A SURVEY ON REVERSE CARLESON MEASURES

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ABSTRACT. This is a survey on reverse Carleson measures for various Hilbert spaces of analytic functions. These spaces include Hardy, Bergman, certain harmonically weighted Dirichlet, Paley-Wiener, Fock, model, and de Branges-Rovnyak spaces.

1. INTRODUCTION

Suppose that $\mathscr{H}$ is a Hilbert space of analytic functions on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ endowed with a norm $\|\cdot\|_{\mathscr{H}}$. If $\mu \in M_+(\mathbb{D}^-)$, the positive finite Borel measures on the closed unit disk $\mathbb{D}^- = \{z \in \mathbb{C} : |z| \leqslant 1\}$, we say that μ is a *Carleson measure* for $\mathscr{H}$ when

$$\|f\|_\mu \lesssim \|f\|_{\mathscr{H}} \quad \forall f \in \mathscr{H}, \tag{1.1}$$

and a *reverse Carleson measure* for $\mathscr{H}$ when

$$\|f\|_{\mathscr{H}} \lesssim \|f\|_\mu \quad \forall f \in \mathscr{H}. \tag{1.2}$$

Here we use the notation

$$\|f\|_\mu := \left(\int_{\mathbb{D}^-} |f|^2 d\mu\right)^{\frac{1}{2}}$$

for the $L^2(\mu)$ norm of f and the notation $\|f\|_\mu \lesssim \|f\|_{\mathscr{H}}$ to mean there is a constant $c_\mu > 0$ such that $\|f\|_\mu \leqslant c_\mu \|f\|_{\mathscr{H}}$ for every $f \in \mathscr{H}$ (similarly for the inequality $\|f\|_{\mathscr{H}} \lesssim \|f\|_\mu$). We will use the notation $\|f\|_\mu \asymp \|f\|_{\mathscr{H}}$ when μ is both a Carleson and a reverse Carleson measure. There is of course the issue of how we define f μ-a.e. on $\mathbb{T} = \partial\mathbb{D}$ so that $\|f\|_\mu$ makes sense; but this will be discussed later.

Carleson measures for many Hilbert (and Banach) spaces of analytic functions have been well studied for many years now. Due to the large literature on this subject, it is probably impossible to give a complete account of these results. Carleson measures make, and continue to make, important connections to many areas of analysis such as operator theory, interpolation, boundary behavior problems, and Bernstein inequalities and they have certainly proved their worth. We will mention a few of these results as they relate to the lesser known topic, and the focus of this survey, of reverse Carleson measures.

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Generally speaking, Carleson measures μ are often characterized by the amount of mass that μ places on a *Carleson window*

$$S_I := \Big\{ z \in \mathbb{D}^- : 1 - |I| \leqslant |z| \leqslant 1, \frac{z}{|z|} \in I \Big\}$$

relative to the length $|I|$ of the side I of that window, i.e., whether or not there exists positive constants C and α such that

$$\mu(S_I) \leqslant C|I|^\alpha. \tag{1.3}$$

for all arcs $I \subset \mathbb{T} = \partial\mathbb{D}$. We will write this as $\mu(S_I) \lesssim |I|^\alpha$.

When $\mathscr{H}$ is a reproducing kernel Hilbert space, it is often the case that the Carleson condition in (1.1) can be equivalently rephrased in terms of the, seemingly weaker, testing condition

$$\|k_\lambda^{\mathscr{H}}\|_\mu \lesssim \|k_\lambda^{\mathscr{H}}\|_{\mathscr{H}} \quad \forall \lambda \in \mathbb{D}, \tag{1.4}$$

where $k_\lambda^{\mathscr{H}}$ is the reproducing kernel function for $\mathscr{H}$. This testing condition (where (1.4) implies (1.1)) is often called the *reproducing kernel thesis* (RKT).

It is natural to ask as to whether or not reverse Carleson measures on $\mathscr{H}$ can be characterized by replacing the conditions in (1.3) and (1.4) with the analogous "reverse" conditions

$$\mu(S_I) \gtrsim |I|^\alpha \quad \text{or} \quad \|k_\lambda^{\mathscr{H}}\|_\mu \gtrsim \|k_\lambda^{\mathscr{H}}\|_{\mathscr{H}}.$$

We will explore when this happens.

Reverse Carleson measures probably first appeared under the broad heading of "sampling measures" for $\mathscr{H}$, in other words, measures μ for which

$$\|f\|_{\mathscr{H}} \asymp \|f\|_\mu \quad \forall f \in \mathscr{H},$$

i.e., μ is both a Carleson *and* a reverse Carleson measure for $\mathscr{H}$. When μ is a discrete measure associated to a sequence of atoms in $\mathbb{D}$, this sequence is often called a "sampling sequence" for $\mathscr{H}$ and there is a large literature on this subject [52]. Equivalent measures have also appeared in the context of "dominating sets". For example, it is often the case that $\mathscr{H}$ is naturally normed by an $L^2(\mu)$ norm, i.e.,

$$\|f\|_{\mathscr{H}} = \|f\|_\mu \quad \forall f \in \mathscr{H},$$

as is the case with the Hardy, Bergman, Paley-Wiener, Fock, and model spaces. For a Borel set E contained in the support of μ, one can ask whether or not the measure $\mu_E = \mu|_E$ satisfies

$$\|f\|_{\mathscr{H}} \asymp \|f\|_{\mu_E} \quad \forall f \in \mathscr{H}. \tag{1.5}$$

Such sets E are called "dominating sets" for $\mathscr{H}$. Historically, for the Bergman, Fock, and Paley-Wiener spaces, the first examples of reverse Carleson measures were obtained via dominating sets which, in these spaces, are naturally related with relative density, meaning that E is never too far from the set on which the norm of the space is evaluated.

Though we will give a survey of reverse Carleson measures considered on a variety of Hilbert spaces, our main effort, and efforts of much recent work, will be on the sub-Hardy Hilbert spaces such as the model spaces and their de Branges-Rovnyak space generalizations. We will also comment on certain Banach space generalizations when appropriate.

2. THE HARDY SPACE

We assume the reader is familiar with the classical *Hardy space* H^2. For those needing a review, three excellent and well-known sources are [16, 20, 28]. Functions in H^2 have radial boundary values almost everywhere on $\mathbb{T}$ and H^2 can be regarded as a closed subspace of L^2 via the "vanishing negative Fourier coefficients" criterion. If m is standard Lebesgue measure on $\mathbb{T}$, normalized so that $m(\mathbb{T}) = 1$, then H^2 is normed by the $L^2(m)$ norm $\|\cdot\|_m$. As expected, the subject of Carleson measures begins with this well-known theorem of Carleson [20, Chap. I, Thm. 5.6].

Theorem 2.1 (Carleson)**.** *For $\mu \in M_+(\mathbb{D})$ the following are equivalent:*

(i) $\|f\|_\mu \lesssim \|f\|_m$ *for all* $f \in H^2$*;*

(ii) $\|k_\lambda\|_\mu \lesssim \|k_\lambda\|_m$ *for all* $\lambda \in \mathbb{D}$*, where* $k_\lambda(z) = (1 - \overline{\lambda}z)^{-1}$ *is the reproducing kernel for* H^2*;*

(iii) $\mu(S_I) \lesssim |I|$ *for all arcs* $I \subset \mathbb{T}$*.*

This theorem can be generalized in a number of ways. First, the theorem works for the H^p classes for $p \in (0, \infty)$ (with nearly the same proof). In particular, the set of Carleson measures for H^p does not depend on p. Furthermore, notice that the original hypothesis of the theorem says that $\mu \in M_+(\mathbb{D})$ and thus places no mass on $\mathbb{T}$. Since $H^2 \cap C(\mathbb{D}^-)$ is dense in H^2 (finite linear combinations of reproducing kernels belong to this set), one can replace the condition $\|f\|_\mu \lesssim \|f\|_m$ for all $f \in H^2$ with the same inequality but with H^2 replaced with $H^2 \cap C(\mathbb{D}^-)$. This enables an extension of Carleson's theorem to measures μ which could possibly place mass on $\mathbb{T}$ where the functions in H^2 are not initially defined. In the end however, this all sorts itself out since the Carleson window condition $\mu(S_I) \lesssim |I|$ implies that $\mu|_\mathbb{T} \ll m$ and so the integral in $\|f\|_\mu$ makes sense when one defines H^2 functions on $\mathbb{T}$ by their m-almost everywhere defined radial limits. Stating this all precisely, we obtain a revised Carleson theorem.

