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COUNTING LATTICE POINTS IN FREE SUMS OF POLYTOPES

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Abstract. We show how to compute the Ehrhart polynomial of the free sum of two lattice polytopes containing the origin P and Q in terms of the enumerative combinatorics of P and Q. This generalizes work of Beck, Jayawant, McAllister, and Braun, and follows from the observation that the weighted h^*-polynomial is multiplicative with respect to the free sum. We deduce that given a lattice polytope P containing the origin, the problem of computing the number of lattice points in all rational dilates of P is equivalent to the problem of computing the number of lattice points in all integer dilates of all free sums of P with itself.

Let P and Q be full-dimensional lattice polytopes containing the origin with respect to lattices $N_P \cong \mathbb{Z}^{\dim P}$ and $N_Q \cong \mathbb{Z}^{\dim Q}$ respectively. The **free sum** (also known as 'direct sum') $P \oplus Q$ is a full-dimensional lattice polytope containing the origin in the lattice $N_P \oplus N_Q$, defined by:

$$P \oplus Q = \operatorname{conv}((P \times 0_Q) \cup (0_P \times Q)) \subseteq (N_P \oplus N_Q)_{\mathbb{R}},$$

where $\operatorname{conv}(S)$ denotes the convex hull of a set S, $N_{\mathbb{R}} := N \otimes_{\mathbb{R}} \mathbb{R}$ for a lattice N, and $0_P, 0_Q$ denote the origin in N_P, N_Q respectively.

The **Ehrhart polynomial** $f(P; m)$ of P is a polynomial of degree $\dim P$ characterized by the property that $f(P; m) = \#(mP \cap N_P)$ for all $m \in \mathbb{Z}_{\geq 0}$ [6]. Our goal is to describe the Ehrhart polynomial of $P \oplus Q$ in terms of the enumerative combinatorics of P and Q.

We first observe that $\{\#(\lambda P \cap N_P) \mid \lambda \in \mathbb{Q}_{\geq 0}\}$ and $\{\#(\lambda Q \cap N_Q) \mid \lambda \in \mathbb{Q}_{\geq 0}\}$ determine $\{\#(\lambda(P \oplus Q) \cap (N_P \oplus N_Q)) \mid \lambda \in \mathbb{Q}_{\geq 0}\}$, and hence the set $\{\#(m(P \oplus Q) \cap (N_P \oplus N_Q)) \mid m \in \mathbb{Z}_{\geq 0}\}$, which is encoded by the Ehrhart polynomial of $P \oplus Q$ (see (8) for a partial converse). Indeed, this follows from the following observation: if $\partial_{\neq 0} P$ denotes the union of the facets of P not containing the origin, then, by definition, for any $\lambda \in \mathbb{Q}_{\geq 0}$:

$$\#(\partial_{\neq 0}(\lambda P) \cap N_P) = \#(\lambda P \cap N_P) - \max_{0 \leq \lambda' < \lambda} \#(\lambda' P \cap N_P),$$

and

$$\partial_{\neq 0}(\lambda(P \oplus Q)) = \bigcup_{\substack{\lambda_P, \lambda_Q \geq 0 \\ \lambda_P + \lambda_Q = \lambda}} \partial_{\neq 0}(\lambda_P P) \times \partial_{\neq 0}(\lambda_Q Q), \tag{1}$$

where the right hand side is a disjoint union.

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It will be useful to express the invariants above in terms of corresponding generating series. Firstly, the Ehrhart polynomial may be encoded as follows:

$$\sum_{m\geq 0} f(P;m)t^m = \frac{h^*(P;t)}{(1-t)^{\dim P+1}},$$

where $h^*(P;t) \in \mathbb{Z}[t]$ is a polynomial of degree at most $\dim P$ with non-negative integer coefficients, called the **h^*-polynomial** of P [9]. Secondly, let $M_P := \mathrm{Hom}(N_P, \mathbb{Z})$ be the dual lattice, and recall that the dual polyhedron $P^\vee$ is defined to be $P^\vee = \{u \in (M_P)_\mathbb{R} \mid \langle u, v\rangle \geq -1 \text{ for all } v \in P\}$. Let

$$r_P := \min\{r \in \mathbb{Z}_{>0} \mid rP^\vee \text{ is a lattice polyhedron }\}. \tag{2}$$

