Interdependent Relationships in Game Theory: A Generalized Model

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Abstract: A generalized model of games is proposed, with which both cooperative games and non-cooperative games can be expressed and analyzed, as well as those games that are neither cooperative nor non-cooperative. The model is based on relationships between players and supposed relationships. A relationship is a numerical value that denotes how one player cares for the payoffs of another player. A supposed relationship is a player's belief about the relationship between two players. The players choose their strategies by taking into consideration not only the material payoffs but also relationships and their change. Two games, a prisoner's dilemma and a repeated ultimatum game, are analyzed as examples of application of this model.

Keywords: game theory, cooperative games, non-cooperative games, relationship.

1. Introduction

There have been two kinds of researches in game theory: cooperative games theory pioneered by Von Neumann and Morgenstern (1953) and non-cooperative games theory developed by Nash (1951). When analyzing a game, one first needs to confirm what type the game is because absolutely different methods will be used for two types of games. Methods for non-cooperative games are based on Nash equilibrium, various perfects of Nash equilibrium (e.g., *strong Nash equilibrium* by Aumann (1959), *Subgame Perfect Nash Equilibrium* and *Trembling Hand Perfect Equilibrium* by Selten (1965, 1975), *Bayesian Nash Equilibrium* and *Strict Nash Equilibrium* by Harsanyi (1967, 1973)) and the folk theorems, while cooperative games are analysed by means of coalitions, core, and Shapley value (Shapley (1953)). The most important work comes from non-cooperative analysis, although some scholars regain interest in cooperative games recently.

A question is whether or not any games can be categorized into these two groups? Consider contract bridge, the card game that is played by four players in two competing partnerships. This game is neither cooperative nor non-cooperative because the relations between individual players are not identical. There are both cooperation between partners and competition between two partnerships.

The players in a non-cooperative game only care for their own payoffs whilst the players in a cooperative game care for the payoffs of other players in the coalition as equal important as their own payoffs. If we use a numerical value to denote how much one player cares for another player's payoff, this value will be zero for non-cooperative games and it will be one for cooperative games. One may naturally ask what if this value is set to be neither zero nor one.

In this paper, we introduce the concepts of relationship and supposed relationship. A relationship is a numerical value denoting how much one player cares for another player's payoff. A supposed relationship is a numerical value denoting a player's belief about how much one player cares for another player's payoff. Relationships and supposed relationships are determined by the players and they are changeable in different stages of games. We propose a relationship model of games, in which strategic interaction among players is determined by the material payoffs, relationships, and the players' belief about relationships. Cooperative games and non-cooperative games, as well as those games that are neither cooperative nor non-cooperative can be expressed and analysed by using this model.

Interpersonal relationship has not attracted interests of research in game theory although it has long been an important topic of research in many social science disciplines such as psychology and politics (Kelley (2013), Heider (2013)). Game theorists take it for granted that the relationships between players are predetermined and they will never change during the strategic interactions of players.

Interdependent preference, which denotes that a player's preference depends on his opponent's payoff as well as his own payoff, has been used to explain cooperation phenomena in experimental economics (e.g. Bolton 1991, 2000, Binmore et al. 2002, Ochs and Roth 1989, Samuelson 2001). It is quite similar to the relationship concept defined in this paper. Reputation effect pioneered by Kreps and Wilson (1982) and Milgrom and Roberts (1982) is introduced into game theory to explain cooperation in repeated non-cooperative games. The relationship is obviously different from reputation in that reputation is independent of the game model and then has less effect in one-shot games. Cooper and et al (1996) and Chan (2000) have suggested that reputation is unnecessarily the unique factor leading to cooperation in either infinitely or finitely repeated games.

In the following paper, we introduce a novel game model taking into consideration the relationship and relationship change among players. We also show that there exists a large set of games that are neither cooperative nor non-cooperative.

2. A Relationship Model

Definition 1: For players *i* and *j* in a game, a ***relationship*** R_{ij} is a numerical value denoting how much player *i* cares for player *j*'s payoff. Specially, there is R_{ii}=1.

