INDEX-p ABELIANIZATION DATA OF p-CLASS TOWER GROUPS, II

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ABSTRACT. Let p be a prime and K be a number field with non-trivial p-class group $\mathrm{Cl}_p(K)$. An important step in identifying the Galois group $G = \mathrm{G}_p^\infty(K)$ of the maximal unramified pro-p extension of K is to determine its two-stage approximation $\mathfrak{M} = \mathrm{G}_p^2(K)$, that is the second derived quotient $\mathfrak{M} \simeq G/G''$. The family of abelian type invariants of the p-class groups $\mathrm{Cl}_p(L)$ of all unramified cyclic extensions $L|K$ of degree p is called the *index-p abelianization data* (IPAD) $\tau_1(K)$ of K and has turned out to be useful for determining the second p-class group $\mathfrak{M}$. In this paper we introduce two different kinds of *generalized* IPADs for obtaining more sophisticated results. The *multi-layered* IPAD $(\tau_1(K), \tau_2(K))$ includes data on unramified abelian extensions $L|K$ of degree p^2 and enables sharper bounds for the order of $\mathfrak{M}$ if $\mathrm{Cl}_p(K) \simeq (p,p,p)$. The *iterated* IPAD of *second order* $\tau^{(2)}(K)$ contains information on non-abelian unramified extensions $L|K$ of degrees p^2 or even p^3 and admits the identification of the p-class tower group G for various series of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$, $d > 0$, with $\mathrm{Cl}_p(K) \simeq (p,p)$ having a p-class field tower of exact length $\ell_p(K) = 3$ as a striking novelty.

1. Introduction

Let p be a prime number. According to the Artin reciprocity law of class field theory [1], the unramified cyclic extensions $L|K$ of relative degree p of a number field K with non-trivial p-class group $\mathrm{Cl}_p(K)$ are in a bijective correspondence to the subgroups of index p in $\mathrm{Cl}_p(K)$. Their number is given by $\frac{p^\varrho - 1}{p-1}$ if ϱ denotes the p-class rank of K [27]. The reason for this fact is that the Galois group $\mathrm{G}_p^1(K) := \mathrm{Gal}(\mathrm{F}_p^1(K)|K)$ of the maximal unramified *abelian* p-extension $\mathrm{F}_p^1(K)|K$, which is called the first Hilbert p-class field of K, is isomorphic to the p-class group $\mathrm{Cl}_p(K)$. The fields L are contained in $\mathrm{F}_p^1(K)$ and each group $\mathrm{Gal}(\mathrm{F}_p^1(K)|L)$ is of index p in $\mathrm{G}_p^1(K) \simeq \mathrm{Cl}_p(K)$.

It was also Artin's idea [2] to leave the abelian setting of class field theory and to consider the second Hilbert p-class field $\mathrm{F}_p^2(K) = \mathrm{F}_p^1(\mathrm{F}_p^1(K))$, that is the maximal unramified *metabelian* p-extension of K, and its Galois group $\mathrm{G}_p^2(K) := \mathrm{Gal}(\mathrm{F}_p^2(K)|K)$, the so-called second p-class group of K [42, p.41], [22], for proving the *principal ideal theorem* that $\mathrm{Cl}_p(K)$ becomes trivial when it is extended to $\mathrm{Cl}_p(\mathrm{F}_p^1(K))$ [16]. Since $K \le L \le \mathrm{F}_p^1(K) \le \mathrm{F}_p^1(L) \le \mathrm{F}_p^2(K)$ is a non-decreasing tower of normal extensions for any assigned unramified abelian p-extension $L|K$, the p-class group of L, $\mathrm{Cl}_p(L) \simeq \mathrm{Gal}(\mathrm{F}_p^1(L)|L) \simeq \mathrm{Gal}(\mathrm{F}_p^2(K)|L) \,/\, \mathrm{Gal}(\mathrm{F}_p^2(K)|\mathrm{F}_p^1(L))$, is isomorphic to the *abelianization* H/H' of the subgroup $H := \mathrm{Gal}(\mathrm{F}_p^2(K)|L)$ of the second p-class group $\mathrm{Gal}(\mathrm{F}_p^2(K)|K)$ which corresponds to L and whose commutator subgroup is given by $H' = \mathrm{Gal}(\mathrm{F}_p^2(K)|\mathrm{F}_p^1(L))$.

In particular, the structure of the p-class groups $\mathrm{Cl}_p(L)$ of all unramified cyclic extensions $L|K$ of relative degree p can be interpreted as the *abelian type invariants* of all abelianizations H/H' of subgroups $H = \mathrm{Gal}(\mathrm{F}_p^2(K)|L)$ of *index* p in the second p-class group $\mathrm{Gal}(\mathrm{F}_p^2(K)|K)$, which has been dubbed the *index-p abelianization data*, briefly IPAD, $\tau_1(K)$ of K by Boston, Bush, and Hajir [11]. This kind of information would have been incomputable and thus useless about twenty years ago. However, with the availability of computational algebra systems like PARI/GP [40] and MAGMA [9, 10, 36] it became possible to compute the class groups $\mathrm{Cl}_p(L)$, collect their structures in the IPAD $\tau_1(K)$, reinterpret them as abelian quotient invariants of subgroups H of $\mathrm{G}_p^2(K)$, and

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to use this information for characterizing a batch of finitely many p-groups, occasionally even a unique p-group, as candidates for the second p-class group $\mathrm{G}_p^2(K)$ of K, which in turn is a *two-stage approximation* of the (potentially infinite) pro-p group $\mathrm{G}_p^\infty(K) := \mathrm{Gal}(\mathrm{F}_p^\infty(K)|K)$ of the maximal unramified pro-p extension $\mathrm{F}_p^\infty(K)$ of K, that is its *Hilbert p-class tower*.

As we recently proved in the main theorem of [33, Thm.5.4], the IPAD is usually unable to permit a decision about the *length* $\ell_p(K)$ of the p-class tower of K when non-metabelian candidates for $\mathrm{G}_p^\infty(K)$ exist. For solving such problems *iterated* IPADs $\tau^{(2)}(K)$ of *second order* are required.

In a previous article with the same title [29], we provided a systematic and rigorous introduction of the concepts of abelian type invariants and iterated IPADs of higher order. These ideas were presented together with impressive numerical applications at the 29th Journées Arithmétiques in Debrecen, July 2015 [30]. The purpose and the organization of the present article, which considerably extends the computational results in [29], is as follows. Basic definitions concerning the *Artin transfer pattern* [29, 31, 33] are recalled in § 2. Then we generally put $p = 3$ and consider 3-class tower groups. First we restate the IPADs of type $(3, 3)$ [29, Thm.3.1–3.2, pp.290–291] in a more succinct and elegant form in § 3, avoiding infinitely many exceptions. Then we characterize all relevant finite 3-groups by IPADs of first and second order in § 4. These groups constitute the candidates for 3-class tower groups $\mathrm{G}_3^\infty(K)$ of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$, $d > 0$, with 3-class group $\mathrm{Cl}_3(K)$ of type $(3, 3)$ in § 5. In § 5.1, statistics of the dominant scenario are given. In §§ 5.2 and 5.3, we provide evidence of unexpected phenomena revealed by real quadratic fields K with 3-principalization types $\varkappa_1(K)$ in Scholz and Taussky's section E [42, p.36]. Their 3-class tower can be of length $2 \le \ell_3(K) \le 3$ and a sharp decision is possible by means of iterated IPADs of second order. We point out that complex quadratic fields with the same 3-principalization types must always have a tower of exact length $\ell_3(K) = 3$ [13, 31, 32]. In § 5.4, resp. § 5.6, results for real quadratic fields K with 3-principalization type H.4, (4111), resp. G.19, (2143), are proved. The latter were discovered in August 2015 and are presented here for the first time.

2. The Artin transfer pattern

Let p be a prime number and G be a pro-p group with finite abelianization G/G', more precisely, assume that the commutator subgroup G' is of index $(G : G') = p^v$ with an integer exponent $v \ge 0$.

Definition 2.1. For each integer $0 \le n \le v$, let $\mathrm{Lyr}_n(G) := \{G' \le H \le G \mid (G : H) = p^n\}$ be the *nth layer* of normal subgroups of G containing G'.

Definition 2.2. For any intermediate group $G' \le H \le G$, we denote by $T_{G,H} : G \to H/H'$ the *Artin transfer* homomorphism from G to H/H' [33, Dfn.3.1].

(1) Let $\tau(G) := [\tau_0(G); \dots ; \tau_v(G)]$ be the *multi-layered transfer target type* (TTT) of G, where $\tau_n(G) := (H/H')_{H \in \mathrm{Lyr}_n(G)}$ for each $0 \le n \le v$.
(2) Let $\varkappa(G) := [\varkappa_0(G); \dots ; \varkappa_v(G)]$ be the *multi-layered transfer kernel type* (TKT) of G, where $\varkappa_n(G) := (\ker(T_{G,H}))_{H \in \mathrm{Lyr}_n(G)}$ for each $0 \le n \le v$.

Definition 2.3. The pair $\mathrm{AP}(G) := (\tau(G), \varkappa(G))$ is called the (restricted) *Artin pattern* of G.

Definition 2.4. The first order approximation $\tau^{(1)}(G) := [\tau_0(G); \tau_1(G)]$ of the TTT, resp. $\varkappa^{(1)}(G) := [\varkappa_0(G); \varkappa_1(G)]$ of the TKT, is called the *index-p abelianization data* (IPAD), resp. *index-p obstruction data* (IPOD), of G.

Definition 2.5. $\tau^{(2)}(G) := [\tau_0(G); (\tau^{(1)}(H))_{H \in \mathrm{Lyr}_1(G)}]$ is called *iterated* IPAD *of* 2^{nd} *order* of G.

Remark 2.1. For the *complete* Artin pattern $\mathrm{AP}_c(G)$ see [33, Dfn.5.3].

(1) Since the 0th layer (top layer), $\mathrm{Lyr}_n(G) = \{G\}$, consists of the group G alone, and $T_{G,G} : G \to G/G'$ is the natural projection onto the commutator quotient with kernel $\ker(T_{G,G}) = G'$, we usually omit the trivial top layer $\varkappa_0(G) = \{G'\}$ and identify the IPOD $\varkappa^{(1)}(G)$ with the first layer $\varkappa_1(G)$ of the TKT.
(2) In the case of an elementary abelianization of rank two, $(G : G') = p^2$, we also identify the TKT $\varkappa(G)$ with its first layer $\varkappa_1(G)$, since the 2nd layer (bottom layer), $\mathrm{Lyr}_2(G) = \{G'\}$, consists of the commutator subgroup G' alone, and the kernel of $T_{G,G'} : G \to G'/G''$ is always *total*, that is $\ker(T_{G,G'}) = G$, according to the *principal ideal theorem* [16].

3. All possible IPADs of 3-groups of type $(3,3)$

Since the abelian type invariants of certain IPAD components of an assigned 3-group G depend on the parity of the nilpotency class c or coclass r, a more economic notation, which avoids the tedious distinction of the cases odd or even, is provided by the following definition [24, § 3].

Definition 3.1. For an integer $n \ge 2$, the *nearly homocyclic abelian* 3*-group* $A(3,n)$ of order 3^n is defined by its type invariants $(q+r,q)\hat{=}(3^{q+r},3^q)$, where the quotient $q \ge 1$ and the remainder $0 \le r < 2$ are determined uniquely by the Euclidean division $n = 2q + r$. Two degenerate cases are included by putting $A(3,1) := (1)\hat{=}(3)$ the cyclic group C_3 of order 3, and $A(3,0) := (0)\hat{=}1$ the trivial group of order 1.

In the following theorem and in the whole remainder of the article, we use the *identifiers* of finite 3-groups up to order 3^7 as they are defined in the SmallGroups Library [6, 7]. They are of the shape $\langle$ *order*, *counter* $\rangle$, where the counter is motivated by the way how the output of descendant computations is arranged in the p-group generation algorithm by Newman [38] and O'Brien [39].

Theorem 3.1. *(D.C. Mayer,* 2014 *[24])*
Let G be a pro-3 group with abelianization G/G' of type $(3,3)$ and metabelianization $\mathfrak{M} = G/G''$ of nilpotency class $c = \mathrm{cl}(\mathfrak{M}) \ge 2$, defect $0 \le k = k(\mathfrak{M}) \le 1$, and coclass $r = \mathrm{cc}(\mathfrak{M}) \ge 1$. Assume that $\mathfrak{M}$ does not belong to the finitely many exceptions in the list below. Then the IPAD $\tau^{(1)}(G) = [\tau_0(G);\tau_1(G)]$ of G in terms of nearly homocyclic abelian 3-groups is given by

$$\tau_0(G) = 1^2;$$

$$\tau_1(G) = \left(\overbrace{\mathrm{A}(3,c-k)}^{\textit{polarization}},\ \overbrace{\mathrm{A}(3,r+1)}^{\textit{co-polarization}},T_3,T_4\right), \tag{3.1}$$

where the polarized first component of $\tau_1(G)$ depends on the class c and defect k, the co-polarized second component increases with the coclass r, and the third and fourth component are completely stable for $r \ge 3$ but depend on the coclass tree of $\mathfrak{M}$ for $1 \le r \le 2$ in the following manner

$$(T_3,T_4) = \begin{cases} (\mathrm{A}(3,r+1)^2) & \textit{if } r=2,\ \mathfrak{M}\in\mathcal{T}^2(\langle 243,8\rangle) \textit{ or } r=1,\\ (1^3,\mathrm{A}(3,r+1)) & \textit{if } r=2,\ \mathfrak{M}\in\mathcal{T}^2(\langle 243,6\rangle),\\ ((1^3)^2) & \textit{if } r=2,\ \mathfrak{M}\in\mathcal{T}^2(\langle 243,3\rangle) \textit{ or } r\ge 3.\end{cases} \tag{3.2}$$

Anomalies of finitely many, precisely 13*, exceptional groups are summarized in the following list.*

$$\begin{aligned} \tau_1(G) &= \big((1)^4\big) \textit{ for } \mathfrak{M}\simeq\langle 9,2\rangle,\ c=1,\ r=1,\\ \tau_1(G) &= \big(1^2,(2)^3\big) \textit{ for } \mathfrak{M}\simeq\langle 27,4\rangle,\ c=2,\ r=1,\\ \tau_1(G) &= \big(1^3,(1^2)^3\big) \textit{ for } \mathfrak{M}\simeq\langle 81,7\rangle,\ c=3,\ r=1,\\ \tau_1(G) &= \big((1^3)^3,21\big) \textit{ for } \mathfrak{M}\simeq\langle 243,4\rangle,\ c=3,\ r=2,\\ \tau_1(G) &= \big(1^3,(21)^3\big) \textit{ for } \mathfrak{M}\simeq\langle 243,5\rangle,\ c=3,\ r=2,\\ \tau_1(G) &= \big((1^3)^2,(21)^2\big) \textit{ for } \mathfrak{M}\simeq\langle 243,7\rangle,\ c=3,\ r=2,\\ \tau_1(G) &= \big((21)^4\big) \textit{ for } \mathfrak{M}\simeq\langle 243,9\rangle,\ c=3,\ r=2,\\ \tau_1(G) &= \big((1^3)^3,21\big) \textit{ for } \mathfrak{M}\simeq\langle 729,44\ldots 47\rangle,\ c=4,\ r=2,\\ \tau_1(G) &= \big((21)^4\big) \textit{ for } \mathfrak{M}\simeq\langle 729,56\ldots 57\rangle,\ c=4,\ r=2.\end{aligned} \tag{3.3}$$

The *polarization* and the *co-polarization* we had in our mind when we spoke about a *bi-polarization* in [26, Dfn.3.2, p.430]. Meanwhile, we have provided yet another proof for the existence of *stable* and *polarized* IPAD components with the aid of a *natural partial order* on the Artin transfer patterns distributed over a descendant tree [33, Thm.6.1–6.2].

Proof. Equations (3.1) and (3.2) are a succinct form of information which summarizes all statements about the first TTT layer $\tau_1(G)$ in the formulas (19), (20) and (22) of [29, Thm.3.2, p.291] omitting the claims on the second TTT layer $\tau_2(G)$. Here we do not need the restrictions arising from lower bounds for the nilpotency class $c = \mathrm{cl}(\mathfrak{M})$ in the cited theorem, since the remaining cases for small values of c can be taken from [29, Thm.3.1, p.290], with the exception of the following 13 anomalies in formula (3.3):

The *abelian* group $\langle 9, 2\rangle \simeq (3, 3)$, the *extra special* group $\langle 27, 4\rangle$, and the group $\langle 81, 7\rangle \simeq \mathrm{Syl}_3(A_9)$ do not fit into the general rules for 3-groups of coclass 1. These three groups appear in the top region of the tree diagram in the Figures 2 and 1.

The four *sporadic* groups $\langle 243, n\rangle$ with $n \in \{4, 5, 7, 9\}$ and the six *sporadic* groups $\langle 729, n\rangle$ with $n \in \{44, \ldots, 47, 56, 57\}$ do not belong to any coclass-2 tree, as can be seen in the Figures 3 and 4, whence the conditions in equation (3.2) cannot be applied to them.

