A GAMMA CLASS FORMULA FOR OPEN GROMOV-WITTEN CALCULATIONS

MATTHEW MAHOWALD

Abstract. For toric Calabi-Yau threefolds, open Gromov-Witten invariants associated to Riemann surfaces with one boundary component can be written as the product of a disk factor and a closed invariant. When the Lagrangian boundary cycle is preserved by the torus action and can be locally described as the fixed locus of an anti-holomorphic involution, we prove a formula that expresses the disk factor in terms of a gamma class and combinatorial data about the image of the Lagrangian cycle in the moment polytope. As a corollary, we construct a generating function for these invariants using Givental's J function. We then verify that this formula encodes the expected invariants obtained from localization by comparing with several examples. Finally, motivated by large N duality, we show that this formula also unexpectedly applies to Lagrangian cycles on $\mathcal{O}_{\mathbb{P}^1}(-1,-1)$ constructed from torus knots.

Contents

1. Introduction

1.1. **Background and motivation.** Gromov-Witten theory has a rich history, both in physics and mathematics. Physically, Gromov-Witten invariants appear in type IIA topological string theory as instanton counts associated to interactions between particles. Mathematically, they are invariants associated to symplectic manifolds that, roughly speaking, count pseudoholomorphic curves in the manifold. The relationship between these two perspectives is conceptually straightforward: as a string moves in time, it sweeps out a compact Riemann surface (its 'worldsheet'). The amplitudes in string theory encode counts of maps from Riemann surfaces into a 3-(complex)-dimensional Calabi-Yau manifold, and Gromov-Witten theory assigns invariants to spaces of such maps.

In general, counting holomorphic maps from Riemann surfaces to a given target space is a difficult problem in enumerative geometry. Gromov-Witten theory has famously benefited from its connections with string dualities, first with mirror symmetry [COGP, Wi1], and more recently, large N duality [GV, OV]. Beginning with [Kon], for toric manifolds, Gromov-Witten invariants associated to maps of closed surfaces have also been systematically computed using localization [GP, CKYZ, KZ]. "Closed" Gromov-Witten theory is a natural mathematical counterpart to closed topological string theory, and, in contrast to the "open" theory (i.e., for maps of Riemann surfaces with boundary), the moduli spaces involved are rigorously defined.

Open Gromov-Witten theory is the subject of this paper. By analogy with the closed case, open Gromov-Witten theory is a mathematical counterpart to open topological string theory: open strings sweep out compact Riemannian surfaces with boundary, and the boundary of the strings are constrained to lie on branes. These boundary constraints are expressed mathematically as Lagrangian submanifolds $L \subset X$, and the string amplitudes are encoded by counts of holomorphic maps $f : \Sigma \to X$, with the image of the boundary constrained to lie on L: $f(\partial\Sigma) \subset L$. However, as observed in [AKV, KL], there are additional subtleties in adapting the methods of the closed theory to the open case. In particular, even for well-behaved Lagrangian boundary cycles, open Gromov-Witten invariants depend on an additional integral parameter (in localization, this parameter corresponds to the weights of the torus action).

In spite of this, the same computational tools of mirror symmetry, large N duality, and localization can still be used. In fact, through these string dualities, open Gromov-Witten theory can be connected to both classical and homological knot theory [OV, DSV, GJKS, MV, Wi2, Wi3]. The primary goal of this paper is to provide a concise and consistent framework for computing open Gromov-Witten invariants directly, via localization. The main result is a formula for open Gromov-Witten invariants expressed in terms of local combinatorial data and a gamma class. As expected from [AKMV], the construction depends only on the local geometry near a vertex of the moment polytope of X. In the case where the associated moduli space of open maps is rigorously defined ([KL]), this formula is proven to be correct. Through large N duality, it is shown that this result applies in a more general situation motivated by knot theory.

The author hopes that the approach described herein will lead to a more general construction of open Gromov-Witten invariants.

1.2. **Organization of the paper.** The paper is organized in the following way. Section 2 reviews some general facts about open Gromov-Witten theory, including deformation theory and localization. Most importantly, this section describes how to express an open Gromov-Witten invariant as the product of a "disk term" and an invariant of closed maps. Section 3 contains the proof of the main result of this paper:

Theorem. *Let X be a Calabi-Yau 3-fold and $L \subset X$ a Lagrangian submanifold which can be described locally as the fixed locus of an anti-holomorphic involution. Let S^1 act on X such that the S^1 action preserves L, and L intersects a rigid circle-invariant curve C. Let $\gamma \in H^2(X;\mathbb{Q})$. Then, the genus g, 1 boundary component, degree d, winding w open Gromov-Witten invariant with Lagrangian boundary L is*

$$\langle \gamma \rangle_{d,w}^{g,1} = \left(\Delta_{X,L} \circ \left\langle \gamma, \frac{\phi_p}{z-\psi} \right\rangle_{g,d} \right) \Bigg|_{z=\alpha},$$

where $\Delta_{X,L}$ is the disk function

$$\Delta_{X,L}(\gamma) := \frac{\pi}{wz\widehat{\Gamma}_X \sin\left(\pi \frac{\lambda}{z}\right)} \cdot \gamma.$$

Here, $\widehat{\Gamma}_X$ is the homogeneous Iritani gamma class, λ is the weight of the torus action along a normal direction to $L \cap C$, $\alpha = c_1(T_0\Delta)$ is the equivariant Chern class of the induced representation of S^1 at the attachment point of the disk, and ϕ_p is the equivariant class of the image $p \in X$ of the disk attachment point.

Section 4 describes how to apply the formula above to several examples where the resulting invariant is already known, and demonstrates that this formula reproduces the expected result. Finally, Section 5 applies this formula to a novel class of Lagrangian cycles motivated by large N duality. These Lagrangian cycles are obtained from the conormal bundles of torus knots in S^3 after the conifold transition, and do not have the same local description required in the above theorem. Nevertheless, the main result of this paper is still found to apply to these cycles. This hints that a version of the theorem may hold for a broader class of Lagrangian cycles.

Acknowledgments. The author thanks H. Gao, H. Jockers, C-C. M. Liu, and P. Zhou for valuable discussions. The author is especially grateful to R. Cavalieri for the suggestion of this project and many related conversations, and to E. Zaslow for guidance and suggestions.

2. Preliminaries

2.1. **Deformation theory for stable maps.** Open Gromov-Witten invariants are enumerative invariants of maps $f : \Sigma \to X$ from Riemann surfaces with boundary into a Calabi-Yau manifold X with a chosen Lagrangian submanifold L, such that $f(\partial\Sigma) \subset L$. By analogy with the definition for closed stable maps appearing in [Kon], [KL] define the open Gromov-Witten invariant in the following way. Fix integers g (the genus of Σ) and h (the number of connected components of $\partial\Sigma$), and a relative homology class $d \in H_2(X, L;\mathbb{Z})$ with $\partial d = \sum w_i \in H_1(L;\mathbb{Z})$. Then, the open Gromov-Witten invariant $GW^{g,h}_{d,w_1,\ldots,w_h}$ is a virtual count of continuous maps $f : (\Sigma, \partial\Sigma) \to (X, L)$ satisfying:

- $(\Sigma, \partial\Sigma)$ is a Riemann surface of genus g with boundary $\partial\Sigma$ consisting of h oriented circles,
- f is holomorphic in the interior of Σ,
- $f_*[\Sigma] = d$, and
- $f_*[\partial\Sigma] = \sum w_i$.

For brevity, $w_1, \ldots, w_h$ will sometimes be denoted by $\overrightarrow{w}$. In order to define such an invariant, [KL] construct a moduli space $\overline{\mathcal{M}}_{g,h}(X, L; d, \overrightarrow{w})$ of stable maps which compactify the maps described above, and give a local description of an orientation and a virtual fundamental class on this moduli space. In particular, the authors generalize the deformation complex in ordinary Gromov-Witten theory to the open case.

Recall that for smooth, closed Σ in ordinary Gromov-Witten theory, there is a normal bundle exact sequence of vector bundles on Σ:

$$0 \longrightarrow T_\Sigma \longrightarrow f^*T_X \longrightarrow N_{\Sigma/X} \longrightarrow 0\ .$$

The corresponding long exact sequence in cohomology is

$$0 \longrightarrow H^0(\Sigma, T_\Sigma) \longrightarrow H^0(\Sigma, f^*T_X) \longrightarrow H^0(\Sigma, N_{\Sigma/X}) \longrightarrow H^1(\Sigma, T_\Sigma) \longrightarrow H^1(\Sigma, f^*T_X) \longrightarrow H^1(\Sigma, N_{\Sigma/X}) \longrightarrow 0$$

The terms in this sequence can be interpreted as infinitesimal automorphisms, deformations, and obstructions to deformations for Σ and f, so this sequence can be re-written as the deformation complex:

$$0 \longrightarrow \mathrm{Aut}(\Sigma) \longrightarrow \mathrm{Def}(f) \longrightarrow \mathrm{Def}(\Sigma, f) \longrightarrow \mathrm{Def}(\Sigma) \longrightarrow \mathrm{Obs}(f) \longrightarrow \mathrm{Obs}(\Sigma, f) \longrightarrow 0. \tag{1}$$

Suitably interpreted, the same sequence holds for nodal, open curves in open Gromov-Witten theory: Over a smooth point (Σ, f) in $\overline{\mathcal{M}}_{g,h}(X, L; d, \overrightarrow{w})$, $H^k(\Sigma, f^*T_X)$ are the cohomology groups associated to sections s of (Σ, f^*T_X) satisfying $s|_{\partial\Sigma} \in \Gamma(\partial\Sigma, f^*T_L)$.

The expected (virtual) dimension of $\overline{\mathcal{M}}_{g,h}(X, L; d, \overrightarrow{w})$ is

$$\begin{aligned} \mathrm{vdim}\ \overline{\mathcal{M}}_{g,h}(X, L; d, \overrightarrow{w}) &= \mathrm{rank\ Def}(\Sigma, f) - \mathrm{rank\ Obs}(\Sigma, f) \\ &= \mu\left(f^*T_X, f|_{\partial\Sigma}^* T_L\right) - (\dim X - 3)\,\chi(\Sigma), \end{aligned}$$

where μ denotes the generalized Maslov index of the real subbundle $(f|_{\partial\Sigma})^* T_L \subset f^*T_X$ [KL]. When X is a complex manifold and L is the fixed locus of an anti-holomorphic involution, $\mu\left(f^*T_X, (f|_{\partial\Sigma})^* T_L\right) = \int_d c_1(T_X)$, and if X is a Calabi-Yau threefold, $\mathrm{vdim}\ \overline{\mathcal{M}}_{g,h}(X, L; d, \overrightarrow{w}) = 0$. When $\overline{\mathcal{M}}_{g,h}(X, L; d, \overrightarrow{w})$ has a well-behaved torus action, [KL] give an explicit description for the localization of the virtual fundamental class to the fixed loci of the torus action. In contrast to closed Gromov-Witten theory, the virtual cycle found in [KL] depends on the torus action. Additionally, the invariants defined in [KL] depend on a choice of orientation. This choice is reflected in the overall sign of the invariant, and the invariant formula proposed in this note has an analogous orientation-dependent sign.

