We call (C,γ) a $\mathbb{G}$-C*-*algebra*. We often drop γ from our notation.

Similarly, a *left coaction* of $\mathbb{G}$ on C is an injective morphism $\gamma\colon C \to A \otimes C$ satisfying a variant of (2.17) and the Podleś condition (2.18).

In this article the we reserve the word “coaction" for right coaction.

A morphism $f\colon C \to D$ between two $\mathbb{G}$-C*-algebras (C,γ) and (D,δ) is $\mathbb{G}$-*equivariant* if $\delta \circ f = (f \otimes \mathrm{id}_A) \circ \gamma$. Let $\mathrm{Mor}^{\mathbb{G}}(C,D)$ be the set of $\mathbb{G}$-equivariant morphisms from C to D. Let $\mathfrak{C}^*\mathfrak{alg}(\mathbb{G})$ be the category with $\mathbb{G}$-C*-algebras as objects and $\mathbb{G}$-equivariant morphisms as arrows.

Definition 2.19. A *covariant representation* of (C,γ,A) on a Hilbert space $\mathcal{H}$ is a pair (U,φ) consisting of a corepresentation $\mathrm{U} \in \mathcal{U}(\mathbb{K}(\mathcal{H}) \otimes A)$ and a representation $\varphi\colon C \to \mathbb{B}(\mathcal{H})$ that satisfy the covariance condition

$$(\varphi \otimes \mathrm{id}_A) \circ \gamma(c) = \mathrm{U}(\varphi(c) \otimes 1_A)\mathrm{U}^* \qquad \text{in } \mathcal{U}(\mathbb{K}(\mathcal{H}) \otimes A) \tag{2.20}$$

for all $c \in C$. A covariant representation is called *faithful* if φ is faithful.

Faithful covariant representations always exist by [6, Example 4.5].

2.5. **Regularity for quantum groups and corepresentations.** Let $\mathbb{G} = (A,\Delta_A)$ be the C*-quantum group generated by a manageable multiplictive unitary $\mathbb{W}$. Let $\widehat{\mathbb{G}} = (\hat{A},\hat{\Delta}_A)$ be its dual, and let $\mathrm{W}^A \in \mathcal{U}(\hat{A} \otimes A)$ be the reduced bicharacter.

Define

$$\mathcal{C} := \{(\mathrm{id}_{\mathcal{H}} \otimes \omega)(\Sigma\mathbb{W}) \mid \omega \in \mathbb{B}(\mathcal{H})_*\}^{\mathrm{CLS}}.$$

The multiplicative unitary $\mathbb{W} \in \mathcal{U}(\mathcal{H}\otimes\mathcal{H})$ is *regular* if $\mathcal{C} = \mathbb{K}(\mathcal{H})$, see [1, Definition 3.3]. By virtue of [1, Proposition 3.2 (b) & Proposition 3.6], this is equivalent to

$$(\hat{A} \otimes 1_A)\mathrm{W}^A(1_{\hat{A}} \otimes A) = \hat{A} \otimes A. \tag{2.21}$$

Now W^A does not depend on the multiplicative unitary generating $\mathbb{G}$, see [14, Theorem 5(3)]. Therefore, regularity is a property of the the quantum group $\mathbb{G}$ and not of a particular multiplicative unitary $\mathbb{W}$ used to construct it.

Moreover, [1, Proposition A.3] shows that the regularity property of $\mathbb{G}$ passes to its corepresentations. More precisely, if $\mathbb{G}$ is regular then every corepresentation $\mathrm{U} \in \mathcal{U}(\mathbb{K}(\mathcal{L}) \otimes A)$ of $\mathbb{G}$ is also regular in the following sense:

$$(\mathbb{K}(\mathcal{L}) \otimes \mathrm{id}_A)\mathrm{U}(1_{\mathbb{K}(\mathcal{L})} \otimes A) = \mathbb{K}(\mathcal{L}) \otimes A. \tag{2.22}$$

We claim that Equation (2.22) is equivalent to

$$(1_{\mathbb{K}(\mathcal{L})} \otimes A)\mathrm{U}(\mathbb{K}(\mathcal{L}) \otimes \mathrm{id}_A) = \mathbb{K}(\mathcal{L}) \otimes A. \tag{2.23}$$

The contragradient of a corepresentation $\mathrm{U} \in \mathcal{U}(\mathbb{K}(\mathcal{L}) \otimes A)$ is defined by $\mathrm{U}^c := \mathrm{U}^{\mathsf{T}\otimes\mathrm{R}_A} \in \mathcal{U}(\mathbb{K}(\overline{\mathcal{L}})\otimes A)$, see [14, Proposition 10]. Here a^{R_A} denotes $\mathrm{R}_A(a)$ for $a \in A$. Regularity of $\mathbb{G}$ implies

$$(\mathbb{K}(\overline{\mathcal{L}}) \otimes \mathrm{id}_A)\mathrm{U}^c(1_{\mathbb{K}(\overline{\mathcal{L}})} \otimes A) = \mathbb{K}(\overline{\mathcal{L}}) \otimes A.$$

Since, $\mathsf{T} \otimes \mathrm{R}_A\colon \mathbb{K}(\overline{\mathcal{L}}) \otimes A \to \mathbb{K}(\mathcal{L}) \otimes A$ is an anti-multiplicative involution, it maps the last Equation to (2.23).

A similar argument replacing the transpose by the unitary antipode $\hat{\mathrm{R}}_A$ and using $(\hat{\mathrm{R}}_A \otimes \mathrm{R}_A)\mathrm{W}^A = \mathrm{W}^A$ (see [14, Lemma 40]), shows that Equation (2.21) is equivalent to

$$(1_{\hat{A}} \otimes A)\mathrm{W}^A(\hat{A} \otimes 1_A) = \hat{A} \otimes A. \tag{2.24}$$

The dual of a regular quantum group is again regular. Therefore, (2.24) is also equivalent to

$$(1_A \otimes \hat{A})\widehat{\mathrm{W}}^A(A \otimes 1_{\hat{A}}) = A \otimes \hat{A}. \tag{2.25}$$

2.6. **Twisted tensor products of Yetter-Drinfeld C*-algebras.**

Definition 2.26 ([9, Definition 3.1])**.** A $\mathbb{G}$-*Yetter-Drinfeld* C^*-algebra is a triple $(C, \gamma, \widehat{\gamma})$ consisting of a C^*-algebra C along with coactions $\gamma\colon C \to C \otimes A$ and $\widehat{\gamma}\colon C \to C \otimes \hat{A}$ of $\mathbb{G}$ and $\hat{\mathbb{G}}$ that satisfy the *Yetter-Drinfeld* compatibility condition

$$(\widehat{\gamma} \otimes \mathrm{id}_A)\gamma(c) = (\mathrm{W}^A_{23})\sigma_{23}\Big((\gamma \otimes \mathrm{id}_{\hat{A}})\widehat{\gamma}(c)\Big)(\mathrm{W}^A_{23})^* \qquad \text{for all } c \in C. \tag{2.27}$$

Example 2.28. Let $\mathbb{G} = (A, \Delta_A)$ be a regular C^*-quantum group. Then $\theta\colon A \to A \otimes \hat{A}$ define by $\theta(a) := \sigma(\mathrm{W}^*(1_{\hat{A}} \otimes a)\mathrm{W})$ for $a \in A$ is a coaction of $\hat{\mathbb{G}}$ on A, and (A, Δ_A, θ) is a $\mathbb{G}$-Yetter-Drinfeld C^*-algebra (see [9, Section 3]).

Let $\mathcal{YDC}^*\mathfrak{alg}(\mathbb{G})$ be the category with $\mathbb{G}$-Yetter-Drinfeld C^*-algebras as objects and $\mathbb{G}$- and $\hat{\mathbb{G}}$-equivariant morphisms as arrows.

Next we briefly recall the monodial structure on $\mathcal{YDC}^*\mathfrak{alg}(\mathbb{G})$.

Let $\mathrm{U}^1 \in \mathcal{U}(\mathbb{K}(\mathcal{L}_1) \otimes A)$ and $\mathrm{V}^2 \in \mathcal{U}(\mathbb{K}(\mathcal{L}_2) \otimes \hat{A})$ be corepresentations of $\mathbb{G}$ and $\hat{\mathbb{G}}$ on $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. The proof of [6, Theorem 4.1] shows that there exists a unique $Z \in \mathcal{U}(\mathcal{L}_1 \otimes \mathcal{L}_2)$ such that

$$\mathrm{U}^1_{1\pi}\mathrm{V}^2_{2\hat{\pi}}Z_{12} = \mathrm{V}^2_{2\hat{\pi}}\mathrm{U}^1_{1\pi} \qquad \text{in } \mathcal{U}(\mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \mathcal{H}) \tag{2.29}$$

for any $\mathbb{G}$-Heisenberg pair $(\pi, \hat{\pi})$ on $\mathcal{H}$. Define ${}^{\mathcal{L}_2}\!\times^{\mathcal{L}_1}\colon \mathcal{L}_2 \otimes \mathcal{L}_1 \to \mathcal{L}_1 \otimes \mathcal{L}_2$ by ${}^{\mathcal{L}_2}\!\times^{\mathcal{L}_1} := Z \circ \Sigma$, and ${}^{\mathcal{L}_1}\!\times^{\mathcal{L}_2}\colon \mathcal{L}_1 \otimes \mathcal{L}_2 \to \mathcal{L}_2 \otimes \mathcal{L}_1$ by ${}^{\mathcal{L}_1}\!\times^{\mathcal{L}_2} := \Sigma \circ Z^*$.

Let $(C_1, \gamma_1, \widehat{\gamma}_1)$ and $(C_2, \gamma_2, \widehat{\gamma}_2)$ be $\mathbb{G}$-Yetter-Drinfeld C^*-algebras. Without loss of generality, assume that $(\mathrm{U}^i, \varphi_i)$ be a faithful covariant representation of (C_i, γ_i) on $\mathcal{L}_i$ and $(\mathrm{V}^i, \widehat{\varphi}_i)$ be a faithful covariant representation of $(C_i, \widehat{\gamma}_i)$ for $i = 1, 2$, respectively.

Define faithful representations of j_1 and j_2 of C_1 and C_2 on $\mathcal{L}_1 \otimes \mathcal{L}_2$ by

$$j_1(c_1) := \varphi_1(c_1) \otimes 1_{\mathcal{L}_2}, \qquad j_2(c_2) := {}^{\mathcal{L}_2}\!\times^{\mathcal{L}_1}(\varphi_2(c_2) \otimes 1_{\mathcal{L}_1})^{\mathcal{L}_1}\!\times^{\mathcal{L}_2} \tag{2.30}$$

Theorem 2.31 ([6, Lemma 3.20, Theorem 4.3, Theorem 4.9])**.** *The subspace*

$$C_1 \boxtimes C_2 := j_1(C_1)j_2(C_2) \subset \mathbb{B}(\mathcal{L}_1 \otimes \mathcal{L}_2)$$

is a nondegenerate C^**-subalgebra. The crossed product* $(C_1 \boxtimes C_2, j_1, j_2)$*, up to equivalence, does not depend on the faithful covariant representations* $(\mathrm{U}^i, \varphi_i)$ *and* $(\mathrm{V}^i, \varphi_i)$ *for* $i = 1, 2$*.*

We call $C_1 \boxtimes C_2$ the *twisted tensor product* of C_1 and C_2.

