On general characterization of Young measures associated with Borel functions

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Abstract

We prove that the Young measure associated with a Borel function f is a probability distribution of the random variable $f(U)$, where U has a uniform distribution on the domain of f. As an auxiliary result, the fact that Young measures associated with simple functions are weak* dense in the set of Young measures associated with measurable functions is proved.

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1 Introduction

One of the major problems in the calculus of variations is minimization of functionals which are bounded from below but do not attain their infima. If the minimize functional $\mathcal{J}$ is bounded, the direct method can be applied: there always exists a minimizing sequence for $\mathcal{J}$, that is a sequence (u_n), $u_n\colon \mathbb{R}^d \to \mathbb{R}^l$, $n \in \mathbb{N}$, such that $\lim\limits_{n\to\infty} \mathcal{J}(u_n) = \inf \mathcal{J}$. Additionally, if $\mathcal{J}$ is coercive, (u_n) is always bounded. However, if $\mathcal{J}$ does not attain its infimum then the elements of (u_n) are functions of highly oscillatory nature. Moreover, weak* convergence in L^∞ of (u_n) to some function u_0 does not guarantee, that the sequence $(\varphi(u_n))$ of compositions of u_n with continuous function φ is weak* convergent in L^∞. Indeed, in general it is not convergent not only to $\varphi(u_0)$, but to any function with domain in $\mathbb{R}^d$.

Laurence Chisholm Young introduced in [11] objects called by him 'generalized curves', nowadays called 'Young measures.' These are the 'generalized limits' of sequences of highly oscillating functions. The 'mature' form of Young's theorem has been proved by J.M.Ball in [3]. According to these theorems, we say that that under their assumptions the considered sequences 'generate' appropriate Young measures. This approach is studied for example in [8] in detail.

Alternatively, we can look at the Young measure as at object associated with *any* measurable function defined on a nonempty, open, bounded subset Ω of $\mathbb{R}^d$ with values in a compact subset K of $\mathbb{R}^l$. Such a conclusion can be derived from the theorem 3.6.1 in [10]. Thank to this theorem it can be proved that the Young measure associated with a simple function is the convex combination of Dirac measures. These Dirac measures are concentrated at the values of the simple function under consideration while coefficients of the convex combinaton are proportional to the Lebesgue measure of the sets on which the respective values are taken on by the function, see [9] for details and more general result.

In this article we significantly generalize the above results. We prove a theorem providing general yet simple description of Young measures associated with Borel functions. As a consequence, the theorem enables one to compute explicit formulae of probability density functions of the Young measures in many interesting cases. This can be done without commonly used to calculate weak* limits of sequences of functions generating Young measures functional analytic apparatus. Since Young measures are widely used in many areas of theoretical and applied sciences (see for example [2], [4], [5], [8]), our result provide a handy tool of obtaining their explicit form.

The main theorem of this article states that the Young measure associated with any Borel function f defined on the set $\Omega \subset \mathbb{R}^d$ with positive Lebesgue measure M and values in a compact set $K \subset \mathbb{R}^l$, is in fact a probability distribution of a random variable $X = f(U)$, where U is uniformly distributed on Ω. Before this, we prove a lemma corresponding to standard measure-theoretic result, that any Borel function is a pointwise limit of the appropriate sequence of simple functions. Relying on this fact we prove, that for any Borel function $f\colon \Omega \to K$, its Young measure is the weak* limit of a sequence of Young measures associated with the elements of the sequence of simple functions convergent pointwise to f.

2 Young measures – necessary information and an auxiliary result

The first part of this section can serve as a very brief introduction to the theory of Young measures. In the second part we state and prove a lemma mentioned at the end of the Introduction.

2.1 An outline of the Young measures theory

We gather now some information about Young measures. An interested reader can find details, together with proofs and further bibliography, for example in [1], [6], [7], [8], [10].