2.2. **Separating the disk term.** Gromov-Witten invariants are, in general, difficult to compute. The primary computational tool is the Atiyah-Bott fixed-point formula [AB]. When applied to computations of Gromov-Witten invariants for toric varieties, this "localizes" integrals over the entire moduli space of stable maps to integrals over only those maps which are fixed by the torus action [Kon, GP, KZ]. The Atiyah-Bott fixed point formula for the integral of a class ϕ over a manifold (or more generally, a Deligne-Mumford stack) M is

$$\int_M \phi = \sum_P \int_P \left(\frac{i_P^* \phi}{e\left(N_P\right)}\right), \tag{2}$$

where the sum is over the fixed point sets P, i_P is the embedding of P into M, and $e\left(N_P\right)$ is the (equivariant) Euler class of the normal bundle of P in M.

Following [GP, Kon], a stable map (Σ, f) can be naturally described as a decorated graph. The vertices v of the graph correspond to contracted components of the nodal curve Σ, and are labeled by the genus $g\left(v\right)$ of that component. The edges e correspond to $\mathbb{P}^1$'s which are not contracted by f, and are labeled by the degree d_e of the map $f|_{\mathbb{P}^1}$. When the fixed stable maps are described as decorated graphs in this way, (2) becomes

$$GW_d^g := \int_{\left[\overline{\mathcal{M}}_{g,0}(X,d)\right]^{vir}} 1 = \sum_\Gamma \frac{1}{|A_\Gamma|} \int_{M_\Gamma} \frac{1}{e\left(N_\Gamma^{vir}\right)}. \tag{3}$$

As observed in [GZ], the graph description of stable maps can be extended to the open stable maps defined in [KL] by treating the open disk component as a "leg" of the graph. A crucial consequence of this is that open Gromov-Witten invariants can be expressed as a closed Gromov-Witten invariant multiplied by a "disk term." For simplicity, restrict attention to surfaces with one boundary component. Let X be a Calabi-Yau manifold equipped with an S^1 action that fixes a Lagrangian submanifold $L \subset X$. Suppose that $f\colon \Sigma \to X$ is a stable map from a genus g Riemann surface with one boundary component such that $f_*\left[\Sigma\right] = d \in H_2\left(X;\mathbb{Z}\right)$ and $f|_{\partial\Sigma}\colon \partial\Sigma \to f\left(\partial\Sigma\right)$ has winding w as a map between homotopy circles. Let $\overline{\mathcal{M}} := \overline{\mathcal{M}}_{g,1,0}\left(X, L; d, w\right)$ denote the the moduli space of such maps (genus g, 1 boundary component, 0 marked points).

The S^1 action on X naturally induces an S^1 action on $\overline{\mathcal{M}}$. If $(\Sigma, f) \in \overline{\mathcal{M}}$ is fixed by the S^1 action, then Σ must take the form

$$\Sigma = \Sigma_0 \cup_\nu \Delta,$$

where Σ_0 is a closed genus g Riemann surface, Δ is a disk, and ν is a simple node on Σ_0 at which Δ is attached. S^1 invariance further requires that $(\Sigma_0, f|_{\Sigma_0}, \nu)$ is fixed by the induced action on $\overline{\mathcal{M}}_{g,1}\left(X, d\right)$.

Then, a virtual localization formula analogous to (3) for the genus g, degree d, winding w open Gromov-Witten invariant would take the form

$$GW_{d,w}^{g,1} = \int_{\overline{\mathcal{M}}^{vir}} 1 = \sum_\Gamma \frac{1}{|\mathbf{A}_\Gamma|} \int_{M_\Gamma} \frac{1}{e\left(N_\Gamma^{vir}\right)}, \tag{4}$$

where $\mathbf{A}_\Gamma = \mathbb{Z}/w\mathbb{Z} \times A_{\Gamma'}$ (Γ' is the graph associated to the closed curve (Σ_0, ν)).

Note that $\overline{\mathcal{M}}_{g,1}\left(X, d\right)$ is equipped with a natural map $e_\nu : \overline{\mathcal{M}}_{g,1}\left(X, d\right) \to X$ given by evaluation at the marked point. The conditions on (Σ, f) specified above (in particular, that $f\left(\nu\right) = p$) imply that the fixed locus M_Γ is isomorphic to the fixed subspace $e_\nu^{-1}(p)^{S^1} \subset \overline{\mathcal{M}}_{g,1}\left(X, d\right)^{S^1}$.

As in the closed case, the equivariant normal bundle $e\left(N_\Gamma^{vir}\right)$ and the virtual fundamental cycle are determined, respectively, by the moving and fixed parts of the deformation complex (1):

$$0 \longrightarrow \mathrm{Aut}(\Sigma) \longrightarrow \mathrm{Def}(f) \longrightarrow \mathrm{Def}(\Sigma, f) \longrightarrow \mathrm{Def}(\Sigma) \longrightarrow \mathrm{Obs}(f) \longrightarrow \mathrm{Obs}(\Sigma, f) \longrightarrow 0$$

This gives the following relationship in the representation ring of S^1:

$$\mathrm{Obs}\,(\Sigma, f) - \mathrm{Def}\,(\Sigma, f) = \mathrm{Aut}\,(\Sigma) + \mathrm{Obs}\,(f) - \mathrm{Def}\,(\Sigma) - \mathrm{Def}\,(f)$$

with $\mathrm{Obs}\,(f) = H^1\,(\Sigma, f^*T_X)$, $\mathrm{Def}\,(f) = H^0\,(\Sigma, f^*T_X)$, $\mathrm{Aut}\,(\Sigma) = \mathrm{Ext}^0\,(\Omega_\Sigma\,(D)\,, \mathcal{O}_\Sigma)$, and $\mathrm{Def}\,(\Sigma) = \mathrm{Ext}^1\,(\Omega_\Sigma\,(D)\,, \mathcal{O}_\Sigma)$. (Here, D is the divisor associated to the nodal points of Σ. When Σ is smooth, these spaces are just $H^0\,(\Sigma, T_\Sigma)$ and $H^1\,(\Sigma, T_\Sigma)$, respectively).

Now, relate the terms in this sequence to the terms concerning Σ_0 and Δ: Let $f_0 := f|_{\Sigma_0}$ and $f_\Delta := f|_\Delta$. Suppose that Δ is parametrized by $\{|t| \leq 1\}$, with ν identified with the point $t = 0$. Then, there is an exact sequence

$$0 \longrightarrow \mathcal{O}_\Sigma \longrightarrow \mathcal{O}_{\Sigma_0} \oplus \mathcal{O}_\Delta \longrightarrow \mathcal{O}_\nu \longrightarrow 0.$$

This becomes the exact sequence on cohomology:

$$0 \longrightarrow \mathrm{Def}(f) \longrightarrow H^0(\Delta, T_{(\Delta, f_\Delta)}) \oplus \mathrm{Def}(f_0) \longrightarrow T_pX \longrightarrow \mathrm{Obs}(f) \longrightarrow H^1(\Delta, T_{(\Delta, f_\Delta)}) \oplus \mathrm{Obs}(f_0) \longrightarrow 0$$

which yields the following relations in the representation ring:

$$\begin{aligned} \mathrm{Obs}(f)^f - \mathrm{Def}(f)^f &= H^1(\Delta, T_{(\Delta, f_\Delta)})^f - H^0(\Delta, T_{(\Delta, f_\Delta)})^f \\ &\quad + \mathrm{Obs}(f_0)^f - \mathrm{Def}(f_0)^f, \end{aligned}$$

$$\begin{aligned} \mathrm{Obs}(f)^m - \mathrm{Def}(f)^m &= H^1(\Delta, T_{(\Delta, f_\Delta)})^m - H^0(\Delta, T_{(\Delta, f_\Delta)})^m \\ &\quad + \mathrm{Obs}(f_0)^m - \mathrm{Def}(f_0)^m + T_pX, \end{aligned}$$

where $p = f\,(\nu) \in X$ and the f, m superscripts denote fixed and moving terms with respect to the S^1 action.

Similarly,

$$\begin{aligned} \mathrm{Aut}(\Sigma)^m &= \mathrm{Aut}\,(\Sigma_0, \nu)^m + \mathrm{Aut}\,(\Delta, 0)^m\,, \\ \mathrm{Aut}(\Sigma)^f &= \mathrm{Aut}\,(\Sigma_0, \nu)^f + \mathrm{Aut}\,(\Delta, 0)^f\,, \\ \mathrm{Def}\,(\Sigma)^f &= \mathrm{Def}\,(\Sigma_0, \nu)^f\,, \end{aligned}$$

and

$$\mathrm{Def}\,(\Sigma)^m = \mathrm{Def}\,(\Sigma_0, \nu)^m + T_\nu\Sigma_0 \otimes T_0\Delta.$$

Note that $\mathrm{Aut}\,(\Delta, 0)$ consists of the infinitesimal automorphisms of Δ preserving the origin $t = 0$, which are generated by the sections $t\partial_t$ over $\mathbb{R}$. Therefore, $\mathrm{Aut}\,(\Delta, 0)^m$ is trivial, and $\mathrm{Aut}\,(\Delta, 0)^f = \mathbb{R}$.

