MINIMAX PERFECT STOPPING RULES FOR SELLING AN ASSET NEAR ITS ULTIMATE MAXIMUM

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Abstract. We study the problem of selling an asset near its ultimate maximum in the minimax setting. The regret-based notion of a perfect stopping time is introduced. The related selling rule improves any earlier one and cannot be improved by further delay. A perfect stopping time is unique and has the following form: one should sell the asset if its price deviates from the running maximum by a certain time-dependent quantity. The result is applicable to a quite general price model.

1. Introduction

Assume that an agent wants to sell an asset before the maturity date T at a price X_τ, which is as close as possible to the ultimate maximum $X_T^* = \max_{0\le t\le T} X_t$. The asset price is a continuous function $t \mapsto X_t(\omega)$, depending on an unknown outcome $\omega \in \Omega$. A selling rule $\tau(\omega)$ may depend on the price history $\{X_s : s \le \tau(\omega)\}$. For such a rule τ the difference $X_T^*(\omega) - X_\tau(\omega)$ can be considered as the agent regret that the selling price X_τ was lower then the maximal price. If the agent is extremely pessimistic, he may try to minimize the value

$$\sup_{\omega\in\Omega} \left(X_T^*(\omega) - X_\tau(\omega)\right) \tag{1.1}$$

over all stopping rules τ. We will see, however, that this approach is somewhat crude. Such optimal selling rule is by no means unique and even a deterministic one (that is, independent of ω) can be optimal in this sense.

To each selling rule we associate the regret over the past, the regret over the future and the overall regret. Based on the latter quantity we introduce the notion of a perfect stopping rule. The idea is that it improves any earlier stopping rule and cannot be improved by further delay. We show that a perfect stopping rule is unique and has the following simple form: one should sell the asset if its price X_t deviates from the running maximum X_t^* by a certain time-dependent quantity.

An optimality of such selling rule ("let profits run but cut losses") was first justified in [1] for a discrete time model. This result was inspired by the paper [7], which studied the case of a divisible asset. The approach of [7, 1] was based on discrete-time specific recurrent dynamic programming formulas. Furthermore, for the problem considered in [1], along with the mentioned optimal selling rule, there exists a deterministic ("non-sequential") selling rule, also minimizing (1.1). The notion of a perfect stopping time,

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introduced below, gives grounds to distinguish between these selling rules and prefer the former.

In continuous-time probabilistic setting the problem of stopping near the ultimate maximum became popular after the stimulating paper [6] and the preceding talk [9]. For instance, the cases of Brownian motion with drift and geometric Browninan motion were studied thoroughly in [4, 8, 5, 2]. In the latter case the ratios X_τ/X_T^*, X_T^*/X_τ were considered instead of (1.1). Typical optimal stopping rules are determined by the processes $X_t^*-X_t$, X_t/X_t^*, or prescribe to sell the asset immediately, or to hold it until the maturity date T.

In Section 2 we introduce a perfect stopping time in a general model and present its explicit description. A model with uniformly Lipschitz continuous trajectories is considered in Section 3. In Section 4 we briefly consider the ratio X_T^*/X_τ performance criterion, reducing the problem to the previous one.

2. Perfect stopping time

Although we do not use any probability measure, the basic terminology comes from probability theory. Possible outcomes are described by a subset Ω of the canonical space $C[0,T]$ of continuous functions ω. Let $X_t:\Omega\mapsto\mathbb{R}$ be the coordinate mappings: $X_t(\omega)=\omega_t$. For each $t\in[0,T]$, $\omega\in\Omega$ we put

$$\mathscr{A}(t,\omega)=\{\omega'\in\Omega: X_s(\omega')=X_s(\omega),\quad s\in[0,t]\}.$$

The set $\mathscr{A}(t,\omega)$ contains all outcomes with the same history as ω up to time t. Let us introduce the *regret over the past*:

$$X_t^*(\omega)-X_t(\omega),\quad X_t^*(\omega)=\sup_{s\in[0,t]}X_s(\omega).$$

and the *regret over the future*:

$$U(t,\omega)=\sup_{\omega'\in\mathscr{A}(t,\omega)}\sup_{s\in[t,T]}(X_s-X_t)(\omega')=\sup_{\omega'\in\mathscr{A}(t,\omega)}\sup_{s\in[t,T]}X_s(\omega')-X_t(\omega).$$

Note, that the regret over the past is calculated along the trajectory observed up to the present time t, and the regret over the future corresponds to a worst future scenario. In the sequel we adopt the following assumptions.

