On twisting real spectral triples by algebra automorphisms

Giovanni Landi, Pierre Martinetti

Abstract

We systematically investigate ways to twist a real spectral triple via an algebra automorphism and in particular, we naturally define a twisted partner for any real graded spectral triple. Among other things we investigate consequences of the twisting on the fluctuations of the metric and possible applications to the spectral approach to the standard model of particle physics.

Contents

1st January 2016

This work was partially supported by the Italian Project “Prin 2010-11 – Operator Algebras, Noncommutative Geometry and Applications”.

1 Introduction

We investigate in a systematic way how to twist a spectral triple, and in particular the consequences of the twisting on the fluctuations of the metric. Twisted spectral triples have been defined by Connes and Moscovici in [7]. They consist in replacing in the definition of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ the condition $[D, a]$ bounded for any $a \in \mathcal{A}$ by the following: there exists an automorphism ρ of $\mathcal{A}$ such that, what is bounded for any $a \in \mathcal{A}$ is the twisted commutator

$$[D, a]_\rho := Da - \rho(a)D \,. \tag{1.1}$$

The original motivation of [7] was to deal with type III operator algebras, for which there is no non trivial trace. The examples there were spectral triples perturbed by a conformal transformation and spectral triples associated to codimension 1 foliations. Twisted spectral triples are relevant for quantum groups (and related spaces) where twisting of the algebra is a natural phenomenon [14], [11]; see [12] for a twisted spectral triple for the quantum group $SU(2)$. They also appear in C^*-dynamical systems [10]. Recently, twisted spectral triples have also occurred in the description of the standard model of elementary particles [8]. Here twisting allows one to build models beyond the standard model without modifying the fermionic content of the theory. This is obtained by twisting the spectral triple of the standard model of [3] while keeping the Hilbert space and the Dirac operator untouched.

In the following we generalize this construction to arbitrary spectral triples. We first show in Sect. 2 how to incorporate the real structure in the twisted framework (Definition 2.1), in a way compatible with the fluctuation of the metric (Proposition 2.6). In Sect. 3 we formalize the idea of *minimal twist*, that is twisting a spectral triple without touching the Hilbert space and Dirac operators (Definition 3.1). A procedure to minimally twist any graded spectral triple is presented in Proposition 3.6, extended to the real case in Proposition 3.7. Next, Sect. 4 deals with commutative and almost commutative geometries with a twisting by grading that is essentially unique. Finally, Sect. 5 is devoted to some applications, notably to study twisted fluctuations of a free Dirac operator and touches on possible uses in the spectral action approach to the standard model with a more thorough analysis of these reported elsewhere.

Acknowledgments. We thank Paolo Antonini, Ludwik Dabrowski, Gianfausto Dell'Antonio, Koen van den Dungen, Alessandro Michelangeli for useful discussions during the seminars in Trieste on 13th and 19th November 2015 where this work was presented.

2 Real twisted spectral triple structure

We first extend the twisting of spectral triples to include the real structure and then introduce twisted-fluctuations of the metric. Proposition 2.6 shows that the picture is coherent: a twisted-fluctuated real spectral triple is a real twisted spectral triple.

2.1 Really twisting

Recall [5] that a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists in an involutive algebra $\mathcal{A}$ acting faithfully [1] by bounded operators on a Hilbert space $\mathcal{H}$ together with a self-adjoint operator D with compact resolvent such that $[D, a]$ is bounded for any $a \in \mathcal{A}$. It is graded (or even) when there exists a grading of $\mathcal{H}$, that is a self-adjoint operator Γ of square $\mathbb{I}$, that commutes with $\mathcal{A}$ and anticommutes with D. Furthermore [6], a real spectral triple of KO-dimension $k \in \{1, 2, \ldots, 7\}$ modulo 8, is a (graded) spectral triple together with an antilinear isometry operator J on $\mathcal{H}$ such that

$$J^2 = \epsilon(k), \quad JD = \epsilon'(k)DJ, \qquad \text{and} \quad J\Gamma = \epsilon''(k)\Gamma J, \tag{2.1}$$

where $\epsilon, \epsilon', \epsilon''$ take value in $\{-1, +1\}$ as a function of k. Furthermore, the conjugate action of J,

$$b \mapsto Jb^*J^{-1} \quad \forall b \in \mathcal{A} \tag{2.2}$$

implements an action of the opposite algebra $\mathcal{A}^\circ$, which is required to commutes with the algebra,

$$[a, Jb^*J^{-1}] = 0 \quad \forall a, b \in \mathcal{A}, \qquad \text{(zero-order condition)} \tag{2.3}$$

as well as to commute with the commutator of D with $\mathcal{A}$,

$$[[D, a], Jb^*J^{-1}] = 0 \quad \forall a, b \in \mathcal{A}, \qquad \text{(first-order condition)}. \tag{2.4}$$

To avoid ambiguity it may be wise occasionally to reintroduce the representation symbol. Thus, if π is the representation of $\mathcal{A}$ on $\mathcal{H}$, then one gets a representation of $\mathcal{A}^\circ$ on $\mathcal{H}$ by

$$\pi^\circ(b) := J\pi(b^*)J^{-1} \tag{2.5}$$

and (2.3) is the statement that the operator algebras $\pi(\mathcal{A})$ and $\pi^\circ(\mathcal{A}^\circ)$ commute. On the other hand, dropping the representation symbols, we shall try and write the above as $b^\circ = Jb^*J^{-1}$.

Twisted and graded twisted spectral triples were defined in [7] by replacing the boundedness of the commutator $[D, a]$ with the requirement that the twisted commutator

$$[D, a]_\rho := Da - \rho(a)D, \tag{2.6}$$

for an automorphism $\rho \in \text{Aut}(\mathcal{A})$, be bounded for any $a \in \mathcal{A}$. Furthermore, the automorphism ρ is not taken to be a $*$-automorphism, but rather to satisfy

$$\rho(a^*) = (\rho^{-1}(a))^*. \tag{2.7}$$

Such an automorphism was named *regular* in [14]. The requirement (2.7) has origin in the additional assumption (coming from considerations in index theory in [7]) that the algebra $\mathcal{A}$ has a 1-parameter group of automorphisms $\{\rho_t\}_{t\in\mathbb{R}}$ and that ρ coincides with the value at $t = \mathrm{i}$ of the analytic extension of $\{\rho_t\}_{t\in\mathbb{R}}$. In typical examples (for instance the spectral triples associated to codimension 1 foliations) the 1-parameter group of automorphisms is the modular automorphism group of a twisted trace. Such twisted traces appear naturally with twisted spectral triples. Indeed, if $(\mathcal{A}, \mathcal{H}, D)$ is a ρ-twisted spectral triple with $D^{-1} \in \mathcal{L}^{n,\infty}$, the Dixmier ideal, from [7, Prop. 3.3] the functional

$$\mathcal{A} \ni a \mapsto \varphi(a) = ⨍ aD^{-n} := \text{Tr}_\omega(aD^{-n}), \tag{2.8}$$

[1] When possible we omit the representation symbol and identify $a \in \mathcal{A}$ with its representation $\pi(a) \in \mathcal{L}(\mathcal{H})$.

with Tr_ω the Dixmier trace, is a ρ^{-n}-trace, that is $\varphi(ab) = \varphi(b\rho^{-n}(a))$ for all $a, b \in \mathcal{A}$.

Now, the algebras $\mathcal{A}$ and $\mathcal{A}^\circ$ have isomorphic automorphism groups. An isomorphism is:

$$\mathrm{Aut}(\mathcal{A}) \ni \rho \to \rho^\circ \in \mathrm{Aut}(\mathcal{A}^\circ), \qquad \rho^\circ(b^\circ) := (\rho^{-1}(b))^\circ, \quad \forall b^\circ \in \mathcal{A}^\circ. \tag{2.9}$$

The use of ρ^{-1} instead of ρ is to parallel condition (2.7). In a sense, the above means

$$\rho^\circ(Jb^*J^{-1}) = J(\rho^{-1}(b))^*J^{-1} = J\rho(b^*)J^{-1}, \tag{2.10}$$

and the second equality is due to condition (2.7). We are then led to the following.

Definition 2.1. *A real twisted spectral triple of KO-dimension k is the datum of a twisted spectral triple* $(\mathcal{A}, \mathcal{H}, D;\, \rho)$ *together with an antilinear isometry operator J satisfying the rule of signs* (2.1)*, the zero-order condition* (2.3)*, and the twisted first-order condition*

$$[[D, a]_\rho, Jb^*J^{-1}]_{\rho^\circ} = 0, \quad \forall a, b \in \mathcal{A}. \tag{2.11}$$

By inserting the representation symbols and with condition (2.7), the above reads as

$$\big(D\pi(a) - \pi(\rho(a))D\big)J\pi(b^*)J^{-1} - J\pi(\rho(b^*))J^{-1}\big(D\pi(a) - \pi(\rho(a))D\big) = 0, \quad \forall a, b \in \mathcal{A}. \tag{2.12}$$

We notice that the condition (2.11) is symmetric in $\mathcal{A}$ and $\mathcal{A}^\circ$. Indeed, an use of the zero-order conditions $[a, Jb^*J^{-1}] = 0$ and $[\rho(a), J(\rho^{-1}(b))^*J^{-1}] = 0$, transforms (2.11) into

$$[[D, Jb^*J^{-1}]_{\rho^\circ}, a]_\rho = 0, \quad \forall a, b \in \mathcal{A}, \tag{2.13}$$

or, for all $a, b \in \mathcal{A}$,

$$\big(DJ\pi(b^*)J^{-1} - J\pi(\rho(b^*))J^{-1}D\big)\pi(a) - \pi(\rho(a))\big(DJ\pi(b^*)J^{-1} - J\pi(\rho(b^*))J^{-1}D\big) = 0. \tag{2.14}$$

Remark 2.2. One could consider twisting also the zero-order condition (2.3), and examples from quantum groups (see for instance [9]), for which the zero-order condition is valid only modulo infinitesimals of arbitrary high order, seems to suggest this possibility. However, from the point of view of the present paper this would introduce unnecessary complication: after all the twist seems to be relevant when the commutator with the operator D is involved. A further, *a posteriori* justification comes from the fluctuation of the metric, as explained below after Lemma 2.3.

2.2 Twisted-fluctuation of the metric

Fluctuations of the metric [6] easily adapt to the twisted case. Given a twisted spectral triple $(\mathcal{A}, \mathcal{H}, D; \rho)$, one defines

$$\Omega^1_D := \Big\{\sum\nolimits_j a_j[D, b_j]_\rho\,, \quad a_j, b_j \in \mathcal{A}\Big\} \tag{2.15}$$

the set of twisted 1-forms. Noticing that

$$[D, ab]_\rho = [D, a]_\rho b + \rho(a)[D, b]_\rho, \tag{2.16}$$

one has [7, Prop. 3.4] that $[D, \cdot]_\rho$ is a derivation of $\mathcal{A}$ in Ω^1_D as soon as the latter is viewed as a $\mathcal{A}$-bimodule with twisted action on the left:

$$a \cdot \xi \cdot b = \rho(a)\,\xi\,b \quad \forall a, b \in \mathcal{A},\ \xi \in \Omega^1_D. \tag{2.17}$$

Lemma 2.3. *For any $A_\rho \in \Omega^1_D$ and any $a, b \in \mathcal{A}$, it holds that*

$$[A_\rho, Jb^* J^{-1}]_{\rho^\circ} = 0 \quad \text{and} \quad [JA_\rho J^{-1}, a]_\rho = 0. \tag{2.18}$$

Proof. If $A_\rho = \sum_j a_j [D, c_j]_\rho$, for $a_j, c_j \in \mathcal{A}$, by linearity, one needs to show that

$$a_j [D, c_j]_\rho \, Jb^* J^{-1} - J\rho(b^*)J^{-1} \, a_j [D, c_j]_\rho = 0,$$

The zero-order condition (2.3) yields $J\rho(b^*)J^{-1} a_j = a_j J\rho(b^*)J^{-1}$ and the l.h.s. becomes

$$a_j \left([D, c_j]_\rho \, Jb^* J^{-1} - J\rho(b^*)J^{-1} \, [D, c_j]_\rho \right)$$

whose vanishing follows from the twisted first-order condition (2.11). Next, by expanding and inserting J^2 and J^{-2} (and using $\epsilon^2 = 1$ from the signs (2.1)) one computes,

$$\begin{aligned} 0 &= A_\rho J b^* J^{-1} - J\rho(b^*)J^{-1} A_\rho = J^2 A_\rho J^{-2} J b^* J^{-1} - J\rho(b^*)J^{-1} J^2 A_\rho J^{-2} \\ &= J \left(JA_\rho J^{-1} b^* - \rho(b^*) JA_\rho J^{-1} \right) J^{-1} = J \left([JA_\rho J^{-1}, b^*]_\rho \right) J^{-1} \end{aligned}$$

and renaming $b^* = a$ we get the second equation above. □

Remark 2.4. We see from the above proof that a twisted first-order condition goes well with a zero-order condition which is not twisted. It is also worth pointing out that, as one would expect, a twisted and an untwisted zero-order condition cannot co-exist. By requiring that

$$[a, Jb^* J^{-1}] = 0 = [a, Jb^* J^{-1}]_{\rho^0} \tag{2.19}$$

for any $a, b \in \mathcal{A}$, a direct computation yields $J(b^* - \rho(b^*))J^{-1} = 0$, that is, ρ has to be the identity. On the other hand, as shown by examples below, for finite matrix geometries a twisted and an untwisted first-order condition are not mutually exclusive.