Collecting the above observations,

$$\begin{aligned}\mathrm{Obs}(\Sigma,f)^f-\mathrm{Def}\,(\Sigma,f)^f &= H^1(\Delta,T_{(\Delta,f_\Delta)})^f - H^0(\Delta,T_{(\Delta,f_\Delta)})^f\\ &+\mathrm{Obs}(f_0)^f-\mathrm{Def}(f_0)^f\\ &+\mathrm{Aut}(\Sigma_0,\nu)^f-\mathrm{Def}(\Sigma_0,\nu)^f\\ &+\mathrm{Aut}(\Delta,0)^f\end{aligned}$$

and

$$\begin{aligned}\mathrm{Obs}(\Sigma,f)^m-\mathrm{Def}\,(\Sigma,f)^m &= H^1(\Delta,T_{(\Delta,f_\Delta)})^m - H^0(\Delta,T_{(\Delta,f_\Delta)})^m\\ &+\mathrm{Obs}(f_0)^m-\mathrm{Def}(f_0)^m\\ &+\mathrm{Aut}(\Sigma_0,\nu)^m-\mathrm{Def}(\Sigma_0,\nu)^m\\ &+T_pX-T_\nu\Sigma_0\otimes T_0\Delta\end{aligned}$$

The first equation implies that the virtual fundamental cycle of the fixed locus is the restriction of the natural virtual cycle of the fixed locus $\left[\overline{\mathcal{M}_{g,1}}\,(X,d)^{S^1}\right]^{vir}$ to the subspace $e_\nu^{-1}(p)^{S^1}$. The second equation yields the following relationship between the normal bundles $e(N_\Gamma^{vir})$ and $e\left(N_{\Gamma'}^{vir}\right)$ (where Γ' is the graph for the closed curve with the disk "leg" removed):

$$\begin{aligned}N_\Gamma^{vir} &= N_{\Gamma'}^{vir}-T_pX+R\mathbb{L}^{-1}\\ &-H^1(\Delta,T_{(\Delta,f_\Delta)})^m+H^0\left(\Delta,T_{(\Delta,f_\Delta)}\right)^m.\end{aligned}$$

Here R is the representation of S^1 on $T_0\Delta\cong\mathbb{C}$ induced by the pullback of the S^1 action on $f\,(\Delta)$, and $\mathbb{L}$ is the tautological cotangent line bundle on $\overline{\mathcal{M}_{g,1}}\,(X,d)$ associated to the marked point ν, i.e., the line bundle whose fiber at the point (f_0,Σ_0,ν) is $T_\nu^*\Sigma_0$. $R\mathbb{L}^{-1}$ is contribution from the term $T_\nu\Sigma_0\otimes T_0\Delta$: $T_\nu\Sigma_0$ is the fiber of $\mathbb{L}$ and $T_0\Delta$ is a constant vector space which carriers the representation R by S^1.

Hence,

$$\int_{M_\Gamma}\frac{1}{e\left(N_\Gamma^{vir}\right)}=\frac{e_{S^1}\left(H^1\left(\Delta,T_{(\Delta,f_\Delta)}\right)\right)e_{S^1}\left(T_pX\right)}{e_{S^1}\left(H^0\left(\Delta,T_{(\Delta,f_\Delta)}\right)\right)}\int_{M_{\Gamma'}}\frac{1}{e\left(N_{\Gamma'}^{vir}\right)(\alpha-\psi)},\tag{5}$$

where $\alpha=c_1\,(R)$ and $\psi=c_1\,(\mathbb{L})$ (so that $e_{S^1}\left(R\mathbb{L}^{-1}\right)=\alpha-\psi$). Denote by $D_{X,L}$ the "disk factor:"

$$D_{X,L}:=\left(\frac{1}{w}\right)\frac{e_{S^1}\left(H^1\left(\Delta,T_{(\Delta,f_\Delta)}\right)\right)}{e_{S^1}\left(H^0\left(\Delta,T_{(\Delta,f_\Delta)}\right)\right)}.\tag{6}$$

Then, (4) becomes

$$GW_{d,w}^g=\int_{\overline{\mathcal{M}}^{vir}}1=D_{X,L}\left(\sum_{\Gamma'}\frac{1}{|A_{\Gamma'}|}\int_{M_{\Gamma'}}\frac{i^*ev^*\left(\phi_p\right)}{e\left(N_{\Gamma'}^{vir}\right)(\alpha-\psi)}\right),$$

where ϕ_p is the equivariant Thom class of the point $p\in X$, and i^* the pullback to the fixed locus $M_{\Gamma'}$. Comparing this formula with (3) shows that the parenthetical quantity is the localization of a closed Gromov-Witten invariant:

$$GW_{d,w}^{g,1}=D_{X,L}\int_{\left[\overline{\mathcal{M}}_{g,1}(X,d)\right]^{vir}}\frac{ev^*\left(\phi_p\right)}{(\alpha-\psi)}.\tag{7}$$

3. Main Results

3.1. **A formula for open Gromov-Witten invariants.** The proposed formula for open Gromov-Witten invariants is obtained by composition of a disk function $\Delta_{X,L}$ and a descendant invariant. $\Delta_{X,L}$ is built from combinatorial data about the moment polytope, and a characteristic class. Recall that if δ_i are the Chern roots of a complex vector bundle E, Iritani's gamma class [Iri] is a characteristic class associated to E defined by

$$\Gamma_E := \prod_{\delta_i} \Gamma\left(1+\delta_i\right).$$

The gamma class appears in quantum cohomology, and can be regarded as a localization contribution from constant maps in Floer theory [GGI]. The inputs of $\Delta_{X,L}$ are the torus weight λ of a normal direction to $f\left(\partial\Sigma\right)$, and the homogenized Iritani gamma class $\hat{\Gamma}_X \in H^*\left(X;\mathbb{Q}\right)(z)$, defined by

$$\widehat{\Gamma}_X := \prod_{\delta_i} \Gamma\left(1+\frac{\delta_i}{z}\right),$$

where δ_i are the Chern roots of the tangent bundle T_X. (When $\deg z = 2$, $\deg \widehat{\Gamma}_X = 0$). With these definitions, the main result is:

Theorem 1. *Let X be a Calabi-Yau 3-fold and $L \subset X$ a Lagrangian submanifold which can be described locally as the fixed locus of an anti-holomorphic involution. Let S^1 act on X such that the S^1 action preserves L, and L intersects a rigid circle-invariant curve C. Let $\gamma \in H^2\left(X;\mathbb{Q}\right)$. Then, the genus g, 1 boundary component, degree d, winding w open Gromov-Witten invariant with Lagrangian boundary L is*

$$\langle\gamma\rangle_{d,w}^{g,1} = \left(\Delta_{X,L} \circ \left\langle \gamma, \frac{\phi_p}{z-\psi}\right\rangle_{g,d}\right)\Bigg|_{z=\alpha}, \tag{8}$$

where $\Delta_{X,L}$ is the disk function

$$\Delta_{X,L}\left(\gamma\right) := \frac{\pi}{wz\widehat{\Gamma}_X \sin\left(\pi\frac{\lambda}{z}\right)}\cdot\gamma. \tag{9}$$

Here, $\widehat{\Gamma}_X$ is the homogeneous Iritani gamma class, λ is the weight of the torus action along a normal direction to $L \cap C$, $\alpha = c_1(T_0\Delta)$ is the equivariant Chern class of the induced representation of S^1 at the attachment point of the disk, and ϕ_p is the equivariant class of the image $p \in X$ of the disk attachment point.

The setup in Theorem 1 is depicted in Figure 1.

In the genus 0 case, the closed Gromov-Witten invariants in formula (7) also appear as terms in Givental's J function [Giv]. Givental's J function is the map on quantum cohomology $J_X\colon H^*\left(X;\mathbb{Q}\right) \longrightarrow H^*\left(X;\mathbb{Q}\right)(z)$ given by

$$J_X\left(\gamma\right) = z + \gamma + \sum_{n=0}^{\infty} \sum_{d\in H_2(X;\mathbb{Z})} \left\langle \gamma^n, \frac{T^\alpha}{z-\psi}\right\rangle_{0,d} T_\alpha,$$

where T^α is a basis for the cohomology of X, T_α is the dual basis with respect to the Poincaré pairing, and

$$\left\langle \gamma^n, \frac{T^\alpha}{z-\psi}\right\rangle_{0,d} = \sum_{k=0}^{\infty} z^{-(k+1)} \left\langle \gamma^n, \tau_k T^\alpha\right\rangle_{0,d}$$

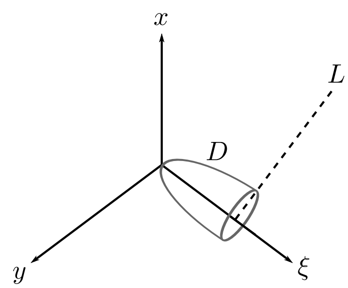

FIGURE 1. A local picture near a vertex in the toric polytope. A Lagrangian L, obtained as the fixed locus of an anti-holomorphic involution, intersects an edge of the toric polytope, labeled by the local coordinate ξ. The only torus-fixed disks are the hemispheres D with $\partial D = L \cap \{x = y = 0\} \cong S^1$. Such a map is given locally by $t \mapsto (\xi = t^w, x = 0, y = 0)$. The x-y hyperplane is normal to the disk. The normal directions to $f(\partial\Sigma) = \partial D$ are spanned by ∂_x and ∂_y, so the weight λ appearing in $\Delta_{X,L}$ (9) can be the weight of any S^1-invariant line spanned by these vectors (for example, λ_x or λ_y).

is a power series of gravitational descendant closed Gromov-Witten invariants. To obtain the genus-g generating function, a higher-genus version of the J-function is needed. Define the genus-g modified J-function $J_X^g \colon H^*(X;\mathbb{Q}) \longrightarrow H^*(X;\mathbb{Q})(z)$ to be

$$J_X^g(\gamma) = z + \gamma + \sum_{n=0}^{\infty} \sum_{d \in H_2(X;\mathbb{Z})} \frac{q^d}{n!} \left\langle \gamma^n, \frac{T^\alpha}{z - \psi} \right\rangle_{g,d} T_\alpha, \tag{10}$$

where

$$q^d = e^{2\pi i \int_d \omega}$$

and ω is the complexified Kähler class of X. From (10) and Theorem 1, it is easy to write a generating function for open Gromov-Witten invariants. The generating function for the one-boundary-component, winding w open Gromov-Witten invariants $\langle \gamma \rangle_{d,w}^{g,1}$ is the function

$$\Phi_w(\gamma) := \sum_{g \geq 0} \sum_{n \geq 0} \sum_{d \in H_2(X;\mathbb{Z})} g_s^{2g-1} \frac{q^d}{n!} \langle \gamma^n \rangle_{d,w}^{g,1},$$

where g_s is the string coupling constant, $\gamma \in H^*(X;\mathbb{Q})$, and the summation is only over combinations of g, n, and d where the summands are defined.