The twisted tensor product $C_1 \boxtimes C_2$ carries diagonal coactions of $\mathbb{G}$ and $\hat{\mathbb{G}}$ defined by

$$\gamma_1 \bowtie \gamma_2\colon C_1 \boxtimes C_2 \to C_1 \boxtimes C_2 \otimes A, \qquad x \mapsto (\mathrm{U}^1 \mathbin{\bigcirc\!\!\!\!\top} \mathrm{U}^2)(x \otimes 1_A)(\mathrm{U}^1 \mathbin{\bigcirc\!\!\!\!\top} \mathrm{U}^2)^*, \tag{2.32}$$

$$\widehat{\gamma}_1 \bowtie \widehat{\gamma}_2\colon C_1 \boxtimes C_2 \to C_1 \boxtimes C_2 \otimes \hat{A}, \qquad x \mapsto (\mathrm{V}^1 \mathbin{\bigcirc\!\!\!\!\top} \mathrm{V}^2)(x \otimes 1_{\hat{A}})(\mathrm{V}^1 \mathbin{\bigcirc\!\!\!\!\top} \mathrm{V}^2)^*. \tag{2.33}$$

Then $(C_1 \boxtimes C_2, \gamma_1 \bowtie \gamma_2, \widehat{\gamma}_1 \bowtie \widehat{\gamma}_2)$ is again a $\mathbb{G}$-Yetter-Drinfeld C^*-algebra.

Theorem 2.34. $(\mathcal{YDC}^*\mathfrak{alg}(\mathbb{G}), \boxtimes)$ *is a monoidal category.*

This theorem has been proved in [9, Section 3] in the presence of Haar weights on $\mathbb{G}$ and in [7, Section 5] in the general framework of modular multiplicative unitaries.

3. Manageable braided multiplicative unitaries

Let $\mathbb{G} = (A, \Delta_A)$ be a C*-quantum group. Let $\widehat{\mathbb{G}} = (\hat{A}, \hat{\Delta}_A)$ be its dual and $\mathrm{W} \in \mathcal{U}(\hat{A} \otimes A)$ be the reduced bicharacter.

The *quantum codouble* $\mathfrak{D}(\mathbb{G})^\frown = (\hat{\mathcal{D}}, \Delta_{\hat{\mathcal{D}}})$ of $\mathbb{G}$ is defined by $\hat{\mathcal{D}} := A \otimes \hat{A}$ and

$$\sigma^{\mathrm{W}} : A \otimes \hat{A} \to \hat{A} \otimes A, \qquad a \otimes \hat{a} \mapsto \mathrm{W}(\hat{a} \otimes a)\mathrm{W}^*,$$
$$\Delta_{\hat{\mathcal{D}}} : \hat{\mathcal{D}} \to \hat{\mathcal{D}} \otimes \hat{\mathcal{D}}, \qquad a \otimes \hat{a} \mapsto \sigma^{\mathrm{W}}_{23}(\Delta_A(a) \otimes \hat{\Delta}_A(\hat{a})),$$

for $a \in A$, $\hat{a} \in \hat{A}$. We may generate $\mathfrak{D}(\mathbb{G})^\frown$ by a manageable multiplicative unitary by [12, Theorem 4.1]. So it is a C*-quantum group.

Let $\mathcal{L}$ be a Hilbert space. A pair of corepresentations (U, V) of $\mathbb{G}$ and $\hat{\mathbb{G}}$ on $\mathcal{L}$ is corepresentations is called $\mathfrak{D}(\mathbb{G})^\frown$*-compatible* if they satisfy the following *Drinfeld compatibility* condition:

$$\mathrm{V}_{12}\mathrm{U}_{13}\mathrm{W}_{23} = \mathrm{W}_{23}\mathrm{U}_{13}\mathrm{V}_{12} \quad \text{in } \mathcal{U}(\mathbb{K}(\mathcal{L}) \otimes \hat{A} \otimes A), \tag{3.1}$$

Let $(\pi, \hat{\pi})$ be the $\mathbb{G}$-Heisenberg pair on $\mathcal{H}$ associated to the manageable multiplicative unitary $\mathbb{W} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$, that is, $(\hat{\pi} \otimes \pi)\mathrm{W}^A = \mathbb{W}$. Define $\hat{\mathrm{V}} \in \mathcal{U}(\hat{A} \otimes \mathbb{K}(\mathcal{L}))$, $\mathbb{U}, \mathbb{V} \in \mathcal{U}(\mathcal{L} \otimes \mathcal{H})$ and $\hat{\mathbb{V}} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{L})$ by

$$\hat{\mathrm{V}} := \sigma(\mathrm{V}^*), \quad \mathbb{U} := (\mathrm{id}_{\mathcal{L}} \otimes \pi)\mathrm{U}, \quad \mathbb{V} := (\mathrm{id}_{\mathcal{L}} \otimes \hat{\pi})\mathrm{V}, \quad \hat{\mathbb{V}} := \Sigma\mathbb{V}^*\Sigma = (\hat{\pi} \otimes \mathrm{id}_{\mathcal{L}})\hat{\mathrm{V}}.$$

Then the Equations (2.29) and (3.1) for U and V are equivalent to

$$Z_{13} = \hat{\mathbb{V}}_{23}\mathbb{U}^*_{12}\hat{\mathbb{V}}^*_{23}\mathbb{U}_{12} \quad \text{in } \mathcal{U}(\mathcal{L} \otimes \mathcal{H} \otimes \mathcal{L}); \tag{3.2}$$
$$\mathbb{U}_{23}\mathbb{W}_{13}\hat{\mathbb{V}}_{12} = \hat{\mathbb{V}}_{12}\mathbb{W}_{13}\mathbb{U}_{23} \quad \text{in } \mathcal{U}(\mathcal{H} \otimes \mathcal{L} \otimes \mathcal{H}). \tag{3.3}$$

As proved in [7, Theorem 5.4], for any $\mathfrak{D}(\mathbb{G})^\frown$-pair (U, V) on $\mathcal{L}$ the unitary ${}^{\mathcal{L}}\!\times^{\mathcal{L}} := Z \circ \Sigma$ is a braiding.

Definition 3.4. Let (U, V) be a $\mathfrak{D}(\mathbb{G})^\frown$-compatible corepresentation on a Hilbert space $\mathcal{L}$. A *braided multiplicative unitary on* $\mathcal{L}$ *over* $\mathbb{G}$ *relative to* (U, V) is a unitary $\mathbb{F} \in \mathcal{U}(\mathcal{L} \otimes \mathcal{L})$ with the following properties:

(1) $\mathbb{F}$ is *invariant* with respect to the right corepresentation $\mathrm{U} \mathbin{\bigcirc\!\!\!\!\top} \mathrm{U} := \mathrm{U}_{13}\mathrm{U}_{23}$ of $\mathbb{G}$ on $\mathcal{L} \otimes \mathcal{L}$:

$$\mathrm{U}_{13}\mathrm{U}_{23}\mathbb{F}_{12} = \mathbb{F}_{12}\mathrm{U}_{13}\mathrm{U}_{23} \quad \text{in } \mathcal{U}(\mathbb{K}(\mathcal{L} \otimes \mathcal{L}) \otimes A); \tag{3.5}$$

(2) $\mathbb{F}$ is *invariant* with respect to the corepresentation $\mathrm{V} \mathbin{\bigcirc\!\!\!\!\top} \mathrm{V} := \mathrm{V}_{13}\mathrm{V}_{23}$ of $\hat{\mathbb{G}}$ on $\mathcal{L} \otimes \mathcal{L}$:

$$\mathrm{V}_{13}\mathrm{V}_{23}\mathbb{F}_{12} = \mathbb{F}_{12}\mathrm{V}_{13}\mathrm{V}_{23} \quad \text{in } \mathcal{U}(\mathbb{K}(\mathcal{L} \otimes \mathcal{L}) \otimes \hat{A}); \tag{3.6}$$

(3) $\mathbb{F}$ satisfies the *braided pentagon equation*

$$\mathbb{F}_{23}\mathbb{F}_{12} = \mathbb{F}_{12}({}^{\mathcal{L}}\!\times^{\mathcal{L}})_{23}\mathbb{F}_{12}({}^{\mathcal{L}}\!\times^{\mathcal{L}})_{23}\mathbb{F}_{23} \quad \text{in } \mathcal{U}(\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L}); \tag{3.7}$$

here the braiding ${}^{\mathcal{L}}\!\times^{\mathcal{L}} \in \mathcal{U}(\mathcal{L} \otimes \mathcal{L})$ and ${}^{\mathcal{L}}\!\times^{\mathcal{L}} = ({}^{\mathcal{L}}\!\times^{\mathcal{L}})^*$ are defined as ${}^{\mathcal{L}}\!\times^{\mathcal{L}} = Z\Sigma$ for the flip Σ, $x \otimes y \mapsto y \otimes x$, and the unique unitary $Z \in \mathcal{U}(\mathcal{L} \otimes \mathcal{L})$ that satisfies (3.2).

From now onwards we fix the $\mathfrak{D}(\mathbb{G})^\frown$-pair (U, V) on $\mathcal{L}$ and say that $\mathbb{F}$ *is a braided multiplicative unitary over* $\mathbb{G}$.