Definition 2.5. *Let $(\mathcal{A}, \mathcal{H}, D; \rho), J$ be a real twisted spectral triple. A twisted-fluctuation of D by $\mathcal{A}$ is any self-adjoint operator of the kind*

$$D_{A_\rho} := D + A_\rho + \epsilon' JA_\rho J^{-1} \tag{2.20}$$

where $A_\rho \in \Omega^1_D$ and the sign ϵ' is given as in (2.1)*.*

Notice that we ask D_{A_ρ} to be self-adjoint, but this is not necessarily the case for A_ρ.

Proposition 2.6. *Any twisted-fluctuation D_{A_ρ} of a real twisted spectral triple $(\mathcal{A}, \mathcal{H}, D; \rho)$ yields a real twisted spectral triple*

$$(\mathcal{A}, \mathcal{H}, D_{A_\rho}; \rho) \tag{2.21}$$

with the same real structure and KO-dimension, and same grading Γ (if any).

Proof. For any $a \in \mathcal{A}$, one has

$$[D_{A_\rho}, a]_\rho = [D, a]_\rho + [A_\rho, a]_\rho + \epsilon' [JA_\rho J^{-1}, a]_\rho. \tag{2.22}$$

The first term in the r.h.s. is bounded since $(\mathcal{A}, \mathcal{H}, D; \rho)$ is a twisted spectral triple. For the same reason, A_ρ is bounded, being the finite sum of products of bounded operators. Thus the second term in the r.h.s. of (2.22) is bounded as well, being the twisted commutator of bounded operators. From Lemma 2.3 the last term in (2.22) vanishes. Hence (2.21) is a twisted spectral triple. It is graded with the same grading Γ as $(\mathcal{A}, \mathcal{H}, D; \rho)$ if the latter is graded: one easily checks that Γ anticommutes with A_ρ and $JA_\rho J^{-1}$, hence with D_{A_ρ}.

To show that the real structure J of $(\mathcal{A}, \mathcal{H}, D; \rho)$ is a real structure for (2.21) with the same KO-dimension, we first check that

$$JD_{A_\rho} = \epsilon' D_{A_\rho} J \tag{2.23}$$

for the same sign ϵ' as in $JD = \epsilon' DJ$. This follows from definition (2.20):

$$\begin{aligned} JD_{A_\rho}J^{-1} &= JDJ^{-1} + JA_\rho J^{-1} + \epsilon' J^2 A_\rho J^{-2}, \\ &= \epsilon' D + JA_\rho J^{-1} + \epsilon' A_\rho, \\ &= \epsilon'(D + \epsilon' JA_\rho J^{-1} + A_\rho) = \epsilon' D_{A_\rho} \end{aligned} \tag{2.24}$$

where we used $\epsilon'^2 = 1$, $J^2 = \epsilon\mathbb{I}$ and $J^{-2} = \epsilon^{-1}\mathbb{I}$.

Finally we must prove the twisted first-order condition

$$[[D_{A_\rho}, a]_\rho, Jb^* J^{-1}]_{\rho 0} = 0 \quad \forall a, b \in \mathcal{A}. \tag{2.25}$$

Writing $b^\circ = Jb^*J^{-1}$, the l.h.s. of the equation above is

$$[[D, a]_\rho, b^\circ]_{\rho^\circ} + [[A_\rho, a]_\rho, b^\circ]_{\rho^\circ} + \epsilon'[[JA_\rho J^{-1}, a]_\rho, b^\circ]_{\rho^\circ}. \tag{2.26}$$

The first term vanishes by the twisted first-order condition for $(\mathcal{A}, \mathcal{H}, D; \rho)$. Next, if $A_\rho \in \Omega^1_D$, it follows that $A'_\rho := A_\rho a - \rho(a)A_\rho$ is in Ω^1_D as well (recall the bimodule structure (2.17)). Then, Lemma 2.3 yields

$$[[A_\rho, a]_\rho, b^\circ]_{\rho^\circ} = [A_\rho a - \rho(a)A_\rho, b^\circ]_{\rho^\circ} = [A'_\rho, b^\circ]_{\rho^\circ} = 0, \tag{2.27}$$

that is, the second term of (2.26) vanishes. For the third term, again from Lemma 2.3 we know that in fact $[JA_\rho J^{-1}, a]_\rho = 0$ and the third term of the r.h.s. of (2.26)is zero as well. □

As in the non-twisted case, a twisted fluctuation of a twisted fluctuation is a twisted fluctuation.

Proposition 2.7. *Let*

$$D_\rho = D + A_\rho + \epsilon' JA_\rho J^{-1} \quad \textit{with} \quad A_\rho \in \Omega^1_D \tag{2.28}$$

be a twisted fluctuation of a real twisted spectral triple $(\mathcal{A}, \mathcal{H}, D; \rho)$*, and*

$$D'_\rho = D_\rho + A'_\rho + \epsilon' JA'_\rho J^{-1} \quad \textit{with} \quad A'_\rho \in \Omega^1_{D_\rho} \tag{2.29}$$

be a fluctuation of $(\mathcal{A}, \mathcal{H}, D_\rho; \rho)$*. Then*

$$D'_\rho = D_\rho + A''_\rho + \epsilon' JA''_\rho J^{-1} \quad \textit{with} \quad A''_\rho = A_\rho + A'_\rho \in \Omega^1_D. \tag{2.30}$$

Proof. In writing $D'_\rho = D + A''_\rho + \epsilon' JA''_\rho J^{-1}$, to $A''_\rho \in \Omega^1_D$. Let $A_\rho = \sum_j a_k[D, b_k]_\rho$, and $A'_\rho = \sum_j a'_k[D_\rho, b'_k]_\rho$ with $a_k, b_k, a'_k, b'_k \in \mathcal{A}$. Omitting the summation indices and symbol, one has

$$\begin{aligned} A'_\rho &= a'[D + A_\rho + \epsilon' JA_\rho J^{-1}, b']_\rho, \\ &= a'[D, b']_\rho + a'[A_\rho, b']_\rho + \epsilon' a'[JA_\rho J^{-1}, b']_\rho. \end{aligned} \tag{2.31}$$

The first term is in Ω^1_D. The second as well from the bimodule structure (2.17). The last term vanishes by Lemma 2.3. Hence A'_ρ is in Ω^1_D, and so is $A''_\rho = A_\rho + A'_\rho$. □

In other terms, in contrast with the fluctuations without first order condition developed in [4], twists do not alter the group structure of the fluctuations of the metric.

3 Minimal twisting for graded spectral triples

In this section, we work out a general procedure to twist a (real) graded spectral triple while keeping the Dirac operator and the Hilbert space unchanged. The twisting uses the grading.

3.1 Minimal twisting

On a manifold there is no room for a twisting; by this we mean the following. Start with the canonical spectral triple of a closed spin manifold $\mathcal{M}$,

$$(C^\infty(\mathcal{M}),\, L^2(\mathcal{M}, S),\, \partial\!\!\!/ := -\mathrm{i}\gamma^\mu \nabla_\mu), \tag{3.1}$$

where $C^\infty(\mathcal{M})$ acts on the Hilbert space $L^2(\mathcal{M}, S)$ of square integrable spinors by multiplication,

$$(\pi_{\mathcal{M}}(f)\psi)(x) := f(x)\psi(x), \tag{3.2}$$

and $\partial\!\!\!/$ is the Dirac operator, with $\nabla_\mu = \partial_\mu + \omega_\mu$ the covariant derivative in the spin bundle. Then any twisted commutator would be of the form

$$[\partial\!\!\!/, f]_\rho = -\mathrm{i}\gamma^\mu(\partial_\mu f) + (f - \rho(f))\,\partial\!\!\!/ \tag{3.3}$$

and it would be bounded for any $f \in C^\infty(\mathcal{M})$ if and only if

$$f - \rho(f) = 0, \tag{3.4}$$

for any function f, which just means that ρ is the identity.

A way to modify (3.1) to allow for non-trivial twistings consists in modifying the Dirac operator, for instance by lifting a conformal transformation like is done in [7]. Having in mind applications to the standard model of elementary particles, we aim however at keeping the Dirac operator and the Hilbert space unchanged, since they encode the fermionic content of the theory that one does not wish to change. Then, the only elements we are allowed to play with are the algebra and/or its representation. Modifying only the latter does not help: if instead of the multiplicative representation (3.2) one let f acts as $(f\psi)(x) = f(x)p(x)\psi(x)$ with p an operator-valued function — for instance p could be the constant projection on a subspace $\mathcal{H}$ of $L^2(\mathcal{M}, S)$, for a reducible representation —, then, the extra term in the twisted commutator as in (3.3) that needs to vanish for any f is $(f - \rho(f))p\,\partial\!\!\!/$, and the conclusion does not change.

More generally, if $(\mathcal{A}, \mathcal{H}, D)$ is a (non-necessarily bounded commutator spectral) triple and ρ is an automorphism of $\mathcal{A}$, simple algebra yields:

$$[D, \pi(a)]_\rho := D\pi(a) - \pi(\rho(a))D = [D, \pi(a)] - \pi(\rho(a) - a)D. \tag{3.5}$$

Thus, if the commutator $[D, \pi(a)]$ is bounded, requiring boundedness of $[D, \pi(a)]_\rho$ is the same as requiring that $\pi(\rho(a) - a)D$ be a bounded operator for all $a \in \mathcal{A}$. If the representation π is faithful the simplest possibility is that $\rho(a) = a$ for all $a \in \mathcal{A}$, that is ρ is the identity.

Therefore, in order to twist the spectral triple (3.1) in a minimal way, that is keeping both $\mathcal{H}$ and D unchanged, one needs to modify the algebra.

Definition 3.1. *Let $\mathcal{B}$ be a unital involutive algebra. A minimal twisting of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is a twisted spectral triple $(\mathcal{A} \otimes \mathcal{B}, \mathcal{H}, D;\, \rho)$ where ρ an automorphism of $\mathcal{A} \otimes \mathcal{B}$. In addition, the representation of $\mathcal{A} \otimes \mathbb{I}_\mathcal{B}$ coincides with the initial representation of $\mathcal{A}$, that is*

$$\pi(a \otimes \mathbb{I}_\mathcal{B}) = \pi_0(a) \quad \forall a \in \mathcal{A} \tag{3.6}$$

where π_0 and π are the representations for $(\mathcal{A}, \mathcal{H}, D)$ and $(\mathcal{A} \otimes \mathcal{B}, \mathcal{H}, D;\, \rho)$.

Let us comment on the condition (3.6). From the representation π of $\mathcal{A} \otimes \mathcal{B}$, one inherits two representations of $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{H}$,

$$\pi_\mathcal{A}(a) := \pi(a \otimes \mathbb{I}_\mathcal{B}), \quad \pi_\mathcal{B}(b) := \pi(\mathbb{I}_\mathcal{A} \otimes b). \tag{3.7}$$

To make meaningful that $(\mathcal{A} \otimes \mathcal{B}, \mathcal{H}, D; \rho)$ is actually a twist of $(\mathcal{A}, \mathcal{H}, D)$ and not simply a twisted spectral triple with the same Hilbert space and Dirac operator, it is natural to impose a relation between $\pi_\mathcal{A}$ and π_0. The most obvious one is (3.6), that is

$$\pi_\mathcal{A} = \pi_0. \tag{3.8}$$

Without any such requirement, Definition 3.1 would not be very helpful: one could call "twist of $(\mathcal{A}, \mathcal{H}, D)$" any twisted spectral triple $(\mathcal{B}, \mathcal{H}, D; \rho)$ with representation $\tilde{\pi}$, by posing $\pi(a \otimes b) := \tilde{\pi}(b)$. In that case, instead of (3.8) one would have

$$\pi_\mathcal{A}(a) = \mathbb{I} \quad \forall a \in \mathcal{A}. \tag{3.9}$$

One could imagine some alternative to Definition 3.1 by imposing a condition less constraining than (3.8) while more significant than (3.9). We shall not explore these possibilities here, also because the requirement (3.8) has the following (easy to establish) consequence that will be of use later on for the Standard Model twisted spectral triple.