Corollary 2. *Let X and L be as in Theorem 1. Then, a generating function for the winding-w open Gromov-Witten invariants of $\overline{\mathcal{M}}_{g,1}(X, L; d, w)$ is given by the formula*

$$\Phi_w(\gamma) = \sum_{g \geq 0} g_s^{2g-1} \left(\Delta_{X,L} \circ J_X^g(\gamma, \phi_p) \right)\big|_{z=\alpha}.$$

Remark. Two observations about this generating function merit mention. First, in [BC], the authors use a similar procedure to obtain a generating function for open invariants from a modification of the J function. The main distinction here is the presentation of the disk term. Second, for $\gamma = 1$, Φ_w has the expression

$$\Phi_w = \sum_{g\geq 0} \sum_{d\in H_2(X;\mathbb{Z})} g_s^{2g-1} q^d GW^{g,1}_{d,w}.$$

As will be discussed in Section 5, [DSV] have found that, for a certain class of Lagrangian cycles originating from torus knots, the expression above encodes the HOMFLY polynomial associated to the original knot.

3.2. **Proof of the main result.** The main content of the proof of Theorem 1 is the comparison of the predicted disk term from (9) with an explicit localization calculation of the open Gromov-Witten invariant.

3.2.1. *Virtual localization of open Gromov-Witten invariants.* First, recall the virtual localization technique: As described in Section 2.2, the open Gromov-Witten invariant $GW^g_{d,w}$ can be expressed as a product of a disk term $D_{X,L}$ and a descendant invariant. The surface Σ can be written as $\Sigma_0 \cup_\nu \Delta$, with Σ_0 is a closed surface, Δ a disk, and ν the point of attachment. In terms of the cohomology of sheaves over Δ, $D_{X,L}$ was found to have the following expression:

$$D_{X,L} = \left(\frac{1}{w}\right) \frac{e_{S^1}\left(H^1\left(\Delta, T_{(\Delta,f_\Delta)}\right)\right)}{e_{S^1}\left(H^0\left(\Delta, T_{(\Delta,f_\Delta)}\right)\right)}, \tag{6}$$

where $f_\Delta : \Delta \to X$ is the restriction of the map $f : \Sigma \to X$ to the disk Δ, $p = f(\nu)$, and $e_{S^1}(\cdot)$ denotes the S^1-equivariant Euler classes of the specified bundles. To compute the disk contribution to $GW^g_{d,w}$, one must compute each of these cohomology groups.

In contrast to the analogous computation of closed invariants, the Lagrangian L imposes boundary conditions on the sections of $T_{(\Delta,f_\Delta)}$. Let f_∂ denote $f_\Delta|_{\partial\Delta}$. Then, $T_{(\Delta,f_\Delta)}$ consists of sections of $f^*_\Delta T_X$ satisfying $s|_{\partial\Delta} \in f^*_\partial T_L$. To obtain an explicit presentation of the boundary conditions, let $Ann(L) \subset T^*_X|_L$ be the subbundle of the cotangent bundle T^*_X which annihilates the tangent bundle $T_L \subset T_X|_L$. Choose a basis of sections $\alpha_1, \alpha_2, \alpha_3$ of $Ann(L)$ along the boundary ∂D of the disk. (The α_i can be obtained by, for example, linearizing the equations defining L). $T_{(\Delta,f_\Delta)}$ consists of the sheaf of germs of holomorphic sections of the bundle $f^*_\Delta T_X$ satisfying the boundary conditions

$$f^*_\partial(\alpha_j)(s|_{\partial\Delta}) = 0, \qquad j = 1,2,3. \tag{11}$$

With this presentation of the boundary conditions, computing $H^i\left(\Delta, T_{(\Delta,f_\Delta)}\right)$ becomes an exercise in Čech cohomology: Let

$$U = \{t : 0 < |t| \leq 1\}, \qquad U' = \{t : 0 \leq |t| < 1\}$$

be an open cover of Δ, and let x, y, ξ be local coordinates such that $f(\nu) = p$ is the origin $(x, y, \xi) = (0,0,0)$. Then, local sections over U and U' are of the form

$$\begin{aligned} s &= \sum_{k\in\mathbb{Z}} \left(a_k t^k \partial_x + b_k t^k \partial_y + c_k t^k \partial_\xi\right), \\ s' &= \sum_{k\geq 0} \left(a'_k t^k \partial_x + b'_k t^k \partial_y + c'_k t^k \partial_\xi\right), \end{aligned}$$

and the coefficients a_k, b_k, and c_k are subject to boundary conditions imposed by (11). Finally, to apply localization, let $\rho_\theta : X \to X$ denote the S^1 action determined by

$$\rho_\theta(x, y, \xi) = \left(e^{i\lambda_x\theta}x, e^{i\lambda_y\theta}y, e^{i\lambda_\xi\theta}\xi\right), \qquad \theta \in S^1.$$

The weights λ_x, λ_y, λ_ξ are required to satisfy $\lambda_x + \lambda_y + \lambda_\xi = 0$ so that the holomorphic volume form is preserved by the S^1 action.

3.2.2. *Boundary conditions.* Now, suppose that L is described locally as the fixed locus of an anti-holomorphic involution σ, and that L intersects a rigid S^1-fixed curve C as depicted in Figure 1. Choose the local coordinates x, y, and ξ such that $L \cap C$ is defined by $x = y = 0$, $|\xi|^2 = 1$. Generically, σ takes the form

$$\sigma(x, y, \xi) = (\sigma_x, \sigma_y, \sigma_\xi),$$

where σ_x, σ_y, and σ_ξ are Laurent series in the variables $\overline{x}$, $\overline{y}$, $\overline{\xi}$:

$$\sigma_x = \sum_{j,k,l\in\mathbb{Z}} X_{jkl}\overline{x}^j\overline{y}^k\overline{\xi}^l,$$
$$\sigma_y = \sum_{j,k,l\in\mathbb{Z}} Y_{jkl}\overline{x}^j\overline{y}^k\overline{\xi}^l,$$
$$\sigma_\xi = \sum_{j,k,l\in\mathbb{Z}} Z_{jkl}\overline{x}^j\overline{y}^k\overline{\xi}^l.$$

In addition, in order to apply localization, L must be fixed by the S^1 action. So, for points $p \in L$, $\rho_\theta(p)$ must also be a point in L. In particular, $p \in L$ must satisfy $\rho_\theta(p) = \sigma(\rho_\theta(p))$. Because $p = \sigma(p)$, this is equivalent to $\rho_\theta \circ \sigma = \sigma \circ \rho_\theta$ on L. This imposes restrictions on the coefficients $X_{j,k,l}$, $Y_{j,k,l}$, $Z_{j,k,l}$. For example, σ_ξ must satisfy

$$\sigma_\xi\left(e^{i\lambda_x\theta}x, e^{i\lambda_y\theta}y, e^{i\lambda_\xi\theta}\xi\right) = e^{i\lambda_\xi\theta}\sigma_\xi(x, y, \xi)$$

for all $\theta \in S^1$. Expanding, this is

$$\sum_{j,k,l} Z_{j,k,l}e^{-i\theta(j\lambda_x + k\lambda_y + l\lambda_\xi)}\overline{x}^j\overline{y}^k\overline{\xi}^l = e^{i\lambda_\xi\theta}\sum_{j,k,l} Z_{j,k,l}\overline{x}^j\overline{y}^k\overline{\xi}^l.$$

Hence, $-(l+1)\lambda_\xi = j\lambda_x + k\lambda_y$. Recalling that $\lambda_x + \lambda_y = -\lambda_\xi$, this forces $j = k = l + 1$. Similar results hold for σ_x and σ_y. Therefore, antiholomorphic involutions commuting with the S^1 action along L must take the form

$$\sigma_x = \sum_{l\in\mathbb{Z}} X_l\left(\overline{\xi y}\right)^{l+1}\overline{x}^l, \qquad \sigma_y = \sum_{l\in\mathbb{Z}} Y_l\left(\overline{x\xi}\right)^{l+1}\overline{y}^l, \qquad \sigma_\xi = \sum_{l\in\mathbb{Z}} Z_l\left(\overline{xy}\right)^{l+1}\overline{\xi}^l.$$

In fact there are further restrictions on σ. Because L is the fixed locus of σ and the intersection $L \cap C$ is defined by $\left\{x = y = 0, |\xi|^2 = 1\right\}$, $X_l = Y_l = 0$ for $l < 0$, and $Z_{-1} = 1$, $Z_l = 0$ for $l < -1$. Substituting these relations into $\sigma^2 = 1$, the equation $x = \sigma_x \circ \sigma(x, y, z)$ becomes

$$x = X_0\overline{Y_0}x + x^2 g(x, y, \xi),$$

where g is a power series in x, y, and ξ. So, $X_0\overline{Y_0} = 1$.

The defining equations of L are

$$x = \sigma_x, \qquad y = \sigma_y, \qquad \xi = \sigma_\xi.$$

Linearizing these equations yields

$$dx = X_0\left(\overline{\xi}d\overline{y} + \overline{y}d\overline{\xi}\right) + \sum_{l\geq 1} X_l\left[l\left(\overline{\xi y}\right)^{l+1}\overline{x}^{l-1}d\overline{x} + (l+1)\,\overline{x}^l\left(\overline{\xi}^{l+1}\overline{y}^l d\overline{y} + \overline{\xi}^l\overline{y}^{l+1}d\overline{\xi}\right)\right],$$

$$dy = Y_0\left(\overline{\xi}d\overline{x} + \overline{x}d\overline{\xi}\right) + \sum_{l\geq 1} Y_l\left[l\left(\overline{\xi x}\right)^{l+1}\overline{y}^{l-1}d\overline{y} + (l+1)\,\overline{y}^l\left(\overline{\xi}^{l+1}\overline{x}^l d\overline{x} + \overline{\xi}^l\overline{x}^{l+1}d\overline{\xi}\right)\right],$$

$$d\xi = -Z_{-1}\left(\overline{\xi}\right)^{-2}d\overline{\xi} + Z_0\left(\overline{x}d\overline{y} + \overline{y}d\overline{x}\right) + \sum_{l\geq 1} Z_l\left[l\left(\overline{xy}\right)^{l+1}\overline{\xi}^{l-1}d\overline{\xi} + (l+1)\,\overline{\xi}^l\left(\overline{x}^l\overline{y}^{l+1}d\overline{x} + \overline{x}^{l+1}\overline{y}^l d\overline{y}\right)\right].$$

At $x = y = 0$, these equations simplify considerably:

$$dx = X_0\overline{\xi}d\overline{y}, \qquad dy = Y_0\overline{\xi}d\overline{x}, \qquad d\xi = -\left(\frac{1}{\overline{\xi}}\right)^2 d\overline{\xi}.$$