Note that since $(P \oplus Q)^\vee$ is the Cartesian product $P^\vee \times Q^\vee$, we have $r_{P\oplus Q} = \mathrm{lcm}(r_P, r_Q)$. Then one may associate a generating series encoding $\{\#(\lambda P \cap N_P) \mid \lambda \in \mathbb{Q}_{\geq 0}\}$:

$$\sum_{\lambda\in\mathbb{Q}_{\geq 0}} \#(\partial_{\neq 0}(\lambda P) \cap N_P)t^\lambda = \frac{\widetilde{h}(P;t)}{(1-t)^{\dim P}}, \tag{3}$$

where $\widetilde{h}(P;t) \in \mathbb{Z}[t^{\frac{1}{r_P}}]$ is a polynomial of degree at most $\dim P$ with fractional exponents and non-negative integer coefficients, called the **weighted h^*-polynomial** of P.

Remark 1. The weighted h^*-polynomial was introduced in [10] and generalized in [13, Section 4.3]. For the specific definition given in (3), see the proof of Proposition 2.6 in [10] with $\lambda \equiv 0$ and $s = t$. For the non-negativity of the coefficients together with a formula to compute $\widetilde{h}(P;t)$, see (15) in [10]. For the fact that $\widetilde{h}(P;t) \in \mathbb{Z}[t^{\frac{1}{r_P}}]$, see Remark 5 below. Note that it follows from (3) that if we write $\widetilde{h}(P;t) = \sum_{j\in\mathbb{Q}} \widetilde{h}_{P,j}t^j$, then the polynomial $\sum_{i\in\mathbb{Z}} \widetilde{h}_{P,i}t^i$ consisting of the terms with integer-valued exponents of t is precisely the h^*-polynomial associated to the lattice polyhedral complex determined by the union of the facets of P not containing the origin.

Moreover, let

$$\Psi : \bigcup_{r\in\mathbb{Z}_{>0}} \mathbb{R}[t^{1/r}] \to \mathbb{R}[t] \tag{4}$$

denote the $\mathbb{R}$-linear map defined by $\Psi(t^j) = t^{\lceil j\rceil}$ for all $j \in \mathbb{Q}_{\geq 0}$. Then we recover the h^*-polynomial of P via the formula (see (14) in [10]):

$$h^*(P;t) = \Psi(\widetilde{h}(P;t)). \tag{5}$$

We also note that when the origin lies in the relative interior of P, we have the symmetry [10, Corollary 2.12]:

$$\widetilde{h}(P;t) = t^{\dim P}\widetilde{h}(P;t^{-1}). \tag{6}$$

Then (1) immediately implies the following multiplicative formula.

Lemma 2. *Let P, Q be full-dimensional lattice polytopes containing the origin with respect to lattices N_P, N_Q respectively. Then*

$$\widetilde{h}(P \oplus Q; t) = \widetilde{h}(P; t)\widetilde{h}(Q; t).$$

Combined with (5), we deduce the following formula for the Ehrhart polynomial of $P \oplus Q$:

$$h^*(P \oplus Q; t) = \Psi(\widetilde{h}(P; t)\widetilde{h}(Q; t)). \tag{7}$$

Remark 3. Let $\Theta : \bigcup_{r \in \mathbb{Z}_{>0}} \mathbb{R}[t^{1/r}] \to \mathbb{R}[t]$ denote the $\mathbb{R}$-linear map defined by $\Theta(t^j) = t^{j - \lfloor j \rfloor}$ for all $j \in \mathbb{Q}_{\geq 0}$. Then [13, Example 4.12] gives an explicit formula for $\Theta(\widetilde{h}(P; t))$ that we will describe below.

Each facet F of P not containing the origin has the form

$$F = P \cap \{v \in (N_P)_{\mathbb{R}} \mid \langle u_F, v \rangle = -m_F\},$$

where $u_F \in M_P$ is a primitive integer vector, and $m_F \in \mathbb{Z}_{>0}$ is the **lattice distance** of F from the origin. Then the vertices of $P^{\vee}$ are precisely $\{\frac{u_F}{m_F} \mid F \text{ facet of } P, 0 \notin F\}$, and hence $r_P = \operatorname{lcm}(m_F \mid F \text{ facet of } P, 0 \notin F)$. Then

$$\Theta(\widetilde{h}(P; t)) = \sum_{\substack{F \text{ facet of } P \\ 0 \notin F}} \operatorname{Vol}(F) \sum_{i=0}^{m_F - 1} t^{\frac{i}{m_F}},$$

where $\operatorname{Vol}(F)$ is defined in Remark 4 below.