The *contragradient* of $\mathrm{U} \in \mathcal{U}(\mathbb{K}(\mathcal{L}) \otimes A)$ is defined by $\mathrm{U}^c := (\mathsf{T} \otimes \mathrm{R}_A)\mathrm{U} \in \mathcal{U}(\mathbb{K}(\overline{\mathcal{L}}) \otimes A)$, see [14, Proposition 10]. There is a unique unitary $\widetilde{Z} \in \mathcal{U}(\overline{\mathcal{L}} \otimes \mathcal{L})$ satisfying

$$\mathrm{U}^c_{1\pi}\mathrm{V}_{2\hat{\pi}}\widetilde{Z}_{12} = \mathrm{V}_{2\hat{\pi}}\mathrm{U}^c_{1\pi} \quad \text{in } \mathcal{U}(\overline{\mathcal{L}} \otimes \mathcal{L} \otimes \mathcal{H}).$$

Definition 3.8. Let $\mathbb{W} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$ be a manageable multiplicative unitary generating $\mathbb{G} = (A, \Delta_A)$, let Q is strictly positive operator in the definition of the manageability of $\mathbb{W}$, and let $Z, \widetilde{\mathbb{Z}}$ be as above. A braided multiplicative unitary $\mathbb{F} \in \mathcal{U}(\mathcal{L} \otimes \mathcal{L})$ over $\mathbb{G}$ is *manageable* if there are a strictly positive operator Q' on $\mathcal{L}$ and a unitary $\widetilde{\mathbb{F}} \in \mathcal{U}(\overline{\mathcal{L}} \otimes \mathcal{L})$ such that

$$\mathbb{U}(Q' \otimes Q)\mathbb{U}^* = Q' \otimes Q, \quad \mathbb{V}(Q' \otimes Q)\mathbb{V}^* = Q' \otimes Q, \quad \mathbb{F}(Q' \otimes Q')\mathbb{F}^* = Q' \otimes Q',$$

and

$$(x \otimes u \mid Z^*\mathbb{F} \mid y \otimes v) = (\overline{y} \otimes Q'(u) \mid \widetilde{\mathbb{F}}\widetilde{Z}^* \mid \overline{x} \otimes (Q')^{-1}(v))$$

for all $x, y \in \mathcal{L}$, $u \in \mathcal{D}(Q')$ and $v \in \mathcal{D}((Q')^{-1})$.

Now we state the main result of this article.

Theorem 3.9. *Let $\mathbb{F} \in \mathcal{U}(\mathcal{L} \otimes \mathcal{L})$ be a manageable braided multiplicative unitary over a regular* C^**-quantum group $\mathbb{G} = (A, \Delta_A)$. Let*

$$B := \{(\omega \otimes \mathrm{id}_{\mathcal{L}})\mathbb{F} \mid \omega \in \mathbb{B}(\mathcal{L})_*\}^{\mathrm{CLS}} \tag{3.10}$$

1. *B is a nondegenerate* C^**-subalgebra of $\mathbb{B}(\mathcal{L})$;*
2. *The morphisms $\beta \in \mathrm{Mor}(B, B \otimes A)$ and $\hat{\beta} \in \mathrm{Mor}(B, B \otimes \hat{A})$ defined by*

$$\beta(b) := \mathrm{U}(b \otimes 1)\mathrm{U}^*, \qquad \hat{\beta}(b) := \mathrm{V}(b \otimes 1)\mathrm{V}^* \tag{3.11}$$

   are coactions of $\mathbb{G}$ and $\hat{\mathbb{G}}$ on B and $(B, \beta, \widehat{\beta})$ is a $\mathbb{G}$-Yetter-Drinfeld C^**-algebra;*
3. *$\mathbb{F} \in \mathcal{U}(\mathbb{K}(\mathcal{L}) \otimes B)$;*

Let $j_1, j_2 \in \mathrm{Mor}(B, B \boxtimes B)$ are the canonical morphisms described by (2.30).

4. *The map $\Delta_B(b) := \mathbb{F}(b \otimes 1_{\mathcal{H}})\mathbb{F}^*$ defines a unique morphism $B \to B \boxtimes B$ that is $\mathbb{G}$- and $\hat{\mathbb{G}}$-equivariant and satisfies*

$$(\mathrm{id} \otimes \Delta_B)\mathbb{F} = (\mathrm{id}_{\mathcal{L}} \otimes j_1)\mathbb{F}(\mathrm{id}_{\mathcal{L}} \otimes j_2)\mathbb{F} \qquad \text{in } \mathcal{U}(\mathbb{K}(\mathcal{L}) \otimes B \boxtimes B). \tag{3.12}$$

   Moreover, Δ_B is coassociative,

$$(\mathrm{id}_B \boxtimes \Delta_B)\Delta_B = (\Delta_B \boxtimes \mathrm{id}_B)\Delta_B, \tag{3.13}$$

   and satisfies

$$j_1(B)\Delta_B(B) = B \boxtimes B = \Delta_B(B)j_2(B). \tag{3.14}$$

We resume the proof of Theorem 3.9 in the next section.

Definition 3.15. The pair (B, Δ_B) in Theorem 3.9 is called a *braided* C^**-quantum group* over $\mathbb{G}$. We say (B, Δ_B) is generated by $\mathbb{F}$.

Let $\mathbb{H} = (C, \Delta_C)$ be a C^*quantum group and let $(\eta, \hat{\eta})$ be a $\mathbb{H}$-Heisenberg pair on a Hilbert space $\mathcal{H}_\eta$. An element $\mathrm{P} \in \mathcal{U}(\hat{C} \otimes C)$ is called a *projection* on $\mathbb{H}$ if it satisfies the following conditions:

1. P is a bicharacter:

$$(\hat{\Delta}_C \otimes \mathrm{id}_C)\mathrm{P} = \mathrm{P}_{23}\mathrm{P}_{13} \qquad (\mathrm{id}_{\hat{C}} \otimes \Delta_C)\mathrm{P} = \mathrm{P}_{12}\mathrm{P}_{13}, \tag{3.16}$$

2. P is an idempotent endomorphism of $\mathbb{H}$:

$$\mathrm{P}_{\hat{\eta}3}\mathrm{P}_{1\eta} = \mathrm{P}_{1\eta}\mathrm{P}_{13}\mathrm{P}_{\hat{\eta}3} \qquad \text{in } \mathcal{U}(\hat{C} \otimes \mathbb{K}(\mathcal{H}_\eta) \otimes C). \tag{3.17}$$

Clearly, $(\hat{\eta} \otimes \eta)\mathrm{P} \in \mathcal{U}(\mathcal{H}_\eta \otimes \mathcal{H}_\eta)$ is a mutliplicative unitary and it is manageable, see [11, Proposition 3.36].

By virtue of [11, Theorem 6.15 & 6.16], a manageable braided multiplicative unitary $\mathbb{F} \in \mathcal{U}(\mathcal{L} \otimes \mathcal{L})$ over $\mathbb{G}$ gives rise to a C^*-quantum group $\mathbb{H} = (C, \Delta_C)$

generated by a manageable multiplicaitive unitary $\mathbb{W}^C \in \mathcal{U}(\mathcal{H}\otimes\mathcal{L}\otimes\mathcal{H}\otimes\mathcal{L})$ defined by

$$\mathbb{W}^C := \mathbb{W}_{13}\mathbb{U}_{23}\hat{\mathbb{V}}^*_{34}\mathbb{F}_{24}\hat{\mathbb{V}}_{34} \qquad \text{in } \mathcal{U}(\mathcal{H}\otimes\mathcal{L}\otimes\mathcal{H}\otimes\mathcal{L}). \tag{3.18}$$

Furthermore, the unitary $\mathbb{P} \in \mathcal{U}(\mathcal{H}\otimes\mathcal{L}\otimes\mathcal{H}\otimes\mathcal{L})$ defined by

$$\mathbb{P} := \mathbb{W}_{13}\mathbb{U}_{23} \qquad \text{in } \mathcal{U}(\mathcal{H}\otimes\mathcal{L}\otimes\mathcal{H}\otimes\mathcal{L}). \tag{3.19}$$

is a projection on $\mathbb{H}$ with the image $\mathbb{G} = (A, \Delta_A)$ see [11, Propositon 2.36 & Theorem 6.17].

Thus, a braided C*-quantum group (B, Δ_B) over a regular C*-quantum group $\mathbb{G}$ gives rise to a C*-quantum group $\mathbb{H} = (C, \Delta_C)$ with projection. Therefore, it is important to encode (C, Δ_C) in terms of (A, Δ_A) and (B, Δ_B) to construct new examples of C*-quantum groups. In the compact case, that is when A and B are unital, this has been already done in [7, Theorem 6.7]. We shall extend this result for locally compact case.

Regularity of $\mathbb{G}$ gives $A \in \mathcal{YDC}^*\mathfrak{alg}(\mathbb{G})$ and by Theorem 3.9(2) $B \in \mathcal{YDC}^*\mathfrak{alg}(\mathbb{G})$. Therefore, $A \boxtimes B := (A \otimes 1_{\mathcal{L}})\hat{\mathbb{V}}^*(1_{\mathcal{H}} \otimes B)\hat{\mathbb{V}}$ as shown in [7, Page 19]. Here we have supressed the faithful representations of A and B on $\mathcal{H}$ and $\mathcal{L}$, respectively.

For any $x \in A \boxtimes B \boxtimes B$ the map

$$x \to \mathbb{W}_{12}\mathbb{U}_{23}\hat{\mathbb{V}}^*_{34}x_{124}\hat{\mathbb{V}}_{34}\mathbb{U}^*_{23}\mathbb{W}^*_{12} \tag{3.20}$$

defines an injective morphism $\Psi\colon A \boxtimes B \boxtimes B \to A \boxtimes B \otimes A \boxtimes B$ (see [7, Proposition 6.5]).

Theorem 3.21. *Let $C = A \boxtimes B$ and define $\Delta_C \in \mathrm{Mor}(C, C \otimes C)$ by $\Delta_C := \Psi \circ (\mathrm{id}_B \boxtimes \Delta_B)$. Then (C, Δ_C) is the C*-quantum group generated by $\mathbb{W}^C$ in* (3.18).

Proof. For any $c \in A \boxtimes B \subset \mathbb{B}(\mathcal{H} \otimes \mathcal{L})$

$$\begin{aligned}\Delta_C(c) &= \Psi \circ (\mathrm{id}_B \boxtimes \Delta_B)(c) = \Psi(\mathbb{F}_{23}(c \otimes 1_{\mathcal{L}})\mathbb{F}^*_{23}) \\ &= \mathbb{W}_{12}\mathbb{U}_{23}\hat{\mathbb{V}}^*_{34}\mathbb{F}_{24}(c \otimes 1_{\mathcal{H}\otimes\mathcal{L}})\mathbb{F}^*_{24}\hat{\mathbb{V}}_{34}\mathbb{U}^*_{23}\mathbb{W}^*_{12} = (\mathbb{W}^C)(c \otimes 1)(\mathbb{W}^C)^*\end{aligned}$$

Therefore, we only need to show

$$A \boxtimes B = \{(\omega \otimes \omega' \otimes \mathrm{id}_{\mathcal{H}\otimes\mathcal{L}})\mathbb{W}^C \mid \ \omega \in \mathbb{B}(\mathcal{H})_*,\, \omega' \in \mathbb{B}(\mathcal{L})_*\}^{\mathrm{CLS}}.$$

Let $L = \{(\omega \otimes \omega' \otimes \mathrm{id}_{\mathcal{H}\otimes\mathcal{L}})\mathbb{W}^C \mid \ \omega \in \mathbb{B}(\mathcal{H})_*,\, \omega' \in \mathbb{B}(\mathcal{L})_*\}^{\mathrm{CLS}}$.