Theorem 2.2. *Suppose $\mu \in M_+(\mathbb{D}^-)$. Then the following are equivalent:*

(i) $\|f\|_\mu \lesssim \|f\|_m$ *for all* $f \in H^2 \cap C(\mathbb{D}^-)$*;*

(ii) $\|k_\lambda\|_\mu \lesssim \|k_\lambda\|_m$ *for all* $\lambda \in \mathbb{D}$*;*

(iii) $\mu(S_I) \lesssim |I|$ *for all arcs* $I \subset \mathbb{T}$*.*

Furthermore, when any of the above equivalent conditions hold, then $\mu|_\mathbb{T} \ll m$*; the Radon-Nikodym derivative* $d\mu|_\mathbb{T}/dm$ *is bounded; and* $\|f\|_\mu \lesssim \|f\|_m$ *for all* $f \in H^2$*.*

We took some time to chase down this technical detail since, for other Hilbert spaces, we need to include the possibility that μ might place mass on the unit circle $\mathbb{T}$ and perhaps even have a non-trivial singular component (with respect to m). In fact, as we will see below when one discusses the works of Aleksandrov and Clark, there are Carleson measures, in fact isometric measures, for model spaces which are singular with respect to m.

The reverse Carleson measure theorem for H^2 is the following [22]. We include the proof since some of the ideas can be used to obtain a reverse Carleson measure for other sub-Hardy Hilbert spaces such as the model or de Branges-Rovnyak spaces (see Section 7).

Theorem 2.3. *Let $\mu \in M_+(\mathbb{D}^-)$. Then the following assertions are equivalent:*

(i) $\|f\|_\mu \gtrsim \|f\|_m$ *for all* $f \in H^2 \cap C(\mathbb{D}^-)$*;*

(ii) $\|k_\lambda\|_\mu \gtrsim \|k_\lambda\|_m$ *for all* $\lambda \in \mathbb{D}$*;*

(iii) $\mu(S_I) \gtrsim |I|$ *for every arc* $I \subset \mathbb{T}$*;*

(iv) ess-inf $d\mu|_{\mathbb{T}}/dm > 0$.

Proof. (i) ⇒ (ii) is clear.

(iii) ⇒ (iv): Define

$$C = \inf_I \frac{\mu(S_I)}{|I|}.$$

Let I be an arc on $\mathbb{T}$ and take any (relatively) open set O in $\mathbb{D}^-$ for which $I \subset O$. Then there exists an integer N such that $h = |I|/N$ satisfies $S_{I,h} \subset O$ where $S_{I,h}$ is the modified Carleson window defined by

$$S_{I,h} = \left\{ z \in \mathbb{D}^- : 1 - h \leqslant |z| \leqslant 1, \frac{z}{|z|} \in I \right\}.$$

Divide I into N sub-arcs I_k (suitable half-open except for the last one) such that $|I_k| = h$ (and hence $S_{I_k,h} = S_{I_k}$). Then

$$\mu(S_{I,h}) = \mu(\bigcup_{k=1}^N S_{I_k,h}) = \sum_{k=1}^N \mu(S_{I_k,h}) \geqslant C \sum_{k=1}^N |I_k| = C|I|.$$

For every (relatively) open set O in $\mathbb{D}^-$ for which $I \subset O$ there exists $h > 0$ such that $S_{I,h} \subset O$. Since $\mu \in M_+(\mathbb{D}^-)$ is outer regular (see [46, Theorem 2.18]) we have

$$\mu(I) = \inf\{\mu(O) : I \subset O \text{ open in } \mathbb{D}^-\} \geqslant \inf_{h>0} \mu(S_{I,h}) \geqslant C|I|.$$

We deduce that m is absolutely continuous with respect to $\mu|_{\mathbb{T}}$ and the corresponding Radon-Nikodym derivative of μ is (essentially) bounded below by C.

(iv) ⇒ (i): Let

$$A = \text{ess-inf } d\mu|_{\mathbb{T}}/dm.$$

For all $f \in H^2 \cap C(\mathbb{D}^-)$,

$$\int_{\mathbb{D}^-} |f|^2 d\mu \geqslant \int_{\mathbb{T}} |f|^2 d\mu \geqslant A \int_{\mathbb{T}} |f|^2 dm.$$

(ii) ⇒ (iii): Let

$$K_\lambda(z) = \frac{k_\lambda(z)}{\|k_\lambda\|_m} \tag{2.4}$$

be the normalized reproducing kernel for H^2 and observe that since

$$\|k_\lambda\|_m = \frac{1}{\sqrt{1-|\lambda|^2}},$$

the quantity

$$|K_\lambda(z)|^2 = \frac{1-|\lambda|^2}{|1-\overline{\lambda}z|^2}$$

is the Poisson kernel for the disk. Let

$$B = \inf_{\lambda\in\mathbb{D}} \|K_\lambda\|_\mu^2$$

and note that $B > 0$ by hypothesis.

Integrating over $S_{I,h}$ with respect to area measure dA on $\mathbb{D}$ we get

$$B|I| \times h \leqslant \int_{S_{I,h}} \int_{\mathbb{D}^-} |K_\lambda|^2 d\mu \, dA(\lambda) = \int_{\mathbb{D}^-} \int_{S_{I,h}} \frac{1-|\lambda|^2}{|1-\overline{\lambda}z|^2} dA(\lambda) d\mu(z). \tag{2.5}$$

Set

$$\varphi_h(z) = \frac{1}{h} \int_{S_{I,h}} \frac{1-|\lambda|^2}{|1-\overline{\lambda}z|^2} dA(\lambda).$$

We claim that

$$\lim_{h\to 0} \varphi_h(z) = \begin{cases} 1 & \text{if } z \in I^\circ \\ \frac{1}{2} & \text{if } z \in \partial I \\ 0 & \text{if } z \in \mathbb{D}^- \setminus I^-, \end{cases}$$

where I^- denotes the closure, I° the interior, and ∂I the boundary of the arc I. Indeed, when $z \notin I^-$, there are constants $\delta, h_0 > 0$ such that for every $h \in (0, h_0)$ and for every $\lambda \in S_{I,h}$, we have $|1-\overline{\lambda}z| \geqslant \delta > 0$. The result now follows from the estimate

$$0 \leqslant \varphi_h(z) = \frac{1}{h} \int_{S_{I,h}} \frac{1-|\lambda|^2}{|1-\overline{\lambda}z|^2} dA(\lambda) \leqslant \frac{1}{\delta^2} \frac{|I| \times h}{h} \times (2h) \lesssim h.$$

When $z = e^{i\theta_0} \in I^\circ$, then setting $\lambda = re^{i\theta}$ for $\lambda \in S_{I,h}$ we have

$$\varphi_h(z) = \frac{1}{h} \int_{S_{I,h}} \frac{1-|\lambda|^2}{|1-\overline{\lambda}z|} dA(\lambda) = \frac{1}{h} \int_{1-h}^{1} \int_I \frac{1-r^2}{|1-re^{-i\theta}z|^2} d\theta r dr.$$

Since $\operatorname{dist}(z, \mathbb{T} \setminus I^\circ) > 0$ we see that when $r \to 1$ we have, via Poisson integrals,

$$\int_I \frac{1-r^2}{|1-re^{-i\theta}z|^2} d\theta = 1 - \int_{\mathbb{T}\setminus I} \frac{1-r^2}{|1-re^{-i\theta}z|^2} d\theta \to 1.$$

Similarly, if can be shown that at the endpoints of I, φ_h converges to $\frac{1}{2}$. Hence φ_h converges pointwise to a function comparable to χ_I, and φ_h is uniformly bounded in h. From (2.5) and the dominated convergence theorem we finally deduce that

$$\mu(I) = \int_{\mathbb{D}^-} \chi_I d\mu \simeq \int_{\mathbb{D}^-} \lim_{h\to 0} \varphi_h(z) d\mu(z) = \lim_{h\to 0} \int_{\mathbb{D}^-} \varphi_h(z) d\mu(z) \gtrsim |I| \,. \qquad \square$$

This theorem was proved in [22] and extends to $1 < p < \infty$ with the same proof. There is a somewhat weaker version of this result in [30], appearing in the context of composition operators on H^2 with closed range, where the authors needed to assume from the onset that μ was a Carleson measure for H^2. Observe that in this theorem we do not require absolute continuity of the restriction $\mu|_{\mathbb{T}}$. However, if we want to extend $\|f\|_\mu \gtrsim \|f\|_m$, originally assumed for $f \in H^2 \cap C(\mathbb{D}^-)$, to all of H^2, then, in order for the integral in $\|f\|_\mu$ to make sense for every function in H^2 (via radial boundary values), we need to impose the condition $\mu|_{\mathbb{T}} \ll m$. Note that we are allowing the possibility that the integral $\|f\|_\mu$ be infinite for certain $f \in H^2$ when the Radon-Nikodym derivative of $\mu|_{\mathbb{T}}$ is unbounded.