As $\xi = \overline{\xi}^{-1}$ along $|\xi|^2 = 1$, a basis for $Ann\,(L)$ along ∂D is given by the 1-forms

$$\alpha_1 = dx - X_0\overline{\xi}d\overline{y}, \qquad \alpha_2 = dy - Y_0\overline{\xi}d\overline{x}, \qquad \alpha_3 = \overline{\xi}d\xi + \xi d\overline{\xi}.$$

Recall that local sections over $U = \{t : 0 < |t| \leq 1\}$ are of the form

$$s = \sum_{k\in\mathbb{Z}}\left(a_k t^k\partial_x + b_k t^k\partial_y + c_k t^k\partial_\xi\right).$$

Parameterizing $|t| = 1$ by $e^{i\theta} = t$, the map f_∂ takes the form $f_\partial\left(e^{i\theta}\right) = \left(x = 0, y = 0, \xi = e^{iw\theta}\right)$. The boundary conditions $f_\partial^*\left(\alpha_j\right)\left(s|_{\partial\Delta}\right) = 0$ impose restrictions on the coefficients a_k, b_k, c_k:

$$\begin{aligned}
f_\partial^*\left(\alpha_1\right)\left(s|_{\partial\Delta}\right) &= \left(dx - X_0 e^{-iw\theta}d\overline{y}\right)\left(\sum_{k\in\mathbb{Z}}\left(a_k e^{ik\theta}\partial_x + b_k e^{ik\theta}\partial_y + c_k e^{ik\theta}\partial_\xi\right)\right)\\
&= \sum_{k\in\mathbb{Z}}\left(a_k e^{ik\theta} - X_0 e^{-iw\theta}\overline{b_k}e^{-ik\theta}\right),\\
f_\partial^*\left(\alpha_2\right)\left(s|_{\partial\Delta}\right) &= \left(dy - Y_0 e^{-iw\theta}d\overline{x}\right)\left(\sum_{k\in\mathbb{Z}}\left(a_k e^{ik\theta}\partial_x + b_k e^{ik\theta}\partial_y + c_k e^{ik\theta}\partial_\xi\right)\right)\\
&= \sum_{k\in\mathbb{Z}}\left(b_k e^{ik\theta} - Y_0 e^{-iw\theta}\overline{a}_k e^{-ik\theta}\right),\\
f_\partial^*\left(\alpha_3\right)\left(s|_{\partial\Delta}\right) &= \left(e^{-iw\theta}d\xi + e^{iw\theta}d\overline{\xi}\right)\left(\sum_{k\in\mathbb{Z}}\left(a_k e^{ik\theta}\partial_x + b_k e^{ik\theta}\partial_y + c_k e^{ik\theta}\partial_\xi\right)\right)\\
&= \sum_{k\in\mathbb{Z}}\left(c_k e^{i(k-w)\theta} + \overline{c}_k e^{-i(k-w)\theta}\right).
\end{aligned}$$

These yield the following equations for the coefficients a_k, b_k, c_k:

$$a_k - X_0\overline{b}_{-k-w} = 0, \qquad b_k - Y_0\overline{a}_{-k-w} = 0, \qquad c_k + \overline{c}_{2w-k} = 0.$$

The first two equations are actually equivalent: after complex conjugation, relabeling of indices, and substituting $X_0\overline{Y_0} = 1$, the second equation becomes the first. So, the Lagrangian boundary conditions on sections over U are

$$a_k = X_0\bar{b}_{-k-w}, \qquad c_k = \bar{c}_{2w-k}. \tag{12}$$

From these boundary conditions, the cohomology groups $H^i\left(\Delta, T_{(\Delta,f_\Delta)}\right)$ can be computed explicitly.

3.2.3. *Computation of cohomology groups and equivariant classes.* $H^0\left(\Delta, T_{(\Delta,f_\Delta)}\right)$ consists of the global sections, i.e., holomorphic sections s on Δ. These take the form

$$s = \sum_{k\geq 0}\left(a_k t^k\partial_x + b_k t^k\partial_y + c_k t^k\partial_\xi\right),$$

with a_k, b_k and c_k subject to the boundary conditions in (12), and $a_k = b_k = c_k = 0$ for $k < 0$. In particular, the equation $a_k = X_0\bar{b}_{-k-w}$ implies that $a_k = 0$ for all k. Shifting indices $k \to -k-w$, this equation also implies that $b_k = 0$ for all k. Finally, from the last boundary equation $c_k = \bar{c}_{2w-k}$, $c_k = 0$ for $k > 2w$. So, $H^0\left(\Delta, T_{(\Delta,f_\Delta)}\right)$ consists of sections of the form

$$s = \sum_{k=0}^{w-1}\left(c_k t^k\partial_\xi + \bar{c}_k t^{2w-k}\partial_\xi\right) + c_w t^w\partial_\xi,$$

where c_w is real. As a vector space, $H^0\left(\Delta, T_{(\Delta,f_\Delta)}\right)$ is isomorphic to

$$\mathbb{R}\left\langle t^w\partial_\xi\right\rangle \oplus \bigoplus_{k=0}^{w-1}\mathbb{C}\left\langle t^k\partial_\xi\right\rangle.$$

The map f_Δ takes $t \mapsto (x=0, y=0, \xi=t^w)$, so S^1 action the section $t^k\partial_\xi$ with weight $\lambda_\xi\,(k/w-1)$. $t^w\partial_\xi$ is fixed by the S^1 action, so $H^0\left(\Delta, T_{(\Delta,f_\Delta)}\right)^m$ is just the complex part of this vector space. Hence,

$$e_{S^1}\left(H^0\left(\Delta, T_{(\Delta,f_\Delta)}\right)\right) = \prod_{k=0}^{w-1}\lambda_\xi\left(\frac{k}{w}-1\right). \tag{13}$$

$H^1\left(\Delta, T_{(\Delta,f_\Delta)}\right)$ consists of the cokernel to the Čech differential. Sections over $U \cap U'$ can be written as

$$\begin{aligned}\delta &= \sum_{k\in\mathbb{Z}}\left(\alpha_k t^k\partial_x + \beta_k t^k\partial_y + \gamma_k t^k\partial_\xi\right)\\ &= \sum_{k\leq -w}\alpha_k t^k\partial_x + \sum_{k=1-w}^{-1}\alpha_k t^k\partial_x + \sum_{k\geq 0}\alpha_k t^k\partial_x\\ &+ \sum_{k<0}\beta_k t^k\partial_y + \sum_{k\geq 0}\beta_k t^k\partial_y\\ &+ \sum_{k<0}\gamma_k t^k\partial_\xi + \sum_{k\geq 0}\gamma_k t^k\partial_\xi.\end{aligned}$$

The image of the Čech differential consists of sections δ of the form $\delta = s - s'$. In terms of the coefficients, this is

$$\alpha_k = \begin{cases} a_k, & k < 0 \\ a_k - a'_k, & k \geq 0 \end{cases}, \quad \beta_k = \begin{cases} b_k, & k < 0 \\ b_k - b'_k, & k \geq 0 \end{cases}, \quad \gamma_k = \begin{cases} c_k, & k < 0 \\ c_k - c'_k, & k \geq 0, \end{cases}$$

where again, a_k, b_k, and c_k are subject to the boundary conditions (12). Solutions always exist for γ_k: set $c_k = \gamma_k$ for $k < 0$, and set $c'_k = \overline{c}_{2w-k} - \gamma_k$ for $k \geq 0$. Similarly, because a'_k and c'_k are completely free, any α_k and β_k for $k \geq 0$ can be solved for. However, to solve for α_k and β_k for $k < 0$, it must be the case that $b_k = \beta_k$. The first boundary equation $a_k = X_0 \overline{b}_{-k-w}$ then implies that a_k is fixed for $-w < k < 0$. So, there are no solutions if $\alpha_k \neq X_0 \beta_{-k-w}$ in $-w < k < 0$. When $k \leq -w$, $-k - w \geq 0$, so setting $b_{-k-w} = \alpha_k$ and $b'_{-k-w} = \alpha_k - \beta_{-k-w}$ will solve these equations. Therefore, the cokernel of the Čech differential is isomorphic to the space of sections δ of the form

$$\delta = \sum_{k=1-w}^{-1} \alpha_k t^k \partial_x.$$

The induced S^1 action on $t^k \partial_x$ has weight $\frac{k}{w}\lambda_\xi - \lambda_x$. As a vector space, $H^1\left(\Delta, T_{(\Delta, f_\Delta)}\right)$ is

$$\bigoplus_{k=1-w}^{-1} \mathbb{C}\left\langle t^k \partial_x \right\rangle,$$

and

$$e_{S^1}\left(H^1\left(\Delta, T_{(\Delta, f_\Delta)}\right)\right) = \prod_{k=1-w}^{-1} \left(\frac{k}{w}\lambda_\xi - \lambda_x\right). \tag{14}$$

3.2.4. *Comparison of disk terms.* Substituting (13) and (14) in (6) yields

$$D_{X,L} = \frac{1}{w} \frac{\prod_{k=1-w}^{-1}\left(\frac{k}{w}\lambda_\xi - \lambda_x\right)}{\prod_{k=0}^{w-1} \lambda_\xi \left(\frac{k}{w} - 1\right)}. \tag{15}$$

The proof will be complete if (15) is equivalent to the claimed expression (9):

$$\frac{1}{w} \frac{\prod_{k=1-w}^{-1}\left(\frac{k}{w}\lambda_\xi - \lambda_x\right)}{\prod_{k=0}^{w-1} \lambda_\xi \left(\frac{k}{w} - 1\right)} = \left.\frac{\pi}{wz\widehat{\Gamma}_X \sin\left(\pi \frac{\lambda}{z}\right)}\right|_{z=\alpha}.$$

First, observe that

$$\prod_{k=0}^{w-1} \lambda_\xi \left(\frac{k}{w} - 1\right) = \left(-\frac{\lambda_\xi}{w}\right)^w \Gamma(w+1).$$

Similarly,

$$\begin{aligned} \prod_{k=1-w}^{-1} \left(\frac{k}{w}\lambda_\xi - \lambda_x\right) &= \left(-\frac{\lambda_\xi}{w}\right)^{w-1} \prod_{k=1}^{w-1} \left(k + w\frac{\lambda_x}{\lambda_\xi}\right) \\ &= \left(-\frac{\lambda_\xi}{w}\right)^{w-1} \frac{\Gamma\left(w\frac{\lambda_x}{\lambda_\xi} + w\right)}{\Gamma\left(w\frac{\lambda_x}{\lambda_\xi} + 1\right)}. \end{aligned}$$