Lemma 3.2. *A grading Γ of the twisted spectral triple $(\mathcal{A} \otimes \mathcal{B}, \mathcal{H}, D; \rho)$ is a grading of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$. On the other hand, a grading Γ of $(\mathcal{A}, \mathcal{H}, D)$ is a grading of $(\mathcal{A} \otimes \mathcal{B}, \mathcal{H}, D; \rho)$ if and only if*

$$[\Gamma, \pi(\mathbb{I}_A \otimes b)] = 0 \quad \forall b \in \mathcal{B}. \tag{3.10}$$

Proof. Since the condition that Γ anticommutes with D is not touched, it is only a matter of checking the commuting of Γ with the relevant representation. If Γ is a grading of $(\mathcal{A} \otimes \mathcal{B}, \mathcal{H}, D; \rho)$, by definition it commutes with π, that is

$$[\Gamma, \pi(A)] = 0 \quad \forall A \in \mathcal{A} \otimes \mathcal{B}. \tag{3.11}$$

For $A = a \otimes \mathbb{I}_\mathcal{B}$, this yields

$$[\Gamma, \pi_0(a)] = 0 \quad \forall a \in \mathcal{A}, \tag{3.12}$$

meaning that Γ is also a grading of $(\mathcal{A}, \mathcal{H}, D)$. On the other hand, for a grading Γ of $(\mathcal{A}, \mathcal{H}, D)$ to be a grading of $(\mathcal{A}\otimes\mathcal{B}, \mathcal{H}, D; \rho)$ one needs $[\Gamma, \pi(A)] = 0$ for any $A = \sum_j a_j \otimes b_j \in \mathcal{A}$. Expanding the commutator, one gets

$$\begin{aligned}[\Gamma, \pi(A)] &= \sum\nolimits_j [\Gamma, \pi(a_j \otimes \mathbb{I}_\mathcal{B})\; \pi(\mathbb{I}_\mathcal{A} \otimes b_j)] \\ &= \sum\nolimits_j \Big(\pi_0(a_j)[\Gamma, \pi(\mathbb{I}_A \otimes b_j)] + [\Gamma, \pi_0(a_j)]\pi(\mathbb{I}_\mathcal{A} \otimes b_j)\Big). \end{aligned} \tag{3.13}$$

The second term vanishes being Γ a grading of $(\mathcal{A}, \mathcal{H}, D)$. The vanishing of (3.13) thus implies (3.10) (take $a_j = \mathbb{I}_\mathcal{A}$). Conversely, (3.10) implies the vanishing of (3.13). Hence the result. □

In addition to the previous result, the requirement (3.8) leads to a necessary condition for a twisted spectral triple $(\mathcal{A}\otimes\mathcal{B}, \mathcal{H}, D; \rho)$ to be a minimal twist of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$.

Lemma 3.3. *Let $(\mathcal{A}\otimes\mathcal{B}, \mathcal{H}, D; \rho)$ be a minimal twist of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$. Then*

$$\pi(a \otimes \mathbb{I}_\mathcal{B} - \rho(a \otimes \mathbb{I}_\mathcal{B}))D \tag{3.14}$$

is a bounded for any $a \in \mathcal{A}$.

Proof. Equation (3.5) for $b = \mathbb{I}_\mathcal{B}$ gives

$$[D, \pi(a \otimes \mathbb{I}_\mathcal{B})]_\rho = [D, \pi(a \otimes \mathbb{I}_\mathcal{B})] - \pi(\rho(a \otimes \mathbb{I}_\mathcal{B}) - a \otimes \mathbb{I}_B)\, D. \tag{3.15}$$

The twisted commutator on the l.h.s. is bounded by hypothesis. From (3.8) and (3.7), the commutator on the r.h.s. is $[D, \pi_0(a)]$, which is also bounded by hypothesis. Hence the result. □

Remark 3.4. A similar conclusion for $\mathbb{I}_\mathcal{A} \otimes b$, namely

$$\pi(\mathbb{I}_\mathcal{A} \otimes b - \rho(\mathbb{I}_\mathcal{A} \otimes b))D \,\in\, \mathcal{L}(\mathcal{H}) \quad \forall b \in \mathcal{B}, \tag{3.16}$$

would follow if $[D, \pi(\mathbb{I}_\mathcal{A} \otimes b)]$ were bounded for any b in $\mathcal{B}$. But this is not implied by Definition 3.1, as illustrated by the twisting of graded spectral triples presented in Sect. 3.2: in (3.36) the commutator $[D, \pi(\mathbb{I}_\mathcal{A} \otimes b)]$ is unbounded. This means that the representation $\pi_\mathcal{B}$ in (3.7) cannot serve to build a spectral triple $(\mathcal{B}, \mathcal{H}, D)$ whose twist by $\mathcal{A}$ would be $(\mathcal{A}\otimes\mathcal{B}, \mathcal{H}, D; \rho)$.

We shall say that a minimal twist is trivial whenever $\pi_\mathcal{B}(\mathcal{B}) = \mathbb{C}$ or — assuming $\pi_\mathcal{B}$ is faithful — when $\mathcal{B} = \mathbb{C}$. Then, condition (3.8) puts a constraint on the type of spectral triples that admit interesting minimal twists: the starting representation π_0 of $\mathcal{A}$ on $\mathcal{H}$ should be reducible. The claim then comes from the following proposition.

Proposition 3.5. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple with representation π_0. Assume $\mathcal{A}$ is a (pre−) C^*-algebra. If π_0 is irreducible, then any minimal twist is trivial.*

Proof. Let $(\mathcal{A}\otimes\mathcal{B}, \mathcal{H}, D; \rho)$ with representation π, be a minimal twist of $(\mathcal{A}, \mathcal{H}, D)$. From

$$\pi(a \otimes b) = \pi(a \otimes \mathbb{I}_\mathcal{B})\, \pi(\mathbb{I}_\mathcal{A} \otimes b) = \pi(\mathbb{I}_\mathcal{A} \otimes b)\, \pi(a \otimes \mathbb{I}_\mathcal{B}). \tag{3.17}$$

one has, denoting with $'$ the commutant in $\mathcal{H}$,

$$\pi_\mathcal{B}(\mathcal{B}) \subset \pi_\mathcal{A}(\mathcal{A})'. \tag{3.18}$$

If π_0 is irreducible then $\pi_0(\mathcal{A})' = \mathbb{C}\mathbb{I}$ [1, Prop. II.6.1.8]. Hence the result. □

A minimal twist is not the tensor product of $(\mathcal{A}, \mathcal{H}, D)$ by a spectral triple for $\mathcal{B}$. A way to see this is to notice that the twisted commutator $[D, a \otimes b]_\rho$ is not symmetric in the exchange of its arguments, and so cannot be written as a usual commutator of $a \otimes b$ with some operator D'. Nevertheless one may argue that a minimal twist is somehow a product of spectral triples where the commutator is then twisted. We shall not elaborate much on this here, but only stress that for this to happen, one needs that the representation π of $\mathcal{A} \otimes \mathcal{B}$ on $\mathcal{H}$ factorizes as the tensor product

$$\pi = \tilde{\pi}_\mathcal{A} \otimes \tilde{\pi}_\mathcal{B} \tag{3.19}$$

of two representations $\tilde{\pi}_\mathcal{A}, \tilde{\pi}_\mathcal{B}$ of $\mathcal{A}, \mathcal{B}$ on Hilbert spaces $\mathcal{H}_\mathcal{A}, \mathcal{H}_\mathcal{B}$ such that $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B} = \mathcal{H}$. On the other hand, from (3.17) the representation π is required to be the product

$$\pi = \pi_\mathcal{A} \, \pi_\mathcal{B} = \pi_\mathcal{B} \, \pi_A \tag{3.20}$$

of two commuting representations of $\mathcal{A}, \mathcal{B}$ on $\mathcal{H}$. There is no reason for (3.19) and (3.20) to be both true at the same time.

An example where one gets from a representation π_0 of $\mathcal{A}$ on $\mathcal{H}$ a representation π of $\mathcal{A} \otimes \mathcal{B}$ on the same $\mathcal{H}$ such that (3.19) and (3.6) both hold, is when

$$\pi_0 = \tilde{\pi}_\mathcal{A} \otimes \mathbb{I}_N \tag{3.21}$$

is the direct sum of N copies of an irreducible representation $\tilde{\pi}_\mathcal{A}$ of $\mathcal{A}$ on an Hilbert space $\tilde{\mathcal{H}}$, and $\mathcal{B} = \mathbb{M}_N(\mathbb{C})$. Indeed, now $\mathcal{H}$ decomposes as $\widetilde{\mathcal{H}} \otimes \mathbb{C}^N$ so that, with $\tilde{\pi}_\mathcal{B}$ the irreducible representation of $\mathbb{M}_N(\mathbb{C})$ on $\mathbb{C}^N$, the representation

$$\pi(a \otimes b) := \tilde{\pi}_\mathcal{A}(a) \otimes \tilde{\pi}_\mathcal{B}(b) \tag{3.22}$$

of $\mathcal{A} \otimes \mathcal{B}$ on $\mathcal{H}$ factorizes as required in (3.19). Equation (3.6) now follows from (3.7):

$$\pi_\mathcal{A}(a) = \pi(a \otimes \mathbb{I}_\mathcal{B}) = \tilde{\pi}_\mathcal{A}(a) \otimes \mathbb{I}_N = \pi_0(a). \tag{3.23}$$

Furthermore, since $\pi_\mathcal{B}(b) = \pi(\mathbb{I}_\mathcal{A} \otimes b) = \mathbb{I}_{\tilde{\mathcal{H}}} \otimes \tilde{\pi}_\mathcal{B}(b)$, one also checks that (3.20) holds.

3.2 Twist by grading

It is not difficult to minimally twist a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ in the sense of Definition 3.1 as soon as the latter is graded. One simply splits the Hilbert space according to the eigenspaces of Γ,

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-, \tag{3.24}$$

and consider the representation of $\mathcal{A} \otimes \mathbb{C}^2 \ni (a, a')$ given by

$$\pi(a, a') := p_+ \pi_0(a) + p_- \pi_0(a') = \begin{pmatrix} \pi_+(a) & 0 \\ 0 & \pi_-(a') \end{pmatrix} \tag{3.25}$$

where

$$p_+ := \tfrac{1}{2}(\mathbb{I} + \Gamma), \quad p_- := \tfrac{1}{2}(\mathbb{I} - \Gamma) \tag{3.26}$$

are projection on the eigenspaces of Γ, while

$$\pi_+(a) := p_+ \pi_0(a)_{|\mathcal{H}_+}, \qquad \pi_-(a) := p_- \pi_0(a)_{|\mathcal{H}_-} \tag{3.27}$$

are the restrictions on $\mathcal{H}_\pm$ of the representation of $\mathcal{A}$ on $\mathcal{H}$.

Proposition 3.6. *Let $(\mathcal{A}, \mathcal{H}, D), \Gamma$ be a graded spectral triple. Then*

$$(\mathcal{A} \otimes \mathbb{C}^2, \mathcal{H}, D\,;\, \rho) \tag{3.28}$$

with representation (3.25) *and automorphism*

$$\rho(a, a') := (a', a), \quad \forall (a, a') \in \mathcal{A} \otimes \mathbb{C}^2 \tag{3.29}$$

is a minimal twist of $(\mathcal{A}, \mathcal{H}, D)$ with grading Γ.

Proof. In agreement with (3.6), one retrieves the initial representation of $\mathcal{A}$ on $\mathcal{H}$ as

$$\pi(a, a) = p_+\pi_0(a) + p_-\pi_0(a) = \begin{pmatrix} \pi_+(a) & 0 \\ 0 & \pi_-(a) \end{pmatrix}. \tag{3.30}$$

Since D anticommutes with Γ, on $\mathcal{H}_+ \oplus \mathcal{H}_-$ it is of the form

$$D = \begin{pmatrix} 0 & \mathcal{D} \\ \mathcal{D}^\dagger & 0 \end{pmatrix} \tag{3.31}$$

where $\mathcal{D}$ is the restriction of D to $\mathcal{H}_-$, with image in $\mathcal{H}_+$. Thus by (3.25)

$$[D, \pi(a, a')]_\rho = \begin{pmatrix} 0 & \mathcal{D}\pi_-(a') - \pi_+(a')\mathcal{D} \\ \mathcal{D}^\dagger\pi_+(a) - \pi_-(a)\mathcal{D}^\dagger & 0 \end{pmatrix}. \tag{3.32}$$

The lower-left term in (3.32) is the restriction to $\mathcal{H}_+$ of the usual commutator

$$[D, \pi(a, a)] = \begin{pmatrix} 0 & \mathcal{D}\pi_-(a) - \pi_+(a)\mathcal{D} \\ \mathcal{D}^\dagger\pi_+(a) - \pi_-(a)\mathcal{D}^\dagger & 0 \end{pmatrix}, \tag{3.33}$$

which is bounded since $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple. Similarly, the upper-right term in (3.32) is bounded, being the restriction of $[D, (a', a')]$ to $\mathcal{H}_-$. Hence (3.32) is bounded and thus (3.28) is a twisted spectral triple.