When $\mu \in M_+(\mathbb{D}^-)$ one can combine Theorem 2.2 and Theorem 2.3 to see that

$$\|f\|_\mu \asymp \|f\|_m \;\; \forall f \in H^2 \iff \|k_\lambda\|_\mu \asymp \|k_\lambda\|_m \;\; \forall \lambda \in \mathbb{D} \iff \mu(S_I) \asymp |I| \;\; \forall I \subset \mathbb{T}.$$

One might ask what are the "isometric measures" for H^2, i.e., $\|f\|_\mu = \|f\|_m$ for all $f \in H^2$. Notice how this is a significantly stronger condition than $\|f\|_m \asymp \|f\|_\mu$. As it turns out, there is only one such isometric measure.

Proposition 2.6. *Suppose $\mu \in M_+(\mathbb{D}^-)$ and $\|f\|_\mu = \|f\|_m$ for all $f \in H^2 \cap C(\mathbb{D}^-)$. Then $\mu = m$.*

Proof. Indeed for each $n \in \mathbb{N} \cup \{0\}$ we have

$$1 = \|z^n\|_m^2 = \int_{\mathbb{D}} |z|^{2n} d\mu + \mu(\mathbb{T}).$$

Clearly, letting $n \to \infty$, we get $\mu(\mathbb{T}) = 1$. When $n = 0$ this yields

$$\mu(\mathbb{D}) = 0 \quad \text{and} \quad \mu = \mu|\mathbb{T}.$$

By Carleson's criterion we see that $\mu \ll m$ and so $d\mu = hdm$, for some $h \in L^1(m)$. To conclude that h is equal to one almost everywhere, apply the fact that μ is an isometric measure to the normalized reproducing kernels K_λ (see (2.4)) to see that

$$1 = \int_{\mathbb{T}} \frac{1 - |\lambda|^2}{|1 - \overline{\zeta}\lambda|^2} h(\zeta) dm(\zeta) \quad \forall \lambda \in \mathbb{D}.$$

If we express the above as a Fourier series, we get

$$1 = \widehat{h}(0) + \sum_{n=1}^{\infty} \widehat{h}(-n)\overline{\lambda}^n + \sum_{n=1}^{\infty} \widehat{h}(n)\lambda^n, \qquad \lambda \in \mathbb{D},$$

and it follows that $h = 1$ m-a.e. on $\mathbb{T}$. Thus $\mu = m$. $\square$

3. BERGMAN SPACES

The *Bergman space* A^2 is the space of analytic functions f on $\mathbb{D}$ with finite norm

$$\|f\|_{A^2} := \left(\int_{\mathbb{D}} |f|^2 dA\right)^{\frac{1}{2}},$$

where $dA = dxdy/\pi$ is normalized area Lebesgue measure on $\mathbb{D}$ [17, 25]. As with the Hardy space, we begin our discussion with the Carleson measures for A^2. This was done by Hastings [23]:

Theorem 3.1. *For $\mu \in M_+(\mathbb{D})$ the following are equivalent:*

(i) $\mu(S_I) \lesssim |I|^2$ for every arc $I \in \mathbb{T}$;

(ii) $\|f\|_{A^2} \lesssim \|f\|_\mu$ for every $f \in A^2$.

We also refer to [25] for further information about Carleson measures in Bergman spaces, including an equivalent restatement of this theorem involving pseudo-hyperbolic disks. In particular (see [25, Theorem 2.15]) condition (i) is replaced by the condition: there exists an $r \in (0, 1)$ such that

$$\mu(D(a, r)) \lesssim A(D(a, r)), \quad a \in \mathbb{D},$$

where

$$D(a, r) = \left\{z \in \mathbb{C} : \left|\frac{z - a}{1 - \overline{z}a}\right| < r\right\}$$

denotes a pseudo-hyperbolic disk of radius r centered at a. Observe that since r is fixed, we have $A(D(z, r)) \asymp (1 - |z|^2)^2$. Again, the geometric condition measures the amount of mass that μ places on a pseudohyperbolic disk with respect to an intrinsic area measure of that disk. Hastings result was generalized by Oleinik and Pavlov, and Stegenga (see [35] for the references).

Reverse Carleson embeddings for the Bergman spaces, and other closely related spaces, were discussed by Luecking [33, 35, 36]. One of his first results in this direction concerns dominating sets, i.e., measures of the type $\chi_G dA$ (see (1.5)). Here we have the following “reverse” of the inequality in Hasting’s result (see [33]).

Theorem 3.2. *Suppose G is a (Lebesgue) measurable subset of $\mathbb{D}$. Then $\mu = \chi_G dA$ is a reverse Carleson measure for A^2 if and only if $\mu(S_I) \gtrsim |I|^2$ for all arcs $I \subset \mathbb{T}$.*

A similar result holds for the harmonic Bergman space [34]. We will discuss dominating sets again later when we cover model spaces (see Definition 6.12).

As it turns out, the general reverse Carleson measure result for Bergman spaces is more delicate [35, Thm. 4.2].

Theorem 3.3. *Let $\delta, \varepsilon > 0$. Then there exists a $\beta > 0$ with the following property: Whenever $\mu \in M_+(\mathbb{D})$ for which*

$$c = \sup_{a \in \mathbb{D}} \frac{\mu(D(a, 1/2))}{A(D(a, 1/2))} < \infty, \tag{3.4}$$

and for which the set

$$G = \{z : \mu(D(z, \beta)) > \varepsilon c A(D(z, \beta))\} \tag{3.5}$$

satisfies

$$m(G \cap S_I) \geqslant \delta |I|^2, \tag{3.6}$$

then $\|f\|_{A^2} \lesssim \|f\|_\mu$ for all $f \in A^2$.

Notice how this theorem requires *a priori* that μ is a Carleson measure for A^2 (via (3.4)). The next two conditions tell us that the reverse Carleson condition (3.5) must be satisfied on a set which is, in a sense, relatively dense. Moreover, the relative density condition in (3.6) should hold close to the unit circle.

For simplicity we stated the results for the A^2 Bergman space. Analogous theorems (with the same proofs) are true for the A^p Bergman spaces for $p \in (0, \infty)$.

4. Fock spaces

We briefly discuss Carleson and reverse Carleson measures for a space of entire functions - the Fock space. Here the conditions are a bit different since the functions are entire and there are no "boundary conditions" or "Carleson boxes".

Let φ be a subharmonic function on $\mathbb{C}$ (often called the weight) such that

$$\frac{1}{c} \leqslant \Delta\varphi \leqslant c$$

for some positive constant c. The *weighted Fock space* $\mathscr{F}^2_\varphi$ is the space of entire functions f with finite norm

$$\|f\|_\varphi = \left(\int_{\mathbb{C}} |f(z)|^2 e^{-2\varphi(z)} dA(z) \right)^{\frac{1}{2}}.$$

Recall that dA is Lebesgue area measure on $\mathbb{C}$. When $\varphi(z) = |z|^2$, this space is often called the Bargmann-Fock space. A good primer for the Fock spaces is [55]. There is also a suitable L^p version of this space denoted by $\mathscr{F}^p_\varphi$ and the results below apply to these spaces as well.

The Carleson measures for $\mathscr{F}^2_\varphi$ were characterized by several authors (for various φ) but the final, most general, result is found in Ortega-Cerdà [40]. Below let $B(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$ be the open ball in $\mathbb{C}$ centered at a with radius r.

Theorem 4.1. *For a locally finite positive Borel measure μ on $\mathbb{C}$, a weight φ as above, and $d\nu = e^{-2\varphi} d\mu$, the following are equivalent:*

(i) $\|f\|_\nu \lesssim \|f\|_\varphi$ *for all* $f \in \mathscr{F}^2_\varphi$;

(ii) $\sup_{z\in\mathbb{C}} \mu(B(z,1)) < \infty$.

The discussion of reverse Carleson measures for Fock spaces was begun by Janson-Peetre-Rochberg [26], again *via* dominating sets.

Theorem 4.2. *For a weight* φ*, a measurable set* $E \subset \mathbb{C}$*, and* $d\nu = e^{-2\varphi}\chi_E dA$*, the following are equivalent:*

(i) $\|f\|_\varphi \lesssim \|f\|_\nu$ *for all* $f \in \mathscr{F}_\varphi$;

(ii) there exists an $R > 0$ *such that* $\inf_{z\in\mathbb{C}} A(E \cap B(z,R)) > 0$.

Condition (ii) is a relative density condition which, in a way, appeared in Theorem 3.2. We will meet such a condition again in Theorem 5.1 below when we discuss the Paley-Wiener space.