Recall that, by assumption, $\lambda_x + \lambda_y + \lambda_\xi = 0$, so $\frac{\lambda_x}{\lambda_\xi} = -1 - \frac{\lambda_y}{\lambda_\xi}$. Therefore, $\Gamma\left(w\frac{\lambda_x}{\lambda_\xi} + w\right) = \Gamma\left(-w\frac{\lambda_y}{\lambda_\xi}\right)$. The above manipulations show that

$$\frac{1}{w}\frac{\prod_{k=1-w}^{-1}\left(\frac{k}{w}\lambda_\xi - \lambda_x\right)}{\prod_{k=0}^{w-1}\lambda_\xi\left(\frac{k}{w}-1\right)} = \left(\frac{1}{w}\right)\frac{\Gamma\left(-w\frac{\lambda_y}{\lambda_\xi}\right)}{\left(-\frac{\lambda_\xi}{w}\right)\Gamma\left(w+1\right)\Gamma\left(w\frac{\lambda_x}{\lambda_\xi}+1\right)}.$$

The induced action on $T_0\Delta$ carries weight $\alpha = \frac{\lambda_\xi}{w}$. Substitute $\frac{\lambda_\xi}{w} = z$ and apply Euler's reflection formula to get

$$\begin{aligned} D_{X,L} &= \left(-\frac{1}{w}\right)\frac{\Gamma\left(-\frac{\lambda_y}{z}\right)}{z\Gamma\left(\frac{\lambda_\xi}{z}+1\right)\Gamma\left(\frac{\lambda_x}{z}+1\right)} \\ &= \left(\frac{1}{w}\right)\frac{\pi}{z\Gamma\left(\frac{\lambda_\xi}{z}+1\right)\Gamma\left(\frac{\lambda_x}{z}+1\right)\Gamma\left(\frac{\lambda_y}{z}+1\right)\sin\left(\pi\frac{\lambda_y}{z}\right)} \\ &= \frac{\pi}{wz\widehat{\Gamma}_X\sin\left(\pi\frac{\lambda_y}{z}\right)}. \end{aligned}$$

Generically, there are two S^1-invariant normal directions to $L \cap C$ in X, given by the tangent vectors ∂_x and ∂_y. In the formula in (9), λ may be the weight of either of these directions, i.e., $\lambda = \lambda_y$ or λ_x. The choice of λ changes the sign of (9) because $\sin\left(\pi\frac{\lambda_y}{z}\right) = -\sin\left(\pi\frac{\lambda_x}{z}\right)$. This sign ambiguity reflects an overall choice of orientation of $\overline{\mathcal{M}}_{g,1,0}(X, L; d, w)$ [AKV, GZ, KL]. This completes the proof of the main result.

4. Comparison to Known Localization Calculations

In this section, (8) is compared to previous virtual localization calculations of open Gromov-Witten invariants. In the interest of brevity, only the geometric setup and final results are stated below; the reader interested in further localization calculations is referred to the original sources, or the computation appearing in the proof of Theorem 1.

4.1. **Simple Lagrangians for $\mathcal{O}_{\mathbb{P}^1}(-1,-1)$.** This situation was first described in [KL]. Let $X = \mathcal{O}_{\mathbb{P}^1}(-1,-1)$. X appears often in Gromov-Witten theory and mirror symmetry: it is the small resolution of the conifold singularity, the normal bundle to a smooth rational line in a Calabi-Yau 3-fold, and it can be obtained from a $U(1)$ gauge theory with 4 chiral fields with charges $(1,1,-1,-1)$. X can be described symplectically using symplectic reduction on $\mathbb{C}^4$, and in this setting it is easiest to obtain the moment polytope of X.

Let S^1 act on $\mathbb{C}^4$ with weights $(1,1,-1,-1)$. Then, the moment map for this action is

$$\begin{aligned} \mu\colon \mathbb{C}^4 &\longrightarrow \mathfrak{s}^1 \cong \mathbb{R} \\ (z_1,z_2,z_3,z_4) &\mapsto \frac{1}{2}\left(|z_1|^2+|z_2|^2-|z_3|^2-|z_4|^2\right), \end{aligned}$$

and it can be seen (for example, by choosing appropriate local coordinates and checking transition functions) that

$$X \cong \mu^{-1}\left(\frac{r}{2}\right)/S^1 = \left\{|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 = r\right\}/S^1$$

for $r \in \mathbb{R}_{>0}$ (r determines the symplectic volume of the base $\mathbb{P}^1$). There's a natural anti-holomorphic involution σ on $\mathbb{C}^4$ given by

$$\sigma\left(z_1, z_2, z_3, z_4\right) = \left(\overline{z_2}, \overline{z_1}, \overline{z_4}, \overline{z_3}\right).$$

The fixed locus of this involution is a Lagrangian submanifold $\tilde{L}$ of $\mathbb{C}^4$ defined by the equations

$$\begin{aligned} |z_1|^2 &= |z_2|^2, \\ |z_3|^2 &= |z_4|^2, \\ \overline{z_1 z_2 z_3 z_4} &= z_1 z_2 z_3 z_4. \end{aligned}$$

Because $\tilde{L}$ is preserved by the S^1 action, $\mu^{-1}\left(\frac{r}{2}\right) \cap \tilde{L}/S^1$ defines a Lagrangian $L \subset X$.

This Lagrangian is easy to visualize in the moment polytope of X. The moment polytope is the image of X in $\mathbb{R}^4$ under the projection $z_i \mapsto |z_i|^2$. Then, L is the intersection of the two planes $|z_1|^2 = |z_2|^2$ and $|z_3|^2 = |z_4|^2$ in the polytope. L intersects the zero section $\mathbb{P}^1$ along its equator, so that

$$L \cap \mathbb{P}^1 = \left\{|z_1|^2 = |z_2|^2, |z_3|^2 = |z_4|^2 = 0, |z_1|^2 + |z_2|^2 = r\right\} \cong S^1,$$

as depicted in Figure 2. There are two unique disks with boundary on L, corre-

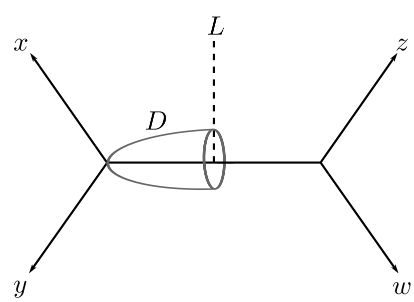

FIGURE 2. The toric polytope for $X = \mathcal{O}_{\mathbb{P}^1}(-1,-1)$ with a Lagrangian. This figure depicts the geometry described in Section 4.1. A Lagrangian L, obtained as the fixed locus of an anti-holomorphic involution, intersects the zero section of X. Local coordinates (x, y) and (z, w) parametrize the fibers of X in a neighborhood of the two vertices of the moment polytope. One possible torus-fixed disk D with boundary $\partial D = L \cap \mathbb{P}^1 \cong S^1$ is depicted.

sponding to the two hemispheres of the $\mathbb{P}^1$. In local coordinates (ξ, x, y) defined by

$$\xi = \frac{z_1}{z_2}, \qquad x = z_2 z_3, \qquad y = z_2 z_4,$$

L is defined by the fixed locus of the antiholomorphic involution

$$\sigma(\xi, x, y) = \left(\frac{1}{\bar{\xi}}, \overline{\xi y}, \overline{\xi x}\right).$$

It is readily checked that at $L \cap \mathbb{P}^1 = \left\{|\xi|^2 = 1\right\}$, and the disk based at the $z_1 = 0$ pole of the $\mathbb{P}^1$ takes the form $|\xi| \leq 1$. The winding w disk map is

$$t \mapsto (\xi = t^w, x = 0, y = 0).$$

In this situation, [KL] computed

$$\left(\frac{1}{w}\right) \frac{e_{S^1}\left(H^1\left(\Delta, T_{(\Delta, f_\Delta)}\right)\right)}{e_{S^1}\left(H^0\left(\Delta, T_{(\Delta, f_\Delta)}\right)\right)} = \left(\frac{1}{w}\right) \frac{\prod_{k=1-w}^{-1}\left(\frac{k}{w}\lambda_\xi - \lambda_x\right)}{\prod_{k=0}^{w-1}\lambda_\xi\left(\frac{k}{w} - 1\right)}.$$

Comparing with (15) in Section 3.2.4, this is $i^*\left(\Delta_{X,L}\right)_{z=\alpha}$, so this result agrees with the proposed formula.

4.2. **The canonical bundle of $\mathbb{P}^2$.** This situation was described in [GZ]. As in the previous example, $X = \mathcal{O}_{\mathbb{P}^2}(-3)$ can be obtained via symplectic reduction. Let S^1 act on $\mathbb{C}^4$ with weights $(1, 1, 1, -3)$. Then,

$$X \cong \mu^{-1}\left(\frac{r}{2}\right)/S^1 = \left\{|z_1|^2 + |z_2|^2 + |z_3|^2 - 3|z_4|^2 = r\right\}/S^1$$

for $r \in \mathbb{R}_{>0}$. [GZ] consider the Lagrangian submanifold $\tilde{L}_c \subset \mathbb{C}^4$ defined by

$$\begin{aligned} |z_1|^2 - |z_3|^2 &= c, \\ |z_2|^2 - |z_4|^2 &= 0, \\ z_1 z_2 z_3 z_4 &= \overline{z_1 z_2 z_3 z_4}, \end{aligned}$$

where $-r < c < r$. Because $\tilde{L}_c$ is preserved by the S^1 action, it descends to a Lagrangian $L_c \subset X$, as depicted in Figure 3. Note that c parametrizes the

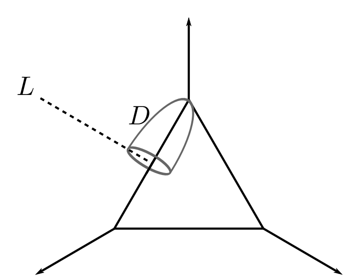

FIGURE 3. The toric polytope for $X = \mathcal{O}_{\mathbb{P}^2}(-3)$ with a Lagrangian. This figure depicts the geometry described in Section 4.2. A Lagrangian L, obtained as the fixed locus of an anti-holomorphic involution, intersects the zero section of X along one edge of the moment polytope. One possible torus-fixed disk D with boundary $\partial D = L \cap \mathbb{P}^2$ is depicted.

intersection of L_c with the $\mathbb{P}^1$ given by the image of $|z_1|^2 + |z_3|^2 = r$ in the quotient space. At $c = 0$, L_c intersects this curve along its equator. For simplicity, restrict attention to $L = L_0$ (locally, other values of c can be obtained by a coordinate transformation). In local coordinates

$$\xi = \frac{z_1}{z_3}, \qquad x = \frac{z_2}{z_3}, \qquad y = z_3^3 z_4, \tag{16}$$

the Lagrangian L is the fixed locus of the anti-holomorphic involution

$$\sigma(\xi, x, y) = \left(\frac{1}{\bar{\xi}}, \overline{\xi y}, \overline{\xi x}\right).$$

The disk is $|\xi|^2 \leq 1$, and the winding w disk map is

$$t \mapsto (\xi = t^w, x = 0, y = 0).$$

So, locally, the situation computed in [GZ] is identical to [KL]. As seen in Section 4.1, this agrees with Theorem 1.