Let $\mathbb{R}^d \supset \Omega$ be nonempty, bounded open set and let $K \subset \mathbb{R}^l$ be compact. Let (f_n) be a sequence of functions from Ω to K, convergent to some function f_0 weakly* in L^∞. Finally, let φ be an arbitrary continuous real valued function on $\mathbb{R}^d$. Then the sequence $(\varphi(f_n))$ is uniformly bounded in L^∞ norm and therefore by the Banach – Alaoglu theorem there exists a subsequence of $(\varphi(f_n))$ weakly* convergent to some function g. In general $g \neq \varphi((f_0))$. L. C. Young proved in [11], that there exists a subsequence of $\varphi((f_n))$, not relabelled, and a family $(\nu_x)_{x\in\Omega}$ of probability measures with supports $\operatorname{supp}\nu_x \subseteq K$, such that $\forall\,\varphi \in C(\mathbb{R}^d)\ \forall w \in L^1(\Omega)$ there holds

$$\lim_{n\to\infty} \int_\Omega \varphi(f_n(x))w(x)dx = \int_\Omega \int_K \varphi(s)\nu_x(ds)w(x)dx := \int_\Omega \overline{\varphi}(x)w(x)dx.$$

This family of probability measures is today called a *Young measure associated with the sequence* (f_n).

In 1989 J. M. Ball proved the following theorem. Let Ω be a measurable subset of $\mathbb{R}^d$, $v\colon [0,+\infty) \to [0,+\infty)$ a continuous, nondecreasing function such that $\lim_{t\to\infty} v(t) = +\infty$. By ψ we denote a function $\psi\colon \Omega \times \mathbb{R}^l \ni (x,\lambda) \to \psi(x,\lambda) \in \overline{\mathbb{R}}$ satisfying Carathéodory conditions: it is measurable with respect to the first, and continuous with respect to the second variable. Consider further a sequence (f_n) of functions on Ω with values in $\mathbb{R}^l$, satisfying the condition

$$\sup_n \int_\Omega v(|f_n(x)|)dx < +\infty.$$

Theorem 2.1 *([3]) Under the above assumptions, there exists a subsequence of (f_n), not relabelled, and a family $(\nu_x)_{x\in\Omega}$ of probability measures, dependent measurably on x, such that if for any Carathéodory function ψ the sequence $(\psi(x, f_n(x))$ is weakly convergent in $L^1(\Omega)$, then its weak limit is a function*

$$\overline{\psi}(x) = \int_{\mathbb{R}^l} \psi(x,\lambda)d\nu_x(\lambda).$$

We now turn our attention to the presentation of the Young measures as in [10]. In general, Young measures can be looked at as the element of the space conjugate to the space $L^1(\Omega, C(K))$ of Bochner integrable functions on $\Omega \subset \mathbb{R}^d$ with values in $C(K)$. The space $L^1(\Omega, C(K))$ is isometrically isomorphic to the space $Car(\Omega, K; \mathbb{R})$ of the Carathéodory functions, equipped with the norm $\|h\|_{Car} := \int_\Omega \sup_{k\in K} |h(x,k)|dx$.

Let $h \in L^1(\Omega, C(K))$. Denote by $\mathcal{U}$ the set of all measurable functions on Ω with values in K. Consider a mapping

$$i\colon \mathcal{U} \to L^1(\Omega, C(K))^*$$

defined by the formula

$$\langle i(f), h\rangle := \int_\Omega h(x, f(x))dx.$$

By $Y(\Omega, K)$ we will denote the weak* closure of the set $i(\mathcal{U})$ in $L^1(\Omega, C(K))^*$:

$$Y(\Omega,K) := \Big\{ L^1(\Omega, C(K))^* \ni \eta : \exists (f_n) \subset \mathcal{U} : i(f_n) \xrightarrow[n\to\infty]{w^*} \eta \Big\}.$$