Assumption 1. All price trajectories start from some fixed value $x\in\mathbb{R}$: $X_0(\omega)=x$, $\omega\in\Omega$, and the set Ω of trajectories is uniformly bounded:

$$\sup\{|X_t|(\omega): t\in[0,T],\ \omega\in\Omega\}<\infty.$$

Assumption 1 guarantees that $U(t,\omega)<\infty$. Clearly, the function U is non-increasing in t. The next assumption strengthens this property.

Assumption 2. The regret over the future $U(t,\omega)$ is continuous and strictly decreasing in t.

Assumption 3. For any $\omega\in\Omega$ and t there exist $\omega'\in\mathscr{A}(t,\omega)$ such that $X_s(\omega')$ is strictly decreasing in s on $[t,T]$.

Assumption 3 means that at any time moment t it is possible that the asset price will permanently go down in the future.

Definition 1. A function $\tau : \Omega \mapsto [0,T]$ is called a *stopping time* if the conditions $\tau(\omega) \le t$, $X_s(\omega') = X_s(\omega)$, $s \le t$ imply that $\tau(\omega') = \tau(\omega)$.

Remark 1. Consider the σ-algebras $\mathscr{F}_t = \sigma(X_s, s \in [0,t])$, generated by the coordinate mappings. It is proved in [3] (Theorem IV.100 (a)) that an $\mathscr{F}_T$-measurable random variable $\tau : C[0,T] \mapsto [0,T]$ is a stopping time in the sense of Definition 1 if and only if $\{\omega : \tau(\omega) \le s\} \in \mathscr{F}_s$, $s \in [0,T]$. Thus, our definition of a stopping time coincides with the usual one except of an additional measurability property, which we do not need.

Remark 2. For any stopping time τ and $\omega \in \Omega$ we have $\tau(\omega') = \tau(\omega)$, $\omega' \in \mathscr{A}(\tau(\omega), \omega)$, since $X_s(\omega') = X_s(\omega)$, $s \le \tau(\omega)$. Furthermore, for any stopping times τ_1, τ_2 such that $\tau_1(\omega) < \tau_2(\omega)$ we have

$$\tau_1(\omega) = \tau_1(\omega') < \tau_2(\omega'), \quad \omega' \in \mathscr{A}(\tau_1(\omega), \omega).$$

Indeed, otherwise, $\tau_2(\omega'') \le \tau_1(\omega'') = \tau_1(\omega)$ for some $\omega'' \in \mathscr{A}(\tau_1(\omega), \omega)$. But $X_s(\omega'') = X_s(\omega)$, $s \le \tau_1(\omega)$, and the Definition 1 gives a contradiction: $\tau_2(\omega'') = \tau_2(\omega) \le \tau_1(\omega)$.

The overall regret related to a stopping time τ, is defined as follows:

$$\mathscr{R}(\tau, \omega) = \max\left\{X^*_\tau(\omega) - X_\tau(\omega), U(\tau(\omega), \omega)\right\}, \quad X_\tau(\omega) = \omega_{\tau(\omega)}.$$

Definition 2. We call a stopping time σ *perfect* if for any stopping time τ and any $\omega \in \Omega$ the following is true:

(B) if $\tau(\omega) < \sigma(\omega)$ then

$$\mathscr{R}(\tau, \omega') > \mathscr{R}(\sigma, \omega') \quad \text{for all } \omega' \in \mathscr{A}(\tau(\omega), \omega),$$

(A) if $\tau(\omega) > \sigma(\omega)$ then

$$\mathscr{R}(\tau, \omega') > \mathscr{R}(\sigma, \omega') \quad \text{for some } \omega' \in \mathscr{A}(\sigma(\omega), \omega).$$

Condition (B) ("before") means that it is not rational to sell the asset before a perfect stopping time σ, since the overall regret can be reduced by waiting until σ. Condition (A) ("after") means that the overall regret can become larger if the asset is not sold at time σ.