Using (2.3) we get

$$\begin{aligned}L &= \{(\omega \otimes \omega' \otimes \mathrm{id}_{\mathcal{H}\otimes\mathcal{L}})\mathbb{W}_{13}\mathbb{U}_{23}\hat{\mathbb{V}}^*_{34}\mathbb{F}_{24}\hat{\mathbb{V}}_{34} \mid \ \omega \in \mathbb{B}(\mathcal{H})_*,\, \omega' \in \mathbb{B}(\mathcal{L})_*\}^{\mathrm{CLS}} \\ &= \{(\omega' \otimes \mathrm{id}_{\mathcal{H}\otimes\mathcal{L}})(1 \otimes a \otimes 1)\mathbb{U}_{12}\hat{\mathbb{V}}^*_{23}\mathbb{F}_{13}\hat{\mathbb{V}}_{23} \mid \omega' \in \mathbb{B}(\mathcal{L})_*,\, a \in A\}^{\mathrm{CLS}}\end{aligned}$$

For $\omega' \in \mathbb{B}(\mathcal{L})_*$ and $\xi \in \mathbb{K}(\mathcal{L})$ define $\omega' \cdot \xi \in \mathbb{B}(\mathcal{L})_*$ by $\omega' \cdot \xi(y) := \omega'(\xi y)$.

Replacing ω' by $\omega' \cdot \xi$ in the last expression gives

$$L = \{(\omega' \otimes \mathrm{id}_{\mathcal{H}\otimes\mathcal{L}})((\xi \otimes a)\mathbb{U}) \otimes 1_{\mathcal{L}}\hat{\mathbb{V}}^*_{23}\mathbb{F}_{13}\hat{\mathbb{V}}_{23} \mid \omega' \in \mathbb{B}(\mathcal{L})_*,\, \xi \in \mathbb{K}(\mathcal{L}),\, a \in A\}^{\mathrm{CLS}}$$

We may replace $(\xi \otimes a)\mathbb{U}$ by $\xi \otimes a$ for $\xi \in \mathbb{K}(\mathcal{L})$, $a \in A$, because $\mathrm{U} \in \mathcal{U}(\mathcal{L} \otimes A)$ and $\mathbb{U} = (\mathrm{id}_{\mathcal{L}} \otimes \pi)\mathrm{U}$. We have

$$\begin{aligned}L &= \{(\omega' \otimes \mathrm{id}_{\mathcal{H}\otimes\mathcal{L}})(\xi \otimes a \otimes 1_{\mathcal{L}})\hat{\mathbb{V}}^*_{23}\mathbb{F}_{13}\hat{\mathbb{V}}_{23} \mid \omega' \in \mathbb{B}(\mathcal{L})_*,\, \xi \in \mathbb{K}(\mathcal{L}),\, a \in A\}^{\mathrm{CLS}} \\ &= \{(\omega' \otimes \mathrm{id}_{\mathcal{H}\otimes\mathcal{L}})(1_{\mathcal{L}} \otimes a \otimes 1_{\mathcal{L}})\hat{\mathbb{V}}^*_{23}\mathbb{F}_{13}\hat{\mathbb{V}}_{23} \mid \omega' \in \mathbb{B}(\mathcal{L})_*,\, a \in A\}^{\mathrm{CLS}}\end{aligned}$$

Finally using (3.10) we obtain

$$\begin{aligned}L &= \{(\omega' \otimes \mathrm{id}_{\mathcal{H}\otimes\mathcal{L}})(1 \otimes a \otimes 1)\hat{\mathbb{V}}^*_{23}\mathbb{F}_{13}\hat{\mathbb{V}}_{23} \mid \xi \in \mathbb{K}(\mathcal{L}),\, a \in A,\, \omega' \in \mathbb{B}(\mathcal{L})_*\}^{\mathrm{CLS}} \\ &= (A \otimes 1_{\mathcal{L}})\hat{\mathbb{V}}^*(1_{\mathcal{H}} \otimes B)\hat{\mathbb{V}}.\end{aligned}$$

□

4. Slices of braided multiplicative unitaries

Let $\mathbb{H} = (C, \Delta_C)$ be a C^*-quantum group and $\mathrm{P} \in \mathcal{U}(\hat{C} \otimes C)$ be a projection on $\mathbb{H}$ with image $\mathbb{G} = (A, \Delta_A)$. Let $\widehat{\mathbb{H}} = (\hat{C}, \hat{\Delta}_C)$ be the dual of $\mathbb{H}$ and $\mathrm{W}^C \in \mathcal{U}(\hat{C} \otimes C)$ be the reduced bicharacter.

Theorem 4.1. *Assume $\mathbb{G}$ is a regular C^*-quantum group. Let $\mathrm{F} := \mathrm{P}^*\mathrm{W}^C \in \mathcal{U}(\hat{C} \otimes C)$. Then*

$$D := \{(\omega \otimes \mathrm{id}_C)\mathrm{F} \mid \omega \in \hat{C}'\}^{\mathrm{CLS}} \subseteq \mathcal{M}(C).$$

is a C^*-algebra.

Remark 4.2. The C^*-algebra D in Theorem 4.1 is independent of the multiplicative unitary generating $\mathbb{H}$. In other words, D depends only on (C, Δ_C), (A, Δ_A) and $\mathrm{P} \in \mathcal{U}(\hat{C} \otimes C)$.

The main tool we use to prove Theorem 4.1 is the Landstad-Vaes theorem for quantum groups.

Let $\gamma\colon C \to A \otimes C$ be a left coaction of $\mathbb{G}$ on a C^*-algebra C and let $i\colon A \to C$ be a morphism. The triple (C, γ, i) is a $\mathbb{G}$-*product* if i is a $\mathbb{G}$-equivariant:

$$\gamma \circ i = (\mathrm{id}_A \otimes i) \circ \Delta_A. \tag{4.3}$$

Let $\widehat{\mathbb{G}} = (\hat{A}, \hat{\Delta}_A)$ be the dual of $\mathbb{G}$ and $\mathrm{W}^A \in \mathcal{U}(\hat{A} \otimes A)$ be the reduced bicharacter. Let $(\pi, \hat{\pi})$ be a $\mathbb{G}$-Heisenberg pair on a Hilbert space $\mathcal{H}$.

Let $X := (\mathrm{id}_{\hat{A}} \otimes i)\mathrm{W}^A \in \mathcal{U}(\hat{A} \otimes C)$. Define $\varphi\colon C \to \mathbb{K}(\mathcal{H}) \otimes C$ by $\varphi(c) := X^*_{\hat{\pi}2}\gamma(c)_{\pi 2}X_{\hat{\pi}2}$ for $c \in C$.

Theorem 4.4 (Landstad–Vaes)**.** *Assume that $\mathbb{G} = (A, \Delta_A)$ is a regular quantum group. Let (C, γ, i) be a $\mathbb{G}$-product. Then there is a unique C^*-subalgebra D of $\mathcal{M}(C)$ with the following properties:*

1. *$D \subseteq \{c \in \mathcal{M}(C) \mid \gamma(c) = 1_A \otimes c\}$;*
2. *$C = i(A)D$;*
3. *$\hat{A} \otimes D = (\hat{A} \otimes 1)\varphi(D) = (\hat{A} \otimes 1)X^*(1 \otimes D)X$.*

The map $\hat{\beta}\colon D \to \mathcal{M}(D \otimes Z)$ defined by $\hat{\beta}(d) := \sigma(\varphi(d))$ takes values in $\mathcal{M}(B \otimes \hat{A})$ and is a (right) coaction of $\hat{\mathbb{G}}$ on B, and $\sigma\varphi$ defines a $\mathbb{G}$-equivariant isomorphism between C and $B \rtimes A$.

The C^*-algebra D is called the Landstad-Vaes algebra for the $\mathbb{G}$-product (C, γ, i).

This theorem is proved in [15, Theorem 6.7] if $\mathbb{G}$ is a regular locally compact quantum group (see [4]) with Haar weights (the conventions in [15] are, however, slightly different), and in [13] in the above generality, assuming only that $\mathbb{G}$ is a regular C^*-quantum group generated by a manageable multiplicative unitary.

By [8, Proposition 2.8] $\mathbb{H} = (C, \Delta_C)$ with projection $\mathrm{P} \in \mathcal{U}(\hat{C} \otimes C)$ with image $\mathbb{G} = (A, \Delta_A)$ is equivalent to a pair (i, Δ_L) consisting of morphisms $i\colon A \to C$ and $\Delta_L\colon C \to A \otimes C$ such that

1. i is a Hopf *-homomorphism: $\Delta_C \circ i = (i \otimes i)\Delta_A$,
2. Δ_L is a left quantum group homomorphism:
$$(\mathrm{id}_A \otimes \Delta_C) \circ \Delta_L = (\Delta_L \otimes \mathrm{id}_C)\Delta_C \qquad (\Delta_A \otimes \mathrm{id}_C)\Delta_L = (\mathrm{id}_A \otimes \Delta_L)\Delta_L,$$
3. i satisfies the following condition:

$$(\mathrm{id}_A \otimes i) \circ \Delta_A = \Delta_L \circ i. \tag{4.5}$$

In particular, Δ_L is a left coaction of $\mathbb{G}$ on C by [5, Lemma 5.8]. Thus (C, Δ_L, i) is a $\mathbb{G}$-product. We shall show that D in Theorem 4.1 is the Landstad-Vaes algebra for the $\mathbb{G}$-product (C, Δ_L, i).