On the other hand, there is no need to list the groups $\langle 27, 3\rangle$ and $\langle 81, 8 \ldots 10\rangle$ in formula (14), the groups $\langle 243, n\rangle$ with $n \in \{3, 6, 8\}$ in formula (15), and the groups $\langle 729, n\rangle$ with $n \in \{34, \ldots, 39\}$ in formula (16) of [29, Thm.3.1, p.290], since they perfectly fit into the general pattern. □

Remark 3.1. The reason why we exclude the second TTT layer $\tau_2(G)$ from Theorem 3.1, while it is part of [29, Thm.3.1–3.2, pp.290–291], is that we want to reduce the exceptions of the general pattern to a *finite* list, whereas the *irregular case* of the abelian quotient invariants of the commutator subgroup G', which forms the single component of $\tau_2(G)$, occurs for each even value of the coclass $r = \mathrm{cc}(\mathfrak{M}) \equiv 0 \pmod 2$ and thus *infinitely* often.

4. Characterization of possible 3-class tower groups

In this section, we shall frequently deal with finite 3-groups G of huge orders $|G| \geq 3^8$ for which no identifiers are available in the SmallGroups database [6, 7]. A workaround for these cases is provided by the *relative identifiers* of the ANUPQ (Australian National University p-Quotient) package [17] which is implemented in our licence of the computational algebra system MAGMA [9, 10, 36] and in the open source system GAP [18].

Definition 4.1. Let p be a prime number and G be a finite p-group with nuclear rank $\nu \geq 1$ [28, eqn.(28), p.178] and immediate descendant numbers $N_1, \ldots, N_\nu$ [28, eqn.(34), p.180]. Then we denote the ith *immediate descendant* of *step size* s of G by the symbol

$$G - \#s; i \tag{4.1}$$

for each $1 \leq s \leq \nu$ and $1 \leq i \leq N_s$.

If a chain of n immediate descendants is formed in identical manner with coinciding values of s and i starting with G, then we use a power with a formal exponent

$$G(-\#s; i)^n \tag{4.2}$$

to denote *iteration* of the process of generating immediate descendants.

Recall that a group with nuclear rank $\nu = 0$ is a *terminal leaf* without any descendants.

All results in this section have been computed by means of the computational algebra system MAGMA [9, 10, 36]. The p-group generation algorithm by Newman [38] and O'Brien [39] was used for the recursive construction of descendant trees $\mathcal{T}(R)$ of finite p-groups G. The tree root (starting group) R was taken to be $\langle 9, 2\rangle$ for Table 1, $\langle 243, 6\rangle$ for Table 2, $\langle 243, 8\rangle$ for Table 3, $\langle 243, 4\rangle$ for Table 4, and $\langle 243, 9\rangle$ for Table 5. For the computation of group theoretic invariants of each tree vertex G we implemented the Artin transfer homomorphism $T_{G,H}$ from finite p-groups G of type $G/G' \simeq (p, p)$ to their maximal subgroups $H < G$ in a MAGMA program script as described in [33, § 4.1].

4.1. 3-groups G of coclass $\mathrm{cc}(G) = 1$.

Table 1. IPOD and iterated IPAD $\tau_*^{(2)}(G)$ of 3-groups G of coclass $\mathrm{cc}(G)=1$

lo	id	w.r.t.	type	$\varkappa_1(G)$	$\tau_0(G)$		$\tau_0(H)$	$\tau_1(H)$	$\tau_2(H)$	
2	2		a.1	0000	1^2	[	1	0	0	$]^4$
3	3		a.1	0000	1^2	[	1^2	$(1)^4$	0	$]^4$
3	4		A.1	1111	1^2		1^2	$(1)^4$	0	
						[	2	1	0	$]^3$
4	7		a.3	2000	1^2		1^3	$(1^2)^{13}$	$(1)^{13}$	
							1^2	$(1^2)^4$	1	
						[	1^2	$1^2,(2)^3$	1	$]^2$
4	8		a.3	2000	1^2		21	$1^2,(2)^3$	$(1)^4$	
							1^2	$(1^2)^4$	1	
						[	1^2	$1^2,(2)^3$	1	$]^2$
4	9		a.1	0000	1^2		21	$1^2,(2)^3$	$(1)^4$	
						[	1^2	$(1^2)^4$	1	$]^3$
4	10		a.2	1000	1^2		21	$1^2,(2)^3$	$(1)^4$	
						[	1^2	$1^2,(2)^3$	1	$]^3$
5	25		a.3	2000	1^2		2^2	$(21)^4$	$1^2,(2)^{12}$	
						[	1^2	$21,(1^2)^3$	1^2	$]^3$
5	26		a.1	0000	IPAD like id 25					
5	27		a.2	1000	IPAD like id 25					
5	$28\ldots30$		a.1	0000	1^2		21	$(21)^4$	$1^2,(2)^3$	
						[	1^2	$21,(1^2)^3$	1^2	$]^3$
6	95		a.1	0000	1^2		32	$2^2,(31)^3$	$(21)^4,(3)^9$	
						[	1^2	$2^2,(1^2)^3$	21	$]^3$
6	96		a.2	1000	IPAD like id 95					
6	$97\|98$		a.3	2000	IPAD like id 95					
6	$99\ldots101$		a.1	0000	1^2		2^2	$2^2,(31)^3$	$(21)^4,(3)^9$	
						[	1^2	$2^2,(1^2)^3$	21	$]^3$
7	386		a.1	0000	1^2		3^2	$(32)^4$	$2^2,(31)^{12}$	
						[	1^2	$32,(1^2)^3$	2^2	$]^3$
7	387		a.2	1000	IPAD like id 386					
7	388		a.3	2000	IPAD like id 386					
7	$389\ldots391$		a.1	0000	1^2		32	$(32)^4$	$2^2,(31)^{12}$	
						[	1^2	$32,(1^2)^3$	2^2	$]^3$
8	$\#1;1$	$\langle 3^7,386\rangle$	a.1	0000	1^2		43	$3^2,(42)^3$	$(32)^4,(41)^9$	
						[	1^2	$3^2,(1^2)^3$	32	$]^3$
8	$\#1;2$	$\langle 3^7,386\rangle$	a.2	1000	IPAD like id $\#1;1$					
8	$\#1;3\|4$	$\langle 3^7,386\rangle$	a.3	2000	IPAD like id $\#1;1$					
8	$\#1;5\ldots7$	$\langle 3^7,386\rangle$	a.1	0000	1^2		3^2	$3^2,(42)^3$	$(32)^4,(41)^9$	
						[	1^2	$3^2,(1^2)^3$	32	$]^3$

Table 1 shows the designation of the transfer kernel type, the IPOD $\varkappa_1(G)$, and the *iterated multi-layered* IPAD of *second order*,

$$\tau_*^{(2)}(G) = [\tau_0(G); [\tau_0(H); \tau_1(H); \tau_2(H)]_{H \in \mathrm{Lyr}_1(G)}], \tag{4.3}$$

for 3-groups G of maximal class up to order $|G| = 3^8$, characterized by the logarithmic order, lo, i.e. $\mathrm{lo}(G) := \log_3(|G|)$, and the SmallGroup identifier, id, resp. the relative identifier for lo $= 8$.

The groups in Table 1 are represented by vertices of the tree diagrams in Figure 2 and 1.

TABLE 2. IPOD and iterated IPAD $\tau_*^{(2)}(G)$ of 3-groups G on $\mathcal{T}^2(\langle 3^5, 6\rangle)$

lo	id	w.r.t.	type	$\varkappa_1(G)$	$\tau_0(G)$		$\tau_0(H)$	$\tau_1(H)$	$\tau_2(H)$	
5	6		c.18	0122	1^2		1^3	$(1^3)^4, (1^2)^9$	$(1^2)^{13}$	
						[	21	$1^3, (21)^3$	$(1^2)^4$	$]^3$
6	48		H.4	2122	1^2		2^2	$(21^2)^4$	$1^3, (21)^{12}$	
							1^3	$21^2, (1^3)^3, (1^2)^9$	$1^3, (21)^3, (1^2)^9$	
						[	21	$21^2, (21)^3$	$1^3, (21)^3$	$]^2$
6	49		c.18	0122	IPAD like id 48					
6	50		E.14	3122	IPAD like id 48					
6	51		E.6	1122	IPAD like id 48					
7	284		c.18	0122	1^2		2^2	$(21^2)^4$	$21^2, (2^2)^{12}$	
							1^3	$(21^2)^4, (1^2)^9$	$21^2, (1^3)^3, (21)^9$	
						[	21	$21^2, (21)^3$	$21^2, (21)^3$	$]^2$
7	285		c.18	0122	1^2		32	$2^21, (31^2)^3$	$(21^2)^4, (31)^9$	
							1^3	$2^21, (\mathbf{1}^3)^3, (1^2)^9$	$21^2, (2^2)^3, (1^2)^9$	
						[	21	$2^21, (\mathbf{21})^3$	$21^2, (2^2)^3$	$]^2$
7	286|287		H.4	2122	IPAD like id 285					
7	**288**		E.6	1122	IPAD like id 285					
7	**289**|**290**		E.14	3122	IPAD like id 285					
7	291		c.18	0122	1^2		2^2	$(21^2)^4$	$21^2, (2^2)^3, (31)^9$	
							1^3	$21^2, (1^3)^3, (1^2)^9$	$21^2, (12)^3, (1^2)^9$	
							21	$21^2, (31)^3$	$21^2, (21)^3$	
							21	$21^2, (21)^3$	$21^2, (21)^3$	
8	$\#2;1$	Q	c.18	0122	1^2		32	$2^21, (31^2)^3$	$(2^21)^4, (32)^9$	
							1^3	$2^21, (\mathbf{21}^2)^3, (1^2)^9$	$2^21, (1^3)^3, (2^2)^3, (21)^6$	
						[	21	$2^21, (\mathbf{31})^3$	$2^21, (2^2)^3$	$]^2$
8	$\#2;2\|3$	Q	H.4	2122	IPAD like id $\#2;1$					
8	$\#2;\mathbf{4}$	Q	E.6	1122	IPAD like id $\#2;1$					
8	$\#2;\mathbf{5}\|\mathbf{6}$	Q	E.14	3122	IPAD like id $\#2;1$					

4.2. **3-groups G of coclass $\mathrm{cc}(G) = 2$ arising from $\langle 3^5, 6\rangle$.** Table 2 shows the designation of the transfer kernel type, the IPOD $\varkappa_1(G)$, and the iterated multi-layered IPAD of second order,

$$\tau_*^{(2)}(G) = [\tau_0(G); [\tau_0(H); \tau_1(H); \tau_2(H)]_{H \in \mathrm{Lyr}_1(G)}],$$

for 3-groups G on the coclass tree $\mathcal{T}^2(\langle 3^5, 6\rangle)$ up to order $|G| = 3^8$, characterized by the logarithmic order, lo, and the SmallGroup identifier, id, resp. the relative identifier for lo $= 8$. To enable a brief reference for relative identifiers we put $Q := \langle 3^6, 49\rangle$, since this group was called the non-CF group Q by J.A. Ascione [3, 4].

The groups in Table 2 are represented by vertices of the tree diagram in Figure 5.

Theorem 4.1. *(D.C. Mayer, April* 2015 [29]*)*
Let G be a finite 3-group with IPAD of first order $\tau^{(1)}(G) = [\tau_0(G); \tau_1(G)]$, where $\tau_0(G) = 1^2$ and $\tau_1(G) = (32, 1^3, (21)^2)$ in ordered form.

If the IPOD of G is given by $\varkappa_1(G) = (1122)$, resp. $\varkappa_1(G) = (3122) \sim (4122)$, then the IPAD of second order $\tau^{(2)}(G) = [\tau_0(G); (\tau_0(H_i); \tau_1(H_i))_{1\le i\le 4}]$, where the maximal subgroups of index 3 in G are denoted by $H_1, \ldots, H_4$, determines the isomorphism type of G in the following way:

1. *$\tau^{(1)}(H_2) = [1^3; (2^21, (\mathbf{1}^3)^3, (1^2)^9)]$ if and only if $\tau^{(1)}(H_i) = [21; (2^21, (\mathbf{21})^3)]$ for $i \in \{3, 4\}$ if and only if $G \simeq \langle 3^7, \mathbf{288}\rangle$, resp. $G \simeq \langle 3^7, \mathbf{289}\rangle$ or $\langle 3^7, \mathbf{290}\rangle$,*
2. *$\tau^{(1)}(H_2) = [1^3; (2^21, (\mathbf{2}1^2)^3, (1^2)^9)]$ if and only if $\tau^{(1)}(H_i) = [21; (2^21, (\mathbf{31})^3)]$ for $i \in \{3, 4\}$ if and only if $G \simeq \langle 3^6, 49\rangle - \#2; \mathbf{4}$, resp. $G \simeq \langle 3^6, 49\rangle - \#2; \mathbf{5}$ or $\langle 3^6, 49\rangle - \#2; \mathbf{6}$,*

whereas the component $\tau^{(1)}(H_1) = [32; (2^21, (31^2)^3)]$ is fixed and does not admit a distinction.

Proof. This is essentially [29, Thm.6.2, pp.297–298]. It is also an immediate consequence of Table 2, which has been computed with MAGMA [36]. As a termination criterion we can now use the more precise [33, Thm.5.1] instead of [13, Cor.3.0.1, p.771]. □

TABLE 3. IPOD and iterated IPAD $\tau_*^{(2)}(G)$ of 3-groups G on $\mathcal{T}^2(\langle 3^5, 8\rangle)$

lo	id	w.r.t.	type	$\varkappa_1(G)$	$\tau_0(G)$		$\tau_0(H)$	$\tau_1(H)$	$\tau_2(H)$	
5	8		c.21	2034	1^2	[	21	$1^3, (21)^3$	$(1^2)^4$	$]^4$
6	52		G.16	2134	1^2		2^2	$(21^2)^4$	$1^3, (21)^{12}$	
						[	21	$21^2, (21)^3$	$1^3, (21)^3$	$]^3$
6	53		E.9	2434	IPAD like id 52					
6	54		c.21	2034	IPAD like id 52					
6	55		E.8	2234	IPAD like id 52					
7	301|305		G.16	2134	1^2		32	$2^21, (31^2)^3$	$(21^2)^4, (31)^9$	
						[	21	$2^21, (\mathbf{21})^3$	$21^2, (2^2)^3$	$]^3$
7	**302|306**		E.9	2334	IPAD like id 301					
7	303		c.21	2034	IPAD like id 301					
7	**304**		E.8	2234	IPAD like id 301					
7	307		c.21	2034	1^2		2^2	$(21^2)^4$	$21^2, (2^2)^{12}$	
							21	$21^2, (31)^3$	$21^2, (21)^3$	
						[	21	$21^2, (21)^3$	$21^2, (21)^3$	$]^2$
7	308		c.21	2034	1^2		2^2	$(21^2)^4$	$21^2, (2^2)^3, (31)^9$	
							21	$21^2, (31)^3$	$21^2, (21)^3$	
						[	21	$21^2, (21)^3$	$21^2, (21)^3$	$]^2$
8	$\#2; 1\|5$	U	G.16	2134	1^2		32	$2^21, (31^2)^3$	$(2^21)^4, (32)^9$	
						[	21	$2^21, (\mathbf{31})^3$	$2^21, (2^2)^3$	$]^3$
8	$\#2; \mathbf{2\|6}$	U	E.9	2334	IPAD like id $\#2; 1$					
8	$\#2; 3$	U	c.21	2034	IPAD like id $\#2; 1$					
8	$\#2; \mathbf{4}$	U	E.8	2234	IPAD like id $\#2; 1$					

4.3. **3-groups G of coclass $\mathrm{cc}(G) = 2$ arising from $\langle 3^5, 8\rangle$.** Table 3 shows the designation of the transfer kernel type, the IPOD $\varkappa_1(G)$, and the iterated multi-layered IPAD of second order,

$$\tau_*^{(2)}(G) = [\tau_0(G); [\tau_0(H); \tau_1(H); \tau_2(H)]_{H\in \mathrm{Lyr}_1(G)}],$$

for 3-groups G on the coclass tree $\mathcal{T}^2(\langle 3^5,8\rangle)$ up to order $|G|=3^8$, characterized by the logarithmic order, lo, and the SmallGroup identifier, id, resp. the relative identifier for lo $=8$. To enable a brief reference for relative identifiers we put $U:=\langle 3^6,54\rangle$, since this group was called the non-CF group U by Ascione [3, 4].

The groups in Table 3 are represented by vertices of the tree diagram in Figure 6.

Theorem 4.2. *(D.C. Mayer, April 2015 [29])*
Let G be a finite 3-group with IPAD of first order $\tau^{(1)}(G)=[\tau_0(G);\tau_1(G)]$, where $\tau_0(G)=1^2$ and $\tau_1(G)=(21,32,(21)^2)$ in ordered form.