Put

$$\tau^*(\omega) = \inf\{t \ge 0 : (X^*_t - X_t)(\omega) \ge U(t, \omega)\}.$$

Note, that τ^* is correctly defined since $U(T, \omega) = 0$. Clearly, τ^* is a stopping time in the sense of Definition 1. By the continuity of U we get

$$\mathscr{R}(\tau^*, \omega) = (X^*_{\tau^*} - X_{\tau^*})(\omega) = U(\tau^*(\omega), \omega).$$

Thus, τ^* balances the regret over the past and over the future.

Theorem 1. *τ^* is the unique perfect stopping time.*

Proof. If a perfect stopping time exists, then it is unique. Indeed, let σ_1, σ_2 be perfect stopping times. If $\sigma_1(\omega) < \sigma_2(\omega)$, then

$$\mathscr{R}(\sigma_1, \omega') > \mathscr{R}(\sigma_2, \omega'), \quad \omega' \in \mathscr{A}(\sigma_1(\omega), \omega),$$

since σ_2 satisfies (B), and

$$\mathscr{R}(\sigma_2, \omega'') > \mathscr{R}(\sigma_1, \omega'') \quad \text{for some } \omega'' \in \mathscr{A}(\sigma_1(\omega), \omega),$$

since σ_1 satisfies (A). This contradiction indicates that $\sigma_1 \geq \sigma_2$. By symmetry, $\sigma_1 = \sigma_2$.

It remains to show that τ^* is perfect. Assume first that $\tau(\omega) < \tau^*(\omega)$ for some stopping time τ and $\omega \in \Omega$. By Remark 2 we have $\tau(\omega') < \tau^*(\omega')$ for all $\omega' \in \mathscr{A}(\tau(\omega), \omega)$. Since U is strictly decreasing in t, we get

$$\mathscr{R}(\tau, \omega') \geq U(\tau(\omega'), \omega') > U(\tau^*(\omega'), \omega') = \mathscr{R}(\tau^*, \omega'), \quad \omega' \in \mathscr{A}(\tau(\omega), \omega).$$

Hence, τ^* satisfies (B) of Definition 2.

Now assume that $\tau(\omega) > \tau^*(\omega)$. By Remark 2 for any $\omega' \in \mathscr{A}(\tau^*(\omega), \omega)$ we have $\tau(\omega') > \tau^*(\omega') = \tau^*(\omega)$. By Assumption 3 there exists $\omega' \in \mathscr{A}(\tau^*(\omega), \omega)$ such that

$$X_\tau(\omega') < X_{\tau^*}(\omega') = X_{\tau^*}(\omega), \quad X^*_\tau(\omega') = X^*_{\tau^*}(\omega') = X^*_{\tau^*}(\omega).$$

Thus,

$$\mathscr{R}(\tau, \omega') \geq X^*_\tau(\omega') - X_\tau(\omega') > X^*_{\tau^*}(\omega') - X_{\tau^*}(\omega') = \mathscr{R}(\tau^*, \omega'),$$

and τ^* satisfies (A). □

Denote by

$$\overline{\mathscr{R}}(\tau) = \sup_{\omega \in \mathscr{A}} \mathscr{R}(\tau, \omega) = \sup_{\omega \in \mathscr{A}} (X^*_T - X_\tau)(\omega)$$

the worst-case overall regret. It is natural to call a stopping time σ *optimal* if

$$\overline{\mathscr{R}}(\sigma) \leq \overline{\mathscr{R}}(\tau)$$

for all stopping times τ.

Theorem 2. *τ^* is optimal.*

Proof. It is trivial to check that for any stopping time τ the function $\tau \vee \tau^*$ is also a stopping time (here $a \vee b = \max\{a, b\}$). If $\tau(\omega) \geq \tau^*(\omega)$, then $(\tau \vee \tau^*)(\omega) = \tau(\omega)$ and $\mathscr{R}(\tau, \omega) = \mathscr{R}(\tau \vee \tau^*, \omega)$. If $\tau(\omega) < \tau^*(\omega)$ then

$$\mathscr{R}(\tau, \omega) > \mathscr{R}(\tau^*, \omega) = \mathscr{R}(\tau \vee \tau^*, \omega)$$

by the property (B) of τ^*. Thus, $\overline{\mathscr{R}}(\tau) \geq \overline{\mathscr{R}}(\tau \vee \tau^*)$.