Before that we prove a technical lemma:

Lemma 4.6. *Let $(\rho, \hat{\rho})$ be an $\mathbb{H}$-anti-Heisenberg pair on a Hilbert space $\mathcal{H}_\rho$. Define $X \in \mathcal{U}(\hat{A} \otimes C)$ by $X := (\mathrm{id}_{\hat{A}} \otimes i)\mathrm{W}^A$. Then*

$$\mathrm{F}_{\hat{\rho}3} X_{13} X_{1\rho} = X_{13} X_{1\rho} \mathrm{F}_{\hat{\rho}3} \qquad \textit{in } \mathcal{U}(\hat{A} \otimes \mathbb{K}(\mathcal{H}_\rho) \otimes C). \tag{4.7}$$

Proof. Since $(\rho, \hat{\rho})$ is an $\mathbb{H}$-anti-Heisenberg pair,

$$\mathrm{W}^C_{1\hat{\rho}} \mathrm{W}^C_{\rho 3} = \mathrm{W}^C_{\rho 3} \mathrm{W}^C_{13} \mathrm{W}^C_{1\hat{\rho}} \qquad \text{in } \mathcal{U}(\hat{C} \otimes \mathbb{K}(\mathcal{H}_\rho) \otimes C). \tag{4.8}$$

Combining (2.5) and (4.8) we can show that

$$(\mathrm{id}_C \otimes \rho)\Delta_C(c) = \sigma(\mathrm{W}^{C}_{\hat{\rho}2}{}^{*}(\rho(c) \otimes 1_C)\mathrm{W}^C_{\rho 2}) \qquad \text{for } c \in C. \tag{4.9}$$

The unitary $X := (\mathrm{id}_{\hat{A}} \otimes i)\mathrm{W}^A \in \mathcal{U}(\hat{A} \otimes C)$ is a bicharacter because i is a Hopf *-homomorphism. Hence $(\mathrm{id}_{\hat{A}} \otimes \Delta_C)X = X_{12}X_{13}$ which is equivalent to

$$X_{1\rho} \mathrm{W}^C_{\hat{\rho}3} = \mathrm{W}^C_{\hat{\rho}3} X_{13} X_{1\rho} \qquad \text{in } \mathcal{U}(\hat{A} \otimes \mathbb{K}(\mathcal{H}_\rho) \otimes C) \tag{4.10}$$

by (4.9). Similarly, replacing Hesienberg pairs by anti-Heisenberg pairs in (3.17) gives

$$\mathrm{P}_{1\rho} \mathrm{P}_{\hat{\rho}3} = \mathrm{P}_{\hat{\rho}3} \mathrm{P}_{13} \mathrm{P}_{1\rho} \qquad \text{in } \mathcal{U}(\hat{C} \otimes \mathbb{K}(\mathcal{H}_\rho) \otimes C).$$

Notice that $\mathrm{P} = (j \otimes i)\mathrm{W}^A$. Since i and j are injective, we apply $j^{-1} \otimes \mathrm{id}_{\mathcal{H}_\rho} \otimes i^{-1}$ on the both sides and obtain

$$X_{1\rho} \mathrm{P}_{\hat{\rho}3} = \mathrm{P}_{\hat{\rho}3} X_{13} X_{1\rho} \qquad \text{in } \mathcal{U}(\hat{A} \otimes \mathbb{K}(\mathcal{H}_\rho) \otimes C). \tag{4.11}$$

The following computation finishes the proof:

$$\begin{aligned} \mathrm{F}_{\hat{\rho}3} X_{13} X_{1\rho} = \mathrm{P}^*_{\hat{\rho}3} \mathrm{W}^C_{\hat{\rho}3} X_{13} X_{1\rho} = \mathrm{P}^*_{\hat{\rho}3} X_{1\rho} \mathrm{W}^C_{\hat{\rho}3} &= X_{13} X_{1\rho} \mathrm{P}^*_{\hat{\rho}3} \mathrm{W}^C_{\hat{\rho}3} \\ &= X_{13} X_{1\rho} \mathrm{F}_{\hat{\rho}3}. \end{aligned}$$ □

Proof of Theorem 4.1. By [5, Theorem 5.5], there is a bicharacter $\chi \in \mathcal{U}(\hat{C} \otimes A)$ such that

$$(\mathrm{id}_{\hat{C}} \otimes \Delta_L)\mathrm{W}^C = \chi_{12} \mathrm{W}^C_{13} \qquad \text{in } \mathcal{U}(\hat{C} \otimes A \otimes C). \tag{4.12}$$

The unitary $\hat{\mathrm{P}} := \sigma(\mathrm{P}^*) \in \mathcal{U}(C \otimes \hat{C})$ is a projection on $\hat{\mathbb{H}}$. This defines an injective Hopf *-homomorphism $j \colon \hat{A} \to \hat{C}$ such that $\mathrm{P} = (j \otimes i)\mathrm{W}^A$. As proved in [8, Proposition 2.8], $\chi := (j \otimes \mathrm{id}_A)\mathrm{W}^A \in \mathcal{U}(\hat{C} \otimes A)$ is the bicharacter satisfying (4.12).

Equation (4.5) gives

$$\begin{aligned} (\mathrm{id}_{\hat{C}} \otimes \Delta_L)\mathrm{P} = (j \otimes \Delta_L \circ i)\mathrm{W}^A &= (j \otimes \mathrm{id}_A \otimes i)\big((\mathrm{id}_{\hat{A}} \otimes \Delta_A)\mathrm{W}^A\big) \\ &= (j \otimes \mathrm{id}_A \otimes i)\big(\mathrm{W}^A_{12}\mathrm{W}^A_{13}\big) = \chi_{12}\mathrm{P}_{13}. \end{aligned} \tag{4.13}$$

Equation (4.12) and the previous computation give

$$(\mathrm{id}_{\hat{C}} \otimes \Delta_L)\mathrm{F} = (\mathrm{id}_{\hat{C}} \otimes \Delta_L)(\mathrm{P}^*\mathrm{W}^C) = \mathrm{P}^*_{13}\chi^*_{12}\chi_{12}\mathrm{W}^C_{13} = \mathrm{F}_{13}.$$

Taking slices on the first leg gives $D \subseteq \{c \in \mathcal{M}(C) \mid \Delta_L(c) = 1_A \otimes c\}$, the first condition in Theorem 4.4.

Now $\chi = (j \otimes \mathrm{id}_A)\mathrm{W}^A \in \mathcal{U}(\hat{C} \otimes A)$ and $\mathrm{P} = (\mathrm{id}_{\hat{C}} \otimes i)\chi \in \mathcal{U}(\hat{C} \otimes C)$. Therefore,

$$(\hat{C} \otimes i(A))\mathrm{P} = (\mathrm{id}_{\hat{C}} \otimes i)\big((\hat{C} \otimes A)\chi\big) = \hat{C} \otimes i(A).$$

The following computation gives the second condition in Theorem 4.4:

$$\begin{aligned} i(A)D &= i(A)\{(\omega \otimes \mathrm{id}_C)\mathrm{F} \mid \omega \in \hat{C}'\}^{\mathrm{CLS}} \\ &= \{(\omega \otimes \mathrm{id}_C)\big((\hat{C} \otimes i(A))\mathrm{F}\big) \mid \omega \in \hat{C}'\}^{\mathrm{CLS}} \\ &= \{(\omega \otimes \mathrm{id}_C)\big((\hat{C} \otimes i(A))\mathrm{PF}\big) \mid \omega \in \hat{C}'\}^{\mathrm{CLS}} \\ &= \{(\omega \otimes \mathrm{id}_C)\big((\hat{C} \otimes i(A))\mathrm{W}^C\big) \mid \omega \in \hat{C}'\}^{\mathrm{CLS}} \\ &= \{(\omega \otimes i(A))\mathrm{W}^C) \mid \omega \in \hat{C}'\}^{\mathrm{CLS}} \\ &= i(A)C = C. \end{aligned}$$

Let $(\rho, \hat{\rho})$ be an $\mathbb{H}$-anti-Heisenberg pair on a Hilbert space $\mathcal{H}_\rho$. Since ρ is faithful,

$$D = \{(\omega \otimes \mathrm{id}_C)\mathrm{F}_{\hat{\rho}2} \mid \omega \in \mathbb{B}(\mathcal{H}_\rho)_*\}^{\mathrm{CLS}}. \tag{4.14}$$

Recall $X \in \mathcal{U}(\hat{A} \otimes C)$ from Lemma 4.6. Equation (4.14) gives

$$(\hat{A} \otimes 1_C)X_{12}^*(1_{\hat{A}} \otimes D)X_{12} = \{(\hat{A} \otimes \omega \otimes \mathrm{id}_C)(X_{13}^*\mathrm{F}_{\hat{\rho}3}X_{13}) \mid \omega \in \mathbb{B}(\mathcal{H}_\rho)_*\}^{\mathrm{CLS}}.$$

Now Lemma 4.6 gives

$$(\hat{A} \otimes 1_C)X_{12}^*(1_{\hat{A}} \otimes D)X_{12} = \{(\hat{A} \otimes \omega \otimes \mathrm{id}_C)(X_{1\rho}\mathrm{F}_{\hat{\rho}3}X_{1\rho}^*) \mid \omega \in \mathbb{B}(\mathcal{H}_\rho)_*\}^{\mathrm{CLS}}.$$

Now $(\hat{A} \otimes \mathbb{K}(\mathcal{H}_\rho))X_{1\rho} = (\hat{A} \otimes \mathbb{K}(\mathcal{H}_\rho)\rho(A))X_{1\rho} = \hat{A} \otimes \mathbb{K}(\mathcal{H}_\rho)\rho(A) = \hat{A} \otimes \mathbb{K}(\mathcal{H}_\rho)$. This implies

$$\begin{aligned} &(\hat{A} \otimes 1_C)X_{12}^*(1_{\hat{A}} \otimes D)X_{12} \\ &= \{(\hat{A} \otimes \omega \otimes \mathrm{id}_C)(X_{1\rho}\mathrm{F}_{\hat{\rho}3}X_{1\rho}^*) \mid \omega \in \mathbb{B}(\mathcal{H}_\rho)_*\}^{\mathrm{CLS}} \\ &= \big\{(\mathrm{id}_{\hat{A}} \otimes \omega \otimes \mathrm{id}_C)\big(((\hat{A} \otimes \mathbb{K}(\mathcal{H}_\rho)X_{1\rho}) \otimes \mathrm{id}_C)\mathrm{F}_{\hat{\rho}3}X_{1\rho}^*\big) \mid \omega \in \mathbb{B}(\mathcal{H}_\rho)_*\big\}^{\mathrm{CLS}} \\ &= \{(\hat{A} \otimes \omega \otimes \mathrm{id}_C)(\mathrm{F}_{\hat{\rho}3}X_{1\rho}^*) \mid \omega \in \mathbb{B}(\mathcal{H}_\rho)_*\}^{\mathrm{CLS}} \\ &= \{(\mathrm{id}_{\hat{A}} \otimes \omega \otimes \mathrm{id}_C)\big(\mathrm{F}_{\hat{\rho}3}((\hat{A} \otimes \mathrm{id}_{\mathcal{H}_\rho})X_{1\rho}^*(1_{\hat{A}} \otimes \mathbb{K}(\mathcal{H}_\rho)) \otimes 1_C)\big) \mid \omega \in \mathbb{B}(\mathcal{H}_\rho)_*\}^{\mathrm{CLS}}. \end{aligned}$$

The regularity condition (2.24) implies

$$(\hat{A} \otimes \mathrm{id}_{\mathcal{H}_\rho})X_{1\rho}^*(1_{\hat{A}} \otimes \mathbb{K}(\mathcal{H}_\rho)) = (\hat{A} \otimes \mathrm{id}_{\mathcal{H}_\rho})X_{1\rho}^*(1_{\hat{A}} \otimes \rho(A)\mathbb{K}(\mathcal{H}_\rho)) = \hat{A} \otimes \mathbb{K}(\mathcal{H}_\rho).$$

This completes the proof

$$(\hat{A} \otimes 1_C)X_{12}^*(1_{\hat{A}} \otimes D)X_{12} = \{(\hat{A} \otimes \omega \otimes \mathrm{id}_C)\mathrm{F}_{\hat{\rho}3} \mid \omega \in \mathbb{B}(\mathcal{H}_\rho)_*\}^{\mathrm{CLS}} = \hat{A} \otimes D. \qquad \square$$

5. Construction of braided C*-quantum groups

Throughut this section we follow the same notations, assumptions and definitions that we introduced and used in Section 3.