Remark 4. For any lattice polytope F, $h^*(F; 1)$ is equal to the **normalized volume** $\operatorname{Vol}(F)$ of F, i.e. after possibly replacing the underlying lattice with a smaller lattice, we may assume that $F \subseteq N_{\mathbb{R}} \cong \mathbb{R}^{\dim F}$ for some lattice N, and then $\operatorname{Vol}(F)$ is $(\dim F)!$ times the Euclidean volume of F. In the formula in Remark 3, to make the connection with [13, Example 4.12] explicit, observe that $\operatorname{Vol}(F') = m_F \operatorname{Vol}(F)$, where F' is the convex hull of F and the origin.

Remark 5. Remark 3 shows that r_P is the minimal choice of denominator in the fractional exponents in $\widetilde{h}(P; t)$ in the sense that $\widetilde{h}(P; t) \in \mathbb{Z}[t^{1/r_P}]$ and if $\widetilde{h}(P; t) \in \mathbb{Z}[t^{1/r}]$, then r_P divides r. For example, $\widetilde{h}(P; t) = h^*(P; t)$ if and only if $r_P = 1$.

Remark 6. If P and Q contain the origin, but are not full-dimensional, then one may apply the results above after replacing N_P and N_Q by their intersections with the linear spans of P and Q respectively. If P contains the origin but not Q, then one may replace Q with $Q' = \operatorname{conv}(Q, 0_Q)$ since $P \oplus Q = P \oplus Q'$.

If neither P nor Q contain the origin, but satisfy the property that the affine spans of P and Q are strict subsets of the linear spans of P and Q respectively, then P, Q and $P \oplus Q$ are the unique facets not containing the origin of $P' = \operatorname{conv}(P, 0_P)$, $Q' = \operatorname{conv}(Q, 0_Q)$ and $P' \oplus Q'$ respectively. In this case, by Remark 1 and Lemma 2, $h^*(P \oplus Q; t)$ is the polynomial consisting of the terms of $\widetilde{h}(P' \oplus Q'; t) = \widetilde{h}(P'; t)\widetilde{h}(Q'; t)$ with integer-valued exponents of t.

We deduce a new proof to the following result of Beck, Jayawant and McAllister [3, Theorem 1.3], which itself generalizes a result of Braun [4].

Corollary 7. *Let P, Q be full-dimensional lattice polytopes containing the origin with respect to lattices N_P, N_Q respectively. Then*

$$h^*(P \oplus Q; t) = h^*(P; t)h^*(Q; t) \iff r_P = 1 \text{ or } r_Q = 1.$$

Proof. If we write $\widetilde{h}(P; t) = \sum_{j \in \mathbb{Q}} \widetilde{h}_{P,j} t^j$, then by (5) and (7),

$$h^*(P \oplus Q; t) = \sum_{j,j' \in \mathbb{Q}} \widetilde{h}_{P,j} \widetilde{h}_{Q,j'} t^{\lceil j+j' \rceil},$$

$$h^*(P; t)h^*(Q; t) = \sum_{j,j' \in \mathbb{Q}} \widetilde{h}_{P,j} \widetilde{h}_{Q,j'} t^{\lceil j \rceil + \lceil j' \rceil},$$

If $r_P = 1$ or $r_Q = 1$, then we have equality. If $r_P, r_Q > 1$, then by Remark 3, there exists $(j, j') \in \mathbb{Q}^2$ such that $\widetilde{h}_{P,j}, \widetilde{h}_{Q,j'} > 0$ and $0 < j - \lfloor j \rfloor, j' - \lfloor j' \rfloor \leq 1/2$. Then $\lceil j \rceil + \lceil j' \rceil = \lceil j + j' \rceil + 1$, and the non-negativity of the coefficients of $\widetilde{h}(P; t)$ and $\widetilde{h}(Q; t)$ implies that $h^*(P \oplus Q; t) \neq h^*(P; t)h^*(Q; t)$. □

Remark 8. A lattice polytope P satisfying $r_P = 1$ and containing the origin in its relative interior is called **reflexive**. These polytopes have received a lot of attention, in particular because of their role in Batyrev and Borisov's construction of mirror pairs of Calabi-Yau varieties [1].