If the IPOD of G is given by $\varkappa_1(G)=(2234)$, resp. $\varkappa_1(G)=(2334)\sim(2434)$, then the IPAD of second order $\tau^{(2)}(G)=[\tau_0(G);(\tau_0(H_i);\tau_1(H_i))_{1\le i\le 4}]$, where the maximal subgroups of index 3 in G are denoted by $H_1,\ldots,H_4$, determines the isomorphism type of G in the following way:

1. *$\tau^{(1)}(H_i)=[21;(2^21,(\mathbf{21})^3)]$ for $i\in\{1,3,4\}$ if and only if $G\simeq\langle 3^7,\mathbf{304}\rangle$, resp. $G\simeq\langle 3^7,\mathbf{302}\rangle$ or $\langle 3^7,\mathbf{306}\rangle$,*
2. *$\tau^{(1)}(H_i)=[21;(2^21,(\mathbf{31})^3)]$ for $i\in\{1,3,4\}$ if and only if $G\simeq\langle 3^6,54\rangle-\#2;\mathbf{4}$, resp. $G\simeq\langle 3^6,54\rangle-\#2;\mathbf{2}$ or $\langle 3^6,54\rangle-\#2;\mathbf{6}$,*

whereas the component $\tau^{(1)}(H_2)=[32;(2^21,(31^2)^3)]$ is fixed and does not admit a distinction.

Proof. This is essentially [29, Thm.6.3, pp.298–299]. It is also an immediate consequence of Table 3, which has been computed with MAGMA [36]. As a termination criterion we can now use the more precise [33, Thm.5.1] instead of [13, Cor.3.0.1, p.771]. □

TABLE 4. IPOD and iterated IPAD $\tau_*^{(2)}(G)$ of sporadic 3-groups G of type H.4

lo	id	w.r.t.	type	$\varkappa_1(G)$	$\tau_0(G)$		$\tau_0(H)$	$\tau_1(H)$	$\tau_2(H)$	
5	4		H.4	4111	1^2		1^3	$(1^3)^4,(1^2)^9$	$(1^2)^{13}$	
						[	1^3	$1^3,(21)^3,(1^2)^9$	$(1^2)^4,(2)^9$	$]^2$
							21	$1^3,(21)^3$	$(1^2)^4$	
6	45		H.4	4111	1^2		1^3	$21^2,(1^3)^3,(1^2)^9$	$1^3,(21)^3,(1^2)^9$	
						[	1^3	$21^2,(21)^{12}$	$21^2,(21)^{12}$	$]^2$
							21	$21^2,(21)^3$	$21^2,(2^2)^3$	
7	270		H.4	4111	1^2		1^3	$(21^2)^4,(1^2)^9$	$21^2,(1^3)^3,(21)^9$	
						[	1^3	$21^2,(21)^{12}$	$21^2,(21)^{12}$	$]^2$
							21	$21^2,(21)^3$	$21^2,(2^2)^3$	
7	271|272		H.4	4111	1^2		1^3	$21^2,(1^3)^3,(1^2)^9$	$21^2,(2^2)^3,(1^2)^9$	
						[	1^3	$21^2,(21)^{12}$	$21^2,(21)^{12}$	$]^2$
							21	$21^2,(31)^3$	$21^2,(21)^3$	
7	273		H.4	4111	1^2		1^3	$21^2,(1^3)^3,(1^2)^9$	$21^2,(21)^3,(1^2)^9$	
							1^3	$21^2,(21)^{12}$	$21^2,(21)^{12}$	
							1^3	$(21^2)^4,(2^2)^9$	$(21^2)^{13}$	
							21	$21^2,(21)^3$	$21^2,(21)^3$	
8	$\#2;1\|2$	N	H.4	4111	1^2		1^3	$(21^2)^4,(1^2)^9$	$2^21,(1^3)^3,(2^2)^3,(21)^6$	
						[	1^3	$(21^2)^4,(2^2)^9$	$2^21,(21^2)^{12}$	$]^2$
							21	$21^2,(31)^3$	$2^21,(2^2)^3$	

4.4. **Sporadic 3-groups G of coclass $\mathrm{cc}(G)=2$.** Table 4 shows the designation of the transfer kernel type, the IPOD $\varkappa_1(G)$, and the iterated multi-layered IPAD of second order,

$$\tau_*^{(2)}(G)=[\tau_0(G);[\tau_0(H);\tau_1(H);\tau_2(H)]_{H\in\mathrm{Lyr}_1(G)}],$$

for sporadic 3-groups G of type H.4 up to order $|G| = 3^8$, characterized by the logarithmic order, lo, and the SmallGroup identifier, id, resp. the relative identifier for lo = 8. To enable a brief reference for relative identifiers we put $N := \langle 3^6, 45\rangle$, since this group was called the non-CF group N by Ascione [3, 4].

The groups in Table 4 are represented by vertices of the tree diagram in Figure 7.

TABLE 5. IPOD and iterated IPAD $\tau_*^{(2)}(G)$ of sporadic 3-groups G of type G.19

lo	id	w.r.t.	type	$\varkappa_1(G)$	$\tau_0(G)$		$\tau_0(H)$	$\tau_1(H)$	$\tau_2(H)$	
5	9		G.19	2143	1^2	[	21	$1^3, (21)^3$	$(1^2)^4$	$]^4$
6	57		G.19	2143	1^2	[	21	$1^4, (21)^3$	$(1^3)^4$	$]^4$
7	311		G.19	2143	1^2		21	$1^4, (21^2)^3$	$1^4, (1^3)^3$	
							21	$1^4, (21)^3$	$(1^4)^4$	
						[	21	$1^4, (21)^3$	$1^4, (1^3)^3$	$]^2$
8	$\#2; 1\ldots6$	W	G.19	2143	1^2	[	21	$1^4, (21^2)^3$	$1^5, (1^4)^3$	$]^4$
9	$\#1; 2$	Φ	G.19	2143	1^2	[	21	$1^4, (21^2)^3$	$(1^5)^4$	$]^4$
9	$\#1; 2$	Ψ	G.19	2143	1^2	[	21	$1^4, (21^2)^3$	$1^5, (21^3)^3$	$]^4$
9	$\#1; 2$	Y	G.19	2143	1^2	[	21	$1^4, (21^2)^3$	$1^6, (1^4)^3$	$]^4$
9	$\#1; 2$	Z	G.19	2143	IPAD like id $Y - \#1; 2$					
9	$\#1; 3$	Z	G.19	2143	IPAD like id $\Psi - \#1; 2$					
10	$\#1; 1$	Y_1	G.19	2143	1^2		21	$1^4, (31^2)^3$	$21^5, (1^4)^3$	
						[	21	$1^4, (21^2)^3$	$21^5, (1^4)^3$	$]^3$
10	$\#1; 1$	Z_1	G.19	2143	1^2		21	$1^4, (2^21)^3$	$21^5, (1^4)^3$	
						[	21	$1^4, (21^2)^3$	$21^5, (1^4)^3$	$]^3$
10	$\#2; 7$	Z	G.19	2143	1^2	[	21	$1^4, (21^2)^3$	$1^6, (21^3)^3$	$]^4$
11	$\#2; 1\|2$	Y_1	G.19	2143	1^2	[	21	$1^4, (31^2)^3$	$2^21^4, (1^4)^3$	$]^4$
11	$\#2; 1\ldots4$	Z_1	G.19	2143	1^2	[	21	$1^4, (2^21)^3$	$2^21^4, (1^4)^3$	$]^4$
11	$\#1; 1$	Z_2	G.19	2143	1^2		21	$1^4, (2^21)^3$	$21^5, (21^3)^3$	
						[	21	$1^4, (21^2)^3$	$21^5, (21^3)^3$	$]^3$
11	$\#1; 5$	Z_2	G.19	2143	1^2		21	$1^4, (21^2)^3$	$1^6, (2^21^2)^3$	
						[	21	$1^4, (21^2)^3$	$1^6, (21^3)^3$	$]^3$
12	$\#2; 1$	Z_2	G.19	2143	1^2	[	21	$1^4, (2^21)^3$	$2^21^4, (21^3)^3$	$]^4$
12	$\#2; 62$	Z_2	G.19	2143	1^2		21	$1^4, (2^21)^3$	$21^5, (2^21^2)^3$	
						[	21	$1^4, (21^2)^3$	$21^5, (21^3)^3$	$]^3$
12	$\#2; 87$	Z_2	G.19	2143	1^2		21	$1^4, (2^21)^3$	$21^5, (21^3)^3$	
							21	$1^4, (21^2)^3$	$21^5, (2^21^2)^3$	
						[	21	$1^4, (21^2)^3$	$21^5, (21^3)^3$	$]^2$
14	$\#4; 1\ldots43$	Z_2	G.19	2143	1^2	[	21	$1^4, (2^21)^3$	$2^21^4, (2^21^2)^3$	$]^4$

Table 5 shows the designation of the transfer kernel type, the IPOD $\varkappa_1(G)$, and the iterated multi-layered IPAD of second order,

$$\tau_*^{(2)}(G) = [\tau_0(G); [\tau_0(H); \tau_1(H); \tau_2(H)]_{H\in\mathrm{Lyr}_1(G)}],$$

for sporadic 3-groups G of type G.19 up to order $|G| = 3^{14}$, characterized by the logarithmic order, lo, and the SmallGroup identifier, id, resp. the relative identifier for lo ≥ 8. To enable a brief

reference for relative identifiers we put
$W := \langle 3^6, 57\rangle$, since this group was called the non-CF group W by Ascione [3, 4],
$\Phi := \langle 3^6, 57\rangle - \#2; 2$, $\Psi := \langle 3^6, 57\rangle - \#2; 4$, and further
$Y := \langle 3^6, 57\rangle - \#2; 5$, $Y_1 := \langle 3^6, 57\rangle - \#2; 5 - \#1; 2$, and
$Z := \langle 3^6, 57\rangle - \#2; 6$, $Z_1 := \langle 3^6, 57\rangle - \#2; 6 - \#1; 2$, $Z_2 := \langle 3^6, 57\rangle - \#2; 6 - \#2; 7$.

The groups in Table 5 are represented by vertices of the tree diagram in Figure 8.

5. 3-class towers of real quadratic fields

Before we focus our attention on the striking novelty of 3-class towers of length $\ell_3(K) = 3$ over *real* quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with positive discriminant $d > 0$, it is illuminating and almost indispensable to recall the historical evolution of the class tower problem for *complex* quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with negative discriminant $d < 0$.

Complex quadratic fields. The investigation of the Hilbert 3-class field tower $\mathrm{F}_3^0(K) \le \mathrm{F}_3^1(K) \le \mathrm{F}_3^2(K) \le \ldots \le \mathrm{F}_3^n(K) \le \ldots \le \mathrm{F}_3^\infty(K)$ was initiated in 1934 by Scholz and Taussky [42]. They focused on complex quadratic fields with 3-class group $\mathrm{Cl}_3(K)$ of type $(3, 3)$ and showed that exactly 19 distinct 3-principalization types $\varkappa_1(K)$, viewed as S_4-orbits, are possible combinatorially for such fields [20, p.80]. They classified closely related types with similar invariants of the second 3-class group $\mathrm{G}_3^2(K)$ into *sections* designated by upper case letters A,D,E,F,G,H, and proved that the 2 nearly constant types, (1211) (two fixed points), (1112) (one fixed point), in section B and the 3 permutation types, (1234) (identity), (1342) (3-cycle), (2341) (4-cycle), in section C are impossible for group theoretic reasons. Further, they also proved that the unique constant type A.1, (1111), of section A can be realized by cyclic cubic fields but not by any quadratic field. So there remain 13 possible types for complex quadratic fields [23, Tbl.6, p.492]. Scholz and Taussky were able to determine the exact length $\ell_3(K) = 2$ of the 3-class tower of fields with either of the two types D.10, (1123), and D.5, (1212), in section D. Towers of length $\ell_3(K) \ge 3$ remained unknown for 80 years until Bush and ourselves proved the existence of towers of exact length $\ell_3(K) = 3$ [13].

Concrete numerical realizations of most of the 3-principalization types were given by Scholz and Taussky in 1934 for D.10 and E.9 (1213) [42], by Heider and Schmithals in 1982 for D.5, E.6, E.14, G.16, G.19, and H.4 [19], and by Brink in 1984 for E.8, F.11, F.12, and F.13 [12]. The mysterious missing type F.7, (2112), and the minimal absolute discriminant $d = -34\,867$ for type E.8, (1231), were found by ourselves in May 2003 and published in 2012 [22, Tbl.3, p.497].

Real quadratic fields. Opposed to the completion of finding 3-principalization types of complex quadratic fields in 2003, only little progress was achieved in the investigation of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with $d > 0$ until 2006. Chang introduced the concept of *capitulation number* in his Ph.D. Thesis 1977 and, in cooperation with Foote, in 1980 [14]. In the present setting of a 3-class group $\mathrm{Cl}_3(K)$ of type $(3, 3)$, the capitulation number of a quadratic field K can be defined as the cardinality $\#\mathcal{H}_0(K)$ of the set

$$\mathcal{H}_0(K) := \{1 \le i \le 4 \mid \ker(j_{L_i|K}) = \mathrm{Cl}_3(K)\}, \tag{5.1}$$

that is the *number of total capitulations*, where $L_1, \ldots, L_4$ denote the unramified cyclic cubic extensions of K with class extension homomorphisms $j_{L_i|K} : \mathrm{Cl}_3(K) \to \mathrm{Cl}_3(L_i)$. A total capitulation, denoted by a 0 in the IPOD $\varkappa_1(K)$, is excluded for a complex quadratic field K, for which we always must have $\#\mathcal{H}_0(K) = 0$.

The modest beginning of the study of real quadratic fields is due to Heider and Schmithals in 1982, where they proved that there are exactly 4 discriminants $d \in \{32\,009, 42\,817, 72\,329, 94\,636\}$ with $\#\mathcal{H}_0(K) = 3$ and a unique $d = 62\,501$ with $\#\mathcal{H}_0(K) = 4$ in the range $0 < d < 10^5$ [19, p.24]. In 1991 [21], we extended the range to $0 < d < 2 \cdot 10^5$ and found $\#\mathcal{H}_0(K) = 3$ for 10 further cases $103\,809 < d < 189\,237$ and $\#\mathcal{H}_0(K) = 4$ for a single further case $d = 152\,949$. So the capitulation number seemed to be confined within the bounds $3 \le \#\mathcal{H}_0(K) \le 4$ for real quadratic fields K. This was our impression from 1991 till 2006. With considerable surprise, we discovered the first real quadratic field K with the unexpected capitulation number $\#\mathcal{H}_0(K) = 0$

in January 2006. It was of type G.19 ($d = 214\,712$, see § 5.6). Further examples appeared in June 2006, E.9 ($d = 342\,664$, see § 5.3), and D.10 ($d = 422\,573$). At the end of 2009, there followed the types D.5 ($d = 631\,769$) and H.4 ($d = 957\,013$, see § 5.4), and at the beginning of 2010, we found the hard-boiled types E.14 ($d = 3\,918\,837$), E.6 ($d = 5\,264\,069$), E.8 ($d = 6\,098\,360$), F.13 ($d = 8\,127\,208$), and G.16 ($d = 8\,711\,453$) in the range $2 \cdot 10^5 < d < 10^7$ [22, Tbl.4, p.498].

Meanwhile, Nebelung [37] had classified all metabelian 3-groups G with abelianization G/G' of type $(3, 3)$ in 1989, and had shown that there are exactly 26 distinct combinatorially possible 3-principalization types $\varkappa_1(K)$ containing at least one total capitulation, which had to be excluded for the 19 types of Scholz and Taussky. Nebelung proved that only 9 of these additional types with capitulation numbers $1 \le \#\mathcal{H}_0(K) \le 4$ are possible for group theoretic reasons. On the new year's day, January 01, 2008, we succeeded in providing evidence of the first occurrence $d = 540\,365$ with $\#\mathcal{H}_0(K) = 1$ of type c.21, (2034). This type was investigated in detail in [34, 35]. The search for capitulation numbers was finished when we discovered the minimal discriminant $d = 710\,652$ with $\#\mathcal{H}_0(K) = 2$ of type b.10, (2100) in November 2009.

5.1. **Real Quadratic Fields of Types a.1, a.2 and a.3.** Numerical investigation of real quadratic fields commenced in 1982, where Heider and Schmithals [19] showed the first examples of Moser's type α of Galois cohomological structure on unit groups (1975) [22, Prop.4.2, p.482], and of IPODs of type a.1 ($d = 62\,501$), a.2 ($d = 72\,329$), a.3 ($d = 32\,009$) [37]. See Figure 1. Our extension in 1991 [21] only produced further examples for these types. For the 15 years from 1991 to 2006 we believed these are the only possible types of real quadratic fields. The absolute frequencies in [22, 24], which should be corrected by the Corrigenda before Remark 5.2, and the extended statistics in Figure 2 underpin the *striking dominance* of types in section a. The distribution of the ground states alone reaches 79.7% for types a.2 and a.3 together, and 6.4% for type a.1.