In this section we shall prove Theorem 3.9. We shall eventually use Theorem 4.1 for the the C*-quantum group $\mathbb{H} = (C, \Delta_C)$ generated by $\mathbb{W}^C$ defined in (3.18) with projection $\mathbb{P}$ defined by (3.19) with the image $\mathbb{G} = (A, \Delta_A)$, which is a regular C*-quantum group. Therefore we must indentify i and Δ_L in order to view C as a $\mathbb{G}$-product.

Lemma 5.1. *Let $(\pi, \hat{\pi})$ be a $\mathbb{G}$-Heisenberg pair on $\mathcal{H}$. There is a faithful representation $\hat{\rho}\colon \hat{A} \to \mathbb{B}(\mathcal{H} \otimes \mathcal{L})$ such that $(\hat{\rho} \otimes \pi)\mathrm{W}^A = \mathbb{W}_{12}\mathbb{U}_{13} \in \mathcal{U}(\mathbb{K}(\mathcal{H} \otimes \mathcal{L} \otimes \mathcal{H})$.*

Proof. Let $(\eta, \hat{\eta})$ be a $\mathbb{G}$-anti-Heisenberg pair on a Hilbert space $\mathcal{H}_\eta$. Hence the corepresentation condition (2.14) for U is equivalent to

$$\mathrm{U}_{1\eta}\mathrm{W}^A_{\hat{\eta}3} = \mathrm{W}^A_{\hat{\eta}3}\mathrm{U}_{13}\mathrm{U}_{1\eta} \qquad \text{in } \mathcal{U}(\mathbb{K}(\mathcal{L} \otimes \mathcal{H}_\eta) \otimes A),$$

by (4.10). Applying σ_{12} on both sides and rearranging gives

$$\hat{\mathrm{U}}^*_{\eta2}\mathrm{W}^A_{\hat{\eta}3}\hat{\mathrm{U}}_{\eta2} = \mathrm{W}^A_{\hat{\eta}3}\mathrm{U}_{23} \qquad \text{in } \mathcal{U}(\mathbb{K}(\mathcal{H}_\eta \otimes \mathcal{L}) \otimes A). \tag{5.2}$$

Here $\hat{\mathrm{U}} := \sigma(\mathrm{U}^*) \in \mathcal{U}(A \otimes \mathbb{K}(\mathcal{L}))$. This yields a representation $\hat{\rho}'$ defined by $\hat{\rho}'(\hat{a}) := \hat{\mathrm{U}}^*_{\eta 2}(\hat{\eta}(\hat{a}) \otimes 1)\hat{\mathrm{U}}_{\eta 2}$. Since $\hat{\eta}$, $\hat{\pi}$ are faithful by [12, Proposition 3.2], we may define $\hat{\rho}(\hat{a}) := (\hat{\pi} \circ \hat{\eta}^{-1} \otimes \mathrm{id}_{\mathcal{L}}) \circ \hat{\rho}'(\hat{a})$. This faithful representation satisfies $(\hat{\rho} \otimes \pi)\mathrm{W}^A = \mathbb{W}_{12}\mathbb{U}_{13}$ by (5.2). □

Let us identify C, $\hat{C}$ with their images inside $\mathbb{B}(\mathcal{H} \otimes \mathcal{L} \otimes \mathcal{H} \otimes \mathcal{L})$ under the representations obtained from the $\mathbb{H}$-Heisenberg pair that arises from the manageable multiplicative unitary $\mathbb{W}^C$ in (3.18).

Define $\rho(a) := \pi(a) \otimes 1_{\mathcal{L}}$. Then $(\hat{\rho} \otimes \rho)\mathrm{W}^A$ is the projection $\mathbb{P}$ in (3.19). Therefore the image of ρ and $\hat{\rho}$ are contained inside the image of C and $\hat{C}$, respectively. The first condition in (3.16) and (2.9) together shows that gives

$$(\hat{\Delta}_C \circ \hat{\rho} \otimes \rho^{-1}\rho)\mathrm{W}^A = (\hat{\Delta}_C \otimes \rho^{-1})\mathbb{P} = (\mathrm{id} \otimes \rho^{-1})\big(\mathbb{P}_{23}\mathbb{P}_{13}\big) = ((\hat{\rho} \otimes \hat{\rho})\hat{\Delta}_A \otimes \rho^{-1}\rho)\mathrm{W}^A.$$

Taking slices on the second legs of the bothsides of the last expression shows that $\hat{\rho}$ is a Hopf *-homomorphism from $\hat{A}$ to $\hat{C}$. Similarly, we can show that ρ is a Hopf *-homomorphism from A to C.

Therefore $\chi := (\hat{\rho} \otimes \mathrm{id}_A)\mathrm{W}^A \in \mathcal{U}(\mathbb{K}(\mathcal{H} \otimes \mathcal{L}) \otimes A)$ is a bicharacter from $\mathbb{H}$ to $\mathbb{G}$ and let $\Delta_L \colon C \to A \otimes C$ be the left quanum groups morphism associated to it. We want to show that the pair (ρ, Δ_L) is the equivalent to the projection $\mathbb{P}$ on $\mathbb{H}$. Hence, we only need to verify (4.5) for this pair. Equation (4.13) and (2.5) gives

$$(\hat{\rho} \otimes \Delta_L \circ \rho)\mathrm{W}^A = (\mathrm{id} \otimes \Delta_L)\mathbb{P} = \chi_{12}\mathbb{P}_{13} = \big(\hat{\rho} \otimes (\mathrm{id} \otimes \rho)\Delta_A)\mathrm{W}^A\big)$$

Injectivity of $\hat{\rho}$ gives (4.5) for (ρ, Δ_L); hence (C, Δ_L, ρ) is a $\mathbb{G}$-product.

Proof of Theorem 3.9. **Ad 1.** The image of $\mathbb{P}$ is $\mathbb{G} = (A, \Delta_A)$, which is regular by assumption (see Section 2.5). Theorem 4.1 shows that

$$\begin{aligned} D &= \{(\omega \otimes \omega' \otimes \mathrm{id}_{\mathcal{K}})\mathbb{P}^*\mathbb{W}^C \mid \omega \in \mathbb{B}(\mathcal{H})_*,\, \omega' \in \mathbb{B}(\mathcal{L})_*\}^{\mathrm{CLS}} \\ &= \{(\omega' \otimes \mathrm{id}_{\mathcal{H}\otimes\mathcal{L}})\hat{\mathbb{V}}^*_{23}\mathbb{F}_{13}\hat{\mathbb{V}}_{23} \mid \omega \in \mathbb{B}(\mathcal{L})_*\}^{\mathrm{CLS}}. \end{aligned}$$

is a C*-algebra. Hence so is

$$B := \{(\omega' \otimes \mathrm{id}_{\mathcal{L}})\mathbb{F} \mid \omega \in \mathbb{B}(\mathcal{L})_*\}^{\mathrm{CLS}} \subseteq \mathbb{B}(\mathcal{L})$$

because $\hat{\mathbb{V}}D\hat{\mathbb{V}}^* = 1_{\mathcal{H}} \otimes B \subseteq \mathbb{B}(\mathcal{H} \otimes \mathcal{L})$.

The second condition in Theorem 4.4 gives $DC = C$. Also $C\mathbb{K}(\mathcal{H}\otimes\mathcal{L}) = \mathbb{K}(\mathcal{H}\otimes\mathcal{L})$ because C is constructed from the manageable multiplicative unitary $\mathbb{W}^C$ in (3.18), and $\hat{\mathbb{V}} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{L})$. Therefore,

$$\hat{\mathbb{V}}^*D\hat{\mathbb{V}}\mathbb{K}(\mathcal{H} \otimes \mathcal{L}) = \hat{\mathbb{V}}^*DC\mathbb{K}(\mathcal{H} \otimes \mathcal{L}) = \hat{\mathbb{V}}^*C\mathbb{K}(\mathcal{H} \otimes \mathcal{L}) = \hat{\mathbb{V}}\mathbb{K}(\mathcal{H} \otimes \mathcal{L}) = \mathbb{K}(\mathcal{H} \otimes \mathcal{L}).$$

Thus B acts nondegenerately on $\mathcal{L}$ and seperability of $\mathbb{B}(\mathcal{L})_*$ implies B is separable.

Ad 2. Define $\hat{\beta}(b) := \mathrm{V}(b \otimes 1_{\hat{A}})\mathrm{V}^*$ for $b \in B$. Clearly, $\hat{\beta}$ is injective.

We have identified the pair (i, γ) in Theorem 4.4 with (ρ, Δ_L). Recall that $(\pi, \hat{\pi})$ is the $\mathbb{G}$-Heisenberg pair on $\mathcal{H}$. The third condition in Theorem 4.4 gives

$$\hat{\pi} \otimes D = \hat{\pi}(\hat{A}) \otimes \hat{\mathbb{V}}^*(1 \otimes B)\hat{\mathbb{V}} = (\hat{\pi}(\hat{A}) \otimes 1_{\mathcal{L}\otimes\mathcal{L}})\mathbb{W}^*_{12}\hat{\mathbb{V}}^*_{23}(1_{\mathcal{H}} \otimes 1_{\mathcal{L}} \otimes B)\hat{\mathbb{V}}_{23}\mathbb{W}_{12}.$$

Now corepresentation condition (2.14) for V is equivalent to

$$\hat{\mathbb{V}}_{23}\mathbb{W}_{12} = \mathbb{W}_{12}\hat{\mathbb{V}}_{13}\hat{\mathbb{V}}_{23} \qquad \text{in } \mathcal{U}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{L}).$$

This gives

$$\begin{aligned}\hat{\mathbb{V}}^*_{23}\big(\hat{\pi}(\hat{A})\otimes 1\otimes B\big)\hat{\mathbb{V}}_{23} &= \hat{\pi}(\hat{A})\otimes\hat{\mathbb{V}}^*(1\otimes B)\hat{\mathbb{V}}\\ &= (\hat{\pi}(\hat{A})\otimes 1_{\mathcal{L}\otimes\mathcal{L}})\hat{\mathbb{V}}^*_{23}\hat{\mathbb{V}}^*_{13}\mathbb{W}^*_{12}(1_{\mathcal{H}}\otimes 1_{\mathcal{L}}\otimes B)\mathbb{W}_{12}\hat{\mathbb{V}}_{13}\hat{\mathbb{V}}_{23}\\ &= (\hat{\pi}(\hat{A})\otimes 1_{\mathcal{L}\otimes\mathcal{L}})\hat{\mathbb{V}}^*_{23}\hat{\mathbb{V}}^*_{13}(1_{\mathcal{H}}\otimes 1_{\mathcal{L}}\otimes B)\hat{\mathbb{V}}_{13}\hat{\mathbb{V}}_{23}\\ &= \hat{\mathbb{V}}^*_{23}\big(\hat{\pi}(\hat{A})\otimes 1)\hat{\mathbb{V}}^*(1\otimes B)\hat{\mathbb{V}}\big)_{13}\hat{\mathbb{V}}_{23}.\end{aligned}$$

This is equivalent to

$$\hat{A}\otimes B = (\hat{\pi}(\hat{A})\otimes 1_{\mathcal{L}})\hat{\mathbb{V}}^*(1_{\mathcal{H}}\otimes B)\hat{\mathbb{V}}; \tag{5.3}$$

this is the Podleś condition for $\hat{\beta}$. Thus $\hat{\beta}\in\mathrm{Mor}(B, B\otimes A)$ and the corepresentation condition (2.14) for V yields (2.17) for $\hat{\beta}$

Similarly, $\beta(b) := \mathrm{U}(b\otimes 1_A)\mathrm{U}^*$ is injective, and it is sufficent to establish the Podleś condition for β. Then $(B,\beta,\hat{\beta})$ is a $\mathbb{G}$-Yetter-Drinfeld C*-algebra because $(\mathbb{U},\hat{\mathbb{V}})$ satisfies the Drinfeld compatibility (3.3).