In [40] Ortega-Cerdà examined the measures μ on $\mathbb{C}$ for which

$$\|f\|^2_\varphi \asymp \int_{\mathbb{C}} |f(z)|^2 e^{-2\varphi(z)} d\mu(z) \qquad \forall f \in \mathscr{F}_{\varphi,2}.$$

in other words, the "equivalent measures" for $\mathscr{F}^2_\varphi$. He called such measures *sampling measures*. A special instance is when

$$\mu = \sum_{n\geqslant 1} \delta_{\lambda_n},$$

where $\Lambda = \{\lambda_n\}_{n\geqslant 1}$ is a sequence in the complex plane. In this case, $\{\lambda_n\}_{n\geqslant 1}$ is called a *sampling sequence*, meaning that

$$\|f\|^2_\varphi \asymp \sum_{n\geqslant 1} |f(\lambda_n)|^2 e^{-2\varphi(\lambda_n)} \qquad \forall f \in \mathscr{F}_{\varphi,2}.$$

Contrary to the approach in Bergman spaces, where Luecking characterized Carleson and reverse Carleson measures which, in turn, yielded information on sampling sequences, Ortega-Cerdà discretized μ to reduce the general case of sampling measures to that of sampling *sequences*. These were characterized in a series of papers by Seip, Seip-Wallstén, Berndtsson-Ortega-Cerdà and Ortega-Cerdà-Seip (see [52] for these references). The main summary theorem is the following:

Theorem 4.3. *A sequence* $\Lambda \subset \mathbb{C}$ *is a sampling sequence for* $\mathscr{F}^2_\varphi$ *if and only if the following two conditions are satisfied:*

(i) Λ *is a finite union of uniformly separated sequences.*

(ii) There is a uniformly separated subsequence $\Lambda' \subset \Lambda$ *such that*

$$\varliminf_{r\to\infty} \inf_{z\in\mathbb{C}} \frac{\#(B(z,r)\cap\Lambda')}{\int_{B(z,r)} \Delta\varphi dA} > \frac{1}{2\pi}.$$

To state the result in terms of sampling measures, we need to introduce some notation. For a large integer N and positive numbers δ and r, decompose $\mathbb{C}$ into big squares S of side-length Nr and each square S is itself decomposed into N^2 little squares of side-length r. Let $n(S)$ denote the number of little squares s contained in S such that $\mu(s) \geqslant \delta$. In terms of sampling measures, we have the following:

Theorem 4.4. *The measure μ is a sampling measure if and only if the following conditions are satisfied:*

(i) $\sup_{z\in\mathbb{C}} \mu(B(z,1)) < \infty$*;*

(ii) There is an $r > 0$ and a grid consisting of squares of side-length r, an integer $N > 0$ and a positive number δ such that

$$\inf_S \frac{n(S)}{\int_S \Delta\varphi \, dA} > \frac{1}{2\pi}, \tag{4.5}$$

where the infimum is taken over all squares S consisting of N^2 little squares from the original grid.

Notice how (i) is a Carleson measure condition while (ii) is a reverse Carleson measure condition.

To deduce Theorem 4.3 from Theorem 4.4, Ortega-Cerdà first showed that it is sufficient to consider the measure μ_1 which is the part of μ supported only on the little squares s for which $\mu(s) \geqslant \delta$ and then he discretized μ_1 by $\mu_1^* = \sum_n \mu_1(s_n)\delta_{a_n}$, where a_n is the center of s_n. In order to show that μ_1 is sampling exactly when μ_1^* is sampling, he used a Bernstein-type inequality. This naturally links the problem of sampling measures to the description of sampling sequences. Note that Bernstein inequalities also appear in the context of Carleson and reverse Carleson measures for model spaces (see Section 6).

5. Paley-Wiener space

Though the Paley-Wiener space enters into the general discussion of model spaces presented in Section 6, we would like to present some older results which will help motivate the more recent ones. The *Paley-Wiener space* PW is the space of entire functions F of exponential type at most π, i.e.,

$$\limsup_{|z|\to\infty} \frac{\log|F(z)|}{|z|} \leqslant \pi,$$

and which are square integrable on $\mathbb{R}$. The norm on PW is

$$\|F\|_{PW} = \left(\int_{\mathbb{R}} |F(t)|^2 dt \right)^{\frac{1}{2}}.$$

A well-known theorem of Paley and Wiener [15] says that PW is the set of Fourier transforms of functions in L^2 which vanish on $\mathbb{R} \setminus [-\pi,\pi]$. Authors such as Kacnelson [27], Panejah [41, 42], and Logvinenko [32] examined Lebesgue measurable sets $E \subset \mathbb{R}$ for which

$$\int_{\mathbb{R}} |F|^2 dt \asymp \int_E |F|^2 dt \quad \forall F \in PW.$$

Following (1.5), such sets will be called *dominating sets* for PW. Clearly we always have

$$\int_E |F|^2 dt \leqslant \int_{\mathbb{R}} |F|^2 dt \quad \forall F \in PW.$$

The issue comes with the reverse lower bound. The summary theorem here is the following:

Theorem 5.1. *For a Lebesgue measurable set $E \subset \mathbb{R}$, the following are equivalent:*

(i) the set E is a dominating set for PW;

(ii) there exists a $\delta > 0$ and an $\eta > 0$ such that

$$|E \cap [x-\eta, x+\eta]| \geqslant \delta, \quad \forall x \in \mathbb{R}. \tag{5.2}$$

Notice how condition (ii) is a relative density condition we have met before when studying the Bergman and Fock spaces.

Lin [31] generalized the above result for measures μ on $\mathbb{R}$. We say that a positive locally finite measure μ on $\mathbb{R}$ is *h-equivalent to Lebesgue measure* if there exists a $K > 0$ such that

$$\mu(x-h, x+h) \asymp h \quad \forall x \in \mathbb{R}, |x| > K.$$

Theorem 5.3. *Suppose μ is a locally finite Borel measure on $\mathbb{R}$.*

(i) There exists a constant $\gamma > 0$ such that if μ is h-equivalent to Lebesgue measure for some $h < \gamma$ then

$$\int_{\mathbb{R}} |F|^2 dt \asymp \int_{\mathbb{R}} |F|^2 d\mu \quad \forall F \in PW.$$

(ii) If

$$\int_{\mathbb{R}} |F|^2 dt \asymp \int_{\mathbb{R}} |F|^2 d\mu \quad \forall F \in PW,$$

then μ is h-equivalent to Lebesgue measure for some $h > 0$.

6. Model spaces

A bounded analytic function Θ on $\mathbb{D}$ is called an *inner function* if the radial limits of Θ (which exist almost everywhere on $\mathbb{T}$ [16]) are unimodular almost everywhere. Examples of inner functions include the Blaschke products B_Λ with (Blaschke) zeros $\Lambda \subset \mathbb{D}$ and singular inner functions with associated (positive) singular measure ν on $\mathbb{T}$. In fact, every inner function is a product of these two basic types [16].

Associated to each inner function Θ is a *model space*

$$\mathcal{K}_\Theta := (\Theta H^2)^\perp = \left\{ f \in H^2 : \int_{\mathbb{T}} f \overline{\Theta g} dm = 0 \ \forall g \in H^2 \right\}.$$

Model spaces are the generic (closed) invariant subspaces of H^2 for the backward shift operator

$$(S^* f)(z) = \frac{f(z) - f(0)}{z}.$$

Moreover, the compression of the shift operator

$$(Sf)(z) = zf(z)$$

to a model space is the so-called "model operator" for certain types of Hilbert space contractions.

It turns out that the Paley-Wiener space PW can be viewed as a certain type of model space. We follow [47]. Let

$$\Psi(z) := \exp\left(2\pi \frac{z+1}{z-1}\right)$$

be the atomic inner function with point mass at $z = 1$ and with weight 2π,

$$(\mathscr{F} f)(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixt} f(t) dt,$$

the Fourier transform on $L^2(\mathbb{R})$, and

$$J : L^2(m) \to L^2(\mathbb{R}), \quad (Jg)(x) = \frac{1}{\sqrt{\pi}} \frac{1}{x+i} f\big(\frac{x-i}{x+i}\big).$$

It is well known that $\mathscr{F}$ is a unitary operator on $L^2(\mathbb{R})$ and a change of variables will show that J is a unitary map from $L^2(m)$ onto $L^2(\mathbb{R})$. It is also known [47, p. 33] that

$$(\mathscr{F} J)\mathcal{K}_\Psi = L^2[0, 2\pi].$$

If

$$T : L^2[0, 2\pi] \to L^2[-\pi, \pi], \quad (Th)(x) = h(x + \pi)$$

is the translation operator then

$$(T\mathscr{F} J)\mathcal{K}_\Psi = L^2[-\pi, \pi]$$

and

$$(\mathscr{F} T \mathscr{F} J)\mathcal{K}_\Psi = PW.$$

Thus the Paley-Wiener space is an isometric copy of a certain model space in a prescribed way.

An important set associated with an inner function is its *boundary spectrum*

$$\sigma(\Theta) := \left\{ \xi \in \mathbb{T} : \varliminf_{z \to \xi} |\Theta(z)| = 0 \right\}. \tag{6.1}$$

Using the factorization of Θ into a Blaschke product and a singular inner function, one can show that when $\sigma(\Theta) \neq \mathbb{T}$, there is a two-dimensional open neighborhood Ω containing $\mathbb{T} \setminus \sigma(\Theta)$ such that Θ has an analytic continuation to Ω.