Player *i*'s attitude toward *j* is non-cooperative when R_{ij}=0, and cooperative when R_{ij}=1. We call it a sub-cooperative attitude when 0<R_{ij}<1, a hostile attitude when R_{ij}<0, or a dedicated attitude when R_{ij}>1.

Definition 2: For players *i*, *j* and *k* in a game, a ***supposed relationship*** ${}_k\overline{R}_{ij}$ is a numerical value denoting how much player *k* thinks player *i* cares for player *j*'s payoff. Specially, there is ${}_k\overline{R}_{ii}$=1.

Player *k* thinks that player *i*'s attitude toward *j* is non-cooperative when ${}_k\overline{R}_{ij}$=0, cooperative when ${}_k\overline{R}_{ij}$=1, sub-cooperative when 0<${}_k\overline{R}_{ij}$<1, hostile when ${}_k\overline{R}_{ij}$<0, or dedicated when ${}_k\overline{R}_{ij}$>1. In a game with complete information, there are $R_{ij} = {}_k\overline{R}_{ij}$, which means that the relationships between players are known to all players. A supposed relationship is not necessarily equivalent to the corresponding relationship in a game of incomplete information.

Obviously cooperative and non-cooperative games are special cases in the set of games defined by different values of $\{R_{ij}\}$ and $\{{}_k\overline{R}_{ij}\}$. For example, a game is non-cooperative when there are $R_{ij} = {}_k\overline{R}_{ij}$ =0 for any $i \neq j$ while a game is cooperative when $R_{ij} = {}_k\overline{R}_{ij}$ =1 for any *i*, *j* and *k*.

Note that $\{R_{ij}\}$ can be considered as a subset of $\{{}_k\overline{R}_{ij}\}$ because every player knows their own relationship and thus there must be ${}_i\overline{R}_{ij} = R_{ij}$. We keep both variables in this paper in order to distinguish a relationship from other player's belief about it.

Definition 3: Relationship model of a *n*-player ***game*** G is a 4-tuple, $G = \{I, S, U, \overline{R}\}$, where $I = \{1, \cdots, n\}$ is a player set, S is a strategy set, U is a payoff set, and $\overline{R}$ is a supposed relationship set.

Definition 4: A ***supposed payoff*** of player *i*, ${}_i\overline{u}_j$, denotes how much player *i* thinks player *j*'s payoff is by taking player *i*'s supposed relationships into consideration.

Given a strategy profile $(s_1, \cdots, s_n)$, player *j*'s payoff $u_j(s_1, \cdots, s_n)$, and player *i*'s supposed relationships $\{{}_i\overline{R}\}$, the supposed payoff, ${}_i\overline{u}_j$, is computed by

$$ {}_i\overline{u}_j(s_1, \cdots, s_n) = \begin{cases} \sum_{k=1}^{n} R_{ik} u_k & \text{if } j = i \\ \sum_{k=1}^{n} {}_i\overline{R}_{jk} u_k & \text{if } j \neq i \end{cases} \tag{1} $$

Under the *relationship model*, the players in a game choose their strategies according to their supposed payoffs. For example, player *i*'s choice is determined by *i*'s supposed payoffs, $\{{}_i\overline{u}\}$. We prove in the following theorem that there must exist a Nash equilibrium for every game when players make choices according to $\{\overline{u}\}$.

Theorem 1: Under relationship model, there must exist a Nash equilibrium for every game.

Proof: Consider an arbitrary player *i* in a *n*-player game ($i \in N$). Let $\{{}_i\overline{u}\}$ denote *i*'s supposed payoffs. Since $\{{}_i\overline{u}\}$ is a complete payoff matrix for *n* players, there must be a Nash equilibrium strategy profile for $\{{}_i\overline{u}\}$, as Nash had proved in [4]. Let s_i be player *i*'s strategy in this profile. The strategies of all players form a new strategy profile $\{s_i\}$, $i = \{1, \cdots, n\}$. It is obvious that $\{s_i\}$ is a Nash equilibrium because every player *i* has no incentive to deviate from s_i. □

Let's analyze the prisoner's dilemma as an example to show how to use the relationship game model. The payoff matrix of a prisoner's dilemma is shown in Fig.1.

		y Player	
		C	*D*
x Player	*C*	3, 3	0, 5
	D	5, 0	1, 1

Figure 1 Two players choose between Cooperate (*C*) and Defect (*D*) in the prisoner's dilemma. The numbers in each cell denote the payoffs of players *x* and *y* respectively.