Furthermore, since $\Omega = \cup_{\omega \in \Omega} \mathscr{A}(\tau^*(\omega), \omega)$, we have

$$\begin{aligned} \overline{R}(\tau \vee \tau^*) &= \sup_{\omega \in \Omega} \sup \{\mathscr{R}(\tau \vee \tau^*, \omega') : \omega' \in \mathscr{A}(\tau^*(\omega), \omega)\} \\ &\geq \sup_{\omega \in \Omega} \mathscr{R}(\tau^*, \omega) = \overline{\mathscr{R}}(\tau^*), \end{aligned}$$

where the inequality is implied by the property (A) of τ^*. □

3. The case of uniformly Lipschitz continuous price trajectories

Consider the set Ω of $\omega \in C[0, T]$ such that $X_0(\omega) = x$ and

$$-l \cdot (t - s) \leq X_t(\omega) - X_s(\omega) \leq u \cdot (t - s), \quad 0 \leq s < t \leq T \tag{3.1}$$

with some constants $l, u > 0$. In particular, $\omega \in \Omega$ are assumed to be uniformly Lipschitz continuous. Note, that any piecewise linear function

$$\omega_t = \omega_{t_i} + \frac{t - t_i}{t_{i+1} - t_i}(\omega_{t_{i+1}} - \omega_{t_i}), \quad t \in [t_i, t_{i+1}],$$

where $0 = t_0 < t_1 < \cdots < t_n = T$ and $-l \leq (\omega_{t_{i+1}} - \omega_{t_i})/(t_{i+1} - t_i) \leq u$, belongs to Ω.

Furthermore, since

$$\sup_{s\in[t,T]} X_s(\omega') - X_t(\omega') \le u\cdot(T-t), \quad \omega'\in\mathscr{A}(t,\omega),$$

we have $U(t,\omega)\le u\cdot(T-t)$. Moreover, as

$$\omega''_s = \begin{cases}\omega_t + u\cdot(s-t), & s\ge t\\ \omega_s, & s<t\end{cases}$$

belongs to $\mathscr{A}(t,\omega)$, from the equality

$$\sup_{s\in[t,T]} X_s(\omega'') - X_t(\omega'') = u\cdot(T-t)$$

it follows that $U(t,\omega) = u\cdot(T-t)$.

Clearly, the Assumptions 1-3 are satisfied. The perfect stopping time is defined by

$$\tau^*(\omega) = \inf\{t\ge 0 : (X^*_t - X_t)(\omega)\ge u\cdot(T-t)\}.$$

Let $\overline{\tau}\le\tau^*$ be a random variable such that $X_{\overline{\tau}} = X^*_{\tau^*}$. Note, that $\overline{\tau}$ need not be a stopping time. By the definition of τ^* and the left inequality (3.1) we get

$$u\cdot(T-\tau^*) = X^*_{\tau^*} - X_{\tau^*} = -(X_{\tau^*} - X_{\overline{\tau}}) \le l(\tau^*-\overline{\tau}) \le l\tau^*. \tag{3.2}$$

Thus,

$$\tau^* \ge \frac{uT}{l+u} \tag{3.3}$$

and the worst-case regret is estimated as follows:

$$\overline{\mathscr{R}}(\tau^*) = \sup_{\omega\in\Omega} U(\tau^*(\omega),\omega) = \sup_{\omega\in\Omega} u\cdot(T-\tau^*(\omega)) \le u\cdot\left(T - \frac{uT}{l+u}\right) = \frac{lu}{l+u}T.$$

In fact, the equality

$$\overline{\mathscr{R}}(\tau^*) = \frac{lu}{l+u}T$$

holds true. Indeed, for $\overline{\omega}_t = x - lt$ we have

$$(X^*_{\tau^*} - X_{\tau^*})(\overline{\omega}) = x - (x - l\tau^*(\overline{\omega})) = l\tau^*(\overline{\omega}) = u\cdot(T-\tau^*(\overline{\omega})).$$

Hence, $\tau^*(\overline{\omega}) = uT/(l+u)$ and

$$\mathscr{R}(\tau^*,\overline{\omega}) = u\cdot(T-\tau^*(\overline{\omega})) = \frac{lu}{l+u}T.$$