Denote by

- $rca(K)$ – the space of regular, countably additive signed measures on K, equipped with the norm $\|m\|_{rca(K)} := |m|(\Omega)$, where $|\cdot|$ stands in this case for the total variation of the measure m. With this norm $rca(K)$ is a Banach space;
- $rca^1(K)$ – the subset of $rca(K)$ with elements being probability measures on K;
- $L^\infty_w(\Omega, \mathrm{rca}(K))$ – the set of the weakly measurable mappings

  $$\nu\colon \Omega \ni x \to \nu(x) \in rca(K).$$

  We equip this set with the norm

  $$\|\nu\|_{L^\infty_w(\Omega,\mathrm{rca}(K))} := \operatorname{ess\,sup}\big\{\|\nu(x)\|_{\mathrm{rca}(K)} : x \in \Omega\big\}.$$

  By the Dunford – Pettis theorem this space is isometrically isomorphic with the space $L^1(\Omega, C(K))^*$.

Now define an element η of $L^1(\Omega, C(K))^*$ by the formula

$$\eta\colon L^1(\Omega, C(K)) \ni h \to \langle \eta, h\rangle := \int_\Omega \Big(\int_K h(x,k)d\nu_x(k)\Big)dx,$$

which in turn will be the value of the mapping

$$\psi\colon L^\infty_w(\Omega, \mathrm{rca}(K)) \ni \nu \to \psi(\nu) := \eta \in L^1(\Omega, C(K))^*$$

Theorem 2.2 *The mapping ψ defined above is an isometric isomorphism between the spaces $L^\infty_w(\Omega, \mathrm{rca}(K))$ and $L^1(\Omega, C(K))^*$.*

The set of the Young measures on the compact set $K \subset \mathbb{R}^l$ will be denoted by $\mathcal{Y}(\Omega, K)$:

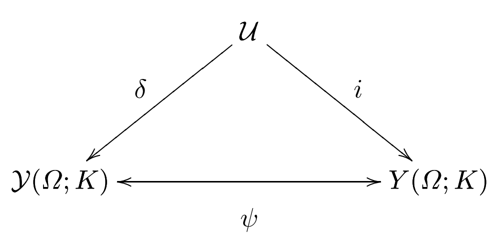
$$\mathcal{Y}(\Omega, K) := \big\{\nu = (\nu(x)) \in L^\infty_w(\Omega, \mathrm{rca}(K)) : \nu_x \in \mathrm{rca}^1(K) \text{ for a.a } x \in \Omega\big\}.$$

We will write ν_x or $(\nu_x)_{x\in\Omega}$ instead of $\nu(x)$.

Finally, we define the Dirac mapping δ: $\forall x \in \Omega$

$$\delta\colon \mathcal{U} \ni f \to [\delta(f)](x) := \delta_{f(x)} \in \mathcal{Y}(\Omega, K)$$

Theorem 2.3 *The diagram*



is commutative.

This means, that for any $f \in \mathcal{U}$ there exists a Young measure associated with it.

2.2 An auxiliary lemma

The notion of quasi-Young measure was introduced in [9]. We state it now in a slightly more general form.

Let Ω be an open subset of $\mathbb{R}^d$ with Lebesgue measure $M > 0$, $d\mu(x) := \frac{1}{M}dx$, where dx is the $d-$dimensional Lebesgue measure on Ω and let $K \subset \mathbb{R}^l$ be compact.

Definition 2.1 *We say that a family of probability measures $\nu = (\nu_x)_{x\in\Omega}$ is a quasi-Young measure associated with the measurable function $f\colon \mathbb{R}^d \supset \Omega \to K \subset \mathbb{R}^l$, if for every continuous function $\beta\colon K \to \mathbb{R}$ there holds an equality*