So it was not astonishing that the first exception ($d = 214\,712$) did not show up earlier than 28 January 2006. See Figure 4.

As mentioned in [29] already:

Theorem 5.1. *The ground state of the types a.2 and a.3 can be separated by means of the iterated IPAD of second order* $\tau^{(2)}(G) = [\tau_0(G); [\tau_0(H); \tau_1(H)]_{H \in \mathrm{Lyr}_1(G)}]$.

Proof. This is essentially [29, Thm.6.1, p.296] but can also be seen directly by comparing the column $\tau_1(H)$ with the IPAD for the rows with lo $= 4$ and id $\in \{7, \dots, 10\}$ in Table 1. Here the column $\tau_2(H)$ containing the second layer of the IPAD does not permit a distinction. □

Unfortunately, we also must state a negative result:

Theorem 5.2. *Even the multi-layered IPAD* $\tau_*^{(2)}(G) = [\tau_0(G); [\tau_0(H); \tau_1(H); \tau_2(H)]_{H \in \mathrm{Lyr}_1(G)}]$ *of second order is unable to separate the excited states of the types a.2 and a.3. It is also unable to distinguish between the three candidates for each state of type a.1, and between the two candidates for each excited state of type a.3.*

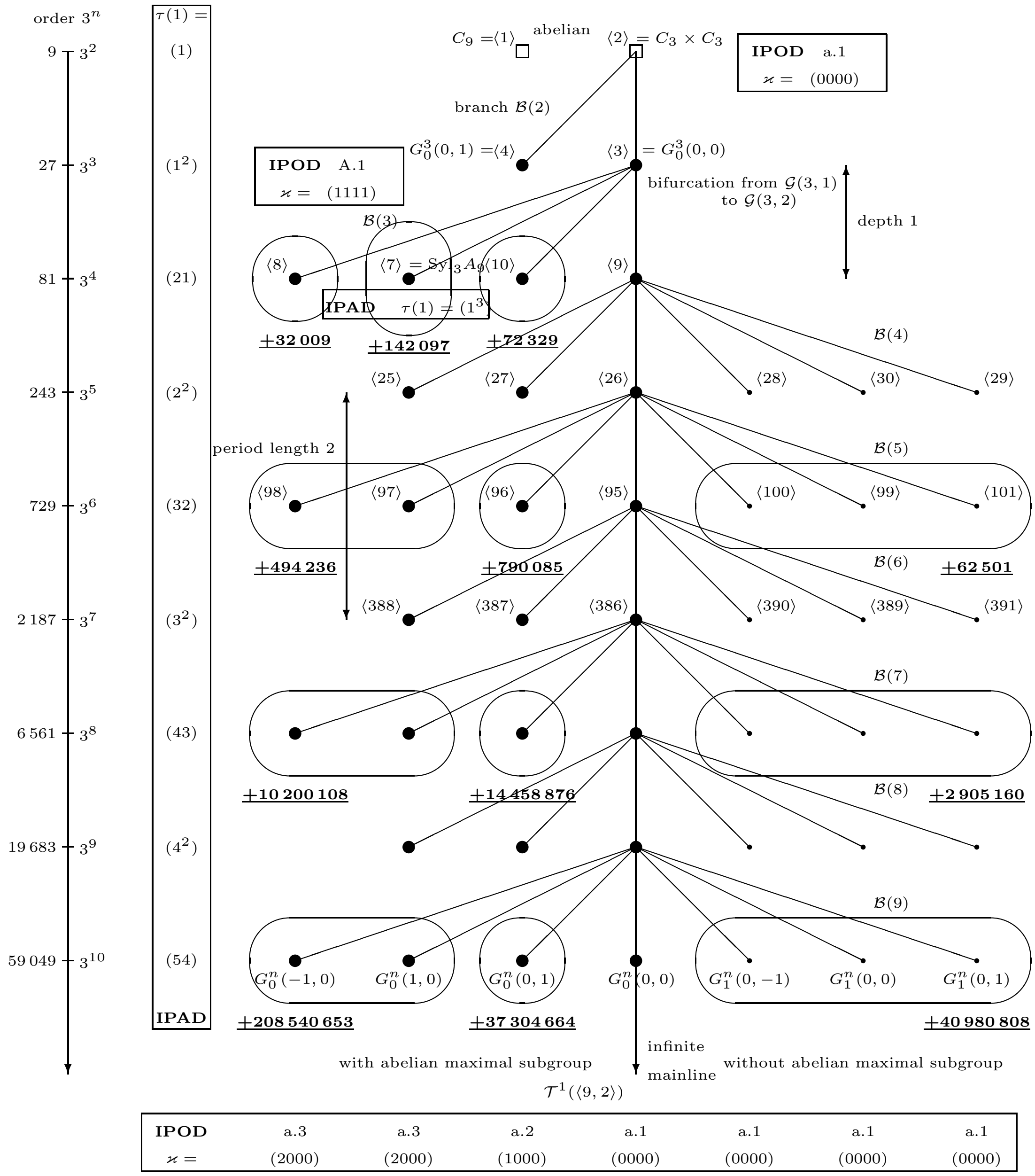
Proof. This is a consequence of comparing both columns $\tau_1(H)$ and $\tau_2(H)$ for the rows with lo $\in \{6, 8\}$ and id $\in \{95, \dots, 101\}$, resp. relative identifier id $\in \{\#1; 1, \dots, \#1; 7\}$ in Table 1. According to the selection rule [26, Thm.3.5, p.420], only every other branch of the tree $\mathcal{T}^1(\langle 3^2, 2\rangle)$ is admissible for second 3-class groups $\mathrm{G}_3^2(K)$ of (real) quadratic fields K. □

Figure 1 visualizes 3-groups which arise as 3-class tower groups $\mathrm{G}_3^\infty(K)$ of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$, $d > 0$, with 3-principalization types a.1, a.2 and a.3 in section § 4.1 and the corresponding minimal discriminants in the sense of Definition 5.1.

Theorem 5.3. *For each (real) quadratic field K with second 3-class group of maximal class the 3-class tower has exact length $\ell_3(K) = 2$.*

Proof. Let G be a 3-group of maximal class. Then G is metabelian [26, Thm.3.7, proof, p.421] or directly [8, Thm.6, p.26]. Suppose that H is a 3-group of derived length $\mathrm{dl}(H) \ge 3$ such that $H/H'' \simeq G$. According to [33, Thm.5.4], the Artin patterns $\mathrm{AP}(H)$ and $\mathrm{AP}(G)$ coincide, in particular, both groups share a common IPOD $\varkappa_1(H) = \varkappa_1(G)$, which contains at least three

FIGURE 1. Distribution of minimal discriminants on the coclass tree $\mathcal{T}^1(\langle 9,2\rangle)$



total kernels, indicated by zeros, $\varkappa_1 = (*000)$ [23]. However, this is a contradiction already, since any non-metabelian 3-group is descendant of one of the five groups $\langle 243, n\rangle$ with $n \in \{3,4,6,8,9\}$ whose IPODs possess at most two total kernels, and a descendant cannot have an IPOD with more total kernels than its parent, by [33, Thm.5.2]. Consequently, the cover of G in the sense of [35, Dfn.5.1] consists of the single element G and the length of the 3-class tower is given by $\ell_3(K) = \mathrm{dl}(G) = 2$. □

Remark 5.1. To the very best of our knowledge, Theorem 5.3 does not appear in the literature, although we are convinced that it is well known to experts, since it can also be proved purely group theoretically with the aid of a theorem by Blackburn [8, Thm.4, p.26]. Here we prefer to give a new proof which uses the structure of descendant trees.

Theorem 5.4. *The mainline vertices of the coclass-1 tree cannot occur as second 3-class groups of (real) quadratic fields (of type a.1).*

Proof. Since periodicity sets in with branch $\mathcal{B}(4)$ in the Figures 1 and 2, all mainline vertices V have p-multiplicator rank $\mu(V) = 4$ and thus relation rank $d_2(V) \geq \mu(V) = 4$. However, a real quadratic field K has torsion free Dirichlet unit rank $r = 1$ and certainly does not contain the (complex) primitive third roots of unity. According to the corrected version [35, Thm.5.1] of the Shafarevich theorem [43], the relation rank $d_2(G)$ of the 3-tower group $G = \mathrm{G}_3^\infty(K)$, which coincides with the second 3-class group $\mathfrak{M} = \mathrm{G}_3^2(K)$ by Theorem 5.3, is bounded by $2 \leq d_2(G) \leq 2+1 = 3$. □

We define two kinds of *arithmetically structured graphs* $\mathcal{G}$ of finite p-groups by mapping each vertex $V \in \mathcal{G}$ of the graph to statistical number theoretic information, e.g. the distribution of second p-class groups $\mathrm{G}_p^2(K)$ or p-class tower groups $\mathrm{G}_p^\infty(K)$, with respect to a given kind of number fields K, for instance real quadratic fields $K = K(d) = \mathbb{Q}(\sqrt{d})$ with discriminant $d > 0$.

Definition 5.1. Let p be a prime number and $\mathcal{G}$ be a subgraph of a descendant tree $\mathcal{T}$ of finite p-groups.

(1) The mapping

$$\mathrm{MD} : \mathcal{G} \to \mathbb{N} \cup \{\infty\}, \quad V \mapsto \inf\{d \mid \mathrm{G}_p^2(K(d)) \simeq V\} \tag{5.2}$$

is called the *distribution of minimal discriminants* on $\mathcal{G}$.

(2) For an assigned upper bound $B > 0$, the mapping

$$\mathrm{AF} : \mathcal{G} \to \mathbb{N} \cup \{0\}, \quad V \mapsto \#\{d < B \mid \mathrm{G}_p^2(K(d)) \simeq V\} \tag{5.3}$$

is called the *distribution of absolute frequencies* on $\mathcal{G}$.

For both mappings, the subset of the graph $\mathcal{G}$ consisting of vertices V with $\mathrm{MD}(V) \neq \infty$, resp. $\mathrm{AF}(V) \neq 0$, is called the *support* of the distribution. The trivial values outside of the support will be ignored in the sequel.

Figure 2 visualizes 3-groups which arise as 3-class tower groups $\mathrm{G}_3^\infty(K)$ of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$, $d > 0$, with 3-principalization types a.1, a.2 and a.3 in section § 4.1 and the corresponding absolute frequencies and percentages (relative frequencies with respect to the total number of 415 698 real quadratic fields with discriminants in the range $0 < d < B$, $B = 10^9$) which occur in Proposition 5.1.

The most extensive computation of data concerning unramified cyclic cubic extensions $L|K$ of the 481 756 real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with discriminant $0 < d < 10^9$ and 3-class rank $r_3(K) = 2$ has been achieved by M.R. Bush in 2015. In the following, we focus on the partial results for 3-class groups of type $(3,3)$, since they extend our own results of 2010 [22].

Proposition 5.1. *(M.R. Bush,* 11 *July* 2015*)*
In the range $0 < d < 10^9$ *with* 415 698 *fundamental discriminants* d *of real quadratic fields* $K = \mathbb{Q}(\sqrt{d})$ *having* 3*-class group of type* $(3,3)$*, there exist precisely*
208 236 *cases (*50.1%*) with IPAD* $\tau^{(1)}(K) = [1^2; (21, (1^2)^3)]$,
122 955 *cases (*29.6%*) with IPAD* $\tau^{(1)}(K) = [1^2; (1^3, (1^2)^3)]$,
26 678 *cases (*6.4%*) with IPAD* $\tau^{(1)}(K) = [1^2; (2^2, (1^2)^3)]$*, and*
11 780 *cases (*2.8%*) with IPAD* $\tau^{(1)}(K) = [1^2; (32, (1^2)^3)]$.

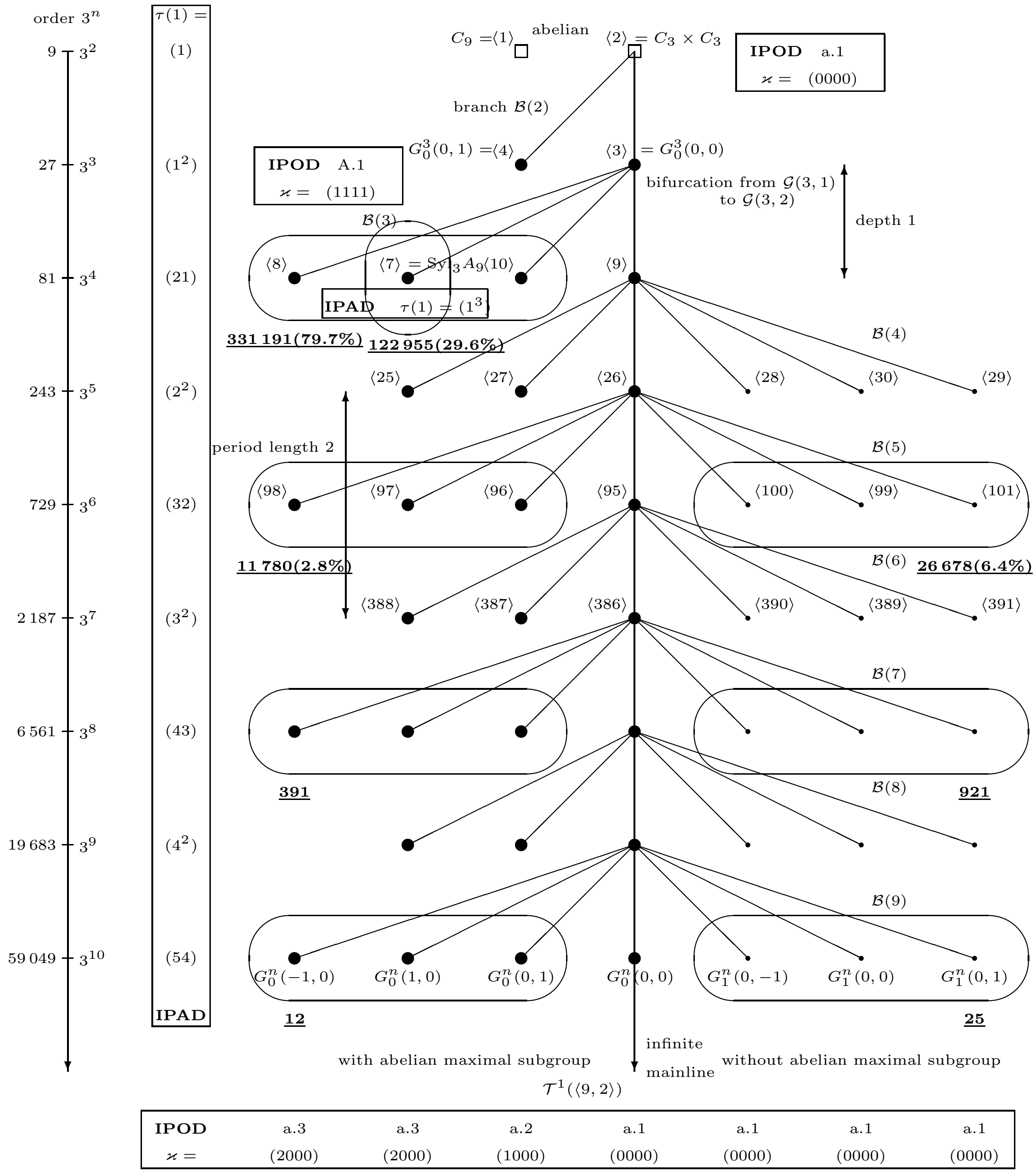
Proof. The results were computed with PARI/GP [40], double-checked with MAGMA [36], and kindly communicated to us privately. □

For establishing the connection between IPADs and IPODs we need the following bridge.

Corollary 5.1. *(D.C. Mayer,* 2014*)*

(1) *A real quadratic field* K *with IPAD* $\tau^{(1)}(K) = [1^2; (21, (1^2)^3)]$ *has IPOD either* $\varkappa_1(K) = (1000)$ *of type a.2 or* $\varkappa_1(K) = (2000)$ *of type a.3.*

FIGURE 2. Distribution of absolute frequencies on the coclass tree $\mathcal{T}^1(\langle 9,2\rangle)$



(2) *A real quadratic field K with IPAD $\tau^{(1)}(K) = [1^2; (1^3, (1^2)^3)]$ has IPOD $\varkappa_1(K) = (2000)$ of type a.3, more precisely a.3*, in view of the exceptional IPAD.*

(3) *A real quadratic field K with IPAD $\tau^{(1)}(K) = [1^2; (2^2, (1^2)^3)]$ has IPOD $\varkappa_1(K) = (0000)$ of type a.1.*

(4) *A real quadratic field K with IPAD $\tau^{(1)}(K) = [1^2; (32, (1^2)^3)]$ has IPOD either $\varkappa_1(K) = (1000)$ of type a.2 or $\varkappa_1(K) = (2000)$ of type a.3.*

Proof. According to Table 1, three (isomorphism classes of) groups G share the common IPAD $\tau^{(1)}(G) = [1^2; (21, (1^2)^3)]$, namely $\langle 81, 8 \ldots 10\rangle$, whereas the IPAD $\tau^{(1)}(G) = [1^2; (1^3, (1^2)^3)]$ unambiguously leads to the group $\langle 81, 7\rangle$ with IPOD $\varkappa_1(G) = (2000)$.