Remark 9. The weighted h^*-polynomial arises naturally in two distinct geometric situations: Firstly, in the computation of dimensions of the graded pieces of orbifold cohomology groups of toric stacks [10, Theorem 4.3] and, more generally, in the computation of motivic integrals on toric stacks [11, Theorem 6.5]. Secondly, in computations of the action of monodromy on the cohomology of the fiber of a degeneration of complex hypersurfaces (or the associated Milnor fiber) [13, Sections 5,6, Corollary 5.12]. In particular, the multiplicative formula in Lemma 2 may be viewed as a Künneth formula for the dimensions of the graded pieces of orbifold cohomology groups of toric stacks.

Example 10. Let $N_P = \mathbb{Z}$ and let $P = [-2, 2]$, $Q = [-1, 3] = P + 1$. Then $r_P = 2, r_Q = 3$, and one may compute:

$$h^*(P; t) = h^*(Q; t) = 1 + 3t.$$

$$\widetilde{h}(P; t) = 1 + 2t^{1/2} + t, \widetilde{h}(Q; t) = 1 + t^{1/3} + t^{2/3} + t,$$

$$h^*(P \oplus P; t) = 1 + 10t + 5t^2, h^*(P \oplus Q; t) = 1 + 8t + 7t^2.$$

$$\widetilde{h}(P \oplus P; t) = 1 + 4t^{1/2} + 6t + 4t^{1/2} + t^2,$$

$$\widetilde{h}(P \oplus Q; t) = 1 + t^{1/3} + 2t^{1/2} + t^{2/3} + 2t^{5/6} + 2t + 2t^{7/6} + t^{4/3} + 2t^{3/2} + t^{5/3} + t^2.$$

Example 11. In order to provide a wider class of examples of weighted h^*-polynomials, we consider a class of examples of lattice polytopes used by Payne in [8]. Consider positive integers $\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_d$ with no common factor and let $N = \mathbb{Z}^{d+1}/(\sum_{i=0}^d \alpha_i e_i = 0)$, where $e_0, \ldots, e_d$ denotes the standard basis of $\mathbb{Z}^{d+1}$. Observe that N is a lattice of rank d and, if $P(\alpha_0, \ldots, \alpha_d)$ denotes the convex hull of the images of $e_0, \ldots, e_d$, then $P(\alpha_0, \ldots, \alpha_d)$ is a lattice polytope containing the origin in its relative interior. The following formula follows from the proof of [12, Lemma 9.1]:

$$\widetilde{h}(P(\alpha_0, \ldots, \alpha_d); t) = \sum_{i=0}^{d} \sum_{j=0}^{\alpha_i - 1} t^{\sum_{0 \leq k \leq d, k \neq i} (\frac{j\alpha_k}{\alpha_i} - \lfloor \frac{j\alpha_k}{\alpha_i} \rfloor) + \sum_{k=i+1}^{d} \varphi(\frac{j\alpha_k}{\alpha_i})},$$

where $\varphi(x) = 1$ if x is an integer and $\varphi(x) = 0$, otherwise.

We now consider a partial converse to (7). We will use the following lemma due to Terence Harris [7].

Lemma 12. *Let $f(t) \in \mathbb{R}[t^{1/r}]$ be a polynomial with non-negative coefficients and fractional exponents for some $r \in \mathbb{Z}_{>0}$. Fix a positive real number x. For any $n \in \mathbb{Z}_{>0}$, let $f_n^*(t) := \Psi(f(t)^n) \in \mathbb{R}[t]$, where Ψ is defined in (4) i.e. $f_n^*(t)$ is obtained from $f(t)^n$ by rounding up exponents in t. Then*

$$f_n^*(x) \leq f(x)^n \leq x^{\frac{1}{r}-1} f_n^*(x) \text{ if } 0 < x \leq 1,$$
$$f(x)^n \leq f_n^*(x) \leq x^{1-\frac{1}{r}} f(x)^n \text{ if } x \geq 1.$$

In particular, given any polynomial $f(t) \in \bigcup_{r \in \mathbb{Z}_{>0}} \mathbb{R}[t^{1/r}]$ with non-negative coefficients and fractional exponents,

$$f(x) = \lim_{n \to \infty} f_n^*(x)^{1/n},$$

and $f(t)$ determines and is determined by $\{f_n^*(t) \mid n \in \mathbb{Z}_{>0}\}$.