Functions in model spaces can have more regularity than generic functions in H^2. Indeed, a result of Moeller [37] says every function in $\mathcal{K}_\Theta$ follows the behavior of its corresponding inner functions and has an analytic continuation to a two dimensional open neighborhood of $\mathbb{T} \setminus \sigma(\Theta)$. In fact, one can say a little bit more. Indeed, for every $\xi \in \mathbb{T} \setminus \sigma(\Theta)$ the evaluation functional $E_\xi f = f(\xi)$ is continuous on $\mathcal{K}_\Theta$ with

$$\|E_\xi\| = \sqrt{|\Theta'(\xi)|}.$$

Thus

$$\sup_{\xi \in W} \|E_\xi\| < \infty \tag{6.2}$$

for any compact set $W \subset \mathbb{D}^- \setminus \sigma(\Theta)$.

In terms of a measure $\mu \in M_+(\mathbb{D}^-)$ being a Carleson measure for $\mathcal{K}_\Theta$, let us make the following simple observation.

Proposition 6.3. *Suppose $\mu \in M_+(\mathbb{D}^-)$ with support contained in $\mathbb{D}^- \setminus \sigma(\Theta)$. Then μ is a Carleson measure for $\mathcal{K}_\Theta$.*

Proof. Let W denote the support of μ. From our previous discussion, every $f \in \mathcal{K}_\Theta$ has an analytic continuation to an open neighborhood of W. Furthermore, using (6.2) we see that

$$\sup_{\xi \in W} |f(\xi)| \lesssim \|f\|_m \quad \forall f \in \mathcal{K}_\Theta.$$

It follows that $\|f\|_\mu \lesssim \|f\|_m$ and hence μ is a Carleson measure for $\mathcal{K}_\Theta$. □

Two observations come from Proposition 6.3. The first is that there are Carleson measures for $\mathcal{K}_\Theta$ which are not Carleson for H^2 since $\mu(S_I) \lesssim |I|$ need not hold for all arcs $I \subset \mathbb{T}$. In fact one could even put point masses on $\mathbb{T} \setminus \sigma(\Theta)$. This is in contrast with the H^2 situation where we have already observed in Theorem 2.2 that if $\mu \in M_+(\mathbb{D}^-)$ is a Carleson measure for H^2, then $\mu|_\mathbb{T} \ll m$. The second observation is that if there is to be a Carleson testing condition like $\mu(S_I) \lesssim |I|$, the focus needs to be on the Carleson boxes S_I which are, in a sense, close to $\sigma(\Theta)$.

So far we have avoided the issue of making sense of the integrals $\|f\|_\mu$ for $f \in \mathcal{K}_\Theta$ when the measure μ could potentially place mass on $\mathbb{T}$. Indeed, we side stepped this in Proposition 6.3 by stipulating that the measure places no mass on $\sigma(\Theta)$, where the functions in $\mathcal{K}_\Theta$ are not well-defined. In order to consider a more general situation, and to adhere to the notation used in [54], we make the following definition.

Definition 6.4. A measure $\mu \in M_+(\mathbb{D}^-)$ will be called Θ-*admissible* if the singular component of $\mu|_\mathbb{T}$ (relative to Lebesgue measure) is concentrated on $\mathbb{T} \setminus \sigma(\Theta)$.

Since functions from $\mathcal{K}_\Theta$ are continuous (even analytic) on this set, it follows that for Θ-admissible measures and functions $f \in \mathcal{K}_\Theta$, the integral $\|f\|_\mu$ makes sense.

As was done with the Hardy spaces in Theorem 2.2, one could state the definition of a Carleson measure for $\mathcal{K}_\Theta$ to be a $\mu \in M_+(\mathbb{D}^-)$ for which

$$\|f\|_\mu \lesssim \|f\|_m \quad \forall f \in \mathcal{K}_\Theta \cap C(\mathbb{D}^-). \tag{6.5}$$

Indeed, an amazing result of Aleksandrov [2] says that $\mathcal{K}_\Theta \cap C(\mathbb{D}^-)$ is dense in $\mathcal{K}_\Theta$ and so this set makes a good "test set" for the Carleson (reverse Carleson) condition. Furthermore, if $\mu \in M_+(\mathbb{D}^-)$ and (6.5) holds, then μ is Θ-admissible, every function in $\mathcal{K}_\Theta$ has radial limits $\mu|_\mathbb{T}$-almost everywhere on $\mathbb{T}$, and $\|f\|_\mu \lesssim \|f\|_m$ for every $f \in \mathcal{K}_\Theta$.

Carleson measures for $\mathcal{K}_\Theta$ were discussed in the papers of Cohn [13] and Treil and Volberg [54]. Their theorem is stated in terms of

$$\Omega(\Theta, \varepsilon) := \{z \in \mathbb{D} : |\Theta(z)| < \varepsilon\}, \qquad 0 < \varepsilon < 1, \tag{6.6}$$

the *sub-level sets* for Θ. Note that boundary spectrum $\sigma(\Theta)$ is contained in the closure of any $\Omega(\Theta, \varepsilon)$, $0 < \varepsilon < 1$.

Theorem 6.7. *Suppose $\mu \in M_+(\mathbb{D}^-)$ and define the following conditions:*

(i) $\mu(S_I) \lesssim |I|$ for all arcs $I \subset \mathbb{T}$ for which $S_I \cap \Omega(\Theta, \varepsilon) \neq \varnothing$;

(ii) μ is a Carleson measure for $\mathcal{K}_\Theta$;

(iii) μ is Θ-admissible and $\|k_\lambda^\Theta\|_\mu \lesssim \|k_\lambda^\Theta\|_m$ holds for every $\lambda \in \mathbb{D}$.

Then $(i) \implies (ii) \implies (iii)$. Moreover, if for some $\varepsilon \in (0, 1)$, the sub-level set $\Omega(\Theta, \varepsilon)$ is connected, then $(i) \iff (ii) \iff (iii)$.

The condition that $\Omega(\Theta, \varepsilon)$ is connected for some $\varepsilon \in (0, 1)$ is often called the *connected level set condition* (CLS). Cohn [13] proved that if $\Omega(\Theta, \varepsilon)$ is connected and $\delta \in (\varepsilon, 1)$, then $\Omega(\Theta, \delta)$ is also connected. Any finite Blaschke product, the atomic inner function

$$\Theta(z) = \exp\left(\frac{z+1}{z-1}\right),$$

and the infinite Blaschke product whose zeros are $\{1 - r^n\}_{n \geqslant 1}$, where $0 < r < 1$, satisfy this connected level set condition.

The sufficient condition appearing in assertion (i) of Theorem 6.7 is, in general, not necessary. More precisely, Treil and Volberg [54] proved that this condition is necessary for the embedding of $\mathcal{K}_\Theta$ into $L^2(\mu)$ if and only if $\Theta \in (CLS)$. Nazarov–Volberg [38] proved that the RKT (reproducing kernel thesis) for Carleson embeddings for $\mathcal{K}_\Theta$ is, in general, not true. In [3], Baranov obtained a significant extension of the Cohn and Volberg–Treil results, introducing a new point of view based on certain Bernstein-type inequalities. Quite recently, in answering a question posed by Sarason [51], Baranov–Besonnov–Kapustin [5] clarified a nice link between Carleson measures for $\mathcal{K}_\Theta$ and an interesting class of operators – the truncated Toeplitz operators – which have received much attention in the last few years [51].

We now state the main reverse embedding results for model spaces from [7]. The first result is a reverse embedding theorem along the lines of Treil-Volberg for which we need the following notation: given an arc $I \subset \mathbb{T}$ and a number $n > 0$, we define the amplified arc nI as the arc with the same center as I but with length $n \times m(I)$.

Theorem 6.8. *Let Θ be inner, $\mu \in M_+(\mathbb{D}^-)$, and $\varepsilon \in (0,1)$. There exists an $N = N(\Theta, \varepsilon) > 1$ such that if*

$$\mu(S_I) \gtrsim m(I) \tag{6.9}$$

for all arcs $I \subset \mathbb{T}$ satisfying

$$S_{NI} \cap \Omega(\Theta, \varepsilon) \neq \varnothing,$$

then

$$\|f\|_m \lesssim \|f\|_\mu \qquad \forall f \in \mathcal{K}_\Theta \cap C(\mathbb{D}^-). \tag{6.10}$$

This theorem is a more general version than the one appearing in [7, Theorem 2.1] and does not require the (direct) Carleson condition. Indeed, it can be checked that the Carleson condition is not really needed in the proof. It was initially proved in [7] for (CLS)-inner function using a perturbation argument from [4, Corollary 1.3 and the proof of Theorem 1.1], but Baranov provided a proof (found in [7]) based on Bernstein inequalities and which does not require the CLS condition.

Corollary 6.11. *Under the hypotheses of Theorem 6.8, and if, moreover, the measure μ is assumed to be Θ-admissible, then* (6.10) *extends to all of $\mathcal{K}_\Theta$.*

Our second reverse Carleson result involves the notion of a dominating set for $\mathcal{K}_\Theta$, defined in (1.5) and discussed earlier for the Bergman and Fock spaces.