It is interesting to mention that the deterministic stopping time $\widehat{\tau} = uT/(l+u)$ is also optimal. Indeed,

$$U(\widehat{\tau},\omega) = u\left(T - \frac{uT}{l+u}\right) = \frac{lu}{l+u}T,$$

$$X^*_{\widehat{\tau}} - X_{\widehat{\tau}} \le l\widehat{\tau} = \frac{lu}{l+u}T,$$

where the inequality is obtained similar to (3.2). It follows that

$$\overline{\mathscr{R}}(\widehat{\tau}) = \sup_{\omega\in\Omega}\max\{(X^*_{\widehat{\tau}} - X_{\widehat{\tau}})(\omega), U(\widehat{\tau},\omega)\} \le \frac{lu}{l+u}T = \overline{\mathscr{R}}(\tau^*).$$

But the strict inequality is impossible, since τ^* is optimal. Optimal stopping times quite similar to τ^*, $\widehat{\tau}$ appeared in [1].

Note, that from the inequality (3.3) and property (B) of the perfect stopping time τ^* it follows that either $\widehat{\tau} = \tau^*(\omega)$ and $\mathscr{R}(\widehat{\tau},\omega) = \mathscr{R}(\tau^*,\omega)$, or $\widehat{\tau} < \tau^*(\omega)$ and

$$\mathscr{R}(\widehat{\tau},\omega) > \mathscr{R}(\tau^*,\omega).$$

We see that the optimality property alone is not sensitive enough to eliminate stopping times which are not Pareto optimal for the family of objective functions $\{\tau \mapsto \mathscr{R}(\tau,\omega) : \omega \in \Omega\}$.

4. Ratio performance criterion for positive price processes

In this section we explicitly require X to be strictly positive on all outcomes $\omega \in \Omega$. Along with (1.1), a natural performance criterion is the ratio X_T^*/X_τ (see, e.g., [5]), which depends only on relative values of the asset price. The related overall regret can be defined as follows

$$\mathscr{R}(t,\omega) = \max\left\{\frac{X_t^*}{X_t}(\omega), \sup_{\omega'\in\mathscr{A}(t,\omega)} \sup_{s\in[t,T]} \frac{X_s}{X_t}(\omega')\right\}.$$

Put $\ln\omega = (\ln\omega_t)_{t=0}^T$ and consider the set of trajectories $\widehat{\Omega} = \{\ln\omega : \omega \in \Omega\}$. We have

$$\ln\mathscr{R}(t,\omega) = \mathscr{R}(t,\ln\omega) = \max\left\{(X_t^* - X_t)(\ln\omega), U(t,\ln\omega)\right\},$$
$$U(t,\ln\omega) = \sup_{\omega'\in\mathscr{A}(t,\ln\omega)} \sup_{s\in[t,T]} (X_s - X_t)(\ln\omega').$$

So, assuming that $\widehat{\Omega}$ satisfies Assumptions 1-3, we may apply the results of Section 2.

Let's calculate the perfect stopping time for

$$\Omega = \{\omega : X_0(\omega) = x > 0,\ e^{-l\cdot(t-s)} \le \frac{X_t}{X_s}(\omega) \le e^{u\cdot(t-s)},\ 0 \le s < t \le T\},$$

where $l, u > 0$. The set $\widehat{\Omega} = \{\ln\omega : \omega \in \Omega\}$ corresponds to the model, considered in Section 3:

$$-l\cdot(t-s) \le X_t(\ln\omega) - X_s(\ln\omega) \le u\cdot(t-s), \quad 0 \le s < t \le T.$$

From the equality $(X_t^* - X_t)(\ln\omega) = \ln(X_t^*/X_t)(\omega)$ we get

$$\tau^*(\ln\omega) = \inf\{t \ge 0 : \ln\frac{X_t^*}{X_t}(\omega) \ge u(T-t)\} = \inf\{t \ge 0 : \frac{X_t^*}{X_t}(\omega) \ge e^{u(T-t)}\}.$$

Qualitatively similar stopping rule was obtained in [5, Theorem 1] for some combinations of drift and volatility parameters of a geometric Brownian motion.

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