Proof. First assume that $x \geq 1$. When $n = 1$, the inequalities $\Psi(f(t))_{t=x} \leq f(x) \leq x^{1-\frac{1}{r}} \Psi(f(t))_{t=x}$ follow from the fact that $x^{\frac{i}{r}} \leq x^{\lceil \frac{i}{r} \rceil} \leq x^{1-\frac{1}{r}} x^{\frac{i}{r}}$ for any $i \in \mathbb{Z}_{\geq 0}$, and the assumption that the coefficients of $f(t)$ are non-negative. When $n \geq 1$, the inequalities follow by replacing $f(t)$ with $f(t)^n$. When $0 < x \leq 1$, $x^{\lceil \frac{i}{r} \rceil} \leq x^{\frac{i}{r}} \leq x^{\frac{1}{r}-1} x^{\lceil \frac{i}{r} \rceil}$ and the result follows similarly. The final statement follows immediately. □

For any positive integer n, let $P^{\oplus n}$ denote the free sum of P with itself n times. By Lemma 2 and (5), one may apply the above lemma with $f(t) = \widetilde{h}(P; t)$, $f(t)^n = \widetilde{h}(P^{\oplus n}; t)$, $f_n^*(t) = h^*(P^{\oplus n}; t)$ and $r = r_P$, to obtain the corollary below.

Corollary 13. *Let P be a full-dimensional lattice polytope containing the origin with respect to a lattice N_P. Fix a positive real number x. For any $n \in \mathbb{Z}_{>0}$, and with r_P as defined in (2),*

$$h^*(P^{\oplus n}; x) \leq \widetilde{h}(P; x)^n \leq x^{\frac{1}{r_P}-1} h^*(P^{\oplus n}; x) \text{ if } 0 < x \leq 1,$$
$$\widetilde{h}(P; x)^n \leq h^*(P^{\oplus n}; x) \leq x^{1-\frac{1}{r_P}} \widetilde{h}(P; x)^n \text{ if } x \geq 1.$$

In particular,

$$\widetilde{h}(P;x) = \lim_{n\to\infty} h^*(P^{\oplus n};x)^{1/n},$$

and $\widetilde{h}(P;t)$ *determines and is determined by* $\{h^*(P^{\oplus n};t) \mid n \in \mathbb{Z}_{>0}\}$.

Note that the final statement above states that the following two sets contain precisely the same information:

$$\{\#(\lambda P \cap N_P) \mid \lambda \in \mathbb{Q}_{\geq 0}\}, \tag{8}$$

$$\{\#(mP^{\oplus n} \cap (N_P \oplus \cdots \oplus N_P)) \mid m \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}_{>0}\}.$$

Remark 14. From the proof of Corollary 13, $\widetilde{h}(P;t)$ determines and is determined by $\{h^*(P^{\oplus n};t) \mid n \in S\}$ for any infinite subset $S \subseteq \mathbb{Z}_{>0}$.

Finally, the above results together with the central limit theorem describe some of the asymptotic behavior of $h^*(P^{\oplus n};t)$ as $n \to \infty$. More precisely, let X_P^*, $\widetilde{X}_P$ be $\mathbb{R}$-valued random variables with probability distributions on $\mathbb{R}$ defined by:

$$\mathbb{P}(X_P^* = i) = \frac{h_{P,i}^*}{\operatorname{Vol}(P)},$$

$$\mathbb{P}(\widetilde{X}_P = i) = \frac{\widetilde{h}_{P,j}}{\operatorname{Vol}(P)},$$

where $h^*(P;t) = \sum_{i\in\mathbb{Z}} h_{P,i}^* t^i$ and $\widetilde{h}(P;t) = \sum_{j\in\mathbb{Q}} \widetilde{h}_{P,j} t^j$, and $h^*(P;1) = \widetilde{h}(P;1) = \operatorname{Vol}(P)$ (see Remark 4). Equivalently, the moment generating functions of X_P^* and $\widetilde{X}_P$ are given by:

$$\mathbb{E}[e^{sX_P^*}] = \frac{1}{\operatorname{Vol}(P)} h^*(P;e^s),$$

$$\mathbb{E}[e^{s\widetilde{X}_P}] = \frac{1}{\operatorname{Vol}(P)} \widetilde{h}(P;e^s).$$

Let $\widetilde{\mu}_P$ and $\widetilde{\sigma}_P$ denote the mean and standard deviation of $\widetilde{X}_P$ respectively, and let $\mathcal{N}(\mu,\sigma)$ denote the normal distribution with mean μ and variance σ.