Since $[\Gamma, \pi(a, a')] = 0$, and $\{\Gamma, D\} = 0$ by hypothesis, the spectral triple (3.28) is Γ-graded. □

It is easy to see that the flip automorphism (3.29) is implemented on the Hilbert space by exchanging the components $\psi_\pm \in \mathcal{H}_\pm$, that is, for all $\alpha \in \mathcal{A} \otimes \mathbb{C}^2$,

$$\pi(\rho(\alpha)) = U_\rho \pi(\rho(\alpha)) U_\rho^*, \quad \text{with} \quad U_\rho \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}. \tag{3.34}$$

To stress the role of ρ, compare the expression of $[D, (a, a')]_\rho$ in (3.32) with the usual commutator

$$[D, \pi(a, a')] = \begin{pmatrix} 0 & \mathcal{D}\pi_-(a') - \pi_+(a)\mathcal{D} \\ \mathcal{D}^\dagger\pi_+(a) - \pi_-(a')\mathcal{D}^\dagger & 0 \end{pmatrix}. \tag{3.35}$$

Then, while the boundedness of the twisted commutator $[D, \pi(a, a')]_\rho$ follows from the boundedness of $[D, \pi(a, a)]$ and $[D, \pi(a', a')]$, there is no reason for the commutator $[D, \pi(a, a')]$ to be bounded.

This is also true for the commutator of D with the representation $\pi_{\mathcal{B}}$ in (3.7), as pointed out after Lemma 3.3. For $b = (z_1, z_2) \in \mathbb{C}^2$, one has

$$[D, \pi_{\mathcal{B}}(b)] = [D, \pi(\mathbb{I}_{\mathcal{A}} \otimes b)] = \begin{pmatrix} z_1\mathbb{I} & 0 \\ 0 & z_2\mathbb{I} \end{pmatrix} \begin{pmatrix} 0 & \mathcal{D} \\ \mathcal{D}^\dagger & 0 \end{pmatrix} = \begin{pmatrix} 0 & (z_1 - z_2)\mathcal{D} \\ (z_2 - z_1)\mathcal{D}^\dagger & 0 \end{pmatrix}, \quad (3.36)$$

which is bounded if and only if $z_2 = z_1$.

The twist-by-grading of Proposition 3.6 passes to the real structure.

Proposition 3.7. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a graded real spectral triple, with grading Γ and real structure J. Then the twisted spectral $(\mathcal{A} \otimes \mathbb{C}^2, \mathcal{H}, D;\, \rho)$ of Proposition 3.6 is a graded real twisted spectral triple with the same real structure J and the same KO-dimension.*

Proof. The operators Γ, D and J are unchanged by the twisting, so the KO-dimension is not modified by passing from $(\mathcal{A}, \mathcal{H}, D)$ to $(\mathcal{A} \otimes \mathbb{C}^2, \mathcal{H}, D;\, \rho)$. One simply needs to check the zero-order and the twisted first-order condition (2.11), being cautious that J commutes or anticommutes with Γ, depending on the KO-dimension. Since the automorphism ρ in (3.29) is just the flip, one has $\rho^2 = \text{id}$ and ρ coincides with its inverse.

Assume J commutes with Γ. On $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ one has

$$J = \begin{pmatrix} J_+ & 0 \\ 0 & J_- \end{pmatrix}. \quad (3.37)$$

For any (a, α), (b, β) in $\mathcal{A} \otimes \mathbb{C}^2$ we write $A = \pi(a, \alpha)$, $B = \pi(b, \beta)$, that is

$$A = \begin{pmatrix} a_+ & 0 \\ 0 & \alpha_- \end{pmatrix}, \quad \rho(A) = \begin{pmatrix} \alpha_+ & 0 \\ 0 & a_- \end{pmatrix}, \quad JB^*J^{-1} = \begin{pmatrix} b_+^\circ & 0 \\ 0 & \beta_-^\circ \end{pmatrix}, \quad J\rho(B^*)J^{-1} = \begin{pmatrix} \beta_+^\circ & 0 \\ 0 & b_-^\circ \end{pmatrix}$$

where

$$a_\pm := \pi_\pm(a), \quad b_\pm^\circ := J_\pm \pi_\pm(b^*) J_\pm^{-1} \quad (3.38)$$

and similarly for α and β. The zero-order condition amounts to

$$[a_+, b_+^\circ] = [\alpha_-, \beta_-^\circ] = 0, \quad (3.39)$$

which follows from the zero-order condition for $(\mathcal{A}, \mathcal{H}, D)$, namely

$$[(a, a), (b^\circ, b^\circ)] = [(\alpha, \alpha), (\beta^\circ, \beta^\circ)] = 0. \quad (3.40)$$

For the twisted first-order condition, one uses (3.32) to get

$$\begin{aligned} [D, A]_\rho\, JB^*J^{-1} &= \begin{pmatrix} 0 & (\mathcal{D}\alpha_- - \alpha_+\mathcal{D})\beta_-^\circ \\ (\mathcal{D}^\dagger a_+ - a_-\mathcal{D}^\dagger)b_+^\circ & 0 \end{pmatrix} \\ J\rho(B^*)J^{-1}[D, A]_\rho &= \begin{pmatrix} 0 & \beta_+^\circ(\mathcal{D}\alpha_- - \alpha_+\mathcal{D}) \\ b_-^\circ(\mathcal{D}^\dagger a_+ - a_-\mathcal{D}^\dagger) & 0 \end{pmatrix}. \end{aligned} \quad (3.41)$$

The lower-left component of

$$\Big[[D, A]_\rho, JB^*J^{-1}\Big]_{\rho^\circ} = [D, A]_\rho\, JB^*J^{-1} - J\rho(B^*)J^{-1}[D, A]_\rho \quad (3.42)$$

using (3.33) is the lower-left component of:

$$\left[[D,(a,a)],(b^\circ,b^\circ)\right] = \begin{pmatrix} 0 & (\mathcal{D}a_- - a_+\mathcal{D})b_-^\circ - b_+^\circ(\mathcal{D}a_- - a_+\mathcal{D}) \\ (\mathcal{D}^\dagger a_+ - a_-\mathcal{D}^\dagger)b_+^\circ - b_-^\circ(\mathcal{D}^\dagger a_+ - a_-\mathcal{D}^\dagger) & 0 \end{pmatrix}$$

which vanishes since $(\mathcal{A},\mathcal{H},D)$ satisfies the first-order condition. Similarly the upper-right component of (3.42) vanishes, being the upper right component of $\left[[D,(\alpha,\alpha)],(\beta^\circ,\beta^\circ)\right]$. Hence the twisted first-order condition is satisfied.

When J anticommutes with Γ, one has

$$J = \begin{pmatrix} 0 & \mathcal{J} \\ \epsilon\,\mathcal{J}^{-1} & 0 \end{pmatrix}, \tag{3.43}$$

so that, writing now $b_+^\circ := \mathcal{J}^{-1}\pi_+(b^*)\mathcal{J}$, and $\beta_-^\circ := \mathcal{J}\pi_-(\beta^*)\mathcal{J}^{-1}$,

$$JB^*J^{-1} = \begin{pmatrix} \beta_-^\circ & 0 \\ 0 & b_+^\circ \end{pmatrix}, \qquad J\rho(B^*)J^{-1} = \begin{pmatrix} b_-^\circ & 0 \\ 0 & \beta_+^\circ \end{pmatrix}. \tag{3.44}$$

The proof is then similar to the previous case. □

Propositions 3.6 and 3.7 give a way to minimally twist a (real) graded spectral triple using its grading. This needs not be the only possibility, although this happens to be the case for an even dimensional manifold, as showed in Proposition 4.2. In particular, while it is important for the construction that the grading Γ commutes with the algebra (otherwise there would be no guarantee that the restrictions $p_\pm\mathcal{A}$ of $\mathcal{A}$ to $\mathcal{H}_\pm$ are algebra representations, unless the $p_\pm$ themselves are elements of the algebra), the condition that Γ anticommutes with the Dirac operator may be slightly relaxed, as illustrated later on in Sect. 4.2. More precisely, given a minimal twist $(\mathcal{A}\otimes\mathbb{C}^2,\mathcal{H},D;\rho)$ of a spectral triple $T := (\mathcal{A},\mathcal{H},D)$, the extended representation π can alway be written as in (3.25) with a suitable unique grading of the Hilbert space $\mathcal{H}$. Indeed, by defining

$$\widetilde{\Gamma} := \pi(\mathbb{I}_\mathcal{A}\otimes(1,-1)) = \pi(\mathbb{I}_\mathcal{A},-\mathbb{I}_\mathcal{A}), \tag{3.45}$$

a direct computation leads to

$$\pi(a,a') = \tfrac{1}{2}(\mathbb{I}+\widetilde{\Gamma})\,\pi_0(a) + \tfrac{1}{2}(\mathbb{I}-\widetilde{\Gamma})\,\pi_0(a') \qquad \forall (a,a')\in\mathcal{A}\otimes\mathbb{C}^2. \tag{3.46}$$

Clearly, the operator $\widetilde{\Gamma}$ defined in (3.45) is a grading of $\mathcal{H}$, that is $\widetilde{\Gamma}^* = \widetilde{\Gamma}$ and $\widetilde{\Gamma}^2 = \mathbb{I}$. It trivially commutes with the representation $\pi_\mathcal{B}$ in (3.7), in fact the latter can be written as

$$\pi_B(z_1,z_2) = \tfrac{1}{2}(z_1+z_2)\mathbb{I} + \tfrac{1}{2}(z_1-z_2)\widetilde{\Gamma} \qquad \forall (z_1,z_2)\in\mathbb{C}^2. \tag{3.47}$$

It also commutes with the representation π_0, since

$$[\widetilde{\Gamma},\pi_0(a)] = [\pi(\mathbb{I}_\mathcal{A}\otimes(1,-1)),\pi(a\otimes\mathbb{I}_\mathcal{B})] = 0. \tag{3.48}$$

Thus from (3.20) it also commutes with $\pi = \pi_0\pi_\mathcal{B}$. However, $\widetilde{\Gamma}$ needs not be a grading of the spectral triple, for $\widetilde{\Gamma}$ may fail to anticommute with D. If it does, the twisted spectral triple $(\mathcal{A}\otimes\mathbb{C}^2,\mathcal{H},D;\rho)$ is the "twist by grading" of the starting spectral triple $(\mathcal{A},\mathcal{H},D)$, obtained by applying Proposition 3.6 with $\Gamma = \widetilde{\Gamma}$. Otherwise the minimal twist does not come from the construction of Proposition 3.6. We come back to this point for the case of almost commutative geometry later on.

Remark 3.8. There is in fact a constraint on the anticommutator of $\widetilde{\Gamma}$ with D coming from the boundedness of $[D,a]_\rho$. From (3.29) one notices that $\rho(\mathbb{I}_{\mathcal{A}}\otimes(1,-1)) = -\mathbb{I}_{\mathcal{A}}\otimes(1,-1)$, so that

$$\begin{aligned}[D,\pi(\mathbb{I}_{\mathcal{A}},0)]_\rho &= \tfrac{1}{2}[D,\,\mathbb{I}+\widetilde{\Gamma}]_\rho = \tfrac{1}{2}\left(D\,\pi\left(\mathbb{I}_{\mathcal{A}}\otimes(1,-1)\right) + \pi\left(\mathbb{I}_{\mathcal{A}}\otimes(1,-1)\right)\,D\right)\\ &= \tfrac{1}{2}\left(D\,\widetilde{\Gamma}+\widetilde{\Gamma}\,D\right).\end{aligned} \tag{3.49}$$

Hence,, in any minimal twist $(\mathcal{A}\otimes\mathbb{C}^2,\mathcal{H},D;\rho)$ with ρ as in (3.29), the anticommutator $\left\{D,\widetilde{\Gamma}\right\}$ is a bounded operator.

4 Unicity of the twist

We show in Sect. 4.1 that twisting by grading as described in Sect. 3.2 is the only way to minimally twist an even dimensional spin manifold. With some conditions of irreducibility, the same is true for almost commutative geometries as soon as one uses the real structure, as shown in Sect. 4.2.