In Theorem 5.4 we have shown that the mainline group $\langle 81, 9\rangle$ cannot occur as the second 3-class group of a real quadratic field. Among the remaining two possible groups, $\langle 81, 8\rangle$ has IPOD $\varkappa_1(G) = (2000)$ and $\langle 81, 10\rangle$ has IPOD $\varkappa_1(G) = (1000)$.

The IPAD $\tau^{(1)}(G) = [1^2; (2^2, (1^2)^3)]$ leads to three groups $\langle 729, 99 \ldots 101\rangle$ with IPOD $\varkappa_1(G) = (0000)$ and defect of commutativity $k = 1$ [26, § 3.1.1, p.412].

Concerning the IPAD $\tau^{(1)}(G) = [1^2; (32, (1^2)^3)]$, Table 1 yields four groups G with SmallGroup identifiers $\langle 729, 95 \ldots 98\rangle$. The mainline group $\langle 729, 95\rangle$ is discouraged by Theorem 5.4, $\langle 729, 96\rangle$ has IPOD $\varkappa_1(G) = (1000)$, and the two groups $\langle 729, 97 \ldots 98\rangle$ have IPOD $\varkappa_1(G) = (2000)$.

By the Artin reciprocity law [1, 2], the Artin pattern $\mathrm{AP}(K)$ of the field K coincides with the Artin pattern $\mathrm{AP}(G)$ of its second 3-class group $G = \mathrm{G}_3^2(K)$. □

Corrigenda.

(1) The restriction of the numerical results in Proposition 5.1 to the range $0 < d < 10^7$ is in perfect accordance with our machine calculations by means of PARI/GP [40] in 2010, thus providing the first independent verification of data in [22, 24, 26].
However, in the manual evaluation of this extensive data material for the ground state of the types a.1, a.2, a.3, and a.3*, a few errors crept in, which must be corrected at three locations: in the tables [22, Tbl.2, p.496] and [24, Tbl.6.1, p.451], and in the tree diagram [26, Fig.3.2, p.422].
The absolute frequency of the ground state is actually given by
1 382 instead of the incorrect 1 386 for the union of types a.2 and a.3,
698 instead of the incorrect 697 for type a.3*,
2 080 instead of the incorrect 2 083 for the union of types a.2, a.3, and a.3*, and
150 instead of the incorrect 147 for type a.1.
(The three discriminants $d \in \{7\,643\,993, 7\,683\,308, 8\,501\,541\}$ were erroneously classified as type a.2 or a.3 instead of a.1.)
In the second table, two relative frequencies (percentages) should be updated:
$\frac{1382}{2303} \approx 60.0\%$ instead of $\frac{1386}{2303} \approx 60.2\%$ and
$\frac{698}{2303} \approx 30.3\%$ instead of $\frac{697}{2303} \approx 30.3\%$.

(2) Incidentally, although it does not concern the section a of IPODs, the single field with discriminant $d = 2\,747\,001$ was erroneously classified as type c.18, $\varkappa_1 = (0313)$, instead of H.4, $\varkappa_1 = (3313)$. This has consequences at four locations: in the tables [22, Tbl.4–5, pp.498–499] and [24, Tbl.6.5, p.452], and in the tree diagram [26, Fig.3.6, p.442].
The absolute frequency of these types is actually given by
28 instead of the incorrect 29 for type c.18 (see also [35, Prop.7.2]),
4 instead of the incorrect 3 for type H.4.
In the first two tables, the total frequencies should be updated, correspondingly:
207 instead of the incorrect 206 in [22, Tbl.4, p.498],
66 instead of the incorrect 67 in [22, Tbl.5, p.499].

Remark 5.2. The huge statistical ensembles underlying these computations admit a prediction of sound and reliable tendencies in the population of the ground state. We have a decrease
$\frac{1382}{2576} \approx 53.6\% \searrow \frac{208236}{415698} \approx 50.1\%$ by 3.5% for the union of types a.2 and a.3,
and increases
$\frac{698}{2576} \approx 27.1\% \nearrow \frac{122955}{415698} \approx 29.6\%$ by 2.5% for type a.3*, and
$\frac{150}{2576} \approx 5.8\% \nearrow \frac{26678}{415698} \approx 6.4\%$ by 0.6% for type a.1.

Of course, the accumulation of types a.2, a.3, and a.3* with absolute frequencies
$1382 + 698 = 2080$, resp. $208236 + 122955 = 331191$, shows a resultant decrease
$\frac{2080}{2576} \approx 80.7\% \searrow \frac{331191}{415698} \approx 79.7\%$ by 1.0%.

For the union of the first excites states of types a.2 and a.3, we have a stagnation
$\frac{72}{2576} \approx 2.8\% \approx \frac{11780}{415698} \approx 2.8\%$ at the same percentage.

Unfortunately, the exact absolute frequency of the ground state of type a.2, resp. type a.3, is unknown. It could be computed using Theorem 5.1. Whereas the values for the range $0 < d < 10^7$ were compatible with the assumption that each of the three types a.2, a.3, and a.3* occurs with equal probability of 27%, the results for the extended range $0 < d < 10^9$ seem to indicate a dominance of type a.3* with 29% as opposed to a (conjectural) equidistribution of 25% of the types a.2 and a.3.

FIGURE 3. Distribution of absolute frequencies on the sporadic coclass graph $\mathcal{G}_0(3,2)$

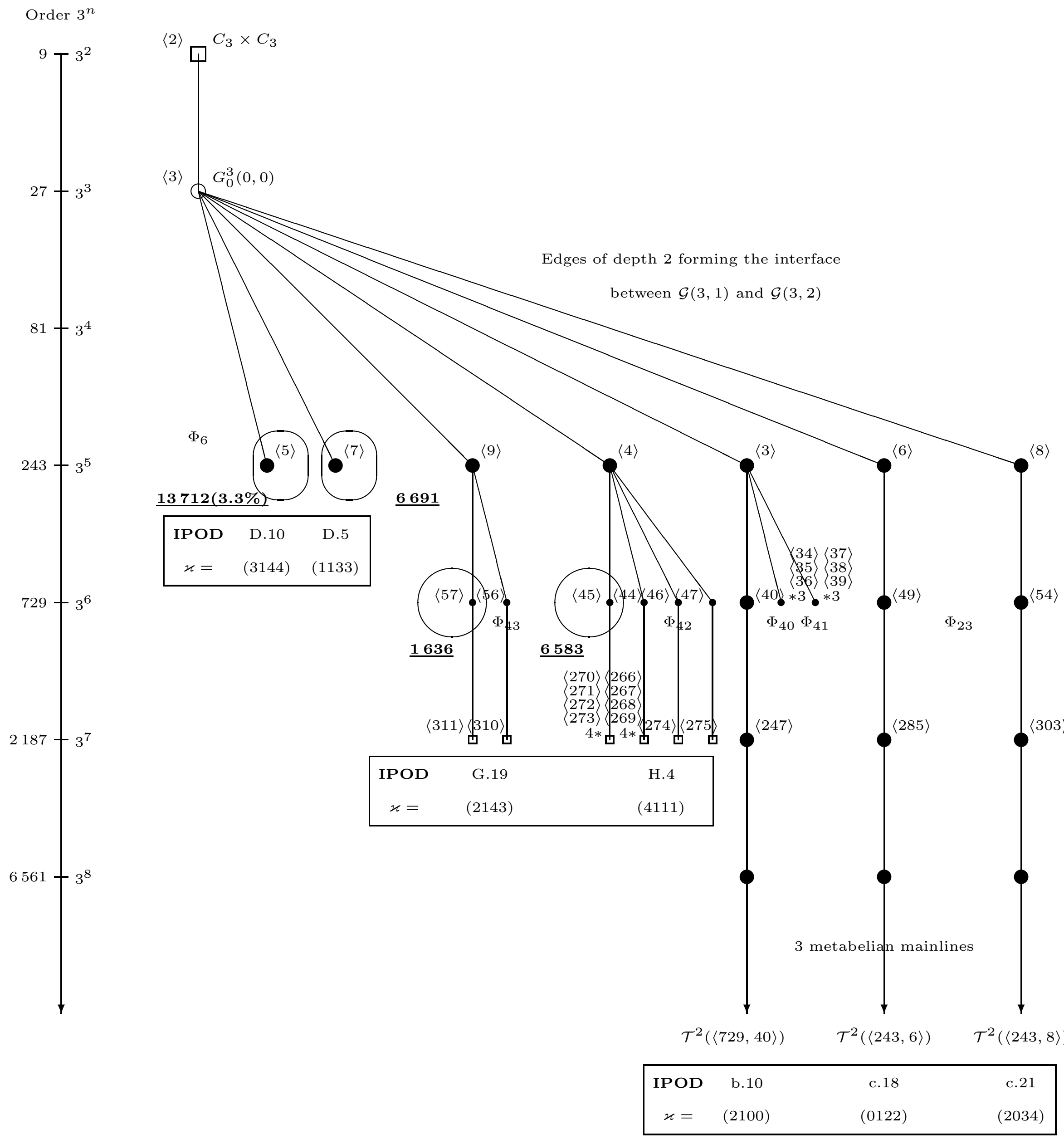
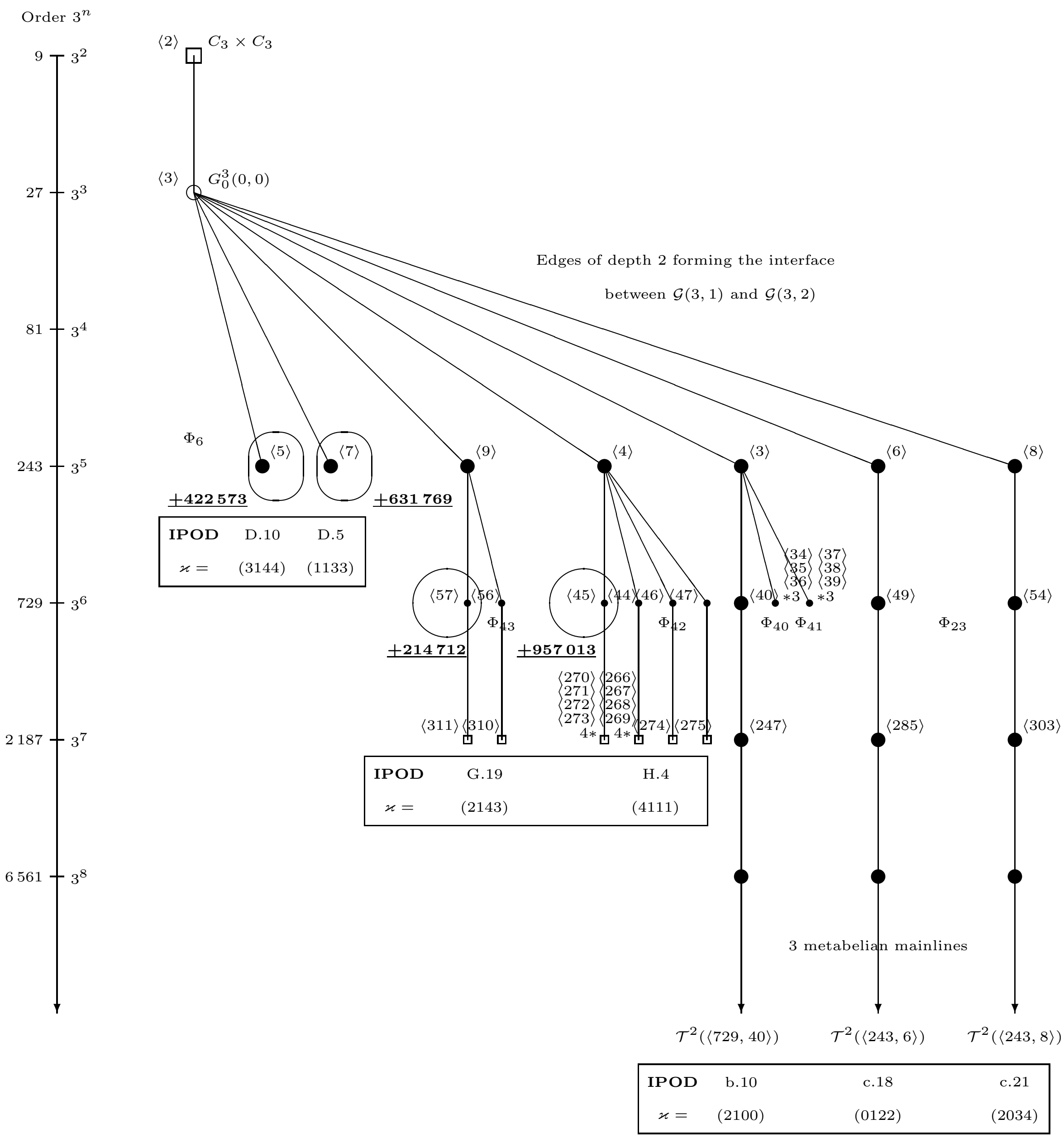


Figure 3, resp. Figure 4, visualizes sporadic 3-groups which arise as second 3-class groups $G_3^2(K)$ of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$, $d > 0$, with 3-principalization types D.10, D.5, G.19 and H.4 in section § 4.4 and the corresponding absolute frequencies, resp. minimal discriminants, which occur in section § 5.6 and § 5.4.

Generally, the vertices of the coclass trees in the Figures 1, 2, 5, 6, of the sporadic parts of coclass graphs in the Figures 3, 4, and of the descendant trees in the Figures 7, 8 represent

FIGURE 4. Distribution of minimal discriminants on the sporadic coclass graph $\mathcal{G}_0(3,2)$



isomorphism classes of finite 3-groups. Two vertices are connected by an edge $H \to G$ if G is isomorphic to the last lower central quotient $H/\gamma_c(H)$, where $c = \mathrm{cl}(H)$ denotes the nilpotency class of H, and either $|H| = 3|G|$, that is, $\gamma_c(H)$ is cyclic of order 3, or $|H| = 9|G|$, that is, $\gamma_c(H)$ is bicyclic of type $(3,3)$. (See also [26, § 2.2, p.410–411] and [27, § 4, p.163–164].)

The vertices of the tree diagrams in Figure 3, 4, 5, and 6 are classified by using different symbols:

(1) big full discs $\bullet$ represent metabelian groups $\mathfrak{M}$ with bicyclic centre of type $(3,3)$ and defect $k(\mathfrak{M}) = 0$ [26, § 3.3.2, p.429],
(2) small full discs $\bullet$ represent metabelian groups $\mathfrak{M}$ with cyclic centre of order 3 and defect $k(\mathfrak{M}) = 1$,

(3) small contour squares □ represent non-metabelian groups $\mathfrak{H}$.

In Figure 1 and 2,

(1) big full discs ● represent metabelian groups $\mathfrak{M}$ with defect $k(\mathfrak{M}) = 0$,
(2) small full discs • represent metabelian groups $\mathfrak{M}$ with defect $k(\mathfrak{M}) = 1$.

In Figure 7 and 8,

(1) big contour squares □ represent groups $\mathfrak{H}$ with p-multiplicator rank $\mu(\mathfrak{H}) \le 3$,
(2) small contour squares □ represent groups $\mathfrak{H}$ with p-multiplicator rank $\mu(\mathfrak{H}) \ge 4$.

A symbol $n*$ adjacent to a vertex denotes the multiplicity of a batch of n immediate descendants sharing a common parent. The groups of particular importance are labelled by a number in angles, which is the identifier in the SmallGroups library [6, 7] of GAP [18] and MAGMA [36], where we omit the orders, which are given on the left hand scale. The IPOD $\varkappa_1$ [23, Thm.2.5, Tbl.6–7], in the bottom rectangle concerns all vertices located vertically above. The first, resp. second, component $\tau_1(1)$, resp. $\tau_1(2)$, of the IPAD [28, Dfn.3.3, p.288] in the left rectangle concerns vertices on the same horizontal level with defect $k(\mathfrak{G}) = 0$. The periodicity with length 2 of branches, $\mathcal{B}(j) \simeq \mathcal{B}(j + 2)$ for $j \ge 4$, resp. $j \ge 7$, sets in with branch $\mathcal{B}(4)$, resp. $\mathcal{B}(7)$, having a root of order 3^4, resp. 3^7, in Figure 1 and 2, resp. 5 and 6. The metabelian skeletons of the Figures 5 and 6 were drawn in [37, p.189ff], the complete trees were given in [3, Fig.4.8, p.76, and Fig.6.1, p.123].

5.2. **Real Quadratic Fields of Types E.6 and E.14.** The range $0 < d < 10^7$ of discriminants d of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ underlying the Theorems 5.5 and 5.6 is just sufficient to prove that each of the possible groups G is actually realized by some field K.