Example 15. When the origin lies in the relative interior of P, (6) implies that $\widetilde{\mu}_P = \frac{\dim P}{2}$.

Corollary 16. *Let P be a full-dimensional lattice polytope containing the origin in a lattice N_P. Then*

$$\frac{X_{P^{\oplus n}}^* - n\widetilde{\mu}_P}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,\widetilde{\sigma}_P),$$

as $n \to \infty$, where convergence means convergence in distribution (see Remark 17).

Proof. Fix $s \in \mathbb{R}$. Then by Corollary 13, for any $n \in \mathbb{Z}_{>0}$,

$$e^{-s\widetilde{\mu}_P\sqrt{n}} h^*(P^{\oplus n}; e^{\frac{s}{\sqrt{n}}}) \le e^{-s\widetilde{\mu}_P\sqrt{n}} \widetilde{h}(P; e^{\frac{s}{\sqrt{n}}})^n \le e^{\frac{s(1-r_P)}{\sqrt{n}r_P}} e^{-s\widetilde{\mu}_P\sqrt{n}} h^*(P^{\oplus n}; e^{\frac{s}{\sqrt{n}}}) \text{ if } s \le 0,$$

$$e^{-s\widetilde{\mu}_P\sqrt{n}} \widetilde{h}(P; e^{\frac{s}{\sqrt{n}}})^n \le e^{-s\widetilde{\mu}_P\sqrt{n}} h^*(P^{\oplus n}; e^{\frac{s}{\sqrt{n}}}) \le e^{\frac{s(r_P-1)}{\sqrt{n}r_P}} e^{-s\widetilde{\mu}_P\sqrt{n}} \widetilde{h}(P; e^{\frac{s}{\sqrt{n}}})^n \text{ if } s \ge 0.$$

If $\widetilde{X}_1, \dots, \widetilde{X}_n$ are iid random variables with distribution $\widetilde{X}_P$, and $\widetilde{Z}_n := \frac{(\widetilde{X}_1 - \widetilde{\mu}_P) + \cdots + (\widetilde{X}_n - \widetilde{\mu}_P)}{\sqrt{n}}$, then either a direct computation or invoking the central limit theorem gives:

$$\lim_{n\to\infty} \mathbb{E}[e^{s\widetilde{Z}_n}] = \lim_{n\to\infty} e^{-s\widetilde{\mu}_P\sqrt{n}} \widetilde{h}(P; e^{\frac{s}{\sqrt{n}}})^n = e^{\frac{(s\widetilde{\sigma}_P)^2}{2}},$$

where $e^{\frac{(s\widetilde{\sigma}_P)^2}{2}}$ is the moment generating function of $\mathcal{N}(0, \widetilde{\sigma}_P)$. If $Z_n^* := \frac{X^*_{P^{\oplus n}} - n\widetilde{\mu}_P}{\sqrt{n}}$, then the above inequalities state that

$$\mathbb{E}[e^{sZ_n^*}] \le \mathbb{E}[e^{s\widetilde{Z}_n}] \le e^{\frac{s(1-r_P)}{\sqrt{n}r_P}} \mathbb{E}[e^{sZ_n^*}] \text{ if } s \le 0,$$

$$\mathbb{E}[e^{s\widetilde{Z}_n}] \le \mathbb{E}[e^{sZ_n^*}] \le e^{\frac{s(r_P-1)}{\sqrt{n}r_P}} \mathbb{E}[e^{s\widetilde{Z}_n}] \text{ if } s \ge 0.$$

Hence $\lim_{n\to\infty} \mathbb{E}[e^{sZ_n^*}] = \lim_{n\to\infty} \mathbb{E}[e^{s\widetilde{Z}_n}] = e^{\frac{(s\widetilde{\sigma}_P)^2}{2}}$ and the result follows since convergence of the moment generating functions of Z_n^* to the moment generating function of $\mathcal{N}(0, \widetilde{\sigma}_P)$ implies convergence of the corresponding distributions [5, Theorem 3] (note that all moment generating functions above converge for all $s \in \mathbb{R}$). ☐