The second condition in Theorem 4.4 gives $C=\rho(A)D=(\pi(A)\otimes 1_{\mathcal{L}})\hat{\mathbb{V}}^*(1_{\mathcal{H}}\otimes B)\hat{\mathbb{V}}$.

Let $\Delta_R\colon C\to C\otimes A$ be the right quantum group homomorphism associated to the projection $\mathbb{P}=\mathbb{W}_{13}\mathbb{U}_{23}$. By [5, Lemma 5.8] $\Delta_R(C)(1\otimes A)=C\otimes A$. Equation (33) in [5] implies

$$\begin{aligned}&(\pi(A)\otimes 1_{\mathcal{L}\otimes\mathcal{H}})\hat{\mathbb{V}}^*_{12}(1_{\mathcal{H}}\otimes B\otimes 1_{\mathcal{H}})\hat{\mathbb{V}}_{12}\mathbb{U}^*_{23}\mathbb{W}^*_{13}(1_{\mathcal{H}\otimes\mathcal{L}}\otimes\pi(A))\\ &\qquad = \mathbb{U}^*_{23}\mathbb{W}^*_{13}(\pi(A)\otimes 1_{\mathcal{L}\otimes\mathcal{H}})\hat{\mathbb{V}}^*_{12}(1_{\mathcal{H}}\otimes B\otimes 1_{\mathcal{H}})\hat{\mathbb{V}}_{12}(1_{\mathcal{H}\otimes\mathcal{L}}\otimes\pi(A)).\end{aligned}$$

Multiplying by $\mathbb{K}(\mathcal{H})$ on the first leg and using $\pi(A)\mathbb{K}(\mathcal{H})=\mathbb{K}(\mathcal{H})$ on the left and right, this gives

$$\begin{aligned}&(\mathbb{K}(\mathcal{H})\otimes 1_{\mathcal{L}\otimes\mathcal{H}})\hat{\mathbb{V}}^*_{12}(1_{\mathcal{H}}\otimes B\otimes 1_{\mathcal{H}})\hat{\mathbb{V}}_{12}\mathbb{U}^*_{23}\mathbb{W}^*_{13}(\mathbb{K}(\mathcal{H})\otimes 1_{\mathcal{L}}\otimes\pi(A))\\ &= (\mathbb{K}(\mathcal{H})\otimes 1_{\mathcal{L}\otimes\mathcal{H}})\mathbb{U}^*_{23}\mathbb{W}^*_{13}(\pi(A)\otimes 1_{\mathcal{L}\otimes\mathcal{H}})\hat{\mathbb{V}}^*_{12}(1_{\mathcal{H}}\otimes B\otimes 1_{\mathcal{H}})\hat{\mathbb{V}}_{12}(\mathbb{K}(\mathcal{H})\otimes 1_{\mathcal{L}}\otimes\pi(A)).\end{aligned}$$

Equation (5.3) now gives

$$\begin{aligned}&(\mathbb{K}(\mathcal{H})\otimes B\otimes 1_{\mathcal{H}})\mathbb{U}^*_{23}\mathbb{W}^*_{13}(\mathbb{K}(\mathcal{H})\otimes 1_{\mathcal{L}}\otimes\pi(A))\\ &\qquad = (\mathbb{K}(\mathcal{H})\otimes 1_{\mathcal{L}\otimes\mathcal{H}})\mathbb{U}^*_{23}\mathbb{W}^*_{13}(\pi(A)\mathbb{K}(\mathcal{H})\otimes B\otimes\pi(A)).\end{aligned}$$

Now $\pi(A)\mathbb{K}(\mathcal{H})=\mathbb{K}(\mathcal{H})$ and $\mathrm{W}^A_{\hat{\pi}2}(\mathbb{K}(\mathcal{H})\otimes A)=\mathbb{K}(\mathcal{H})\otimes A$. The regularity (2.24) of $\mathbb{G}$ gives $(\mathbb{K}(\mathcal{H})\otimes 1_{\mathcal{H}})\mathbb{W}(1_{\mathcal{H}}\otimes\pi(A))=\mathbb{K}(\mathcal{H})\otimes\pi(A)$. Thus

$$\mathbb{K}(\mathcal{H})\otimes\big((B\otimes 1_{\mathcal{H}})\mathbb{U}^*(1_{\mathcal{L}}\otimes\pi(A)\big)=\mathbb{K}(\mathcal{H})\otimes\big(\mathbb{U}^*_{23}(\otimes B\otimes\pi(A)\big).$$

Taking slices by $\omega\in\mathbb{B}(\mathcal{H})$ on the first leg and rearranging $\mathbb{U}$ now gives

$$\mathbb{U}(B\otimes 1_{\mathcal{H}})\mathbb{U}^*(1_{\mathcal{L}}\otimes\pi(A))=B\otimes\pi(A).$$

This is equivalent to the Podleś condition for β.

Ad 3. Now we show that $\mathbb{F}\in\mathcal{U}(\mathbb{K}(\mathcal{L})\otimes B)$. The second condition in the Landstad theorem 4.4 shows that $C=\pi(A)\otimes 1_{\mathcal{L}})\hat{\mathbb{V}}^*(1\otimes B)\hat{\mathbb{V}}$ and C is *-invariant. Since $\mathbb{W}^C$ is a unitary multiplier of $\mathbb{K}(\mathcal{H}\otimes\mathcal{L})\otimes C$ we have $(\mathbb{K}(\mathcal{H})\otimes\mathbb{K}(\mathcal{L})\otimes C)\mathbb{W}^C=\mathbb{K}(\mathcal{H})\otimes\mathbb{K}(\mathcal{L})\otimes C)$. For two C*-algebras D and D' let $D_1D'_2=D\otimes D'$. This gives

$$\hat{\mathbb{V}}^*_{34}B_4\hat{\mathbb{V}}_{34}\mathbb{K}(\mathcal{H})_1\mathbb{K}(\mathcal{L})_2\pi(A)_3\mathbb{W}_{13}\mathbb{U}_{23}\hat{\mathbb{V}}^*_{34}\mathbb{F}_{24}\hat{\mathbb{V}}_{34}=\mathbb{K}(\mathcal{H})_1\mathbb{K}(\mathcal{L})_2\pi(A)_3\hat{\mathbb{V}}^*_{34}B_4\hat{\mathbb{V}}_{34}.$$

Now $(\mathbb{K}(\mathcal{H})\otimes\pi(A)\mathbb{W}=\mathbb{K}(\mathcal{H})\otimes\pi(A)$ and $\mathbb{U}=(\mathrm{id}_{\mathcal{L}}\otimes\pi)\mathrm{U}$. Therefore, $(\mathbb{K}(\mathcal{H})\otimes\mathbb{K}(\mathcal{L})\otimes\pi(A))\mathbb{W}_{13}\mathbb{U}_{23}=\mathbb{K}(\mathcal{H})\otimes\mathbb{K}(\mathcal{L})\otimes\pi(A)$. This gives

$$\hat{\mathbb{V}}^*_{34}B_4\hat{\mathbb{V}}_{34}\mathbb{K}(\mathcal{H})_1\mathbb{K}(\mathcal{L})_2\pi(A)_3\hat{\mathbb{V}}^*_{34}\mathbb{F}_{24}\hat{\mathbb{V}}_{34}=\mathbb{K}(\mathcal{H})_1\mathbb{K}(\mathcal{L})_2\pi(A)_3\hat{\mathbb{V}}^*_{34}B_4\hat{\mathbb{V}}_{34}.$$

Multiplying by $\mathbb{K}(\mathcal{H})$ on the third leg from the left and using (5.3) on bothsides, this gives

$$\mathbb{K}(\mathcal{H})_1\mathbb{K}(\mathcal{L})_2\mathbb{K}(\mathcal{H})_3\pi(A)_3B_4\hat{\mathbb{V}}^*_{34}\mathbb{F}_{24}\hat{\mathbb{V}}_{34} = \mathbb{K}(\mathcal{H})_1\mathbb{K}(\mathcal{L})_2\mathbb{K}(\mathcal{H})_3\pi(A)_3B_4.$$

Now $\pi\colon A \to \mathbb{B}(\mathcal{H})$ is nondegenerate: $\pi(A)\mathbb{K}(\mathcal{H}) = \mathbb{K}(\mathcal{H})\pi(A) = \mathbb{K}(\mathcal{H})$. This simplifies the last expression and gives

$$\mathbb{K}(\mathcal{H})_1\mathbb{K}(\mathcal{L})_2\mathbb{K}(\mathcal{H})_3B_4\hat{\mathbb{V}}^*_{34}\mathbb{F}_{24}\hat{\mathbb{V}}_{34} = \mathbb{K}(\mathcal{H})_1\mathbb{K}(\mathcal{L})_2\mathbb{K}(\mathcal{H})_3B_4.$$

The invariant condition (3.6) is equivalent to $\hat{\mathbb{V}}^*_{23}\mathbb{F}_{13}\hat{\mathbb{V}}_{23} = \mathbb{V}_{12}\mathbb{F}_{13}\mathbb{V}^*_{12}$. This gives

$$\mathbb{K}(\mathcal{H})_1\mathbb{K}(\mathcal{L})_2\mathbb{K}(\mathcal{H})_3B_4\mathbb{V}_{23}\mathbb{F}_{24}\mathbb{V}^*_{23} = \mathbb{K}(\mathcal{H})_1\mathbb{K}(\mathcal{L})_2\mathbb{K}(\mathcal{H})_3B_4.$$