Proposition 5.2. *(D.C. Mayer,* 2010 *and* 2015*)*
In the range $0 < d < 10^7$ *of fundamental discriminants* d *of real quadratic fields* $K = \mathbb{Q}(\sqrt{d})$*, there exist precisely* **3***, resp.* **4***, cases with* 3*-principalization type E.6,* $\varkappa_1(K) = (1122) \sim (1313)$*, resp. E.14,* $\varkappa_1(K) = (3122) \sim (2313)$*.*

Proof. The results of [24, Tbl.6.5, p.452], where the entry in the last column freq. should be 28 instead of 29 in the first row and 4 instead of 3 in the fourth row, were computed in 2010 by means of the free number theoretic computer algebra system PARI/GP [40] using an implementation of our own principalization algorithm in a PARI script, as described in detail in [24, § 5, pp.446–450]. The accumulated frequency 7 for the second and third row was recently split into 3 and 4 with the aid of the computational algebra system MAGMA [36]. □

Remark 5.3. The minimal discriminant $d = 5\,264\,069$ of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ of type E.6, resp. $d = 3\,918\,837$ of type E.14, is indicated in boldface font adjacent to an oval surrounding the vertex, resp. batch of two vertices, which represents the associated second 3-class group $G_3^2(K)$, on the branch $\mathcal{B}(6)$ of the coclass tree $\mathcal{T}^2(\langle 243, 6\rangle)$ in Figure 5.

Figure 5 visualizes 3-groups which arise as second 3-class groups $\mathrm{G}_3^2(K)$ of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$, $d > 0$, with 3-principalization types E.6 and E.14 in section § 4.2 and the corresponding minimal discriminants.

Theorem 5.5. *(D.C. Mayer,* 2015*)*

(1) *The* **2** *real quadratic fields* $K = \mathbb{Q}(\sqrt{d})$ *with the following discriminants* d *(***67***% of* 3 *fields with type E.6),*

$$5\,264\,069,\ 6\,946\,573,$$

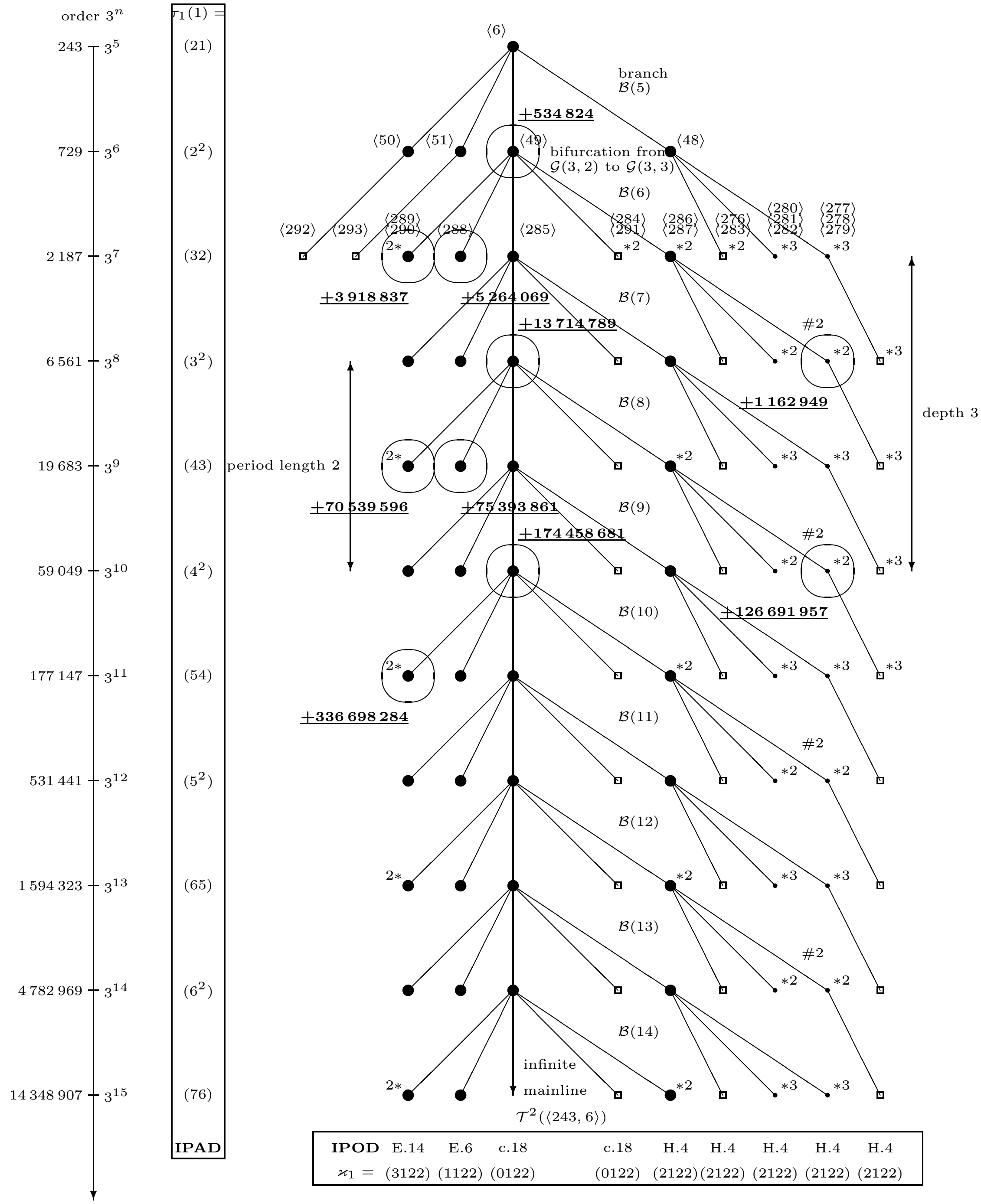
have the unique 3*-class tower group* $G \simeq \langle 3^6, 49\rangle - \#\mathbf{2}; \mathbf{4}$ *and* 3*-tower length* $\ell_3(K) = 3$*.*

(2) *The* ***single*** *real quadratic field* $K = \mathbb{Q}(\sqrt{d})$ *with discriminant* $d = 7\,153\,097$ *(***33***% of* 3 *fields with type E.6) has the unique* 3*-class tower group* $G \simeq \langle 3^7, \mathbf{288}\rangle = \langle 3^6, 49\rangle - \#\mathbf{1}; \mathbf{4}$ *and* 3*-tower length* $\ell_3(K) = 2$*.*

(3) *The* **3** *real quadratic fields* $K = \mathbb{Q}(\sqrt{d})$ *with the following discriminants* d *(***75***% of* 4 *fields with type E.14),*

$$3\,918\,837,\ 8\,897\,192,\ 9\,991\,432,$$

Figure 5. Distribution of minimal discriminants on the coclass tree $\mathcal{T}^2(\langle 243, 6\rangle)$



either have the 3*-class tower group* $G \simeq \langle 3^7, \mathbf{289}\rangle = \langle 3^6, 49\rangle - \#\mathbf{1}; \mathbf{5}$ *or* $G \simeq \langle 3^7, \mathbf{290}\rangle = \langle 3^6, 49\rangle - \#\mathbf{1}; \mathbf{6}$ *and* 3*-tower length* $\ell_3(K) = 2$.

(4) *The* ***single*** *real quadratic field* $K = \mathbb{Q}(\sqrt{d})$ *with discriminant* $d = 9\,433\,849$ *(***25***% of* 4 *fields with type E.14) either has the* 3*-class tower group* $G \simeq \langle 3^6, 49\rangle - \#\mathbf{2}; \mathbf{5}$ *or* $G \simeq \langle 3^6, 49\rangle - \#\mathbf{2}; \mathbf{6}$*, both of order* $|G| = 3^8$*, and* 3*-tower length* $\ell_3(K) = 3$.

Proof. Since all these real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ have 3-capitulation type $\varkappa_1(K) = (1122)$ or (3122) and 1st IPAD $\tau^{(1)}(K) = [1^2; \mathbf{32}, 1^3, (21)^2]$, and the 4 fields with $d \in \{3\,918\,837,\ 7\,153\,097,$

8 897 192, 9 991 432} have 2nd IPAD

$\tau_1(L_1) = (2^21, (31^2)^3),\ \tau_1(L_2) = (2^21, (\mathbf{1^3})^\mathbf{3}, (1^2)^9),\ \tau_1(L_3) = (2^21, (\mathbf{21})^\mathbf{3}),\ \tau_1(L_4) = (2^21, (\mathbf{21})^\mathbf{3}),$

whereas the 3 fields with $d \in \{5\,264\,069,\ 6\,946\,573,\ 9\,433\,849\}$ have 2nd IPAD

$\tau_1(L_1) = (2^21, (31^2)^3),\ \tau_1(L_2) = (2^21, (\mathbf{21^2})^\mathbf{3}, (1^2)^9),\ \tau_1(L_3) = (2^21, (\mathbf{31})^\mathbf{3}),\ \tau_1(L_4) = (2^21, (\mathbf{31})^\mathbf{3}),$

the claim is a consequence of Theorem 4.1. □

Remark 5.4. The 3-principalization type E.14 of the field with $d = 9\,433\,849$ resisted all attempts with MAGMA versions up to V2.21-7. Due to essential improvements in the change from relative to absolute number fields by the staff of the Magma group at the University of Sydney, it actually became feasible with V2.21-8 [36] for UNIX/LINUX machines.

5.3. **Real Quadratic Fields of Types E.8 and E.9.**

Proposition 5.3. *(D.C. Mayer,* 2010 *and* 2015*)*
In the range $0 < d < 10^7$ *of fundamental discriminants* d *of real quadratic fields* $K = \mathbb{Q}(\sqrt{d})$*, there exist precisely* **3***, resp.* **11***, cases with* 3*-principalization type E.8,* $\varkappa_1(K) = (2234) \sim (1231)$*, resp. E.9,* $\varkappa_1(K) = (2334) \sim (2231)$*.*

Proof. The results of [24, Tbl.6.7, p.453] were computed in 2010 by means of PARI/GP [40] using an implementation of our principalization algorithm, as described in [24, § 5, pp.446–450]. The accumulated frequency 14 in the last column “freq.” for the second and third row was recently split into 3 and 11 with the aid of MAGMA [36]. □

Remark 5.5. The minimal discriminant $d = 6\,098\,360$ of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ of type E.8, resp. $d = 342\,664$ of type E.9, is indicated in boldface font adjacent to an oval surrounding the vertex, resp. batch of two vertices, which represents the associated second 3-class group $G_3^2(K)$, on the branch $\mathcal{B}(6)$ of the coclass tree $\mathcal{T}^2(\langle 243, 8\rangle)$ in Figure 6.

Figure 6 visualizes 3-groups which arise as second 3-class groups $\mathrm{G}_3^2(K)$ of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$, $d > 0$, with 3-principalization types E.8 and E.9 in section § 4.3 and the corresponding minimal discriminants.

Theorem 5.6. *(D.C. Mayer,* 2015*)*

(1) *The* **2** *real quadratic fields* $K = \mathbb{Q}(\sqrt{d})$ *with the following discriminants* d *(***67***% of* 3 *fields with type E.8),*

$$6\,098\,360,\ 7\,100\,889,$$

have the unique 3*-class tower group* $G \simeq \langle 3^6, 54\rangle - \#\mathbf{2}; \mathbf{4}$ *and* 3*-tower length* $\ell_3(K) = 3$*.*

(2) *The* ***single*** *real quadratic field* $K = \mathbb{Q}(\sqrt{d})$ *with discriminant* $d = 8\,632\,716$ *(***33***% of* 3 *fields with type E.8) has the unique* 3*-class tower group* $G \simeq \langle 3^7, \mathbf{304}\rangle = \langle 3^6, 54\rangle - \#\mathbf{1}; \mathbf{4}$ *and* 3*-tower length* $\ell_3(K) = 2$*.*

(3) *The* **7** *real quadratic fields* $K = \mathbb{Q}(\sqrt{d})$ *with the following discriminants* d *(***64***% of* 11 *fields with type E.9),*

$$342\,664,\quad 1\,452\,185,\quad 1\,787\,945,\quad 4\,861\,720,$$
$$5\,976\,988,\quad 8\,079\,101,\quad 9\,674\,841,$$

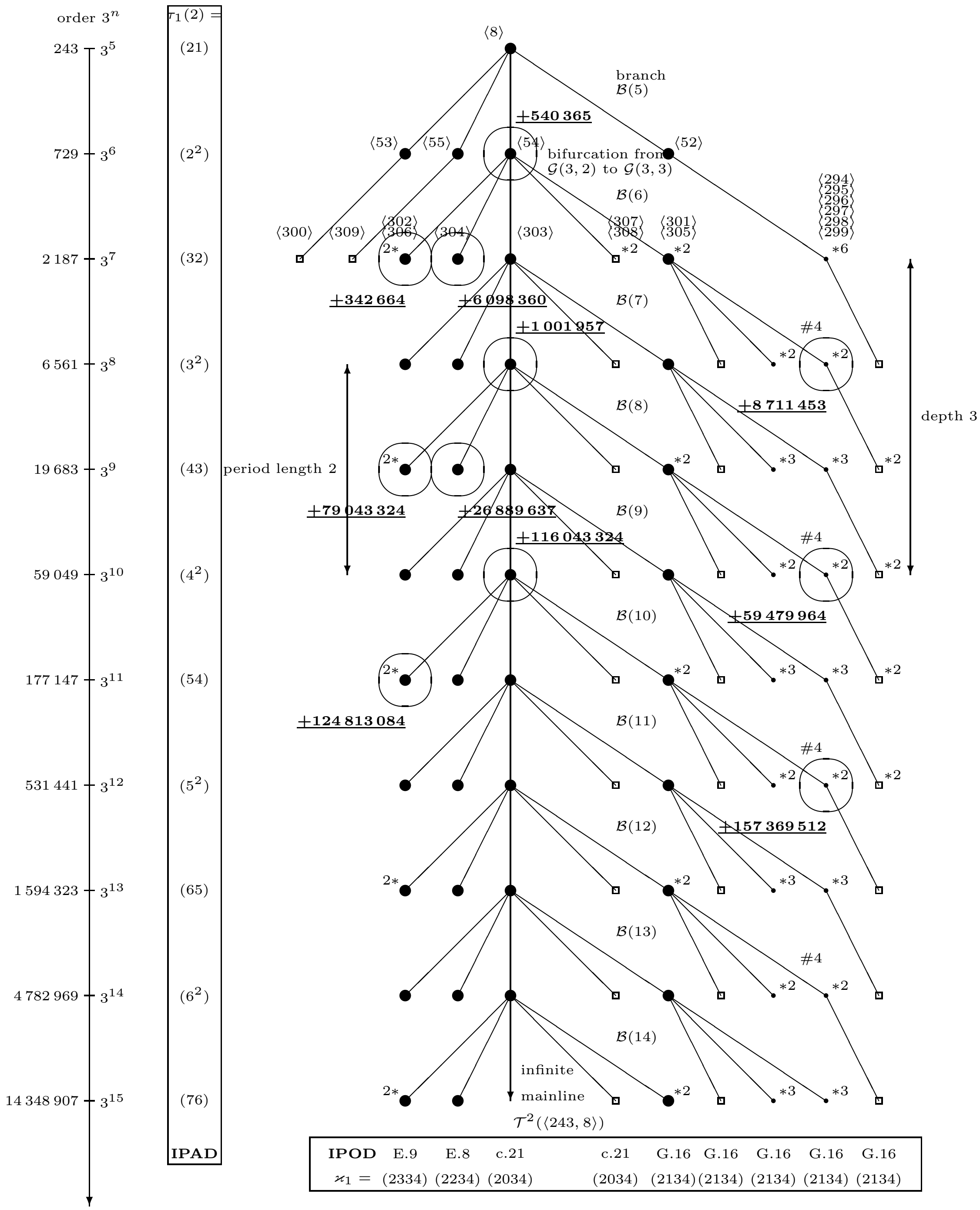
either have the 3*-class tower group* $G \simeq \langle 3^6, 54\rangle - \#\mathbf{2}; \mathbf{2}$ *or* $G \simeq \langle 3^6, 54\rangle - \#\mathbf{2}; \mathbf{6}$*, both of order* $|G| = 3^8$*, and* 3*-tower length* $\ell_3(K) = 3$*.*

(4) *The* **4** *real quadratic fields* $K = \mathbb{Q}(\sqrt{d})$ *with the following discriminants* d *(***36***% of* 11 *fields with type E.9),*

$$4\,760\,877,\ 6\,652\,929,\ 7\,358\,937,\ 9\,129\,480,$$

either have the 3*-class tower group* $G \simeq \langle 3^7, \mathbf{302}\rangle = \langle 3^6, 54\rangle - \#\mathbf{1}; \mathbf{2}$ *or* $G \simeq \langle 3^7, \mathbf{306}\rangle = \langle 3^6, 54\rangle - \#\mathbf{1}; \mathbf{6}$ *and* 3*-tower length* $\ell_3(K) = 2$*.*

FIGURE 6. Distribution of minimal discriminants on the coclass tree $\mathcal{T}^2(\langle 243, 8\rangle)$



Proof. Since all these real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ have 3-capitulation type $\varkappa_1(K) = (2234)$ or (2334) and 1st IPAD $\tau^{(1)}(K) = [1^2; \mathbf{32}, (21)^3]$, and the 5 fields with $d \in \{4\,760\,877,\ 6\,652\,929,\ 7\,358\,937,\ 8\,632\,716,\ 9\,129\,480\}$ have 2nd IPAD

$$\tau_1(L_1) = (2^21, (31^2)^3),\ \tau_1(L_2) = (2^21, (\mathbf{21})^{\mathbf{3}}),\ \tau_1(L_3) = (2^21, (\mathbf{21})^{\mathbf{3}}),\ \tau_1(L_4) = (2^21, (\mathbf{21})^{\mathbf{3}}),$$

whereas the 9 fields with $d \in \{342\,664,\ 1\,452\,185,\ 1\,787\,945,\ 4\,861\,720,\ 5\,976\,988,\ 6\,098\,360,\ 7\,100\,889,\ 8\,079\,101,\ 9\,674\,841\}$ have 2nd IPAD

$$\tau_1(L_1) = (2^21, (31^2)^3),\ \tau_1(L_2) = (2^21, (\mathbf{31})^{\mathbf{3}}),\ \tau_1(L_3) = (2^21, (\mathbf{31})^{\mathbf{3}}),\ \tau_1(L_4) = (2^21, (\mathbf{31})^{\mathbf{3}}),$$

the claim is a consequence of Theorem 4.2. □

Remark 5.6. The 3-principalization type E.9 of the field with $d = 9\,674\,841$ could not be computed with MAGMA versions up to V2.21-7. Finally, we succeeded in the determination by means of V2.21-8 [36].