Remark 17. The convergence in Corollary 16 is defined in terms of the corresponding cumulative distribution functions as follows: for all $x \in \mathbb{R}$, if we write $h^*(P^{\oplus n}; t) = \sum_{i \in \mathbb{Z}} h^*_{P^{\oplus n}, i} t^i$,

$$F_n(x) = \mathbb{P}\big(\frac{X^*_{P^{\oplus n}} - n\widetilde{\mu}_P}{\sqrt{n}} \le x\big) = \frac{1}{\operatorname{Vol}(P)^n} \sum_{\substack{i \in \mathbb{Z} \\ i \le \sqrt{n}x + n\widetilde{\mu}_P}} h^*_{P^{\oplus n}, i},$$

and $\Phi_{\widetilde{\sigma}_P}(x) = \frac{1}{\sqrt{2\pi}\widetilde{\sigma}_P} \int_{-\infty}^{x} e^{-(\frac{s}{\widetilde{\sigma}_P})^2/2} ds$, then $\lim\limits_{n\to\infty} F_n(x) = \Phi_{\widetilde{\sigma}_P}(x)$.

Example 18. A lattice polytope P containing the origin is a **standard simplex** if its non-zero vertices form a basis of N_P. In this case, $\widetilde{h}(P; t) = h^*(P; t) = 1$, $\widetilde{\mu}_P = \widetilde{\sigma}_P = 0$ and $P^{\oplus n}$ is a standard simplex for all n. This is the only case when $\widetilde{\sigma}_P = 0$.

Example 19. Fix a positive integer n and consider the lattice $N = \mathbb{Z}[e^{\frac{2\pi i}{n}}]$. The **n-th cyclotomic polytope** $\mathcal{C}_n$ is the convex hull of all n-th roots of unity in $N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{\varphi(n)}$, where φ is the Euler totient function. In [2, Theorem 7,Lemma 8,Corollary 9], Beck and Hoşten prove that the lattice points of $\mathcal{C}_n$ consist of the n-th roots of unity, which are vertices, together with the origin, which is the unique interior lattice point, and they identify $\mathcal{C}_n$ with $\mathcal{C}_{\operatorname{sqf}(n)}^{\oplus \frac{n}{\operatorname{sqf}(n)}}$, where $\operatorname{sqf}(n)$ denotes the square-free part of n i.e. the product of the prime divisors of n. Moreover, they prove that $\mathcal{C}_n$ is reflexive if n is divisible by at most two odd primes, and they show how to compute $h^*(\mathcal{C}_n; t) = \widetilde{h}(\mathcal{C}_n; t)$ for $n \le 104$. The smallest value of n for which $h^*(\mathcal{C}_n; t)$ is unknown is $n = 105 = 3 \cdot 5 \cdot 7$. We refer the reader to [2] for further results and details.

By (7), and using Beck and Hoşten's result above, for any positive integer n,

$$h^*(\mathcal{C}_n;t) = h^*(\mathcal{C}_{\mathrm{sqf}(n)}^{\oplus\frac{n}{\mathrm{sqf}(n)}};t) = \Psi(\widetilde{h}(\mathcal{C}_{\mathrm{sqf}(n)};t)^{\frac{n}{\mathrm{sqf}(n)}}).$$

It follows from this observation and Remark 14 that for any product of distinct primes a, the problem of computing $\{h^*(\mathcal{C}_n;t) \mid \mathrm{sqf}(n) = a\}$ is equivalent to the problem of computing $\widetilde{h}(\mathcal{C}_a;t)$. More precisely, consider a strictly increasing sequence of positive integers $\{n_k\}_{k\in\mathbb{Z}_{>0}}$ satisfying $\mathrm{sqf}(n_k) = a$ for all k. Then by Remark 14, $\{h^*(\mathcal{C}_{n_k};t) = h^*(\mathcal{C}_a^{\oplus\frac{n_k}{a}};t) \mid k \in \mathbb{Z}_{>0}\}$ determines and is determined by $\widetilde{h}(\mathcal{C}_a;t)$. Moreover, since $\widetilde{\mu}_{\mathcal{C}_a} = \frac{\dim \mathcal{C}_a}{2} = \frac{\varphi(a)}{2}$ by Example 15, setting $P = \mathcal{C}_a$ and $n = \frac{n_k}{a}$ in Corollary 16 implies that

$$\frac{X^*_{\mathcal{C}_{n_k}} - \frac{n_k\varphi(a)}{2a}}{\sqrt{\frac{n_k}{a}}} \xrightarrow{d} \mathcal{N}(0, \widetilde{\sigma}_{\mathcal{C}_a}),$$

as $k \to \infty$. We note that it is an open problem to compute $\widetilde{\sigma}_{\mathcal{C}_a}$ for any product of distinct primes a.

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