Now $\mathbb{V}_{23}$ commutes with B_4 and $(\mathbb{K}(\mathcal{L}\otimes\mathbb{K}(\mathcal{H}))\mathbb{V} = \mathbb{K}(\mathcal{L})\otimes\mathcal{H}$. This gives

$$\mathbb{K}(\mathcal{H})_1\mathbb{K}(\mathcal{L})_2\mathbb{K}(\mathcal{H})_3B_4\mathbb{F}_{24}\mathbb{V}^*_{23} = \mathbb{K}(\mathcal{H})_1\mathbb{K}(\mathcal{L})_2\mathbb{K}(\mathcal{H})_3B_4.$$

Multiplying bothside by $\mathbb{V}_{23}$ from the right and using $(\mathbb{K}(\mathcal{L}\otimes\mathbb{K}(\mathcal{H}))\mathbb{V} = \mathbb{K}(\mathcal{L})\otimes\mathcal{H}$, this gives

$$\mathbb{K}(\mathcal{H})_1\mathbb{K}(\mathcal{L})_2\mathbb{K}(\mathcal{H})_3B_4\mathbb{F}_{24} = \mathbb{K}(\mathcal{H})_1\mathbb{K}(\mathcal{L})_2\mathbb{K}(\mathcal{H})_3B_4.$$

Taking the slices on the first and third legs by $\omega \in \mathbb{B}(\mathcal{H})_*$ gives $(\mathbb{K}(\mathcal{L})\otimes B)\mathbb{F} = \mathbb{K}(\mathcal{L})\otimes B$. This shows that $\mathbb{F}$ is a unitary right multiplier of $\mathbb{K}(\mathcal{L})\otimes B)$. Multiplying bothsides by $\mathbb{F}^*$ of the above equation from the right gives gives $\mathbb{K}(\mathcal{L})\otimes B = (\mathbb{K}(\mathcal{L})\otimes B)\mathbb{F}^*$; hence $\mathbb{F}$ is also a left multiplier of $\mathbb{K}(\mathcal{L})\otimes B)$.

Ad 4. The unitary $Z \in \mathcal{U}(\mathcal{L}\otimes\mathcal{L})$ is characterised by (3.2); hence (2.30) gives $j_1(b) := b\otimes 1_{\mathcal{L}}$, $j_2(b) := {}^{\mathcal{L}}\!\times^{\mathcal{L}}(b\otimes 1_{\mathcal{L}})^{\mathcal{L}}\!\times^{\mathcal{L}}$, and $B\boxtimes B = j_1(B)j_2(B) \subseteq \mathbb{B}(\mathcal{L}\otimes\mathcal{L})$.

Define $\Delta_B(b) := \mathbb{F}(b\otimes 1_{\mathcal{L}})\mathbb{F}^*$ for all $b \in B$. The braided pentagon equation (3.7) gives (3.12):

$$(\mathrm{id}_{\mathcal{L}}\otimes\Delta_B)\mathbb{F} = \mathbb{F}_{23}\mathbb{F}_{12}\mathbb{F}^*_{23} = \mathbb{F}_{12}{}^{\mathcal{L}}\!\times^{\mathcal{L}}{}_{23}\mathbb{F}_{12}{}^{\mathcal{L}}\!\times^{\mathcal{L}}{}_{23}.$$

Since $\mathbb{F} \in \mathcal{U}(\mathbb{K}(\mathcal{L}\otimes B)$, taking slices on the first leg of the both sides of (3.12) shows that $\Delta_B\colon B \to \mathcal{M}(B\boxtimes B)$ is the unique *-homomorphism satisfying (3.12).

The diagonal coaction $\beta\bowtie\beta$ of $\mathbb{G}$ on $B\boxtimes B$ is described by (2.32) as

$$\beta\bowtie\beta\colon B\boxtimes B \to B\boxtimes B\otimes A, \qquad x \mapsto \mathrm{U}_{13}\mathrm{U}_{23}(x\otimes 1_A)\mathrm{U}^*_{23}\mathrm{U}^*_{13}.$$

The invariance condition (3.5) for $\mathbb{F}$ gives

$$\begin{aligned}\beta\bowtie\beta\circ\Delta_B(b) &= \mathbb{U}_{13}\mathbb{U}_{23}\mathbb{F}_{12}(b\otimes 1_{\mathcal{L}\otimes\mathcal{H}})\mathbb{F}^*_{12}\mathbb{U}^*_{23}\mathbb{U}^*_{13}\\ &= \mathbb{F}_{12}\mathbb{U}_{13}\mathbb{U}_{23}(b\otimes 1_{\mathcal{L}\otimes\mathcal{H}})\mathbb{U}^*_{23}\mathbb{U}^*_{13}\mathbb{F}^*_{12}\\ &= (\Delta_B\otimes\mathrm{id}_A)\circ\beta(b);\end{aligned}$$

hence Δ_B is $\mathbb{G}$-equivariant. Similarly, we may show that Δ_B is $\hat{\mathbb{G}}$-equivariant.

The coassociativity of Δ_B follows from the top-braided pentagon equation (3.7):

$$\begin{aligned}(\Delta_B\boxtimes\mathrm{id}_B)\Delta_B(b) = \mathbb{F}_{12}{}^{\mathcal{K}}\!\times^{\mathcal{L}}{}_{23}\Delta_B(b)_{12}{}^{\mathcal{L}}\!\times^{\mathcal{L}}{}_{23}\mathbb{F}^*_{12} &= \mathbb{F}_{12}{}^{\mathcal{L}}\!\times^{\mathcal{L}}{}_{23}\mathbb{F}_{12}b_1\mathbb{F}^*_{12}{}^{\mathcal{L}}\!\times^{\mathcal{L}}{}_{23}\mathbb{F}^*_{12}\\ &= \mathbb{F}_{23}\mathbb{F}_{12}b_1\mathbb{F}^*_{12}\mathbb{F}_{23}\\ &= (\mathrm{id}_B\boxtimes\Delta_B)\circ\Delta_B(b).\end{aligned}$$

Now$(\mathbb{K}(\mathcal{L})\otimes B)\mathbb{F} = \mathbb{K}(\mathcal{L})\otimes B$. Then (3.12) gives

$$\begin{aligned}(\mathbb{K}(\mathcal{L})\otimes j_1(B))(\mathrm{id}_{\mathcal{L}}\otimes\Delta_B)\mathbb{F} &= (\mathrm{id}_{\mathcal{L}}\otimes j_1)\big((\mathbb{K}(\mathcal{L})\otimes B)\mathbb{F}\big)(\mathrm{id}_{\mathcal{L}}\otimes j_2)\mathbb{F}\\ &= (\mathbb{K}(\mathcal{L})\otimes j_1(B))(\mathrm{id}_{\mathcal{L}}\otimes j_2)\mathbb{F}.\end{aligned}$$

Slicing the first leg by $\omega \in \mathbb{B}(\mathcal{L})_*$ on both sides gives $j_1(B)\Delta_B(B) = j_1(B)j_2(B) = B\boxtimes B$. A similar computation gives $\Delta_B(B)j_2(B) = B\boxtimes B$. ☐

References

[1] Saad Baaj and Georges Skandalis, *Unitaires multiplicatifs et dualité pour les produits croisés de C^*-algèbres*, Ann. Sci. École Norm. Sup. (4) **26** (1993), no. 4, 425–488, available at http://www.numdam.org/item?id=ASENS_1993_4_26_4_425_0. MR **1235438**

[2] David Bücher and Sutanu Roy, *Regularity for braided multiplicative unitaries* (2014), eprint. arXiv: 1411.4611.

[3] Paweł Kasprzak and Piotr M. Sołtan, *Quantum groups with projection on von Neumann algebra level*, J. Math. Anal. Appl. **427** (2015), no. 1, 289–306, doi: 10.1016/j.jmaa.2015.02.047. MR **3318200**

[4] Johan Kustermans and Stefaan Vaes, *Locally compact quantum groups*, Ann. Sci. École Norm. Sup. (4) **33** (2000), no. 6, 837–934, doi: 10.1016/S0012-9593(00)01055-7. MR **1832993**

[5] Ralf Meyer, Sutanu Roy, and Stanisław Lech Woronowicz, *Homomorphisms of quantum groups*, Münster J. Math. **5** (2012), 1–24, available at http://nbn-resolving.de/urn:nbn:de:hbz:6-88399662599. MR **3047623**

[6] ———, *Quantum group-twisted tensor products of* C^**-algebras*, Internat. J. Math. **25** (2014), no. 2, 1450019, 37, doi: 10.1142/S0129167X14500190. MR **3189775**

[7] ———, *Quantum group-twisted tensor products of* C^**-algebras II*, J. Noncommut. Geom. (2015), accepted. arXiv: 1501.04432.

[8] ———, C^**-quantum groups with projection*, in preparation.

[9] Ryszard Nest and Christian Voigt, *Equivariant Poincaré duality for quantum group actions*, J. Funct. Anal. **258** (2010), no. 5, 1466–1503, doi: 10.1016/j.jfa.2009.10.015. MR **2566309**

[10] David E. Radford, *The structure of Hopf algebras with a projection*, J. Algebra **92** (1985), no. 2, 322–347, doi: 10.1016/0021-8693(85)90124-3. MR **778452**

[11] Sutanu Roy, C^**-Quantum groups with projection*, Ph.D. Thesis, Georg-August Universität Göttingen, 2013, http://hdl.handle.net/11858/00-1735-0000-0022-5EF9-0.

[12] ———, *The Drinfeld double for C^*-algebraic quantum groups*, J. Operator Theory **74** (2015), no. 2, 485–515, available at http://www.mathjournals.org/jot/2015-074-002/2015-074-002-010.html.

[13] Sutanu Roy and Stanisław Lech Woronowicz, *Landstad-Vaes theory for locally compact quantum groups*, in preparation.

[14] Piotr M. Sołtan and Stanisław Lech Woronowicz, *From multiplicative unitaries to quantum groups. II*, J. Funct. Anal. **252** (2007), no. 1, 42–67, doi: 10.1016/j.jfa.2007.07.006. MR **2357350**

[15] Stefaan Vaes, *A new approach to induction and imprimitivity results*, J. Funct. Anal. **229** (2005), no. 2, 317–374, doi: 10.1016/j.jfa.2004.11.016. MR **2182592**

[16] Stanisław Lech Woronowicz, *From multiplicative unitaries to quantum groups*, Internat. J. Math. **7** (1996), no. 1, 127–149, doi: 10.1142/S0129167X96000086. MR **1369908**

E-mail address: sutanu.roy@carleton.ca

School of Mathematics and Statistics, Carleton University, 1125 Colonel By Drive, K1S 5B6 Ottawa, Canada.