Definition 6.12. A (Lebesgue) measurable subset $\Sigma \subset \mathbb{T}$, with $m(\Sigma) < 1$, is called a *dominating set* for $\mathcal{K}_\Theta$ if

$$\int_{\mathbb{T}} |f|^2 dm \lesssim \int_{\Sigma} |f|^2 dm \quad \forall f \in \mathcal{K}_\Theta.$$

This is equivalent to saying that the measure $d\mu = \chi_\Sigma dm$ is a reverse Carleson measure for $\mathcal{K}_\Theta$. Here we list some observations concerning dominating sets for model spaces. We will use the following notation for sets A, B and a point x:

$$d(A,B) := \inf\{|a-b| : a \in A, b \in B\}, \quad d(x,A) := d(\{x\}, A).$$

Throughout the list below we will assume that Θ is inner and $\sigma(\Theta)$ is its boundary spectrum from (6.1). All of these results can be found in [7, Section 5].

(i) If Σ is a dominating set for $\mathcal{K}_\Theta$ then, for every $\zeta \in \sigma(\Theta)$, we have $d(\zeta, \Sigma) = 0$.

(ii) If Σ is a dominating set for $\mathcal{K}_\Theta$ then $d(\Sigma, \sigma(\Theta)) = 0$.

(iii) Let $\zeta \in \sigma(\Theta)$ and Σ dominating. Then there exists an $\alpha > 0$ such that for every sequence $\lambda_n \to \zeta$ with $\Theta(\lambda_n) \to 0$, there is an integer N with

$$m(\Sigma \cap I^\alpha_{\lambda_n}) \gtrsim m(I^\alpha_{\lambda_n}), \quad n \geqslant N.$$

In the above, I_λ^α is the subarc of $\mathbb{T}$ centered at $\frac{\lambda}{|\lambda|}$ with length $\alpha(1-|\lambda|)$.

(iv) Every open subset Σ of $\mathbb{T}$ such that $\sigma(\Theta) \subset \Sigma$ and $m(\Sigma) < 1$ is a dominating set for $\mathcal{K}_\Theta$.

(v) Let Θ be an inner function such that $m(\sigma(\Theta)) = 0$. Then for every $\varepsilon \in (0,1)$ there is a dominating set Σ for $\mathcal{K}_\Theta$ such that $m(\Sigma) < \varepsilon$. In particular, this is true for (CLS)-inner functions.

(vi) If $\sigma(\Theta) = \mathbb{T}$ and if Σ is a dominating set for $\mathcal{K}_\Theta$ then Σ is dense in $\mathbb{T}$.

(vii) There exists a Blaschke product B with $\sigma(B) = \mathbb{T}$ and an open subset $\Sigma \subsetneq \mathbb{T}$ dominating for $\mathcal{K}_B$.

(vi) Every model space admits a dominating set.

Theorem 6.8 shows, in the special case of the Paley-Wiener space, that when (5.2) is satisfied for sufficiently small η, then E is a dominating set for PW.

For reverse Carleson measures there is the following result from [7].

Theorem 6.13. *Let Θ be an inner function, Σ be a dominating set for $\mathcal{K}_\Theta$, and $\mu \in M_+(\mathbb{D}^-)$. Suppose that*

$$\inf_I \frac{\mu(S_I)}{m\,(I)} > 0,$$

where the above infimum is taken over all arcs $I \subset \mathbb{T}$ such that $I \cap \Sigma \neq \varnothing$. Then

$$\|f\|_m \lesssim \|f\|_\mu \qquad \forall f \in \mathcal{K}_\Theta \cap C(\mathbb{D}^-). \tag{6.14}$$

Corollary 6.15. *Under the hypotheses of Theorem 6.13, and if moreover the measure μ is assumed to be Θ-admissible, then the inequality in* (6.14) *extends to all of $\mathcal{K}_\Theta$.*

For the Hardy space, the reverse Carleson measures were characterized by the reverse reproducing kernel thesis, i.e., $\|k_\lambda\|_m \lesssim \|k_\lambda\|_\mu$ for all $\lambda \in \mathbb{D}$. For model spaces, however, the reverse reproducing kernel thesis is a spectacular failure [22].

Theorem 6.16. *Let Θ be an inner function that is not a finite Blaschke product. Then there exists a measure $\mu \in M_+(\mathbb{T})$ such that μ is a Carleson measure for $\mathcal{K}_\Theta$, the reverse estimate on reproducing kernels k_λ^Θ,*

$$\|k_\lambda^\Theta\|_\mu \gtrsim \|k_\lambda^\Theta\|_m \qquad \forall \lambda \in \mathbb{D},$$

is satisfied, but μ is not a reverse Carleson measure for $\mathcal{K}_\Theta$.

Let us see this counterexample worked out in the special case of the Paley-Wiener space PW, which, recall from our earlier discussion, is isometrically isomorphic to the model space $\mathcal{K}_\Theta$ with

$$\Theta(z) = \exp\big(2\pi \frac{z+1}{z-1}\big).$$

Consider the sequence $S = \{x_n\}_{n\in\mathbb{Z}\setminus\{0\}}$, where

$$x_n = \begin{cases} n + 1/8 & \text{if } n \text{ is even} \\ n - 1/8 & \text{if } n \text{ is odd.} \end{cases}$$

By the Kadets-Ingham theorem [39, Theorem D4.1.2], S is a minimal sampling (or complete interpolating) sequence if we include the point 0. Since S is not sampling, the discrete measure

$$\mu := \sum_{n\neq 0} \delta_{x_n}$$

does not satisfy the reverse inequality

$$\|f\|_{L^2(\mathbb{R})} \lesssim \|f\|_{L^2(\mu)} \quad \forall f \in PW.$$

However, the $L^2(\mu)$-norm of the normalized reproducing kernels

$$K_\lambda(z) = c_\lambda \operatorname{sinc}(\pi(z-\lambda)) = c_\lambda \frac{\sin(\pi(z-\lambda))}{\pi(z-\lambda)}, \qquad c_\lambda^2 \simeq (1+|\operatorname{Im}\lambda|)e^{-2\pi|\operatorname{Im}\lambda|},$$

is uniformly bounded from below. Indeed, if λ is such that $|\operatorname{Im}\lambda| > 1$ then

$$|\sin(\pi(x_n-\lambda))| \simeq e^{\pi|\operatorname{Im}\lambda|},$$

and hence

$$\int_{\mathbb{C}} |K_\lambda(x)|^2 d\mu(x) = \sum_{n\neq 0} c_\lambda^2 \left|\frac{\sin(\pi(x_n-\lambda))}{\pi(x_n-\lambda)}\right|^2 \simeq \sum_{n\neq 0} \frac{|\operatorname{Im}\lambda|}{|x_n-\lambda|^2} \simeq 1.$$

Thus it is enough to consider points $\lambda \in \mathbb{C}$ with $|\operatorname{Im}\,\lambda| \leqslant 1$. Let x_{n_0} be the point of S closest to λ. Then there is $\delta > 0$, independent of λ, such that

$$\int_{\mathbb{C}} |K_\lambda(x)|^2 d\mu(x) = \sum_{n\neq 0} |K_\lambda(x_n)|^2 \geqslant \left|\frac{\sin(\pi(x_{n_0}-\lambda))}{\pi(x_{n_0}-\lambda)}\right|^2 \geqslant \delta.$$

It is interesting to point out that μ is a Carleson measure for PW since S is in a strip and separated.

As was asked for the Paley-Wiener space PW, what are the $\mu \in M_+(\mathbb{T})$ for which

$$\|f\|_m \asymp \|f\|_\mu \quad \forall f \in \mathcal{K}_\Theta?$$

In [53] Volberg generalized the previous results and gave a complete answer for general model spaces and absolutely continuous measures $d\mu = wdm$, where $w \in L^\infty(\mathbb{T})$, $w \geqslant 0$. Let

$$\widehat{w}(z) = \int_{\mathbb{T}} w(\zeta) \frac{1-|z|^2}{|z-\zeta|^2}\, dm(\zeta), \qquad z \in \mathbb{D},$$

be the Poisson integral of w and note that $\widehat{w}$ is harmonic (and positive) on $\mathbb{D}$ and has radial boundary values equal to w m-almost everywhere [16].