$$\int\limits_K \beta(k)d\nu_x(k) = \int\limits_\Omega \beta(f(x))d\mu(x).$$

In [9] it was proved that in many application important cases the quasi-Young measures associated with functions satysfying appropriate assumptions were identical with the Young measures with them. In particular, quasi-Young measures associated with simple functions are Young measures associated with them. Denote by μ the normalized Lebesgue measure on a nonempty, bounded subset Ω of $\mathbb{R}^d$ with positive measure M: $d\mu(x) := \frac{1}{M}dx$ with a d-dimensional Lebesgue measure dx. Let $\{\Omega\}_{i=1}^n$ be a partition of Ω into open, pairwise disjoint subsets Ω_i with Lebesgue measure $m_i > 0$, such that $\bigcup\limits_{i=1}^n cl(\Omega_i) = cl(\Omega)$, where '$cl$' stands for 'closure'. By $\mathbf{1}_A$ we denote the characteristic function of the set A.

Theorem 2.4 *[9] Choose and fix points $p_i \in \mathbb{R}^l$, $i = 1, 2, \ldots, n$, and let f be a simple function:*

$$f := \sum_{i=1}^n p_i \mathbf{1}_{\Omega_i}.$$

Then the Young measure associated with f is of the form

$$\nu_x = \frac{1}{M}\sum_{i=1}^n m_i \delta_{p_i}.$$

Remark 2.1 *Observe that ν_x does not depend on the variable $x \in \Omega$; this is a* homogeneous Young measure*.*

Definition 2.2 *The (quasi-)Young measure associated with simple function will be called a* simple (quasi-)Young measure*.*

We now recall the notion of weak* convergence of measures on compact sets.

Definition 2.3 *We say that a sequence (ν_n) of bounded measures on a compact set $K \subset \mathbb{R}^l$ converges weakly* to a measure ν_0, if $\forall \beta \in C(K, \mathbb{R})$ there holds*

$$\lim_{n\to\infty} \int\limits_K \beta(k)d\nu_n(k) = \int\limits_K \beta(k)d\nu_0(k).$$

We now prove a lemma which needed further.

Lemma 2.1 *Let $f\colon \Omega \to K$ be a measurable function and let (f_n) be a pointwise convergent to f sequence of simple functions. Then the Young measure ν^f associated with f is a weak* limit of the sequence of the simple Young measures associated with respective elements of (f_n).*

Proof. Choose and fix $\varepsilon > 0$. Using change of variable theorem, continuity of the function β and the finiteness of the measure of Ω, we infer the existence of $n_0 \in \mathbb{N}$ such that $\forall m,\, n > n_0$ we have

$$\Big|\int_K \beta(k)d\nu_n - \int_K \beta(k)d\nu_m\Big| = \Big|\int_\Omega \beta(f_n(x))d\mu - \int_\Omega \beta(f_m(x))d\mu\Big| \leq$$
$$\leq \int_\Omega |\beta(f_n(x)) - \beta(f_m(x))|d\mu \leq \varepsilon \cdot \mu(\Omega).$$

This means that (ν_n) is a weak* Cauchy sequence, so its weak* accumulation point belongs to $Y(\Omega, K)$. By the injectivity of ψ, there is exactly one Young measure $\rho = (\rho_x)_{x\in\Omega} \in \mathcal{Y}(\Omega, K)$ corresponding to this accumulation point. Then we have

$$\Big|\int_K \beta(k)d\rho_x - \int_K \beta(k)d\nu_x^f\Big| \leq \Big|\int_K \beta(k)d\rho_x - \int_K \beta(k)d\nu_n\Big| +$$
$$+ \Big|\int_K \beta(k)d\nu_n - \int_K \beta(k)d\nu_x^f\Big|.$$

Since ρ is an accumulation point of (ν_n) and (f_n) converges pointwise to f, the result follows. ■

Remark 2.2 *Observe that ν_f need not be a homogeneous Young measure.*

Corollary 2.1 *The set of all simple Young measures is weak* dense in the set of the Young measures associated with functions from $\mathcal{U}$.*