4.1 Even dimensional manifold

Let $\mathcal{M}$ be a closed spin manifold of even dimension $n=2m$, $m\geq 1$. The Euclidean Dirac matrices $\gamma_{[2m]}$ in the chiral basis are the $p := 2^m$ dimensional square matrices defined recursively by

$$\gamma^1_{[2]} = \sigma_1 \quad \gamma^2_{[2]} = \sigma_2 \quad \gamma_{(2)} = -\mathrm{i}\gamma^1_{[2]}\,\gamma^2_{[2]} = \sigma_3 \tag{4.1}$$

where

$$\sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2 = \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix},\quad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{4.2}$$

are the Pauli matrices, and

$$\begin{aligned}\gamma^k_{[2m+2]} &= \begin{pmatrix}0_{2m} & \gamma^k_{[2m]}\\ \gamma^k_{[2m]} & 0_{2m}\end{pmatrix} \qquad \text{for } k=1,...,2m\\ \gamma^{2m+1}_{[2m+2]} &= \begin{pmatrix}0_{2m} & \gamma_{(2m)}\\ \gamma_{(2m)} & 0_{2m}\end{pmatrix},\quad \gamma^{2m+2}_{[2m+2]} = \begin{pmatrix}0_{2m} & -\mathrm{i}\,\mathbb{I}_{2m}\\ \mathrm{i}\,\mathbb{I}_{2m} & 0_{2m}\end{pmatrix}\end{aligned} \tag{4.3}$$

where $\gamma_{(2m)}$ is the grading operator

$$\gamma_{(2m)} := (-\mathrm{i})^m\,\gamma^1_{[2m]}\,\gamma^2_{[2m]}\cdots\gamma^{2m}_{[2m]} = \begin{pmatrix}\mathbb{I}_{2m} & 0_{2m}\\ 0_{2m} & -\mathbb{I}_{2m}\end{pmatrix}. \tag{4.4}$$

Lemma 4.1. *Let $A,B\in\mathbb{M}_{2^{m+1}}(\mathbb{C})$, be such that*

$$\gamma^\mu_{[2m+2]}\,A = B\,\gamma^\mu_{[2m+2]} \quad \forall\mu = 1,...,m+2. \tag{4.5}$$

Then, there exist $\lambda,\lambda'\in\mathbb{C}$ such that

$$A = \begin{pmatrix}\lambda\,\mathbb{I}_{2^m} & 0_{2^m}\\ 0_{2^m} & \lambda'\,\mathbb{I}_{2^m}\end{pmatrix},\qquad B = \begin{pmatrix}\lambda'\,\mathbb{I}_{2^m} & 0_{2^m}\\ 0_{2^m} & \lambda\,\mathbb{I}_{2^m}\end{pmatrix}. \tag{4.6}$$

Proof. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad B = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, \tag{4.7}$$

be non zero matrices whose entries are 2^m-square matrices. For $\mu = 2m+2$ requiring (4.5) implies

$$b' = -c, \quad c' = -b, \quad \text{and} \quad a' = d, \quad d' = a. \tag{4.8}$$

Then, for $k = 1, ..., 2m+1$, one obtains

$$\gamma^k_{[2m]} a = a\gamma^k_{[2m]}, \quad \gamma^k_{[2m]} d = d\gamma^k_{[2m]} \qquad \text{and} \qquad \gamma^k_{[2m]} c = -c\gamma^k_{[2m]}, \quad \gamma^k_{[2m]} b = -b\gamma^k_{[2m]} \tag{4.9}$$

and similar relations with $\gamma_{(2m)}$. Thus b and c should anticommute with all the $\gamma^k_{[2m]}$ as well as with their product $\gamma_{(2m)}$, which is not possible, unless $b = c = 0$. Meanwhile a and d should commute with all the $\gamma^k_{[2m]}$, which is possible only if a and d are multiple of the identity. Hence the result. □

The twist by grading of Sect. 3.2 turns out to be the only way to minimally twist the spectral triple of a manifold (3.1) by a finite dimensional algebra, provided the latter acts faithfully.

Proposition 4.2. *Let $\mathcal{M}$ be a closed manifold of dimension $2m$; $\mathcal{B}$ be a finite dimensional C^*-algebra and ρ a non-trivial automorphism of $C^\infty(\mathcal{M}) \otimes \mathcal{B}$ such that*

$$(C^\infty(\mathcal{M}) \otimes \mathcal{B}, L^2(\mathcal{M}, S), \not{\partial};\ \rho) \tag{4.10}$$

is a minimal twist of the canonical triple $(C^\infty(\mathcal{M}), L^2(\mathcal{M}, S), \not{\partial})$, with $\pi_{\mathcal{B}}$ as defined in (3.7) taken to be faithful. Then $\mathcal{B} = \mathbb{C}^2$ and

$$\rho(f, g) = (g, f) \quad \forall (f, g) \in C^\infty(\mathcal{M}) \otimes \mathbb{C}^2. \tag{4.11}$$

Moreover the representation π of $C^\infty(\mathcal{M}) \otimes \mathcal{B}$ on $L^2(\mathcal{M}, S)$ is given by (3.25)-(3.27) with $\Gamma = \gamma_{(2m)}$ the grading of the canonical spectral triple of $\mathcal{M}$ in (3.1).

Proof. Let $\mathbb{I}_{\mathcal{M}}$ denote the identity of $C^\infty(\mathcal{M})$. Any element $\mathbb{I}_{\mathcal{M}} \otimes b$ acts on $\mathcal{H}$ as a constant matrix

$$B := \pi(\mathbb{I}_{\mathcal{M}} \otimes b) = \pi_{\mathcal{B}}(b) \tag{4.12}$$

of dimension at most 2^m. Thus $\pi_{\mathcal{B}}(\mathcal{B})$ is a subalgebra of $\mathbb{M}_{2^m}(\mathbb{C})$, and since $\pi_{\mathcal{B}}$ is faithful the same is true for the algebra $\mathcal{B}$. For any $b \in \mathcal{B}$, one finds for the twisted commutator

$$[\not{\partial}, B]_\rho = -\mathrm{i}\gamma^\mu[\omega_\mu, B] - \mathrm{i}[\gamma^\mu, B]_\rho \nabla_\mu, \tag{4.13}$$

using $\not{\partial} = -\mathrm{i}\gamma^\mu\nabla_\mu = -\mathrm{i}\gamma^\mu(\partial_\mu + \omega_\mu)$. Clearly, this is bounded if and only if

$$\gamma^\mu B - \rho(B)\gamma^\mu = 0, \quad \forall \mu = 1, ..., 2m. \tag{4.14}$$

Then by Lemma 4.1, the algebra $\mathcal{B}$ is isomorphic either to the algebra of block-diagonal matrices

$$\mathrm{diag}\,(\lambda\,\mathbb{I}_{2^m}, \lambda'\,\mathbb{I}_{2^m}), \tag{4.15}$$

with ρ the permutation of the two-blocks, or to a subalgebra of it. The first case yields $\mathcal{B} \simeq \mathbb{C}^2$ resulting into an automorphism of $C^\infty(\mathcal{M}) \otimes \mathcal{B}$ ρ as given in (4.11). The second case means $\mathcal{B} = \mathbb{C}$ with ρ the trivial identity automorphism, excluded by hypothesis.

To establish the last point of the proposition, it suffices to show that the operator $\widetilde{\Gamma}$ defined in (3.48) coincides with the grading $\gamma_{(2m)}$, possibly up to an irrelevant global sign. From (4.15) and (4.12) one indeed gets

$$\pi_{\mathcal{B}}(\lambda_1, \lambda_2) = \pm \mathrm{diag}\,(\lambda_1, \lambda_2) \otimes \mathbb{I}_{2^m} \qquad \forall (\lambda_1, \lambda_2) \in \mathbb{C}^2, \tag{4.16}$$

hence $\widetilde{\Gamma} = \pm\gamma_{(2m)}$ as stated. □

4.2 Almost commutative geometries

For $\mathcal{M}$ a closed spin manifold of even dimension $2m$, the product of the canonical spectral triple (3.1), with grading $\gamma_{(2m)}$ and real structure $\mathcal{J}$, by a finite dimensional unital spectral triple $(\mathcal{A}_F, \mathcal{H}_F, D_F)$ is the spectral triple

$$\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathcal{A}_F, \quad \mathcal{H} = L^2(\mathcal{M}, S) \otimes \mathcal{H}_F, \quad D = \partial\!\!\!/ \otimes \mathbb{I}_F + \gamma_{(2m)} \otimes D_F \tag{4.17}$$

where $\mathbb{I}_F$ is the identity on $\mathcal{H}_F$, and the representation

$$\pi_0 = \pi_{\mathcal{M}} \otimes \pi_F \tag{4.18}$$

of $\mathcal{A}$ on $\mathcal{H}$ is the tensor product of the multiplicative representations (3.2) of $C^\infty(\mathcal{M})$ on spinors, by the representation π_F of $\mathcal{A}_F$ on $\mathcal{H}_F$. In addition, when $(\mathcal{A}_F, \mathcal{H}_F, D_F)$ has grading Γ_F and real structure J_F, then the product $(\mathcal{A}, \mathcal{H}, D)$ is graded and real with

$$\Gamma = \gamma_{(2m)} \otimes \Gamma_F, \quad J = \mathcal{J} \otimes J_F. \tag{4.19}$$

As for the canonical spectral triples for manifolds, there is no room for twisting the product spectral triple (4.17) while keeping $\mathcal{A}$, $\mathcal{H}$ and D unchanged. As in (3.5), with $\rho \in \mathrm{Aut}(\mathcal{A})$ one has

$$[D, \pi_0(a)]_\rho = [D, \pi_0(a)] - \pi_0(\rho(a) - a)D, \tag{4.20}$$

with the commutator $[D, \pi(a)]$ which is bounded by hypothesis. On the other hand, the image of the faithful representation $\pi_0 = \pi_{\mathcal{M}} \otimes \pi_F$ is made of finite matrices of multiplicative operators (elements of the algebra $C^\infty(\mathcal{M})$). Thus the only way to get a bounded operator $[D, \pi_0(a)]_\rho$ is that $\rho(a) = a$ for all $a \in \mathcal{A}$, that is ρ is forced to be the identity.

From now one we assume that $\mathcal{A}_F$ is a C^*-algebras, which is the case in all the models of almost-commutative geometries applied to physics. The possibilities to minimally twist an almost commutative geometry are a bit larger than the ones for manifolds, due to possible degeneracies of the representation of $\mathcal{A}_F$ on $\mathcal{H}_F$. Before proving this, let us begin with a lemma showing that the (minimal) twisting automorphism ρ actually acts only on the extra algebra $\mathcal{B}$.

Lemma 4.3. *Let $(\mathcal{A} \otimes \mathcal{B}, \mathcal{H}, D\,;\, \rho)$ be a non-trivial minimal twist of the spectral triple* (4.17) *by a finite dimensional C^*-algebra $\mathcal{B}$. Then there exists $\rho' \in \mathrm{Aut}(\mathcal{A}_F \otimes \mathcal{B})$ such that*

$$\rho(f \otimes T) = f \otimes \rho'(T) \quad \forall f \in C^\infty(\mathcal{M}),\, T \in \mathcal{A}_F \otimes \mathcal{B}. \tag{4.21}$$

Proof. By Lemma 3.3, one has that

$$\pi(a \otimes \mathbb{I}_{\mathcal{B}} - \rho(a \otimes \mathbb{I}_{\mathcal{B}}))\,(\partial\!\!\!/ \otimes \mathbb{I}_F + \gamma^5 \otimes D_F) \\ = \pi(a \otimes \mathbb{I}_{\mathcal{B}} - \rho(a \otimes \mathbb{I}_{\mathcal{B}}))(\partial\!\!\!/ \otimes \mathbb{I}_F) + \pi(a \otimes \mathbb{I}_{\mathcal{B}} - \rho(a \otimes \mathbb{I}_{\mathcal{B}}))(\gamma^5 \otimes D_F)$$

is bounded for any a. The second term in the r.h.s. is always bounded. By the same argument as after (4.20), the first term is bounded if and only if $\pi(a \otimes \mathbb{I}_{\mathcal{B}} - \rho(a \otimes \mathbb{I}_{\mathcal{B}})) = 0$, that is

$$\rho(a \otimes \mathbb{I}_{\mathcal{B}}) = a \otimes \mathbb{I}_{\mathcal{B}} \quad \forall a \in \mathcal{A}. \tag{4.22}$$

Thus in particular, ρ preserve the center $C^\infty(\mathcal{M})$ of $\mathcal{A}\otimes\mathcal{B}$ and by [13] is a function from $\mathcal{M}$ to inner automorphisms of the finite part algebra $\mathcal{A}_F \otimes \mathcal{B}$. We next show that for our case this function has to be a constant one. Let $k := \dim \mathcal{H}_F$. Now, for any $T \in \mathcal{A}_F \otimes \mathcal{B}$, the element $\mathbb{I}_{\mathcal{M}} \otimes T$ acts on $\mathcal{H}$ as a constant $2^m k \times 2^m k$ matrix

$$M := \pi\left(\mathbb{I}_{\mathcal{M}} \otimes T\right) = \left\{M_{jl}\right\}_{j,l=1,\dots,k} \tag{4.23}$$

where each block M_{jl} is in $\mathbb{M}_{2^m}(\mathbb{C})$. On the other hand, if we write

$$\rho(\mathbb{I}_A \otimes T) = \sum\nolimits_j f^j \otimes T_j \tag{4.24}$$

for some $f^j \in C^\infty(\mathcal{M})$ and $T_j \in \mathcal{A}_F \otimes \mathcal{B}$, its representation $\pi(\rho(\mathbb{I}_A \otimes T))$ is a matrix $\widetilde{M} := \left\{\widetilde{M}_{jl}\right\}$ where each block $\widetilde{M}_{jl}$ is a priori a function on $\mathcal{M}$, that is an element in $C^\infty(\mathcal{M}, M_{2^m}(\mathbb{C}))$. The operator $\partial\!\!\!/ \otimes \mathbb{I}_F$ acts as $\text{diag}\,(\partial\!\!\!/, \partial\!\!\!/, ..., \partial\!\!\!/)$, so that

$$(\partial\!\!\!/ \otimes \mathbb{I}_F)\,\pi(\mathbb{I}_{\mathcal{M}} \otimes T) - \pi(\rho(\mathbb{I}_{\mathcal{M}} \otimes T)(\partial\!\!\!/ \otimes \mathbb{I}_F) = (\gamma^\mu M - \widetilde{M}\gamma^\mu)\partial_\mu \tag{4.25}$$

is bounded in and only if

$$\gamma^\mu M = \widetilde{M}\gamma^\mu. \tag{4.26}$$

This means that all the $\widetilde{M}_{jl}$'s are constant or, given the nature of the representation π, that all f^j's in (4.24) are constant Therefore (4.24) reads

$$\rho(\mathbb{I}_{\mathcal{M}} \otimes T) = \mathbb{I}_{\mathcal{M}} \otimes \rho'(T) \tag{4.27}$$

where the automorphism $\rho' \in \text{Aut}\,(\mathcal{A}_F \otimes \mathcal{B})$ is defined by

$$\rho'(T) := f^j T_j. \tag{4.28}$$

Using (4.22) it is straightforward that $\rho(f \otimes T) = \rho(f \otimes \mathbb{I}_{\mathcal{A}_F\otimes\mathcal{B}})\,\rho(\mathbb{I}_{\mathcal{M}} \otimes T) = f \otimes \rho'(T)$, for all $f \in C^\infty(\mathcal{M})$, $T \in \mathcal{A}_F \otimes \mathcal{B}$, which proves the statement of the lemma. ☐