Theorem 6.17. *Let $d\mu = wdm$, with $w \in L^\infty(\mathbb{T})$, $w \geqslant 0$, and let Θ be an inner function. Then the following assertions are equivalent:*

(i) $\|f\|_m \asymp \|f\|_\mu$ *for all* $f \in \mathcal{K}_\Theta$*;*

(ii) *if* $\{\lambda_n\}_{n\geqslant 1} \subset \mathbb{D}$*, then*

$$\lim_{n\to\infty} \widehat{w}(\lambda_n) = 0 \implies \lim_{n\to\infty} |\Theta(\lambda_n)| = 1;$$

(iii) $\inf\{\widehat{w}(\lambda) + |\Theta(\lambda)| : \lambda \in \mathbb{D}\} > 0.$

In particular, this theorem applies to the special case when $d\mu = \chi_\Sigma dm$, with Σ a Borel subset of $\mathbb{T}$. However the conditions obtained from Volberg's theorem are not expressed directly in terms of a density condition as was the case for PW (see Theorem 5.1). It is natural to ask if we can obtain a characterization of dominating sets for $\mathcal{K}_\Theta$ in terms of a relative density. Dyakonov answered this question in [18]. In the following result, $\mathscr{H}^2$ is the Hardy space of the upper-half plane $\{\operatorname{Im} z > 0\}$, Ψ is an inner function on $\{\operatorname{Im} z > 0\}$, and $\mathscr{K}_\Psi = (\Psi\mathscr{H}^2)^\perp$ is a model space for the upper-half plane.

Theorem 6.18. *For an inner function* Ψ *on* $\{\operatorname{Im} z > 0\}$ *the following are equivalent:*

(i) $\Psi' \in L^\infty(\mathbb{R})$*;*

(ii) Every Lebesgue measurable set $E \subset \mathbb{R}$ *for which these exists an* $\delta > 0$ *and an* $\eta > 0$ *such that*

$$|E \cap [x-\eta, x+\eta]| \geqslant \delta \quad \forall x \in \mathbb{R}$$

is dominating for the model space $\mathscr{K}_\Psi$*.*

In the case corresponding to the Paley-Wiener space PW, $\Psi(z) = e^{2i\pi z}$ and thus $|\Psi'(x)| = 2\pi$ on $\mathbb{R}$. As was shown by Garnett [20], the condition $\Psi' \in L^\infty(\mathbb{R})$ is equivalent to one of the following two conditions:

(i) $\exists h > 0$ such that

$$\inf\{|\Psi(z)| : 0 < \operatorname{Im}(z) < h\} > 0;$$

(ii) Ψ is invertible in the Douglas algebra $[H^\infty, e^{-ix}]$ (the algebra generated by H^∞ and the space of bounded uniformly continuous functions on $\mathbb{R}$).

For instance, the above conditions are satisfied when $\Psi(z) = e^{iaz}B(z)$, where $a > 0$ and B is an interpolating Blaschke product satisfying $\operatorname{dist}(B^{-1}(\{0\}), \mathbb{R}) > 0$ (e.g., the zeros of B are $\{n+i\}_{n\in\mathbb{Z}}$).

What happens if we were to replace the condition

$$\|f\|_m \asymp \|f\|_\mu \quad \forall f \in \mathcal{K}_\Theta$$

with the stronger condition

$$\|f\|_m = \|f\|_\mu \quad \forall f \in \mathcal{K}_\Theta.$$

Such "isometric measures" were characterized by Aleksandrov [1] (see also [7]).

Theorem 6.19. *For* $\mu \in M_+(\mathbb{T})$ *the following assertions are equivalent:*

(i) $\|f\|_\mu = \|f\|_m$ *for all* $f \in \mathcal{K}_\Theta$*;*

(ii) Θ *has non-tangential boundary values* μ*-almost everywhere on* $\mathbb{T}$ *and*

$$\int_{\mathbb{T}} \left| \frac{1-\overline{\Theta(z)}\Theta(\zeta)}{1-\overline{z}\zeta} \right|^2 d\mu(\zeta) = \frac{1-|\Theta(z)|^2}{1-|z|^2}, \quad z \in \mathbb{D};$$

(iii) *there exists a* $\varphi \in H^\infty$ *such that* $\|\varphi\|_\infty \leqslant 1$ *and*

$$\int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta-z|^2} d\mu(\zeta) = \operatorname{Re}\left(\frac{1+\varphi(z)\Theta(z)}{1-\varphi(z)\Theta(z)}\right), \quad z \in \mathbb{D}. \tag{6.20}$$

The condition in (6.20) says that μ is one of the so-called *Aleksandrov-Clark measures* for $b = \varphi\Theta$. It is known that the operator $V_b : L^2(\mu) \longrightarrow \mathscr{H}(b) = K_\Theta \oplus \Theta\mathscr{H}(\varphi)$ introduced in (7.4) below is an onto partial isometry, which is isometric on $H^2(\mu)$, the closure of the polynomials in $L^2(\mu)$ (see Section 7 for more on $\mathscr{H}(b)$-spaces and Aleksandrov-Clark measures). By a result of Poltoratski [44], $V_b g = g$ μ_S-a.e. where μ_S is the singular part of μ with respect to m. In particular, when φ is inner, then $\mathscr{H}(b) = \mathcal{K}_{\Theta\varphi} = \mathcal{K}_\Theta \oplus \Theta\mathcal{K}_\varphi$ and $\mu = \mu_S$ is singular, and hence for every $f = V_b g \in \mathcal{K}_\Theta$, where $g \in H^2(\mu)$, we have

$$\|f\|_m = \|V_b g\|_m = \|g\|_\mu = \|f\|_\mu.$$

When φ is not inner, Aleksandrov proves Theorem 6.19 by using the above fact for inner functions along with the fact that the isometric measures form a closed subset of the Borel measures $M(\mathbb{T})$ in the topology $\sigma(M(\mathbb{T}), C(\mathbb{T}))$.

L. de Branges [15] proved a version of Theorem 6.19 for meromorphic inner functions and Krein [21] obtained a characterization of isometric measures for $\mathcal{K}_\Theta$ using more operator theoretic langage.

7. DE BRANGES-ROVNYAK SPACES

These spaces are generalizations of the model spaces. Let

$$H_1^\infty = \{f \in H^\infty : \|f\|_\infty \leqslant 1\}$$

be the *closed unit ball in* H^∞. Recall that when Θ is inner, the model space $\mathcal{K}_\Theta$ is a closed subspace of H^2 with reproducing kernel function

$$k_\lambda^\Theta(z) = \frac{1-\overline{\Theta(\lambda)}\Theta(z)}{1-\overline{\lambda}z}, \qquad \lambda, z \in \mathbb{D}.$$

Using this as a guide, one can, for a given $b \in H_1^\infty$, define the *de Branges-Rovnyak space* $\mathscr{H}(b)$ to be the unique reproducing kernel Hilbert space of analytic functions on $\mathbb{D}$ for which

$$k_\lambda^b(z) = \frac{1-\overline{b(\lambda)}b(z)}{1-\overline{\lambda}z}, \qquad \lambda, z \in \mathbb{D},$$

is the reproducing kernel [43]. Note that the function $K(z,\lambda) := k_\lambda^b(z)$ is positive semi-definite on $\mathbb{D}$, i.e.,

$$\sum_{i,j=1}^{n} \overline{a_i} a_j K(\lambda_i, \lambda_j) \geqslant 0,$$

for all finite sets $\{\lambda_1, \ldots, \lambda_n\}$ of points in $\mathbb{D}$ and all complex numbers $a_1, \ldots, a_n$. Hence, we can associate to it a reproducing kernel Hilbert space and the above definition makes sense. There is an equivalent definition of $\mathscr{H}(b)$ via defects of certain Toeplitz operators [48].

It is well known that though these spaces play an important role in understanding contraction operators, the norms on these $\mathscr{H}(b)$ spaces, along with the elements contained in these spaces, remain mysterious. When $\|b\|_\infty < 1$ (i.e., b belongs to the interior of H_1^∞), then $\mathscr{H}(b) = H^2$ with an equivalent norm. When b is an inner function, then $\mathscr{H}(b) = \mathcal{K}_b$ with the H^2 norm. For general $b \in H_1^\infty$, $\mathscr{H}(b)$ is contractively contained in H^2 and this space is often called a "sub-Hardy Hilbert space" [48]. The analysis of these $\mathscr{H}(b)$ spaces naturally splits into two distinct cases corresponding as to whether or not b is an extreme function for H_1^∞, equivalently, $\log(1-|b|) \notin L^1(m)$.

When $b \in H_1^\infty$ is non-extreme, there is a unique outer function $a \in H_1^\infty$ such that $a(0) > 0$ and

(7.1) $$|a(\xi)|^2 + |b(\xi)|^2 = 1 \quad m\text{-a.e. } \xi \in \mathbb{T}.$$

Such a is often called the *Pythagorean mate* for b and the pair (a,b) is called a *Pythagorean pair*.