Slightly extending the computation in [GZ], Theorem 1 can also be used to compute the invariants associated to a Lagrangian cycle intersecting an external leg of the moment polytope. Let $\tilde{L}$ be the submanifold of $\mathbb{C}^4$ defined by

$$\begin{aligned} |z_1|^2 - |z_2|^2 &= 0, \\ |z_3|^2 - |z_4|^2 &= 0, \\ z_1 z_2 z_3 z_4 &= \overline{z_1 z_2 z_3 z_4}. \end{aligned}$$

Again, these equations are preserved by the S^1 action, so the image of $\tilde{L}$ in the quotient space X is a well-defined Lagrangian submanifold L. L can be equivalently described as the fixed locus of the anti-holomorphic involution $(z_1, z_2, z_3, z_4) \mapsto (\overline{z_2}, \overline{z_1}, \overline{z_4}, \overline{z_3})$. In local coordinates (ξ, x, y) (16), L is defined by

$$\begin{aligned} |y|^2 &= 1, \\ |x|^2 &= |\xi|^2, \\ \xi x y &= \overline{\xi x y}. \end{aligned}$$

(This Lagrangian cycle is shown in Figure 4). The disk is $|y|^2 \leq 1$, and the winding

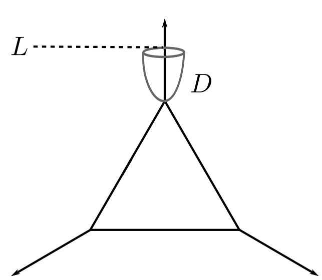

FIGURE 4. $X = \mathcal{O}_{\mathbb{P}^2}(-3)$ with a Lagrangian brane on an external leg. A Lagrangian cycle intersects an external leg of the moment polytope of X. The Lagrangian is obtained as the fixed locus of an anti-holomorphic involution. There is only one torus-fixed disk D with boundary on L, as depicted above.

w disk map is $t \mapsto (\xi = 0, x = 0, y = t^w)$. So, $\alpha = \lambda_y / w$. Applying Theorem 1,

$$\begin{aligned}
\left(\frac{1}{w}\right) \frac{e_{S^1}\left(H^1\left(\Delta, T_{(\Delta, f_\Delta)}\right)\right)}{e_{S^1}\left(H^0\left(\Delta, T_{(\Delta, f_\Delta)}\right)\right)} &= \left.\frac{\pi}{wz\widehat{\Gamma}_X \sin\left(\pi \frac{\lambda_\xi}{z}\right)}\right|_{z=\alpha} \\
&= \frac{-\pi \Gamma\left(-w\frac{\lambda_\xi}{\lambda_y}\right)}{\lambda_y \Gamma\left(w\frac{\lambda_x}{\lambda_y}+1\right)\Gamma(w+1)} \\
&= \frac{1}{w}\frac{\prod_{k=1-w}^{-1}\left(\frac{k}{w}\lambda_y - \lambda_\xi\right)}{\prod_{k=0}^{w-1}\lambda_y\left(\frac{k}{w}-1\right)}.
\end{aligned}$$

Here, the normal direction weight $\lambda = \lambda_\xi$ has been chosen in (9). Choosing $\lambda = \lambda_x$ instead changes the sign, reflecting the overall dependence of these counts on the choice of torus weights. This can be seen from the product identity:

$$\prod_{k=1-w}^{-1}\left(\frac{k}{w}\lambda_y - \lambda_\xi\right) = (-1)^{w-1}\prod_{k=1-w}^{-1}\left(\frac{k}{w}\lambda_y - \lambda_x\right).$$

5. Lagrangian Cycles in Large N Duality

5.1. **Lagrangian cycles and the conifold transition.** In addition to Lagrangians appearing as the fixed loci of anti-holomorphic involutions, there is another family of Lagrangians on $X = \mathcal{O}_{\mathbb{P}^1}(-1,-1)$ motivated by large N duality and knot theory. Recent work in this area has yielded many connections between knot theory and Gromov-Witten theory ([BEM, DSV, GJKS]); this section reviews the geometric relationship between knots on S^3 and open Gromov-Witten theory on X.

Recall that X can be identified with the resolved conifold—X is the small resolution of the conifold singularity

$$xy - zw = 0$$

in $\mathbb{C}^4$. In particular, by blowing up the subspace $y = z = 0$, X can be described by the equations

$$xy - zw = 0, \qquad x\lambda = w\rho, \qquad y\lambda = z\rho,$$

where $(x, y, z, w) \in \mathbb{C}^4$ and $[\lambda : \rho] \in \mathbb{P}^1$. The conifold singularity is also the singular limit of the smooth hypersurface threefold $Y_\mu \subset \mathbb{C}^4$ defined by

$$xy - zw = \mu,$$

where $\mu \in \mathbb{R}_{>0}$. As described in [DSV], Y_μ is symplectomorphic to the cotangent bundle $T^*_{S^3}$. The base $S_\mu \cong S^3$ is the fixed locus of the anti-holomorphic involution $\sigma(x, y, z, w) = (\overline{z}, -\overline{w}, \overline{x}, -\overline{y})$, expressed by the equations $|x|^2 + |y|^2 = \mu$.

The large N duality conjecture states that the large N limit of the topological A-model on Y_μ with N Lagrangian branes wrapping S_μ is equivalent the topological A-model on X [GV]. This has been checked in several ways. First, according to [Wi2], the topological A-model on Y_μ with N Lagrangian branes wrapping S_μ is equivalent to the $U(N)$ Chern-Simons theory on S_μ. Then, in the large N expansion, the partition function $Z_{CS}(k, N)$ is equivalent to the topological A-model partition function $Z_X(g_s, t)$ [GV]. The parameters determining the A-model theory on X are the string coupling constant g_s and the symplectic area t of the zero section $\mathbb{P}^1 \subset X$, which are related to the Chern-Simons parameters k and N by

$$g_s = \frac{2\pi}{k+N}, \qquad t = -\frac{2\pi i N}{k+N}.$$

Large N duality is extended to incorporate Wilson loops in [OV]. Following [Wi3], Wilson loop observables in the Chern-Simons theory on S^3 correspond to colored HOMFLY polynomials of knots $K \subset S^3$. The conormal bundle N_K^* to a knot $K \subset S^3$ is Lagrangian submanifold of $T^*_{S^3}$. The main difficulty in extending large N duality in this manner is determining the corresponding A-model on X: N_K^* intersects the zero section S^3 in the knot K, which becomes contracted after the conifold transition. To remedy this difficulty, the Lagrangian cycle N_K^* must be lifted to a new Lagrangian $\tilde{L}$ disjoint from the zero section before performing the conifold transition [AMV, MV], as depicted in Figure 5.

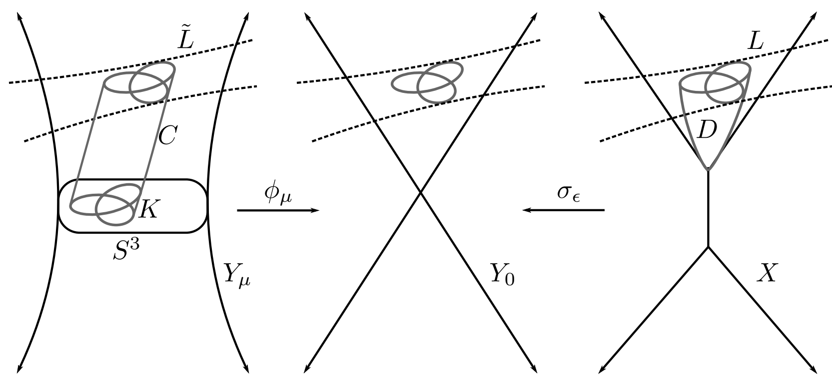

Figure 5. The conifold transition for lifted Lagrangian cycles. This figure depicts the conifold transition. The Lagrangian $\tilde{L} \subset Y_\mu \cong T^*_{S^3}$ is constructed by shifting the conormal bundle of a knot $K \subset S^3$ off of the zero section. This lift introduces a holomorphic cylinder C connecting the knot on S^3 to its image in $\tilde{L}$. Y_0 is the conifold singularity $xz - yw = 0$ in $\mathbb{C}^4$. The map $\phi_\mu : Y_\mu \to Y_0$ is a symplectomorphism away from the zero section, so $\phi_\mu\left(\tilde{L}\right)$ is a Lagrangian submanifold of Y_0. $X \cong \mathcal{O}_{\mathbb{P}^1}(-1,-1)$ is the small resolution of the conifold singularity, and $\sigma_\epsilon : X \to Y_0$ is the corresponding natural map. In fact, there are a family of such symplectomorphisms, where ϵ parametrizes the symplectic form on the zero section $\mathbb{P}^1 \subset X$. Hence, $L := \sigma_\epsilon^{-1} \circ \phi_\mu\left(\tilde{L}\right)$ is a Lagrangian submanifold of X. The holomorphic disk D is the image of C under the conifold transition.