Since $\mathcal{A}_F$ is a C^*-algebra, it is a sum of matrix algebras,

$$\mathcal{A}_F = \bigoplus_{i=1}^{q} \mathbb{M}_{n_j}(\mathbb{C}) \quad n_j \in \mathbb{N}^*. \tag{4.29}$$

The representation π_F is faithful, and we assume that each of the $\mathbb{M}_{n_j}(\mathbb{C})$ acts faithfully on $\mathcal{H}_F$ as the direct sum of d_j copies of the fundamental representation. The dimension k of $\mathcal{H}_F$ is $k = \sum_j n_j d_j$ and we denote

$$d := \min\left\{d_1, d_2, ..., d_q\right\}. \tag{4.30}$$

Proposition 4.4. *Let* $(\mathcal{A}\otimes\mathcal{B}, \mathcal{H},\, D\,;\, \rho)$ *be a non-trivial minimal twist of the almost commutative spectral triple* (4.17) *with* $\mathcal{A}_F$ *as above, and* $\mathcal{B}$ *a finite dimensional* C^**-algebra such that* $\pi_{\mathcal{B}}$ *in* (3.7) *faithful. Then*

$$\mathcal{B} = \mathbb{C}^l \otimes \mathbb{C}^2 \qquad \textit{for some} \quad l \in [1, d], \tag{4.31}$$

with d *defined in* (4.30) *and, for all* $(a_1, ..., b_1, ...) \in \mathcal{A}\otimes\mathbb{C}^2\otimes\mathbb{C}^l$ *the automorphism is*

$$\rho(a_1, a_2, ..., a_l, b_1, b_2, ..., b_l) = (b_1, b_2, ..., b_l, a_1, a_2, ..., a_l) \tag{4.32}$$

Proof. The notations are those of Lemma 4.3. By Lemma 4.1, Equation (4.26) implies that the matrices M_{jl} and $\widetilde{M}_{jl}$ are of the form

$$M_{jl} = \begin{pmatrix} \alpha_{jl} & 0 \\ 0 & \beta_{jl} \end{pmatrix} \otimes \mathbb{I}_{2^{m-1}}, \quad \widetilde{M}_{jl} = \begin{pmatrix} \beta_{jl} & 0 \\ 0 & \alpha_{jl} \end{pmatrix} \otimes \mathbb{I}_{2^{m-1}}\,, \qquad \text{with} \quad \alpha_{jl}, \beta_{jl} \in \mathbb{C}. \tag{4.33}$$

Let

$$\mathfrak{A} \simeq \mathbb{M}_k(\mathbb{C}) \otimes \mathbb{C}^2 \tag{4.34}$$

be the C^*-algebra generated by all the matrices as in (4.23), with blocks satisfying (4.33). Since π is faithful, $\mathcal{A}_F\otimes\mathcal{B} \simeq \mathbb{I}_{\mathcal{M}}\otimes(\mathcal{A}_F\otimes\mathcal{B})$ is isomorphic to a subalgebra of $\mathfrak{A}$.

If $\mathcal{A}_F$ is a single matrix algebra, then $\mathcal{A}_F = \mathbb{M}_k(\mathbb{C})$ since π_F is faithful. By (4.34), one obtains $\mathcal{B} = \mathbb{C}^2$ with ρ given by (4.32) and $l = 1$. This is the statement (4.31) where $l = d_1 = 1$ and all the other d_j's vanishing.

Otherwise, with $M_j^{(r)}$ denoting the r-th copy in $\pi_F(\mathcal{A}_F)$ of the fundamental representation of the matrix algebra $\mathbb{M}_{n_j}(\mathbb{C})$, one has

$$\pi_0(\mathbb{I}_{\mathcal{M}}\otimes\mathcal{A}_F) = \text{diag}\left(M_1^{(1)}, \ldots, M_1^{(d_1)}, M_2^{(1)}, \ldots, M_q^{(d_q)}\right)\otimes\mathbb{I}_{2^m}. \tag{4.35}$$

For any $b\in\mathcal{B}$, the operator $\pi(\mathbb{I}_{\mathcal{A}}\otimes b)$ commutes with the operator $\pi(\mathcal{A}\otimes\mathbb{I}_{\mathcal{B}})$ hence, by (3.23), with $\pi(\mathbb{I}_{\mathcal{M}}\otimes\mathcal{A}_F\otimes\mathbb{I}_{\mathcal{B}}) = \pi_0(\mathbb{I}_{\mathcal{M}}\otimes\mathcal{A}_F)$. This means

$$\pi(\mathbb{I}_{\mathcal{A}}\otimes b) = \text{diag}\left(\lambda_1^{(1)}\mathbb{I}_{n_1}, \ldots, \lambda_1^{(d_1)}\mathbb{I}_{n_1}, \lambda_2^{(1)}\mathbb{I}_{n_2}, \ldots, \lambda_q^{(d_q)}\mathbb{I}_{n_q}\right)\otimes T \tag{4.36}$$

for $\{\lambda_j^{(t)}\}\in\mathbb{C}^{d'}$, with $d' := \sum_j d_j$, and T an arbitrary matrix in $\mathbb{M}_{2^m}(\mathbb{C})$. But $\pi(\mathbb{I}_{\mathcal{A}}\otimes b)$ belonging to $\mathfrak{A}$ forces T to be of the form

$$T = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\otimes\mathbb{I}_{2^{m-1}} \tag{4.37}$$

for some $\alpha, \beta\in\mathbb{C}$. Hence $\mathcal{B}$ is isomorphic to a subalgebra of the algebra

$$\mathfrak{B} = \mathbb{C}^{d'}\otimes\mathbb{C}^2 \tag{4.38}$$

generated by all elements (4.36) with (4.37). The automorphism ρ is defined as in Lemma 4.1 by the permutation of α_{jl} and β_{jl} in (4.33). Thus it acts only on the $\mathbb{C}^2$ factor of $\mathfrak{B}$. Since ρ is non trivial by hypothesis, this forbids to consider any subalgebra $\mathbb{C}^l\otimes\mathbb{C}$ of (4.38). Hence

$$\mathcal{B}\simeq\mathbb{C}^l\otimes\mathbb{C}^2 \qquad \text{for some } l\leq d'. \tag{4.39}$$

Next, for any $b \in \mathcal{B}$, and $S \in \mathbb{M}_{n_1}(\mathbb{C})$ viewed as an element of $\mathcal{A}_F$, equations (4.35)-(4.37) lead to

$$\begin{aligned}\pi(\mathbb{I}_\mathcal{M} \otimes S \otimes b) &= \pi_0(\mathbb{I}_\mathcal{M} \otimes S)\, \pi(\mathbb{I}_\mathcal{A} \otimes b) \\ &= \mathrm{diag}\left(\begin{pmatrix} M_1 & 0 \\ 0 & N_1 \end{pmatrix}, \ldots, \begin{pmatrix} M_{d_1} & 0 \\ 0 & N_{d_1} \end{pmatrix}, 0, \ldots, 0 \right). \end{aligned} \tag{4.40}$$

Thus $\pi(\mathbb{I}_\mathcal{M} \otimes \mathcal{A}_F \otimes \mathcal{B})$ contains at most $2d_1$ independent representations of $\mathbb{M}_{n_1}(\mathbb{C})$. So if $l > d_1$, the representation π of $\mathbb{I}_\mathcal{M} \otimes \mathcal{A}_F \otimes (\mathbb{C}^l \otimes \mathbb{C}^2)$ is not faithfull, which is excluded by hypothesis. Therefore

$$l \leq d_1. \tag{4.41}$$

The same is true for all the d_j's, hence the result that $d = \min\{d_1, d_2, ..., d_q\}$. □

Unlike the case of the canonical triple of a manifold, a minimal twist of an almost commutative geometry is not necessarily by $\mathbb{C}^2$. However, although the algebra $\mathcal{B} = \mathbb{C}^l \otimes \mathbb{C}^2$ may be bigger than $\mathbb{C}^2$, the twisting automorphism ρ always results in permuting the two components of spinors like in (3.34). Thus ρ is an automorphism of the $\mathbb{C}^2$ factor of $\mathcal{B}$, which forms the "irreducible" part of the twist, in contrast with the $\mathbb{C}^l$ factor which reflects the reducibility of the representation π_F of the finite dimensional algebra. By adding a condition of irreducibility for the finite part representation π_F Proposition 4.4 yields the same unicity result as for manifolds.

Corollary 4.5. *Let $(\mathcal{A}, \mathcal{H}, D)$ be an almost commutative geometry as in Proposition 4.4, such that the representation π_F of $\mathcal{A}_F$ is irreducible. Then any non-trivial minimal twist $(\mathcal{A} \otimes \mathcal{B}, \mathcal{H}, D\,;\, \rho)$ is by $\mathcal{B} = \mathbb{C}^2$ with automorphism $\rho(a, a') = (a', a)$ for any $a, a' \in \mathcal{A} \otimes \mathbb{C}^2$.*

Proof. This is Proposition 4.4 with all the d_i's equal to 1, so that $l = 1$ and $\mathcal{B} = \mathbb{C}^2 \otimes \mathbb{C} = \mathbb{C}^2$. □

Nevertheless, there is still a degree of freedom in the representation $\pi_\mathcal{B}$ of the auxiliary algebra $\mathbb{C}^2$, and thus in the grading operator $\widetilde{\Gamma}$ as defined in (3.45). This freedom could lead to twisting of almost commutative geometries which are not of the 'the twist by grading' type, in contrast to what happens for manifolds as shown in Proposition 4.2. Restricting to the irreducible case where all the d_j's are equal to 1, from (4.36) one has:

$$\widetilde{\Gamma} := \pi(\mathbb{I}_\mathcal{A} \otimes (1, -1)) = \bigoplus_{j=1}^{q} T_j \tag{4.42}$$

where each T_j is one of the two possible representations of $(1, -1)$ on $L^2(\mathcal{M}, S)$ allowed by (3.27), that is $T_j = \gamma_{(2m)}$ or $-\gamma_{(2m)}$. In other terms, one has

$$\widetilde{\Gamma} = \gamma_{(2m)} \otimes \widetilde{\Gamma}_F \tag{4.43}$$

where $\widetilde{\Gamma}_F$ is a diagonal matrix with entries ± 1. As stressed at the end of Sect. 3.2, the point is whether the operator $\widetilde{\Gamma}$ is a grading of the twisted almost commutative geometry or not. If yes, the only minimal twist of any such a geometry by $\mathbb{C}^2$ would be by grading as in Sect. 3.2; otherwise, there would be alternative ways to minimally twist an almost commutative geometry by $\mathbb{C}^2$, even in the irreducible case.

Remark 4.6. *In case $q = 1$, that is when $\mathcal{A}_F = M_n(\mathbb{C})$, the operator $\widetilde{\Gamma}$ is either $\gamma_{(2m)} \otimes \mathbb{I}_F$ or $-\gamma_{(2m)} \otimes \mathbb{I}_F$. This it is not a grading of the almost commutative geometry since it does not anticommutes with $\gamma_{(2m)} \otimes D_F$. This simply reflects the fact that there is no grading for $\mathcal{A}_F = M_n(\mathbb{C})$ acting irreducibly on $\mathcal{H}_F$, for the only operator that commutes with $\pi_F(\mathcal{A}_F)$ is the identity.*

So far, we are able to answer this question in the real case, adding the assumption that $\widetilde{\Gamma}$ behaves well with respect to the real structure J, that is

$$\widetilde{\Gamma} J = \tilde{\epsilon}\, J\Gamma \qquad \text{for some } \tilde{\epsilon} = 1 \text{ or } -1. \tag{4.44}$$

Proposition 4.7. *Let $(\mathcal{A} \otimes \mathbb{C}^2, \mathcal{H}, D; \rho)$ with ρ as in (3.29) be a minimal twist of an almost commutative geometry $(\mathcal{A}, \mathcal{H}, D)$ with $\mathcal{A}_F$ as in (4.29). Assume in addition that the twisted spectral triple is real, with real structure J. If (4.44) holds true, then*

$$\widetilde{\Gamma}\, D + D\, \widetilde{\Gamma} = 0, \tag{4.45}$$

meaning that $\widetilde{\Gamma}$ is a grading of both the starting and the twisted spectral triples.