Let R_{xy} and R_{yx} denote the relationships between the players x and y, and ${}_y\bar{R}_{xy}$ and ${}_x\bar{R}_{yx}$ the corresponding supposed relationships of y and x. The supposed payoffs of player x can be computed according to (1). They are expressed as a matrix as shown in Fig. 2,

		y Player	
		C	*D*
x Player	*C*	$3+3R_{xy}$, $3+3\,{}_x\bar{R}_{yx}$	$5R_{xy}$, 5
	D	5, $5\,{}_x\bar{R}_{yx}$	$1+R_{xy}$, $1+{}_x\bar{R}_{yx}$

Figure 2 The supposed payoffs of player *x*.

What player x chooses between C and D depends on the values of R_{xy} and ${}_x\bar{R}_{yx}$. C is the dominant strategy for player x when $R_{xy} \geq \frac{2}{3}$, while D is dominant when $R_{xy} \leq \frac{1}{4}$. In the case of $\frac{1}{4} < R_{xy} < \frac{2}{3}$, C is dominant when ${}_x\bar{R}_{yx} \leq \frac{1}{4}$ while D is dominant when ${}_x\bar{R}_{yx} \geq \frac{2}{3}$. No strategy is dominant for player x when there are $\frac{1}{4} < R_{xy}, {}_x\bar{R}_{yx} < \frac{2}{3}$.

Similarly, the supposed payoffs of player y are computed and expressed as a matrix as shown in Fig. 3,

		y Player	
		C	*D*
x Player	*C*	$3+3\,{}_y\bar{R}_{xy}$, $3+3R_{yx}$	$5\,{}_y\bar{R}_{xy}$, 5
	D	5, $5R_{yx}$	$1+{}_y\bar{R}_{xy}$, $1+R_{yx}$

Figure 3 The supposed payoffs of player *y*.

Comparing Fig. 3 with Fig. 2, the supposed payoffs of two players will be identical if ${}_y\bar{R}_{xy}$ and R_{yx} are replaced by R_{xy} and ${}_x\bar{R}_{yx}$. C is the dominant strategy for player y when $R_{yx} > \frac{2}{3}$, while D is dominant when $R_{yx} < \frac{1}{4}$. In the case of $\frac{1}{4} \leq R_{yx} \leq \frac{2}{3}$, C is dominant when ${}_y\bar{R}_{xy} < \frac{1}{4}$ while D is dominant when ${}_y\bar{R}_{xy} > \frac{2}{3}$. No strategy is dominant when there are $\frac{1}{4} < R_{yx}, {}_y\bar{R}_{xy} < \frac{2}{3}$.

If we consider the prisoner's dilemma as a game of complete information, there should be $R_{xy} = {}_y\bar{R}_{xy}$ and $R_{yx} = {}_x\bar{R}_{yx}$, which means that the relationships are known to both players. The supposed payoffs of different players are identical in a game of complete information. The players' choices depend on the exact values of R_{xy} and R_{yx}. For example, the strategy profile (C, C) is dominant when $R_{xy} > \frac{2}{3}$ and $R_{yx} > \frac{2}{3}$; (D, D) is dominant when $R_{xy} < \frac{1}{4}$ and $R_{yx} < \frac{1}{4}$; (C, D) is dominant when $R_{xy} > \frac{2}{3}$ and $R_{yx} < \frac{1}{4}$; and (D, C) is dominant when $R_{xy} < \frac{1}{4}$ and $R_{yx} > \frac{2}{3}$. The game has multiple equilibria and no strategy profile is dominant when there are $\frac{1}{4} < R_{yx}, R_{xy} < \frac{2}{3}$.