3 Some necessary notions from probability theory and notation

To set up notation, we recall now standard probabilistic notions needed in the sequel. If Σ is a σ-algebra of subsets of a nonempty set A and P – a measure on Σ, then the triple (A, Σ, P) is called a measure space, and a *probability space* if P is a probability measure. A *random variable* (or a *random vector*) $X\colon A \to \mathbb{R}^d$ is a function such that for any Borel set $B \subseteq \mathbb{R}^d$ there holds $X^{-1}(B) \in \Sigma$. Obviously, if $\varphi\colon \mathbb{R}^d \to \mathbb{R}^l$ is a Borel function, then $\varphi(X)$ is a random variable. The *probability distribution* on $\mathbb{R}^d$ is any probability measure

P on the σ-algebra $\mathcal{B}(\mathbb{R}^d)$ of Borel subsets of $\mathbb{R}^d$. The probability distribution of a random variable X with values in $\mathbb{R}^d$ is a probability measure P_X on $\mathbb{R}^d$ defined for any $B \in \mathcal{B}(\mathbb{R}^d)$ by the equality $P_X(B) := P(X^{-1}(B))$. Consequently, for the distribution of the random variable $\varphi(X)$ we have: for any $C \in \mathcal{B}(\mathbb{R}^l)$

$$P_{\varphi(X)}(C) = P(\varphi(X)^{-1}(C)) = P_X(\varphi^{-1}(C)).$$

If P is a probability distribution on $\mathbb{R}^d$ and for some Lebesgue integrable function $g\colon \mathbb{R}^d \to \mathbb{R}$ there holds: $\forall\, A \in \mathcal{B}(\mathbb{R}^d)\ P(A) = \int\limits_A g(x)dx$, then the function g is called a *density* of P.

Let Ω be a Borel subset of $\mathbb{R}^d$ with Lebesgue measure $M > 0$. We say that random variable $U\colon \mathbb{R}^d \to \mathbb{R}^l$ is *uniform* on Ω, if its density g_u is of the form

$$g_U(x) = \begin{cases} \frac{1}{M}, & x \in \Omega \\ 0, & x \notin \Omega. \end{cases}$$

The probability distribution P_U is then called the *uniform distribution.*

4 Main result

As in the previous sections, let Ω be an open subset of $\mathbb{R}^d$ with Lebesgue measure $M > 0$, $d\mu(x) := \frac{1}{M}dx$, where dx is the $d-$dimensional Lebesgue measure on Ω and let $K \subset \mathbb{R}^l$ be compact. Denote $P := \frac{1}{M}dx$.

Finally, we are ready to formulate the main theorem of the article.

Theorem 4.1 *Let $f\colon \mathbb{R}^d \supset \Omega \to K \subset \mathbb{R}^l$ be a Borel function with Young measure μ^f. Then μ^f is the probability distribution of the random variable $Y = f(U)$, where U has a uniform distribution on Ω.*

Proof. The distribution of a random variable Y is of the form: $\forall\, C \in \mathcal{B}(K)$, $P_{f(U)}(C) = P_U(f^{-1}(C))$. Let f be constant on Ω with value p and vanish on the complement of Ω. By lemma 2.1 we have $\mu^f = \delta_p$. For any $C \subseteq K$ we have

$$\mu^f(C) = \int\limits_{\mathbb{R}^l} \mathbf{1}_C(p) d\delta_p = \begin{cases} 1, & p \in C \\ 0, & p \notin C. \end{cases}$$

On the other hand,

$$P_U(f^{-1}(C)) = \int\limits_{f^{-1}(C)} g_u dP = \frac{1}{M} \int\limits_{\{x: f(x) \in C\}} dx = \begin{cases} \frac{1}{M} \cdot M = 1, & p \in C \\ \frac{1}{M} \cdot 0 = 0, & p \notin C. \end{cases}$$

Thus $P_Y = \mu^f$. This equality also holds when f is a simple function, due to the linearity of the integral. Since functions under consideration have values in the compact set K, the dominated convergence theorem yields the result for any Borel f. ∎

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