Proof. We only sketch the proof that goes along the lines of the proofs of Propositions 3.7 and 3.7 since in a sense the present proposition goes in the inverse direction of those. The key is to decompose $\mathcal{H} = \widetilde{\mathcal{H}}_+ \oplus \widetilde{\mathcal{H}}_-$ into the eigenbasis of $\widetilde{\Gamma}$ and then all operators accordingly.

Firstly, the boundedness of the twisted commutator $[D, \pi(a, \alpha)]_\rho$ for any $(a, \alpha) \in \mathcal{A} \otimes \mathbb{C}^2$, restricts to requiring only the boundedness of

$$[-\mathrm{i}\gamma^\mu \partial_\mu \otimes \mathbb{I}_F, \pi(a, \alpha)]_\rho \tag{4.46}$$

since, the twisted commutators of $\pi(a, \alpha)$ with $\gamma_{(2m)} \otimes D_F$ and $-\mathrm{i}\gamma^\mu \omega_\mu \otimes \mathbb{I}_F$ are trivially bounded. That the twisted commutator in (4.46) be bounded leads, with a direct computation to

$$(\gamma^\mu \otimes \mathbb{I}_F)\, \widetilde{\Gamma} + \widetilde{\Gamma}\, (\gamma^\mu \otimes \mathbb{I}_F) = 0 \qquad \forall \mu = 1, ..., 2m. \tag{4.47}$$

This shows that $\widetilde{\Gamma}$ anticommutes with $\gamma^\mu \partial_\mu \otimes \mathbb{I}_F$, as well as with $\gamma^\mu \omega_\mu \otimes \mathbb{I}_F$, as can be seen using the local form of the spin connection $\omega_\mu = \frac{1}{4} \Gamma^{\nu\rho}_\mu \gamma_\rho \gamma_\nu$. Hence:

$$(\partial\!\!\!/ \otimes \mathbb{I}_F)\, \widetilde{\Gamma} + \widetilde{\Gamma}\, (\partial\!\!\!/ \otimes \mathbb{I}_F) = 0. \tag{4.48}$$

On the other hand, the condition on the finite part $\gamma_{(2m)} \otimes D_F$, that is

$$\left(\gamma_{(2m)} \otimes D_F\right) \widetilde{\Gamma} + \widetilde{\Gamma} \left(\gamma_{(2m)} \otimes D_F\right) = 0, \tag{4.49}$$

follows from the zero-order and the twisted first-order conditions. For this one uses again a decomposition of the operator J on the eigenbasis of $\widetilde{\Gamma}$; this being possible once requiring (4.44). □

5 Applications

A twisted spectral triple for the standard model of elementary particles has been proposed in [8], whose twisted fluctuations yield the extra-scalar field σ required to stabilize the electroweak vacuum as pointed out in [2], together with an unexpected additional vector field X_μ. It has been shown in [15] that for $\mathcal{M}$ a four dimensional manifold, the appearance of X_μ is not due to the peculiar structure of the standard model, but is a consequence of the twist on the commutative part of the almost commutative geometry. We generalize this result to any even dimensional manifolds of KO-dimension 0 or 4 in Sect. 5.1 below. Then we study in Sect. 5.2 to what extend the twisted spectral triple of [8] enters in the framework of minimal twisting introduced in the present paper.

5.1 Twisted fluctuations of the free Dirac operator

Let us consider the minimal twist of a even dimensional closed Riemannian manifold $\mathcal{M}$ as described in Proposition 4.2, that is

$$(C^\infty(\mathcal{M})\otimes\mathbb{C}^2, L^2(M,S), \partial\!\!\!/; \rho) \quad \text{where} \quad \rho(f,g)=(g,f) \quad \forall f,g\in C^\infty(\mathcal{M}), \tag{5.1}$$

with grading $\gamma_{(2m)}$ and real structure J (the 'charge conjugation' operator). Recall that via a real structure, equation (2.5), that is $\pi^\circ(b) := J\pi(b^*)J^{-1}$, yields a representation of the opposite algebra. Since a commutative algebra can be identify with its opposite, for such an algebra one simply writes $J\pi(b^*)J^{-1} = \pi(b)$. In particular, for the above minimal twists, when restricting to $f\in C^\infty(\mathcal{M})$, one identifies

$$J\pi_0(f)J^{-1} = \pi_0(\bar f). \tag{5.2}$$

Clearly the twisting automorphism in (5.1) is such that $\rho^2 = \text{id}$ and this in particular means that

$$\rho(a^*) = (\rho(a))^* \tag{5.3}$$

from (2.7). We denote by KO-dim the KO-dimension of a triple.

Lemma 5.1. *For the minimal twisted spectral triple in* (5.1) *one has*

$$J\pi(a)J^{-1} = \begin{cases} \pi(a^*) & \text{if } \textit{KO-dim} = 0,\ 4 \\ \\ \pi(\rho(a^*)) & \text{if } \textit{KO-dim} = 2,\ 6 \end{cases}. \tag{5.4}$$

Proof. For any $a=(f,g)\in C^\infty(\mathcal{M})\otimes\mathbb{C}^2$, Proposition 4.2 yields

$$J\pi(a)J^{-1} = J\,p_+\pi_0(f)\,p_+\,J^{-1} + J\,p_-\pi_0(f)\,p_-\,J^{-1} \tag{5.5}$$

where π_0 is the usual representation of $C^\infty(\mathcal{M})$ on spinors and $p_\pm = \frac{1}{2}(\mathbb{I}\pm\gamma_{(2m)})$.

If the KO-dimension is 0 or 4, the operator J commutes with $\gamma_{(2m)}$, hence with p_+ and p_-. Thus, using (5.2),

$$\begin{aligned} J\pi(a)J^{-1} &= p_+J\pi_0(f)J^{-1}p_+ + p_-J\pi_0(f)J^{-1}p_- \\ &= p_+\pi_0(\bar f) + p_-\pi_0(\bar g) = \pi(\bar f,\bar g) = \pi(a^*). \end{aligned} \tag{5.6}$$

In KO-dimension 2 or 6, the operator J anticommutes with $\gamma_{(2m)}$, meaning that $Jp_+ = p_-J$ and $\mathcal{J}p_- = p_+J$. Hence, using now (5.3),

$$\begin{aligned} J\pi(a)J^{-1} &= p_-J\pi_0(f)J^{-1}p_- + p_+J\pi_0(f)J^{-1}p_+ \\ &= p_-\pi_0(\bar f) + p_+\pi_0(\bar g) = \pi(\bar g,\bar f) = \pi(\rho(a^*)). \end{aligned} \tag{5.7}$$

Thus the statement (5.4). ∎

Now, if $\dim\mathcal{M}=2m$, any $a=(f,g)\in C^\infty(\mathcal{M})\otimes\mathbb{C}^2$, one has

$$\pi(a)=\begin{pmatrix} f\mathbb{I}_{2^{m-1}} & 0\\ 0 & g\mathbb{I}_{2^{m-1}}\end{pmatrix},\qquad \pi(\rho(a))=\begin{pmatrix} g\mathbb{I}_{2^{m-1}} & 0\\ 0 & f\mathbb{I}_{2^{m-1}}\end{pmatrix}. \tag{5.8}$$

Using the fact that the spin connection commutes with the representation (and omitting the symbol of representation) a direct computation leads to

$$\begin{aligned}[\partial\!\!\!/, a]_\rho &= -\mathrm{i}\gamma^\mu[\partial_\mu, a] + \left(\gamma^\mu\, a - \rho(a)\,\gamma^\mu\right)\partial\!\!\!/\\ &= -\mathrm{i}\gamma^\mu(\partial_\mu a),\end{aligned} \tag{5.9}$$

since from Lemma 4.1 for the particular automorphism ρ in (5.8) one has

$$\gamma^\mu a = \rho(a)\,\gamma^\mu. \tag{5.10}$$

Using again this, any twisted 1-form as defined in (2.15) can thus be written as

$$A_\rho = -\mathrm{i}\sum\nolimits_j a_j\,\gamma^\mu(\partial_\mu b_j) =: -\mathrm{i}\gamma^\mu\sum\nolimits_j \rho(a_j)(\partial_\mu b_j)\qquad \text{for}\quad a_j, b_j\in C^\infty(\mathcal{M})\otimes\mathbb{C}^2. \tag{5.11}$$

Lemma 5.2. *For the minimal twisted spectral triple in* (5.1) *one has*

$$JA_\rho J^{-1} = \begin{cases} -\rho(A_\rho^*) & \textit{if KO-dim} = 0,\ 4\\ \\ -A_\rho^* & \textit{if KO-dim} = 2,\ 6\end{cases}. \tag{5.12}$$

Proof. In even dimensions the real structure J commutes with the Dirac operator, $JD=DJ$ so that from the signs in (2.1) one has $\epsilon'=1$. Being J antilinear this means that

$$J\gamma^\mu = -\gamma^\mu J \tag{5.13}$$

since usual gamma matrice algebra yields that J commutes with the covariant spin derivatives ∇_μ. By Lemma 5.1, since J is antilinear, it commutes with ∂_μ and ρ is a $*$-automorphism from (5.3), direct computations yields

$$JA_\rho J^{-1} = \begin{cases} -\mathrm{i}\gamma^\mu\sum_j \rho(a_j^*)(\partial_\mu b_j^*) & \text{if KO-dim} = 0{,}4\\ \\ -\mathrm{i}\gamma^\mu\sum_j a_j^*(\partial_\mu\rho(b_j^*)) & \text{if KO-dim} = 2{,}6\end{cases}. \tag{5.14}$$

On the other hand, using (5.3) and (5.10), one computes:

$$A_\rho^* = \mathrm{i}\sum\nolimits_j(\partial_\mu b_j^*)\rho(a_j^*)\,\gamma^\mu = \mathrm{i}\gamma^\mu\sum\nolimits_j(\partial_\mu\rho(b_j^*))a_j^* = \mathrm{i}\gamma^\mu\sum\nolimits_j a_j^*(\partial_\mu\rho(b_j^*)), \tag{5.15}$$

since $(\partial_\mu\rho(b_j^*))\in C^\infty(\mathcal{M})$ commutes with $a_j^*\in C^\infty(\mathcal{M})$. With a slight abuse of notation due to the omission of the symbol of representation, we denote the first line of the r.h.s. of (5.14) as $\rho(A_\rho^*)$. The results in (5.12) follows by comparison. ☐

Proposition 5.3. *There are no twisted fluctuations of the Dirac operator $\partial\!\!\!/$ if the KO-dimension is* 2 *or* 6*. On the other hand, for KO-dimension* 0 *or* 4*, the twisted fluctuations are of the form*

$$\partial\!\!\!/_\rho = \partial\!\!\!/ - \mathrm{i}\gamma^\mu\, f_\mu\gamma_{(2m)}, \tag{5.16}$$

where $f_\mu=(f_1,\dots,f_{2m})$ are arbitrary real functions in $C^\infty(\mathcal{M})$.

Proof. From Lemma 5.2 one has

$$\partial\!\!\!/_\rho = \partial\!\!\!/ + A_\rho + JA_\rho J^{-1} = \partial\!\!\!/ + A_\rho - \begin{cases} \rho(A_\rho^*) & \text{if KO-dim} = 0,\, 4 \\ \\ A_\rho^* & \text{if KO-dim} = 2,\, 6 \end{cases} . \tag{5.17}$$

By requiring that $\partial\!\!\!/_\rho$ be self-adjoint one sees that for the KO-dimension 2 or 6, the additional term $A_\rho - A_\rho^*$ equals its opposite, hence it vanishes. For KO-dimension 0 or 4, let us write

$$Y_\mu := \sum\nolimits_j \rho(a_j)\,(\partial_\mu b_j), \qquad \rho(Y_\mu) := \sum\nolimits_j a_j\,(\partial_\mu \rho(b_j)) \tag{5.18}$$

so that (5.11) and (5.15) yields

$$A_\rho = -\mathrm{i}\gamma^\mu Y_\mu, \quad A_\rho^* = \mathrm{i}\gamma^\mu Y_\mu^*. \tag{5.19}$$

Therefore $\partial\!\!\!/$ is selfadjoint if and only if

$$A_\rho - \rho(A_\rho^*) = -\mathrm{i}\gamma^\mu (Y_\mu + Y_\mu^*) \tag{5.20}$$

is self-adjoint. By (5.10) this is equivalent to

$$\gamma^\mu \left(\rho(Y_\mu + Y_\mu^*) + Y_\mu + Y_\mu^*\right) = 0. \tag{5.21}$$

With $a_j = (f_j, g_j)$ and $b_j = (f_j', g_j')$ in $C^\infty(\mathcal{M}) \otimes \mathbb{C}^2$, one has

$$Y_\mu := \mathrm{diag}\left(f_\mu\, \mathbb{I}_{2^{m-1}},\, g_\mu\, \mathbb{I}_{2^{m-1}}\right), \quad \rho(Y_\mu) := \mathrm{diag}\left(g_\mu\, \mathbb{I}_{2^{m-1}},\, f_\mu\, \mathbb{I}_{2^{m-1}}\right), \tag{5.22}$$

where $f_\mu := \sum_j g_j\, \partial_\mu f_j'$ and $g_\mu := \sum_j f_j\, \partial_\mu g_j'$. Both $Y_\mu + Y_\mu^*$ and $\rho(Y_\mu + Y_\mu^*)$ are block diagonal matrices with block $C^\infty(\mathcal{M})$-proportional to $\mathbb{I}_{2^{m-1}}$, so the l.h.s. of (5.21) is block off-diagonal, with blocks $C^\infty(\mathcal{M})$-linear combinations of Pauli matrices. Hence (5.22) is equivalent to

$$Y_\mu + Y_\mu^* = -\rho(Y_\mu + Y_\mu^*). \tag{5.23}$$

This means

$$g_\mu + g_\mu^* = -(f_\mu + f_\mu^*) \tag{5.24}$$

which is the same as

$$Y_\mu + Y_\mu^* = 2(\mathrm{Re}\, f_\mu)\, \gamma_{(2m)}. \tag{5.25}$$

The latter is of the form in (5.16). This concludes the proof. □

In the non-twisted case, that is when ρ the identity automorphism, then (5.17) shows that the fluctuations of $\partial\!\!\!/$ also vanish in KO-dimension $0, 4$. This can also be read in (5.23), which for $\rho = \mathrm{Id}$ implies that $Y_\mu + Y_\mu^*$ equals its opposite, hence is zero. One retrieves the well known result that (non-twisted) fluctuations of the Dirac operator in the commutative case always vanish.