The strategies of players in a game of incomplete information can be much diverse and complex because the relationships of one player are unknown to others. The supposed relationships are not necessarily equivalent to the corresponding relationship in a game of incomplete information. Since the supposed relationships are private information, the players would take advantage of them in strategic interactions. For example, player *i* attempts to exploit the opponent by setting $R_{ij} < {}_i\overline{R}_{ji}$. On the other hand, $R_{ij} > {}_i\overline{R}_{ji}$ reflects an altruistic attitude of player *i* toward *j*. There should be $R_{ij} = {}_i\overline{R}_{ji}$ if player *i* adopts a tit-for-tat strategy, or in other words, player *i* wants to treat the opponent exactly same as what the opponent treats him/her.

If we consider the prisoner's dilemma as a game of incomplete information, equilibrium analysis can be made given the values of R_{xy}, R_{yx}, ${}_y\overline{R}_{xy}$, and ${}_x\overline{R}_{yx}$. Suppose that there are $R_{xy} = R_{yx} = \frac{1}{3}$ and ${}_y\overline{R}_{xy} = {}_x\overline{R}_{yx} = \frac{1}{5}$, the supposed payoffs can be computed as shown in Fig. 4. From Fig. 4(*a*), *D* is dominant strategy for player *y*. Player *x* will choose *C* given that he/she believes that the other player will choose *D*. Similarly, player *y* will also choose *C* given that he/she thinks the opponent would choose *D*. Thus, strategy profile (*C*, *C*) will be the outcome.

		y Player	
		C	*D*
x Player	*C*	$4, 3\frac{3}{5}$	$1\frac{2}{3}, 5$
	D	$5, 1$	$1\frac{1}{3}, 1\frac{1}{5}$

(*a*)

		y Player	
		C	*D*
x Player	*C*	$3\frac{3}{5}, 4$	$1, 5$
	D	$5, 1\frac{2}{3}$	$1\frac{1}{5}, 1\frac{1}{3}$

(*b*)

Figure 4 (*a*) Supposed payoffs of player *x*. (*b*) Supposed payoffs of player *y*.

3. Repeated games

A repeated game G^T is a 4-tuple, $G = \{I, S, U, \overline{R}^T\}$, where T is the number of iteration. The players in a repeated game will have to take relationship change into consideration when choosing their strategies. Relationship change reflects the complexity of intelligent decision making. It could be a complex action depending on how the players retrieve information from previous interactions with other players and how they update their supposed relationships.

One reason for relationship change in repeated games lies in the fact that previous strategic interactions provide new information about relationships so that the players should update their supposed relationships. Take the iterated prisoner's dilemma as an example. Suppose that two players play the prisoner's dilemma with payoff matrix as shown in Fig. 1 repeatedly and there are $R_{xy} = R_{yx} = \frac{1}{3}$ and ${}_y\overline{R}_{xy} = {}_x\overline{R}_{yx} = \frac{1}{5}$ at the beginning of game. According to two players' supposed payoffs shown in Fig. 4, they choose (*C*, *C*) in the first round. After playing the first round, two players realize that they have underestimated ${}_y\overline{R}_{xy}$ and ${}_x\overline{R}_{yx}$. If the original values of ${}_y\overline{R}_{xy}$ and ${}_x\overline{R}_{yx}$ matched the corresponding relationships exactly, each player should have chosen *D* instead of *C*. Two players should then increase their supposed relationships to some values greater than $\frac{1}{4}$.

Relationship change may take place in games of complete information as well. Assume that there are $R_{xy} \neq R_{yx}$ in the above example. A tit-for-tat player will update his/her relationship to make sure $R_{xy} = R_{yx}$.

In the following example we analyze a repeated ultimatum game in order to show how players take into consideration relationship change in game playing.

Let's consider an infinitely repeated ultimatum game. In each round, Row player proposes an offer of dividing one dollar between two players. If Column player accepts the offer, they receive the corresponding share. Otherwise, both players receive nothing. The minimum division of one dollar is one cent. Fig. 5 shows the payoff matrix of this game. The supposed payoffs of Column player are shown in Fig. 6.