Such a lift is easy to construct: define coordinates $(\overrightarrow{u}, \overrightarrow{v})$ for $T^*_{S^3}$ by

$$T^*_{S^3} = \left\{(\overrightarrow{u}, \overrightarrow{v}) \in \mathbb{R}^4 \times \mathbb{R}^4 : |\overrightarrow{u}| = 1, \overrightarrow{u} \cdot \overrightarrow{v} = 0\right\}.$$

Any knot $K \subset S^3$ is given by a parametrization $\overrightarrow{u} = f(\theta)$. Then, the conormal bundle N_K^* can be expressed as

$$N_K^* = \left\{(\overrightarrow{u}, \overrightarrow{v}) \in T^*S^3 : \overrightarrow{u} = f(\theta), \frac{df}{d\theta} \cdot \overrightarrow{v} = 0\right\}.$$

Lifts of N_K^* can be specified by maps $g : S^1 \to T^*_{f(\theta)}S^3$ such that $\frac{df}{d\theta} \cdot g(\theta) \neq 0$: for such a g, define the lifted conormal bundle $\tilde{L}$ to be

$$\tilde{L} := \left\{ (\overrightarrow{u}, \overrightarrow{v}) \in T^*S^3 : \overrightarrow{u} = f(\theta), \frac{df}{d\theta} \cdot (\overrightarrow{v} - g(\theta)) = 0 \right\}.$$

The image of $\tilde{L}$ under the conifold transition will be a Lagrangian $L \subset X$, and the open A-model on X with this Lagrangian boundary can be computed. Shifting N_K^* off of the zero section modifies large N duality in the following ways: The lift of N_K^* to $\tilde{L}$ introduces corrections to the Wilson loop observables in the Chern-Simons theory proportional to the area of the holomorphic cylinder C connecting the lift of the knot to its image in the zero section [DSV]. Instead of the closed A-model on X, the corresponding theory should be an open A-model with Lagrangian boundary L. This statement of large N duality is found to be true for torus knots in [DSV], and their construction provides a novel source of Lagrangians.

5.2. **Toric Lagrangian cycles and Theorem 1.** It is important to note that the Lagrangians considered in [DSV] are not obtained as the fixed loci of anti-holomorphic involutions, so there is no a priori reason to expect that the formula proposed in Theorem 1 should apply in this situation. For the (r, s) torus knot, the corresponding Lagrangian L is found to be fixed under the torus action

$$\rho_\theta\left((x, y, z, w), [\lambda : \rho]\right) = \left(e^{is\theta}x, e^{ir\theta}y, e^{-is\theta}z, e^{-ir\theta}w\right), \left[e^{-i(r+s)\theta}\lambda : \rho\right].$$

There is only one holomorphic disk in X fixed by this S^1 action, and it lies entirely in the x-y face of the moment polytope, as depicted in Fig. 6. A neighborhood of

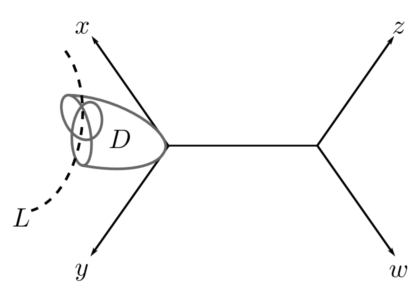

Figure 6. A moment polytope picture of torus knot Lagrangians. This figure depicts the geometry described in Section 5.2. The Lagrangian L is the image of a shifted conormal bundle to a knot in S^3 under the conifold transition. Local coordinates (x, y) and (z, w) parametrize the fibers of X in a neighborhood of the two vertices of the moment polytope. The boundary of the disk D is symplectomorphic to the torus knot (in the depiction above, the trefoil). D is contained entirely in the x-y face of the polytope, and the disk map can be written in local coordinates as $t \mapsto (\xi = 0, x = b_1^s t^{ws}, y = b_1^r t^{wr})$.

the disk can be described by local coordinates x, y, $\xi = \lambda/\rho$. In these coordinates, the disk map is

$$t \mapsto \left(\xi = 0, x = b_1^s t^{ws}, y = b_1^r t^{wr}\right),$$

where $|t| \leq 1$ and $b_1 \in \mathbb{R}_{>0}$ is a constant obtained from the geometric construction in [DSV]. After a lengthy localization calculation, [DSV] compute the winding-1 open Gromov-Witten invariants with Lagrangian boundary L. This computation readily generalizes to higher winding [GJKS], and gives the following expression for $D_{X,L}$:

$$D_{X,L} = (-1)^{ws} \frac{\prod_{k=1}^{ws-1} \left(r + s - \frac{k}{w}\right)}{w \prod_{k=0}^{ws-1} \left(s - \frac{k}{w}\right)}. \tag{17}$$

This can be re-written in terms of gamma functions in the following way:

$$\frac{\prod_{k=1}^{ws-1} \left(r + s - \frac{k}{w}\right)}{\prod_{k=0}^{ws-1} \left(s - \frac{k}{w}\right)} = \frac{\Gamma\left(wr + ws\right)}{\left(\frac{1}{w}\right) \Gamma\left(ws + 1\right) \Gamma\left(wr + 1\right)}.$$

Locally (Figure 6), the weights of the torus action are $\lambda_\xi = -r - s$, $\lambda_x = s$, $\lambda_y = r$. The induced torus action on Δ is $t \mapsto e^{i\theta/w} t$, so $\alpha = \frac{1}{w}$. Replacing $\alpha = z$ and substituting these weights into the above formula yields

$$= \frac{\Gamma\left(-\frac{\lambda_\xi}{z}\right)}{wz\Gamma\left(\frac{\lambda_x}{z} + 1\right)\Gamma\left(\frac{\lambda_y}{z} + 1\right)} = \frac{\pi}{wz\widehat{\Gamma}_X \sin\left(\pi \frac{\lambda_\xi}{z}\right)}$$

after Euler's reflection identity. As remarked above, the Lagrangian L is not the fixed locus of an anti-holomorphic involution. However, λ_ξ is the weight of an S^1-invariant normal direction to ∂D: the boundary of the disk is entirely contained in the x-y plane of the moment polytope. The author finds it curious that the result of Theorem 1 appears to apply in this situation, and hopes that this is evidence that, properly formulated, a more general version of this theorem exists.

References

[AB] M. F. Atiyah and R. Bott. "The moment map and equivariant cohomology." Topology **23** (1984) 1–28.

[AKMV] M. Aganagic, A. Klemm, M. Marino, C. Vafa. "The topological vertex." Comm. Math. Phys. **254** (2005), 425–478. arXiv:hep-th/0305132

[AKV] M. Aganagic, A. Klemm, C. Vafa. "Disk instantons, mirror symmetry, and the duality web." Z. Naturforsch. **A57** (2002) 1–28 MR 1906661. arXiv:hep-th/0105045 [hep-th]

[AMV] M. Aganagic, M. Mariño, C. Vafa. "All loop topological string amplitudes from Chern-Simons theory." Comm. Math. Phys. **247** (2004), 467–512.

[BC] A. Brini and R. Cavalieri, "Open orbifold Gromov-Witten invariants of $[\mathbb{C}^3/\mathbb{Z}_n]$: Localization and Mirror Symmetry." arXiv:1007.0934 [math.AG]

[BEM] A. Brini, B. Eynard, M. Mariño. "Torus knots and mirror symmetry." Annales Henri Poincare **13** (2012) 1873–1910. arXiv:1105.2012 [hep-th]

[CKYZ] T.-M. Chiang, A. Klemm, S.-T. Yau, E. Zaslow. "Local Mirror Symmetry: Calculations and Interpretations," Adv. Theor. Math. Phys. **3** (2000), 495–565. arXiv:hep-th/9903053v4

[COGP] P. Candelas, X. de la Ossa, P. Green, and L. Parkes. "A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory." Nucl. Phys. B **359** (1991) no. 1, 21–74.

[DSV] D.-E. Diaconescu, V. Shende, C. Vafa. "Large N Duality, Lagrangian Cycles, and Algebraic Knots." Comm. Math. Phys. **319** (2013), 813–863. arXiv:1111.6533v1 [hep-th]

[GGI] S. Galkin, V. Golyshev, H. Iritani. "Gamma Classes and Quantum Cohomology of Fano Manifolds: Gamma Conjectures." arXiv:1404.6407

[Giv] A. Givental. "Equivariant Gromov-Witten invariants." *Internat. Math. Res. Notices* (1996), 613–663. arXiv:alg-geom/9603021

[GJKS] J. Gu, H. Jockers, A. Klemm, M. Soroush. "Knot Invariants from Topological Recursion on Augmentation Varieties." Comm. Math. Phys. **336** (2015), 987–1051. arXiv:1401.5095v3 [hep-th]

[GP] T. Graber and R. Pandharipande. "Localization of Virtual Classes." Invent. Math. **135** (1999) 487–518. arXiv:alg-geom/9708001v2

[GV] R. Gopakumar and C. Vafa. "On the Gauge Theory/Geometry Correspondence." Adv. Theor. Math. Phys. (2002) B646:498–523. arXiv:hep-th/9811131

[GZ] T. Graber and E. Zaslow. "Open-String Gromov-Witten Invariants: Calculations and a Mirror 'Theorem"' in *Orbifolds in Mathematics and Physics*, 107–121, Contemp. Math. **310**, Amer. Math. Soc. (Providence) 2002. arXiv:hep-th/0109075

[Iri] H. Iritani. "An Integral Structure in Quantum Cohomology and Mirror Symmetry for Toric Orbifolds," Adv. Math. **222** (2009), 1016–1079. arXiv:0903.1463 [math.AG]

[KL] S. Katz and C.-C. M. Liu. "Enumerative Geometry of Stable Maps with Lagrangian Boundary conditions and Multiple Covers of the Disc," Adv. Theor. Math. Phys. **5** (2001), 1–49. arXiv:math/0103074 [math.AG]

[Kon] M. Kontsevich. "Enumeration of Rational Curves via Torus Actions," in *The Moduli Space of Curves*, Dijkgraaf et al eds., Progress in Mathematics **129**, Birkhauser (Boston) 1995. arXiv:hep-th/9405035v2

[KZ] A. Klemm and E. Zaslow. "Local Mirror Symmetry at Higher Genus." arXiv:hep-th/9906046

[MV] M. Mariño and C. Vafa. "Framed Knots at Large N." arXiv:hep-th/0108064

[OV] H. Ooguri and C. Vafa. "Knot Invariants and Topological Strings." *Nucl. Phys.*, B577:419–438, 2000. arXiv:hep-th/9912123

[Wi1] E. Witten. "Mirror Manifolds and Topological Field Theory," arXiv:hep-th/9112056

[Wi2] E. Witten. "Chern-Simons Gauge Theory as a String Theory." In *The Floer Memorial Volume*, volume 133 of *Progr. Math.*, 637–678. Birkhäuser (Basel) 1995. arXiv:hep-th/9207094

[Wi3] E. Witten. "Quantum Field Theory and the Jones Polynomial," Comm. Math. Phys. **121,** 351 (1989).

Matthew Mahowald, Department of Mathematics, Northwestern University, Evanston, IL 60208, USA

E-mail address: mamahowa@math.northwestern.edu