5.4. Real Quadratic Fields of Type H.4.

Proposition 5.4. *(D.C. Mayer,* 2010 [22]*)*
In the range $0 < d < 10^7$ *of fundamental discriminants* d *of real quadratic fields* $K = \mathbb{Q}(\sqrt{d})$*, there exist precisely* **27** *cases with* 3*-principalization type H.4,* $\varkappa_1(K) = (4111)$*, and IPAD* $\tau^{(1)}(K) = [1^2; ((1^3)^3, 21)]$*. They share the common second* 3*-class group* $\mathrm{G}_3^2(K) \simeq \langle 3^6, 45\rangle$*.*

Proof. The results of [24, Tbl.6.3, p.452] were computed in 2010 by means of PARI/GP [40] using an implementation of our principalization algorithm, as described in [24, § 5, pp.446–450]. The frequency 27 in the last column "freq." for the fourth row concerns type H.4. □

Remark 5.7. To discourage any misinterpretation, we point out that there are four other real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with discriminants $d \in \{1\,162\,949,\ 2\,747\,001,\ 3\,122\,232,\ 4\,074\,493\}$ in the range $0 < d < 10^7$ which possess the same 3-principalization type H.4. However their second 3-class group $\mathrm{G}_3^2(K)$ is isomorphic to either $\langle 3^7, 286\rangle - \#1; 2$ or $\langle 3^7, 287\rangle - \#1; 2$ of order 3^8, which is not a sporadic group located on the coclass tree $\mathcal{T}^2(\langle 3^5, 6\rangle)$, and has a different IPAD $\tau^{(1)}(K) = [1^2; (32, 1^3, (21)^2)]$.

Theorem 5.7. *(D.C. Mayer,* 2015*)*

(1) *The* **11** *real quadratic fields* $K = \mathbb{Q}(\sqrt{d})$ *with the following discriminants* d *(***41**% *of* 27 *fields with type H.4),*

$$957\,013,\quad 1\,571\,953,\quad 1\,734\,184,\quad 3\,517\,689,\quad 4\,025\,909,\quad 4\,785\,845,$$
$$4\,945\,973,\quad 5\,562\,969,\quad 6\,318\,733,\quad 7\,762\,296,\quad 8\,070\,637,$$

have the unique 3*-class tower group* $G \simeq \langle 3^7, \mathbf{273}\rangle = \langle 3^6, 45\rangle - \#1; 4$ *and* 3*-tower length* $\ell_3(K) = 3$.

(2) *The* **8** *real quadratic fields* $K = \mathbb{Q}(\sqrt{d})$ *with the following discriminants* d *(***29**% *of* 27 *fields with type H.4),*

$$2\,023\,845,\ 4\,425\,229,\ 6\,418\,369,\ 6\,469\,817,\ 6\,775\,224,\ 6\,895\,612,\ 7\,123\,493,\ 9\,419\,261,$$

have 3*-class tower group either* $G \simeq \langle 3^7, \mathbf{271}\rangle = \langle 3^6, \mathbf{45}\rangle - \#1; 2$
or $G \simeq \langle 3^7, \mathbf{272}\rangle = \langle 3^6, \mathbf{45}\rangle - \#1; 3$ *and* 3*-tower length* $\ell_3(K) = 3$.

(3) *The* **5** *real quadratic fields* $K = \mathbb{Q}(\sqrt{d})$ *with the following discriminants* d *(***19**% *of* 27 *fields with type H.4),*

$$2\,303\,112,\ 3\,409\,817,\ 3\,856\,685,\ 5\,090\,485,\ 6\,526\,680,$$

have the unique 3*-class tower group* $G \simeq \langle 3^7, \mathbf{270}\rangle = \langle 3^6, \mathbf{45}\rangle - \#1; 1$ *and* 3*-tower length* $\ell_3(K) = 3$.

(4) *The* **3** *real quadratic fields* $K = \mathbb{Q}(\sqrt{d})$ *with the following discriminants* d *(***11**% *of* 27 *fields with type H.4),*

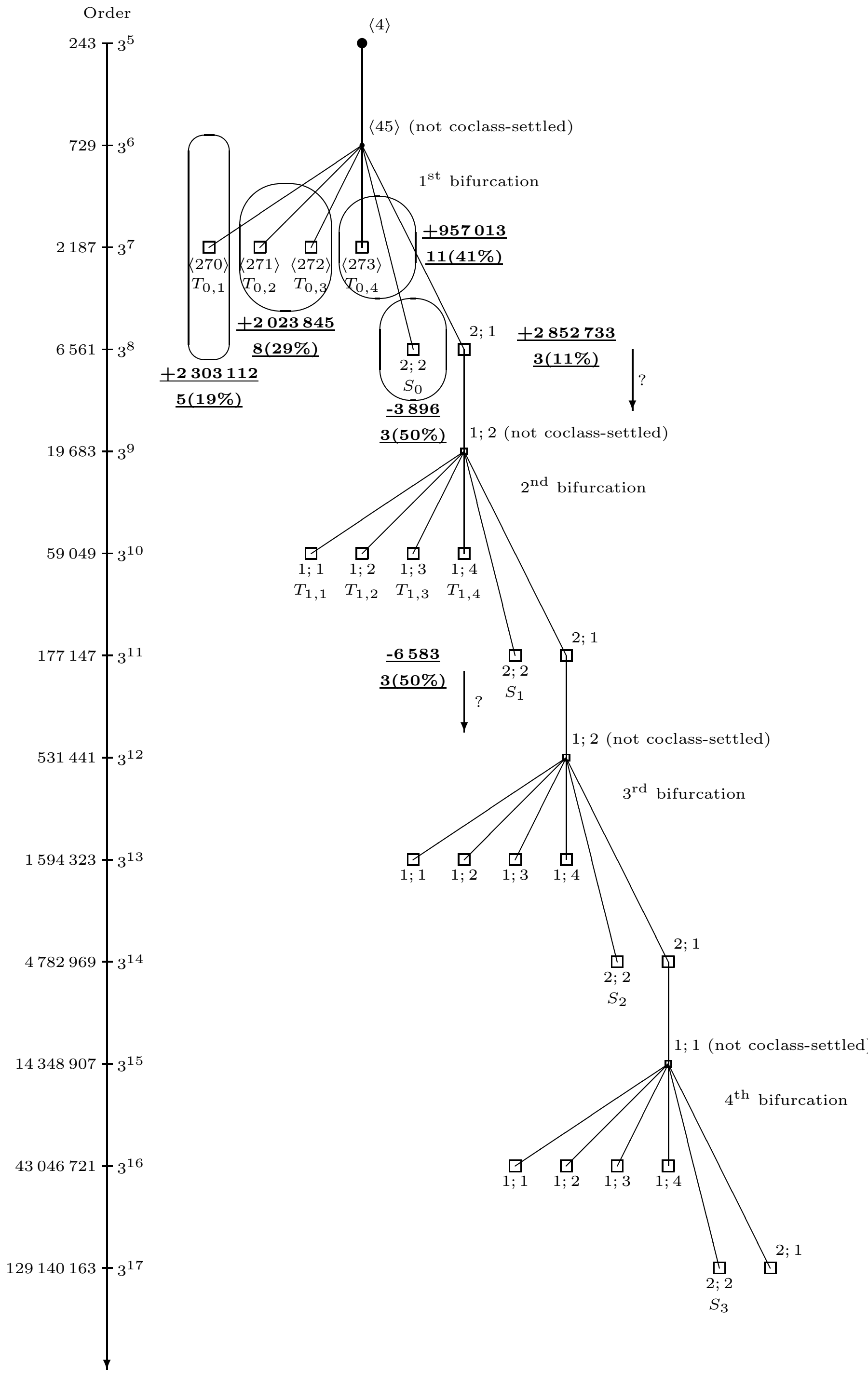
$$2\,852\,733,\ 8\,040\,029,\ 8\,369\,468,$$

have a 3*-class tower group of order at least* 3^8 *and* 3*-tower length* $\ell_3(K) \geq 3$.

Proof. Extensions of absolute degrees 6 and 18 were constructed in steps, using the technique of C. Fieker [15], with MAGMA [36]. The resulting iterated IPADs of second order $\tau^{(2)}(K)$ were used for the identification, according to Table 4, which is also contained in the more extensive theorem [29, Thm.6.5, pp.304–306]. □

Figure 7 visualizes sporadic 3-groups which arise as 3-class tower groups $G_3^\infty(K)$ of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$, $d > 0$, with 3-principalization type H.4 in section § 4.4 and the corresponding minimal discriminants. The tree is infinite, according to [5] and [29, Cor.6.2, p.301].

FIGURE 7. Distribution of 3-class tower groups on the descendant tree $\mathcal{T}_*(\langle 243, 4\rangle)$



5.5. **Complex Quadratic Fields of Type H.4.**

Proposition 5.5. *(D.C. Mayer,* 1989 [20] *and* 2009 [22]*)*
In the range $-30\,000 < d < 0$ *of fundamental discriminants* d *of complex quadratic fields* $K = \mathbb{Q}(\sqrt{d})$*, there exist precisely* **6** *cases with* 3*-principalization type H.4,* $\varkappa_1(K) = (4111)$*, and IPAD* $\tau^{(1)}(K) = [1^2; ((1^3)^3, 21)]$*. They share the common second* 3*-class group* $\mathrm{G}_3^2(K) \simeq \langle 3^6, 45\rangle$*.*

Proof. In the table of suitable base fields [20, p.84], the row Nr.4 contains 7 discriminants $-30\,000 < d < 0$ of complex quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with type H.4. It was computed in 1989 by means of our implementation of the principalization algorithm by Scholz and Taussky, described in [20, pp.80–83]. We recognized (in 1989 already) that only for the discriminant $d = -21\,668$ one of the four absolute cubic subfields L_i, $1 \le i \le 4$, of the unramified cyclic cubic extensions N_i of K has 3-class number $h_3(L_i) = 9$, which is not the case for the other 6 cases of type H.4 in the table [20, pp.78–79]. According to [22, Prop.4.4, p.485] or [22, Thm.4.2, p.489] or [41], the exceptional cubic field L_i is contained in a sextic field N_i with 3-class number $h_3(N_i) = 3 \cdot h_3(L_i)^2 = 243$, which discourages an IPAD $\tau^{(1)}(K) = [1^2; ((1^3)^3, 21)]$. □

Remark 5.8. The complex quadratic field with discriminant $d = -21\,668$ possesses the same 3-principalization type H.4, but its second 3-class group $\mathrm{G}_3^2(K)$ is isomorphic to either $\langle 3^7, 286\rangle - \#1; 2$ or $\langle 3^7, 287\rangle - \#1; 2$ of order 3^8, and has the different IPAD $\tau^{(1)}(K) = [1^2; (32, 1^3, (21)^2)]$.

Theorem 5.8. *(D.C. Mayer,* 2015*)*

(1) *The* **3** *complex quadratic fields* $K = \mathbb{Q}(\sqrt{d})$ *with the following discriminants* d *(***50**% *of* 6 *fields with type H.4),*
$$-3\,896,\ -25\,447,\ -27\,355,$$
have the unique 3*-class tower group* $G \simeq S_0 = \langle 3^6, 45\rangle - \#2; 2$ *of order* 3^8 *and* 3*-tower length* $\ell_3(K) = 3$*.*

(2) *The* **3** *complex quadratic fields* $K = \mathbb{Q}(\sqrt{d})$ *with the following discriminants* d *(***50**% *of* 6 *fields with type H.4),*
$$-6\,583,\ -23\,428,\ -27\,991,$$
have a 3*-class tower group of order at least* 3^{11} *and* 3*-tower length* $\ell_3(K) \ge 3$*.*

Proof. Extensions of absolute degrees 6 and 54 were constructed in two steps, using the technique of Fieker [15], squeezing MAGMA [36] close to its limits. The resulting multi-layered iterated IPADs of second order $\tau_*^{(2)}(K)$ were used for the identification, according to Table 4, resp. the more detailed theorem [29, Thm.6.5, pp.304–306]. □

5.6. Real Quadratic Fields of Type G.19.

Proposition 5.6. *(D.C. Mayer,* 2010 [22]*)*
In the range $0 < d < 10^7$ *of fundamental discriminants* d *of real quadratic fields* $K = \mathbb{Q}(\sqrt{d})$*, there exist precisely* **11** *cases with* 3*-principalization type G.19,* $\varkappa_1(K) = (2143)$*. Their IPAD is uniformly given by* $\tau^{(1)}(K) = [1^2; ((21)^4)]$*, in this range.*

Proof. The results of [24, Tbl.6.3, p.452] were computed in 2010 by means of PARI/GP [40] using an implementation of our principalization algorithm, as described in [24, § 5, pp.446–450]. The frequency 11 in the last column "freq." for the first row concerns type G.19. □

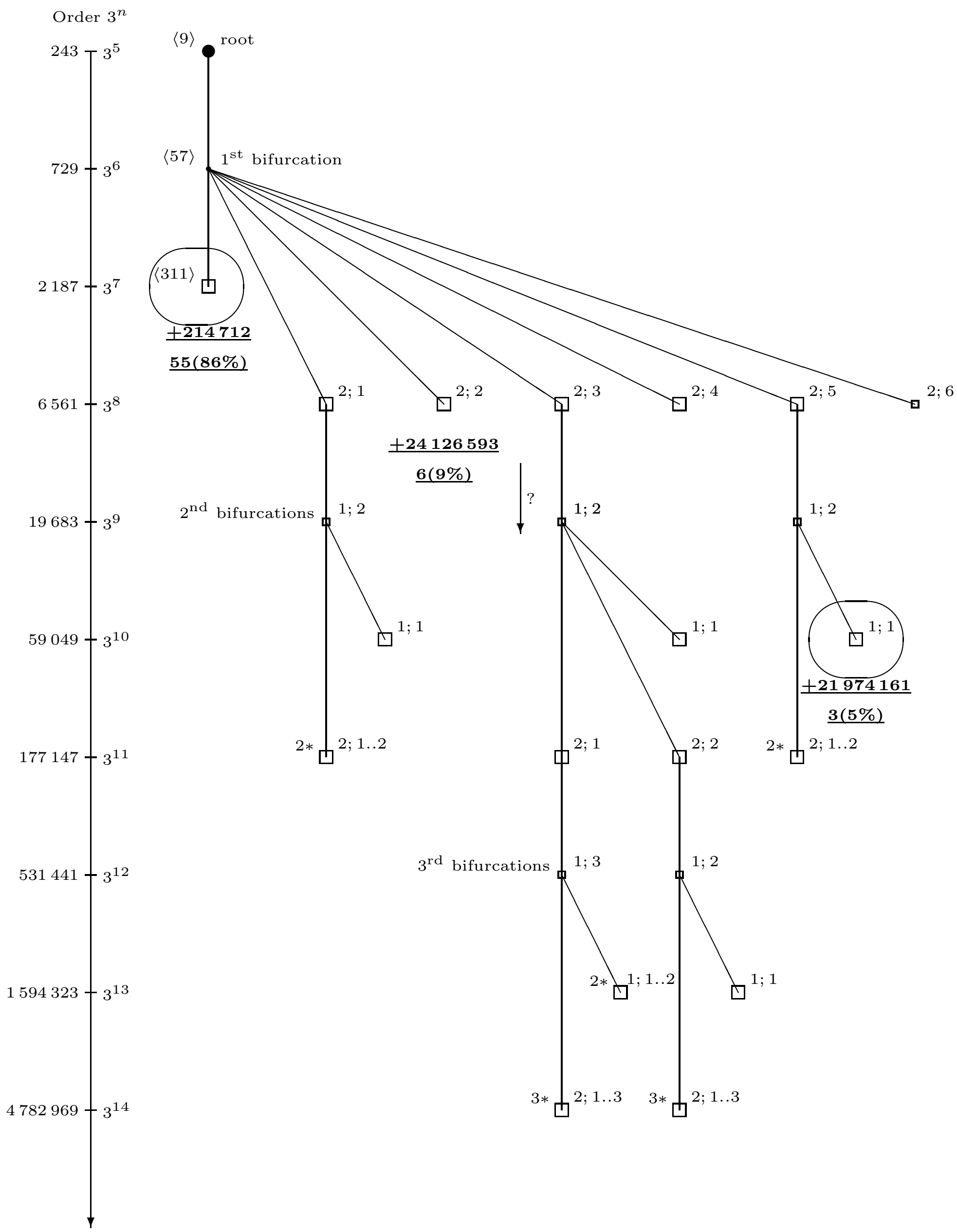
Theorem 5.9. *(D.C. Mayer,* 2015*)*
The **11** *real quadratic fields* $K = \mathbb{Q}(\sqrt{d})$ *with the following discriminants* d *(***100**% *of* 11 *fields with type G.19),*
$$\begin{array}{rrrr} 214\,712, & 943\,077, & 1\,618\,493, & 2\,374\,077, \\ 3\,472\,653, & 4\,026\,680, & 4\,628\,117, & 5\,858\,753, \\ 6\,405\,317, & 7\,176\,477, & 7\,582\,988, & \end{array}$$
have the unique 3*-class tower group* $G \simeq \langle 3^7, \mathbf{311}\rangle = \langle 3^6, 57\rangle - \#1; 1$ *and* 3*-tower length* $\ell_3(K) = 3$*.*