		Column player	
		Accept	*Reject*
Row player	*0*	1, 0	0, 0
	0.01	0.99, 0.01	0, 0
	...	…	0, 0
	α	1-*α*, *α*	0, 0
	...	…	0, 0
	0.99	0.01, 0.99	0, 0

Figure 5 A ultimatum game in which Row player makes an offer of dividing a dollar to Column player. If Column player accepts the offer, two players share the dollar as the offer suggests. Otherwise, both receive zero.

		Column player	
		Accept	*Reject*
Row player	*0*	1, 0	0, 0
	...	…	0, 0
	α	$1-\alpha+{}_C\bar{R}_{RC}\alpha,\ \alpha+(1-\alpha)\,R_{CR}$	0, 0
	...	…	0, 0

Figure 6 The supposed payoffs of Column player.

The supposed payoffs of two players can be computed according to the relationships and supposed relationships of R_{RC}, R_{CR}, ${}_R\bar{R}_{CR}$, and ${}_C\bar{R}_{RC}$. For an offer α, Row player's supposed payoff profile is ($1-\alpha+R_{RC}\alpha$, $\alpha+(1-\alpha)\ {}_R\bar{R}_{CR}$) while Column player's supposed payoff profile is ($1-\alpha+{}_C\bar{R}_{RC}\alpha$, $\alpha+(1-\alpha)\,R_{CR}$).

In order for Column player to accept α, there must be $\alpha+(1-\alpha)\,R_{CR}>0$, or $\alpha > \frac{-R_{CR}}{1-R_{CR}}$. On the other hand, for Row player there must be $1-\alpha+R_{RC}\alpha>0$, or $\frac{1}{1-R_{RC}} > \alpha$. Taking into consideration the supposed relationships of two players, Row player will offer α satisfying $\frac{1}{1-R_{RC}} > \alpha > \frac{-{}_R\bar{R}_{CR}}{1-{}_R\bar{R}_{CR}}$ and Column player will accept α satisfying $\frac{1}{1-{}_C\bar{R}_{RC}} > \alpha > \frac{-R_{CR}}{1-R_{CR}}$.

When ${}_R\bar{R}_{CR} \geq 0$, Row player thinks Column player would accept any offer $\alpha>0$ so Row player would offer $\alpha=0.01$ to maximize the payoff. This is what non-cooperative game theory has predicted. However, Column player could choose R_{CR}<0 and would probably reject the offer of α=0.01. In an infinitely repeated game, two players will reach an agreement with the offer in range of $\frac{1}{1-R_{RC}} > \alpha > \frac{-R_{CR}}{1-R_{CR}}$ even if two players cannot communicate with each other. Consider the case of $R_{RC}=R_{CR}$=-0.5 and each player does not know the exactly relationship of the other player, for example. Column player will reject any offer with $\frac{1}{3} > \alpha$, which transfers the information about Column player's relationship to Row player who will have to increase the offer. This process is similar to a two-player bargaining game in which rational players will reach an agreement with $\frac{2}{3} > \alpha > \frac{1}{3}$ in the bargaining.

Relationships have a significant influence on strategic interaction in games. Players could choose relationships in order to maximize payoffs. For example, Column player in the above game chooses a minus relationship. How players take advantage of relationships in game playing depends on specific material payoffs of the game and it could be a complex problem.

6. Conclusions

A generalized model of games that takes into consideration the relationships between players is proposed. Cooperative games and non-cooperative games are special cases in this model. There exists a significant set of games that are neither cooperative nor non-cooperative, which have not been investigated in game theory.

We prove that there must exist Nash equilibrium for every game under the relationship model so that equilibrium analysis developed in non-cooperative game theory can be applied. A prisoner's dilemma and a repeated Ultimatum game are analyzed.

One advantage of the relationship model lies in that it provides an accurate description of the players' attitudes toward others in game playing. A player's attitude toward another player could be sub-cooperative, hostile, dedicated, as well as cooperative and non-cooperative. By taking supposed relationships into consideration, the attitudes of altruism and exploitation can be considered. Relationships and relationship changes make the strategies of players interdependent. How the players take advantage of relationships and relationship changes will be my future research.

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