5.2 On twisting the spectral standard model

We investigate how the twisted spectral triple for the standard model of elementary particles proposed in [8] fits the framework of the present paper. The (non-twisted) spectral triple of the standard model [3] is the almost commutative geometry

$$\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathcal{A}_{sm}, \quad \mathcal{H} = L^2(\mathcal{M}, S) \otimes \mathcal{H}_F, \quad D = \partial\!\!\!/ \otimes \mathbb{I}_F + \gamma_{(2m)} \otimes D_F \tag{5.26}$$

where

$$\mathcal{A}_{sm} := \mathbb{C} \oplus \mathbb{H} \oplus \mathbb{M}_3(\mathbb{C}) \tag{5.27}$$

acts on the finite dimensional space $\mathcal{H}_F$ whose dimension is the number of elementary fermions. Then D_F is a matrix acting on $\mathcal{H}_F$ whose coefficients encode the masses of these fermions. As in [8] we work with one generation only, so that $\mathcal{H}_F \simeq \mathbb{C}^{32}$ splits as

$$\mathcal{H}_F = \mathcal{H}_L \oplus \mathcal{H}_R \oplus \mathcal{H}_L^c \oplus \mathcal{H}_R^c \tag{5.28}$$

with each of the summands isomorphic to $\mathbb{C}^8$ (8 is for one pair of colored quarks and one pair electron/neutrino). The index L/R is for left/right particles, and the exponent c is for antiparticles. The (real) algebra of quaternion acts only on $\mathcal{H}_L$, the algebra $\mathbb{M}_3(\mathbb{C})$ only on $\mathcal{H}_L^c \oplus \mathcal{H}_R^c$ and $\mathbb{C}$ on $\mathcal{H}_R \oplus \mathcal{H}_L^c \oplus \mathcal{H}_R^c$, namely for $c \in \mathbb{C}, q \in \mathbb{H}$ and $m \in \mathbb{M}_3(\mathbb{C})$ one has

$$\pi_F(c, q, m) = \pi_L(q) \oplus \pi_R(c) \oplus \pi_L^c(c, m) \oplus \pi_R^c(c, m). \tag{5.29}$$

Explicitly, identifying a quaternion q with its usual representation as 2×2 complex matrix, one has

$$\pi_L(q) := q \otimes \mathbb{I}_4, \quad \pi_R(c) := \mathrm{diag}(c, \bar{c}) \otimes \mathbb{I}_4, \quad \pi_L^c(c, m) = \pi_R^c(c, m) := \mathbb{I}_2 \otimes \mathrm{diag}(c, m). \tag{5.30}$$

The identity $\mathbb{I}_4$ in the particle sector means that $\mathbb{C}$ and $\mathbb{H}$ preserve the color, and do not mix leptons with quarks. The identity $\mathbb{I}_2$ in the antiparticle sector means that $\mathbb{C}$ and $\mathbb{M}_3(\mathbb{C})$ preserves the flavour: c acts by multiplication on antileptons while $\mathbb{M}_3(\mathbb{C})$ mixes the color of the antiquarks. The representation of $\mathcal{A}$ on $\mathcal{H}$ is thus

$$\pi_0(f \otimes a_F) = \pi_{\mathcal{M}}(f) \otimes \pi_F(a_F) \quad \forall f \in C^\infty(\mathcal{M}),\, a_F \in \mathcal{A}_{sm}. \tag{5.31}$$

The twisted spectral triple $(\widetilde{\mathcal{A}}, \mathcal{H}, D; \rho)$ of the standard model given in [8] is obtained by letting $\mathbb{C} \oplus \mathbb{H}$ act independently on the left/right components of spinors, in the particle sector only. Explicitly one takes $\widetilde{\mathcal{A}} = C^\infty(\mathcal{M}) \otimes \widetilde{\mathcal{A}}_{sm}$ where

$$\widetilde{\mathcal{A}}_{sm} := \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{H} \oplus \mathbb{H} \oplus \mathbb{M}_3(\mathbb{C}), \tag{5.32}$$

whose generic element is denoted

$$A = (c^r, c^l, q^r, q^l, m) \quad \text{with} \quad (c^r, c^l) \in \mathbb{C}^2,\ (q^r, q^l) \in \mathbb{H}^2, m \in \mathbb{M}_3(\mathbb{C}). \tag{5.33}$$

The representation is

$$\pi(f \otimes A) := p_+ \pi_{\mathcal{M}}(f) \otimes (\pi_L(q^r) + \pi_R(c^r)) \ + \ p_- \pi_{\mathcal{M}}(f) \otimes \Big(\pi_L(q^l) + \pi_R(c^l)\Big) \\ + \pi_M(f) \otimes (\pi_L^c(c^r, m) + \pi_R^c(c^r, m)) \tag{5.34}$$

with $p_\pm := \frac{1}{2}(\mathbb{I}_\mathcal{M} \pm \gamma_{(2m)})$. The automorphism ρ is given by the permutation of q^l, c^l with q^r, c^r in the particle sector only, that is

$$\rho\left(\pi(f\otimes A)\right) = p_+\pi_\mathcal{M}(f)\otimes\left(\pi_L(q^l)+\pi_R(c^l)\right) \;+\; p_-\pi_\mathcal{M}(f)\otimes\left(\pi_L(q^r)+\pi_R(c^r)\right) \\ +\pi_M(f)\otimes\left(\pi^c_L(c^r,m)+\pi^c_R(c^r,m)\right). \qquad (5.35)$$

There are two main differences with the minimal twisting introduced earlier in this paper:

i. By passing from (5.27) to (5.32), only a part of the finite dimensional algebra is doubled.

ii. The automorphism ρ in (5.35) is an automorphism of $\pi(\widetilde{A})$ that does not commute with the representation, i.e. there is no elements $\widetilde{A}\in\widetilde{\mathcal{A}}_{sm}$ such that $\rho(\pi(f\otimes A)) = \pi(f\otimes\widetilde{A})$.

The first point can be fixed by refining our earlier definition of minimal twist as follows.

Definition 5.4. *Let $(\mathcal{A},\mathcal{H},D)$ be a spectral triple whose algebra*

$$\mathcal{A} = \mathcal{A}'\oplus\mathcal{A}'' \qquad (5.36)$$

is the direct sum of two (pre-) C^* algebras $\mathcal{A}'$ and $\mathcal{A}''$. A partial minimal twist of $(\mathcal{A},\mathcal{H},D)$ by the algebra $\mathcal{B}$ is a twisted spectral triple

$$((\mathcal{A}'\otimes\mathcal{B})\oplus\mathcal{A}'',\mathcal{H},D;\rho) \qquad (5.37)$$

such that the initial representation π_0 of $\mathcal{A}'\oplus\mathcal{A}''$ on $\mathcal{H}$ is retrieved from the representation π of the algebra $(\mathcal{A}'\otimes\mathcal{B})\oplus\mathcal{A}''$ as

$$\pi_0(a'\oplus a'') = \pi\big((a'\otimes\mathbb{I}_\mathcal{B})\oplus a''\big) \quad \forall a'\in\mathcal{A}',\, a''\in\mathcal{A}'',\, b\in\mathcal{B}. \qquad (5.38)$$

In our case, by setting $\mathcal{A}' = C^\infty(\mathcal{M})\otimes\mathbb{C}\oplus\mathbb{H}$ and $\mathcal{A}'' = C^\infty(\mathcal{M})\otimes\mathbb{M}_3(\mathbb{C})$ so that

$$C^\infty(\mathcal{M})\otimes\mathcal{A}_{sm} = \mathcal{A}'\oplus\mathcal{A}'', \qquad (5.39)$$

one gets as expected

$$C^\infty(\mathcal{M})\otimes\widetilde{\mathcal{A}}_{sm} = (\mathcal{A}'\otimes\mathcal{B})\oplus\mathcal{A}'' \qquad (5.40)$$

with twisting algebra $\mathcal{B}=\mathbb{R}^2$ — one cannot consider $\mathcal{B}=\mathbb{C}^2$, for $\mathbb{H}$ is not a complex algebra. In addition, the representations π in (5.34) and π_0 in (5.31) satisfy (5.38).

The second point requires extra care and its investigated will be reported elsewhere. One option is to generalize the results of the present paper to automorphisms that do not commute with the representation, so as to fit the twisted spectral triple of [8] in the scheme. A second possibility is to minimally twist the standard model in the sense of Definition 3.1 or Definition 5.4, and see whether twisted fluctuations still generate the extra-scalar field σ needed in the standard model.

References

[1] B. Blackadar. *Operator algebras. Theory of C^*-algebras and von Neumann algebras.* Encyclopaedia of Mathematical Sciences vol. 122, Springer 2006.

[2] A.H. Chamseddine and A. Connes. Resilience of the spectral standard model. *JHEP*, 09 (2012) 104.

[3] A.H. Chamseddine, A. Connes, M. Marcolli. Gravity and the standard model with neutrino mixing. *Adv. Theor. Math. Phys.*, 11 (2007) 991–1089.

[4] A.H. Chamseddine, A. Connes, W.D. van Suijlekom. Inner fluctuations in noncommutative geometry without first order condition. *J. Geom. Phy.*, 73 (2013) 222–234.

[5] A. Connes, M. Marcolli. *Noncommutative geometry, quantum fields and motives.* Colloquium Publications, vol. 55, AMS, 2008.

[6] A. Connes. Gravity coupled with matter and the foundations of noncommutative geometry. *Commun. Math. Phys.*, 182 (1996) 155–176.

[7] A. Connes, H. Moscovici. Type III and spectral triples. In: *Traces in number theory, geometry and quantum fields, Aspects Math. Friedt. Vieweg, Wiesbaden*, E38 (2008) 57–71.

[8] A. Devastato, P. Martinetti. Twisted spectral triple for the standard and spontaneous breaking of the grand symmetry. *arXiv 1411.1320 [hep-th].*

[9] L. Dabrowski, G. Landi, M. Paschke, A. Sitarz. The Spectral Geometry of the Equatorial Podleś Sphere. *C. R. Acad. Sci. Paris*, Ser. I 340 (2005) 819–822.

[10] F. Fathizadeh and O. Gabriel. On the Chern-Gauss-Bonnet theorem and conformally twisted spectral triples for C*-dynamical systems. *arXiv 1506.07913 [math.OA].*

[11] T. Hadfield, U. Krähmer. On the Hochschild homology of quantum $SL(N)$. *C. R. Acad. Sci. Paris*, Ser. I 343 (2006) 9–13.

[12] J. Kaad, R. Senior. A twisted spectral triple for quantum $SU(2)$. *J. Geom. Phys.*, 62 (2012) 731–739.

[13] R. Kadison, J. Ringrose. Derivations and Automorphisms of Operator Algebras. *Commun. Math. Phys.*, 4 (1967) 32–63.

[14] J. Kustermans, G.J. Murphy, L. Tuset. Differential calculi over quantum groups and twisted cyclic cocycles. *J. Geom. Phys.*, 44 (2003) 570–594.

[15] P. Martinetti, *Twisted spectral geometry for the standard model*, J. Phys.: Conf. Ser. 626 (2015) 012044.

—— —— —— —— ——

Giovanni Landi, Università di Trieste, Trieste, Italy and I.N.F.N. Sezione di Trieste, Trieste, Italy.
email: landi@units.it

Pierre Martinetti, Università di Trieste, Trieste and Università di Genova, Genova, Italy.
email: martinetti@dima.unige.it