Proof. Extensions of absolute degrees 6 and 18 were constructed, using Fieker's technique [15], with MAGMA [36]. The resulting iterated IPADs of second order $\tau^{(2)}(K)$ were used for the identification, according to Table 5. □

Since real quadratic fields of type G.19 seemed to have a very rigid behaviour with respect to their 3-class field tower, admitting no variation at all, we were curious about the continuation of these discriminants beyond the range $d < 10^7$. Fortunately, M.R. Bush granted access to his extended numerical results for $d < 10^9$, and so we are able to state the following unexpected answer to our question "Is the 3-class tower group G of real quadratic fields with type G.19 and IPAD $\tau^{(1)}(K) = [1^2; ((21)^4)]$ always isomorphic to $\langle 3^7, \mathbf{311}\rangle$?"

Figure 8 visualizes sporadic 3-groups which arise as 3-class tower groups $\mathrm{G}_3^\infty(K)$ of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$, $d > 0$, with 3-principalization type G.19 in section § 4.4 and the corresponding minimal discriminants. The subtrees $\mathcal{T}(W-\#2; i)$ are finite and drawn completely for $i \in \{1, 3, 5\}$, but they are omitted in the complicated cases $i \in \{2, 4, 6\}$, where they reach beyond order 3^{20}.

FIGURE 8. Distribution of 3-class tower groups on the descendant tree $\mathcal{T}_*(\langle 243, 9\rangle)$



Proposition 5.7. *(M.R. Bush, 2015)*
In the range $0 < d < 5 \cdot 10^7$ of fundamental discriminants d of real quadratic fields $K = \mathbb{Q}(\sqrt{d})$, there exist precisely **64** *cases with 3-principalization type G.19, $\varkappa_1(K) = (2143)$ and IPAD $\tau^{(1)}(K) = [1^2; ((21)^4)]$.*

Proof. Private communication on July 11, 2015. □

Theorem 5.10. *(D.C. Mayer, 2015)*

(1) *The* **11** *real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with the discriminants d in Theorem 5.9 and the* **44** *real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with the following discriminants d (that is, together* **55** *fields or* **86**% *of* 64 *fields with type G.19),*

10 169 729,	11 986 573,	14 698 056,	14 836 573,	16 270 305,
16 288 424,	18 195 889,	19 159 368,	21 519 660,	21 555 097,
22 296 941,	22 431 068,	24 229 337,	25 139 461,	26 977 089,
27 696 973,	29 171 832,	29 523 765,	30 019 333,	31 921 420,
32 057 249,	33 551 305,	35 154 857,	35 846 545,	36 125 177,
36 409 821,	37 344 053,	37 526 493,	37 796 984,	38 691 433,
39 693 865,	40 875 944,	42 182 968,	42 452 445,	42 563 029,
43 165 432,	43 934 584,	44 839 889,	44 965 813,	45 049 001,
46 180 124,	46 804 541,	46 971 381,	48 628 533,	

have the unique 3*-class tower group $G \simeq \langle 3^7, \mathbf{311}\rangle = \langle 3^6, 57\rangle - \#1; 1$ and* 3*-tower length $\ell_3(K) = 3$.*

(2) *The* **3** *real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with the following discriminants d (***5**% *of* 64 *fields with type G.19),*

$$21\,974\,161,\ 22\,759\,557,\ 35\,327\,365,$$

have the unique 3*-class tower group $G \simeq \langle 3^6, \mathbf{57}\rangle - \#2; 5 - \#1; 2 - \#1; 1$ of order 3^{10} and* 3*-tower length $\ell_3(K) = 3$.*

(3) *The* **6** *real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with the following discriminants d (***9**% *of* 64 *fields with type G.19),*

$$24\,126\,593,\ 29\,739\,477,\ 31\,353\,229,\ 35\,071\,865,\ 40\,234\,205,\ 40\,706\,677,$$

have a 3*-class tower group of order at least 3^8 and* 3*-tower length $\ell_3(K) \geq 3$.*

Proof. Similar to the proof of Theorem 5.9, but now applied to the more extensive range of discriminants. □

6. Complex quadratic fields of type $(3, 3, 3)$

In the final section § 7 of [29], we proved that the second 3-class groups $\mathrm{G}_3^2(K)$ of the 14 complex quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with fundamental discriminants $-10^7 < d < 0$ and 3-class group $\mathrm{Cl}_3(K)$ of type $(3, 3, 3)$ are pairwise non-isomorphic [29, Thm.7.1, p.307]. For the proof of this theorem in [29, § 7.3, p.311], the IPADs of the 14 fields were not sufficient, since the three fields with discriminants

$$d \in \{-4\,447\,704, -5\,067\,967, -8\,992\,363\}$$

share the common accumulated (unordered) IPAD

$$\tau^{(1)}(K) = [\tau_0(K); \tau_1(K)] = [1^3; (32^21; (21^4)^5, (2^21^2)^7)].$$

To complete the proof we had to use information on the occupation numbers of the accumulated (unordered) IPODs,
$\varkappa_1(K) = [1, 2, 6, (8)^6, 9, (10)^2, 13]$ with maximal occupation number 6 for $d = -4\,447\,704$,
$\varkappa_1(K) = [1, 2, (3)^2, (4)^2, 6, (7)^2, 8, (9)^2, 12]$ with maximal occupation number 2 for $d = -5\,067\,967$,
$\varkappa_1(K) = [(2)^2, 5, 6, 7, (9)^2, (10)^3, (12)^3]$ with maximal occupation number 3 for $d = -8\,992\,363$.

Meanwhile we succeeded in computing the *second layer* of the transfer target type, $\tau_2(K)$, for the three critical fields with the aid of the computational algebra system MAGMA [36] by determining the structure of the 3-class groups $\mathrm{Cl}_3(L)$ of the 13 unramified bicyclic bicubic extensions $L|K$ with relative degree $[L : K] = 3^2$ and absolute degree 18. In accumulated (unordered) form the second layer of the TTT is given by
$\tau_2(K) = [32^51^2; 4321^5; 2^51^3, (3^221^5)^2; 2^41^4, 32^21^5; (2^21^7)^3, (2^31^5)^3]$ for $d = -4\,447\,704$,
$\tau_2(K) = [3^22^21^4; (3^221^5)^3; 32^21^5; (2^31^5)^8]$ for $d = -5\,067\,967$, and
$\tau_2(K) = [32^21^6, (3^221^5)^3; 2^41^4, 32^21^5; 2^21^7, (2^31^5)^6]$ for $d = -8\,992\,363$.

These results admit incredibly powerful conclusions, which bring us closer to the ultimate goal to determine the precise isomorphism type of $\mathrm{G}_3^2(K)$. Firstly, they clearly show that the second 3-class groups of the three critical fields are pairwise non-isomorphic without using the IPODs. Secondly, the component with the biggest order establishes an impressively sharpened estimate for the order of $\mathrm{G}_3^2(K)$ from below. The background is explained by the following lemma.

Lemma 6.1. *Let G be a finite p-group with abelianization G/G' of type (p, p, p) and denote by $\mathrm{lo}_p(G) := \log_p(\mathrm{ord}(G))$ the logarithmic order of G with respect to the prime number p. Then the abelianizations H/H' of subgroups $H < G$ in various layers of G admit lower bounds for $\mathrm{lo}_p(G)$:*

(1) $\mathrm{lo}_p(G) \ge 1 + \max\{\mathrm{lo}_p(H/H') \mid H \in \mathrm{Lyr}_1(G)\}$.
(2) $\mathrm{lo}_p(G) \ge 2 + \max\{\mathrm{lo}_p(H/H') \mid H \in \mathrm{Lyr}_2(G)\}$.
(3) $\mathrm{lo}_p(G) \ge 3 + \mathrm{lo}_p(G'/G'')$, *and in particular we have an equation* $\mathrm{lo}_p(G) = 3 + \mathrm{lo}_p(G')$ *if G is metabelian.*

Proof. The Lagrange formula for the order of G in terms of the index of a subgroup $H \le G$ reads

$$\mathrm{ord}(G) = (G : H) \cdot \mathrm{ord}(H),$$

and taking the p-logarithm yields

$$\mathrm{lo}_p(G) = \log_p((G : H)) + \mathrm{lo}_p(H).$$

In particular, we have $\log_p((G : H)) = \log_p(p^n) = n$ for $H \in \mathrm{Lyr}_n(G)$, $0 \le n \le 3$, and again by the Lagrange formula

$$\mathrm{ord}(H) = (H : H') \cdot \mathrm{ord}(H') \ge (H : H'),$$

respectively

$$\mathrm{lo}_p(H) = \log_p((H : H')) + \mathrm{lo}_p(H') \ge \mathrm{lo}_p(H/H'),$$

with equality if and only if $H' = 1$, that is, H is abelian.
Finally, G is metabelian if and only if G' is abelian. □

Let us first draw weak conclusions from the first layer of the TTT, i.e. the IPAD, with the aid of Lemma 6.1.

Theorem 6.1. *(Coarse estimate, D.C. Mayer, July 2014)*
The order of $G := \mathrm{G}_3^2(K)$ for the three critical fields K is bounded from below by $\mathrm{ord}(G) \ge 3^9$.
If the maximal subgroup $H < G$ with the biggest order of H/H' is abelian, i.e. $H' = 1$, then the precise logarithmic order of G is given by $\mathrm{lo}_3(G) = 9$.

Proof. The three critical fields with discriminants $d \in \{-4\,447\,704, -5\,067\,967, -8\,992\,363\}$ share the common accumulated IPAD $\tau^{(1)}(K) = [\tau_0(K); \tau_1(K)] = [1^3; (32^21; (21^4)^5, (2^21^2)^7)]$.

Consequently, Lemma 6.1 yields a uniform lower bound for each of the three fields:

$$\mathrm{lo}_3(G) \ge 1 + \max\{\mathrm{lo}_3(H/H') \mid H \in \mathrm{Lyr}_1(G)\} = 1 + \mathrm{lo}_3(32^21) = 1 + 3 + 2 \cdot 2 + 1 = 9.$$

The assumption that a maximal subgroup $U < G$ having not the biggest order of U/U' were abelian (with $U/U' \simeq U$) immediately yields the contradiction that

$$\mathrm{lo}_3(G) = \log_3((G : U)) + \mathrm{lo}_3(U) = 1 + \mathrm{lo}_3(U/U') < 1 + \max\{\mathrm{lo}_3(H/H') \mid H \in \mathrm{Lyr}_1(G)\} \le \mathrm{lo}_3(G).$$

□

It is illuminating that much stronger estimates and conclusions are possible by applying Lemma 6.1 to the second layer of the TTT.

Theorem 6.2. *(Finer estimates, D.C. Mayer, March 2015)*
None of the maximal subgroups of $G := \mathrm{G}_3^2(K)$ for the three critical fields K can be abelian.
The logarithmic order of G is bounded from below by
$\mathrm{lo}_3(G) \geq 17$ *for* $d = -4\,447\,704$,
$\mathrm{lo}_3(G) \geq 16$ *for* $d = -5\,067\,967$,
$\mathrm{lo}_3(G) \geq 15$ *for* $d = -8\,992\,363$.

Proof. As mentioned earlier already, computations with MAGMA [36] have shown that the accumulated second layer of the TTT is given by
$\tau_2(K) = [32^5 1^2; 4321^5; 2^5 1^3, (3^2 21^5)^2; 2^4 1^4, 32^2 1^5; (2^2 1^7)^3, (2^3 1^5)^3]$ for $d = -4\,447\,704$,
$\tau_2(K) = [3^2 2^2 1^4; (3^2 21^5)^3; 32^2 1^5; (2^3 1^5)^8]$ for $d = -5\,067\,967$, and
$\tau_2(K) = [32^2 1^6, (3^2 21^5)^3; 2^4 1^4, 32^2 1^5; 2^2 1^7, (2^3 1^5)^6]$ for $d = -8\,992\,363$.

Consequently the maximal logarithmic order $M := \max\{\mathrm{lo}_3(H/H') \mid H \in \mathrm{Lyr}_2(G)\}$ is
$M = \mathrm{lo}_3(32^5 1^2) = 3 + 5 \cdot 2 + 2 \cdot 1 = 15$ for $d = -4\,447\,704$,
$M = \mathrm{lo}_3(3^2 2^2 1^4) = 2 \cdot 3 + 2 \cdot 2 + 4 \cdot 1 = 14$ for $d = -5\,067\,967$,
$M = \mathrm{lo}_3(32^2 1^6) = 3 + 2 \cdot 2 + 6 \cdot 1 = 13$ for $d = -8\,992\,363$.
According to Lemma 6.1, we have $\mathrm{lo}_3(G) \geq 2 + \max\{\mathrm{lo}_3(H/H') \mid H \in \mathrm{Lyr}_2(G)\} = 2 + M$.

Finally, if one of the maximal subgroups of G were abelian, then Theorem 6.1 would give the contradiction that $\mathrm{lo}_3(G) = 9$. □

Unfortunately, it was impossible for any of the three critical fields K to compute the third layer of the TTT, $\tau_3(K)$, that is the structure of the 3-class group of the Hilbert 3-class field $\mathrm{F}_3^1(K)$ of K, which is of absolute degree 54. This would have given the precise order of the metabelian group $G = \mathrm{G}_3^2(K) = \mathrm{Gal}(\mathrm{F}_3^2(K)|K)$, according to Lemma 6.1, since $G' = \mathrm{Gal}(\mathrm{F}_3^2(K)|\mathrm{F}_3^1(K)) \simeq \mathrm{Cl}_3(\mathrm{F}_3^1(K))$.

We also investigated whether the complete iterated IPAD of second order, $\tau^{(2)}(G)$, is able to improve the lower bounds in Theorem 6.2 further. It turned out that, firstly none of the additional non-normal components of $(\tau_1(H))_{H \in \mathrm{Lyr}_1(G)}$ seems to have bigger order than the normal components of $\tau_2(G)$, and secondly, due to the huge 3-ranks of the involved groups, the number of required class group computations enters astronomic regions.

To give an impression, we show the results for five of the 13 maximal subgroups in the case of $d = -4\,447\,704$:
$\tau^{(1)}(H_1) = [2^2 1^2; (32^5 1^2; (2^3 1^5)^3; (3^2 21^2)^3; (321^4)^9, (32^2 1^2)^{24})]$, with 40 components,
$\tau^{(1)}(H_2) = [21^4; (32^5 1^2; 2^5 1^3; 2^4 1^4; 2^2 1^7; (31^6)^3, (321^4)^{33}; (321^2)^{81})]$, with 121 components,
$\tau^{(1)}(H_3) = [2^2 1^2; (32^5 1^2; 32^2 1^5; (2^2 1^7)^2; (321^5)^3, (32^2 1^3)^6, (3^2 21^2)^3, (32^2 1^2)^{24})]$, with 40 comp.,
$\tau^{(1)}(H_4) = [32^2 1; (32^5 1^2; 4321^5; (3^2 21^5)^2; (4321^3)^6; (431^4)^6, (3^2 21^3)^6, (4321^2)^9, (3^3 1^2)^9)]$, 40 comp.,
$\tau^{(1)}(H_5) = [2^2 1^2; (3^2 21^5; 32^2 1^5, 2^4 1^4; 2^3 1^5; (321^3)^{36})]$, with 40 components.

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