There is the, now familiar, issue of boundary behavior of $\mathscr{H}(b)$ functions when defining the integrals $\|f\|_\mu$ in the Carleson and reverse Carleson testing conditions. With the model spaces (and with H^2) there is a dense set of continuous functions for which one can sample in order to test the Carleson ($\|f\|_\mu \lesssim \|f\|_m$) and reverse Carleson conditions ($\|f\|_m \lesssim \|f\|_\mu$). For a general $\mathscr{H}(b)$ space however, it is not quite clear whether or not $\mathscr{H}(b) \cap C(\mathbb{D}^-)$ is even non-zero. In certain circumstances, for example when b is non-extreme or when b is an inner function, $\mathscr{H}(b) \cap C(\mathbb{D}^-)$ is actually dense in $\mathscr{H}(b)$. For general extreme b, this remains unknown. Thus we are forced to make some definitions.

Definition 7.2. For $\mu \in M_+(\mathbb{D}^-)$ we say that an analytic function f on $\mathbb{D}$ is *μ-admissible* if the non-tangential limits of f exist μ-almost everywhere on $\mathbb{T}$. We let $\mathscr{H}(b)_\mu$ denote the set of μ-admissible functions in $\mathscr{H}(b)$.

With this definition in mind, if $f \in \mathscr{H}(b)_\mu$, then defining f on the carrier of $\mu|_\mathbb{T}$ via its non-tangential boundary values, we see that $\|f\|_\mu$ is well defined with a value in $[0, +\infty]$.

Of course when μ is carried on $\mathbb{D}$, i.e., $\mu(\mathbb{T}) = 0$, then $\mathscr{H}(b)_\mu = \mathscr{H}(b)$. Hence Definition 7.2 only comes into play when μ has part of the unit circle $\mathbb{T}$ in its carrier. Note that $\mathscr{H}(b) = \mathscr{H}(b)_m$ since $\mathscr{H}(b) \subset H^2$. However, there are often other μ, even ones with non-trivial singular parts on $\mathbb{T}$ with respect to m, for which $\mathscr{H}(b) = \mathscr{H}(b)_\mu$. The Clark measures associated with an inner function b have this property [7, 12].

Definition 7.3. A measure $\mu \in M_+(\mathbb{D}^-)$ is a *Carleson measure* for $\mathscr{H}(b)$ if $\mathscr{H}(b)_\mu = \mathscr{H}(b)$ and $\|f\|_\mu \lesssim \|f\|_b$ for all $f \in \mathscr{H}(b)$.

When $b \equiv 0$, i.e., when $\mathscr{H}(b) = H^2$ then, as a consequence of Carleson's theorem (see Theorem 2.2) for H^2, we see that when μ satisfies $\mu(S_I) \lesssim |I|$ for all arcs I, then $\mu|_{\mathbb{T}} \ll m$ and so $\mathscr{H}(b)_\mu = \mathscr{H}(b)$. When b is an inner function, recall a discussion following (6.5) which says that if the Carleson testing condition $\|f\|_\mu \lesssim \|f\|_m$ holds for all $f \in \mathscr{H}(b) \cap C(\mathbb{D}^-)$, then $\mathscr{H}(b)_\mu = \mathscr{H}(b)$. So in these two particular cases, the delicate issue of defining the integrals in $\|f\|_\mu$ for $f \in \mathscr{H}(b)$ seems to sort itself out. For general b, we do not have this luxury.

Lacey et al. [29] solved the longstanding problem of characterizing the two-weight inequalities for Cauchy transforms. Let us take a moment to indicate how their results can be used to discuss Carleson measures for $\mathscr{H}(b)$. Let σ be the Aleksandrov-Clark measure associated with b, that is the unique $\sigma \in M_+(\mathbb{T})$ satisfying

$$\frac{1 - |b(z)|^2}{|1 - b(z)|^2} = \int_{\mathbb{T}} \frac{1 - |z|^2}{|z - \zeta|^2}\, d\sigma(\zeta), \quad z \in \mathbb{D}.$$

Let $V_b : L^2(\sigma) \longrightarrow \mathscr{H}(b)$ be the operator defined by

$$(V_b f)(z) = (1 - b(z)) \int_{\mathbb{T}} \frac{f(\zeta)}{1 - \bar{\zeta} z}\, d\sigma(\zeta) = (1 - b(z))(C_\sigma f)(z), \tag{7.4}$$

where C_σ is the Cauchy transform

$$(C_\sigma f)(z) = \int_{\mathbb{T}} \frac{f(\zeta)}{1 - \bar{\zeta} z} d\sigma(\zeta).$$

It is known [48] that V_b is a partial isometry from $L^2(\sigma)$ onto $\mathscr{H}(b)$ and

$$\operatorname{Ker} V_b = \operatorname{Ker} C_\sigma = (H^2(\sigma))^\perp.$$

Here $H^2(\sigma)$ denotes the closure of polynomials in $L^2(\sigma)$ and the $\perp$ is in $L^2(\sigma)$. As a consequence, since every function $f \in \mathscr{H}(b)$ can be written as $f = V_b g$ for some $g \in H^2(\sigma)$, μ is a Carleson measure for $\mathscr{H}(b)$ if and only if

$$\|V_b g\|_\mu = \|f\|_\mu \lesssim \|f\|_b = \|V_b g\|_b = \|g\|_\sigma \quad \forall g \in H^2(\sigma).$$

Setting $\nu_{b,\mu} := |1 - b|^2 \mu$, we have

$$\|V_b g\|_\mu^2 = \int_{\mathbb{D}^-} |1 - b|^2 |C_\sigma g|^2\, d\mu = \|C_\sigma g\|_{\nu_{b,\mu}}^2.$$

This yields the following:

Theorem 7.5. *Let $\mu \in M_+(\mathbb{D}^-)$, b a μ-admissible function in H_1^∞, and $\nu_{b,\mu} := |1 - b|^2 \mu$. Then the following are equivalent:*

- *(i) μ is a Carleson measure for $\mathscr{H}(b)$;*
- *(ii) The Cauchy transform C_σ is a bounded operator from $L^2(\sigma)$ into $L^2(\mathbb{D}^-, \nu_{b,\mu})$, where σ is the Aleksandrov-Clark measure associated with b.*

We refer the reader to [29, Theorem 1.7] for a description of the boundedness of the Cauchy transform operator C_σ. However, it should be noted that the characterization of Carleson measures for $\mathscr{H}(b)$, obtained combining Theorem 7.5 and [29, Theorem 1.7], is not purely geometric.

The following result from [6], similar in flavor to Theorem 6.7, discusses the Carleson measures for $\mathscr{H}(b)$.

Theorem 7.6. *For $b \in H_1^\infty$ and $\varepsilon \in (0,1)$ define*

$$\Omega(b,\varepsilon) := \{z \in \mathbb{D} : |b(z)| < \varepsilon\},$$

$$\Sigma(b) := \left\{\zeta \in \mathbb{T} : \varliminf_{z \to \zeta} |b(z)| < 1\right\},$$

$$\widetilde{\Omega}(b,\varepsilon) := \Omega(b,\varepsilon) \cup \Sigma(b).$$

Let $\mu \in M_+(\mathbb{D}^-)$ and define the following conditions:

(i) $\mu(S_I) \lesssim |I|$ for all arcs $I \subset \mathbb{T}$ for which $I \cap \widetilde{\Omega}(b,\varepsilon) \neq \varnothing$;

(ii) $\mathscr{H}(b)_\mu = \mathscr{H}(b)$ and $\|f\|_\mu \lesssim \|f\|_b$ for all $f \in \mathscr{H}(b)$;

(iii) $\mathscr{H}(b)_\mu = \mathscr{H}(b)$ and $\|k_\lambda^b\|_\mu \lesssim \|k_\lambda^b\|_b$ for all $\lambda \in \mathbb{D}$.

Then $(i) \implies (ii) \implies (iii)$. Moreover, suppose there exists an $\varepsilon \in (0,1)$ such that $\Omega(b,\varepsilon)$ is connected and its closure contains $\Sigma(b)$. Then $(i) \iff (ii) \iff (iii)$.

It should be noted here that, contrary to the inner case, the containment $\Sigma(b) \subset \operatorname{clos}(\Omega,\varepsilon)$ is not, in general, automatic. Indeed, when $b(z) = (1+z)/2$, one can easily check that the above containment is not satisfied.

Here is a complete description of the Carleson measures for a very specific b [8]. Note that if b is a non-extreme rational function (e.g., rational but not a Blaschke product), one can show that the Pythagorean mate a from (7.1) is also a rational function.

Theorem 7.7. *Let $b \in H_1^\infty$ be rational and non-extreme and let $\mu \in M_+(\mathbb{D}^-)$. Then the following assertions are equivalent:*

(1) μ is a Carleson measure for $\mathscr{H}(b)$;

(2) $|a|^2\, d\mu$ is a Carleson measure for H^2.

If $b(z) = (1+z)/2$ then $a(z) = (1-z)/2$ and, if μ is the measure supported on $(0,1)$ defined by $d\mu(t) = (1-t)^{-\beta} dt$, for $\beta \in (0,1]$, we can use Theorem 7.7 to see that μ is Carleson measure for $\mathscr{H}(b)$. However, μ is not a Carleson measure for H^2. One can see this by considering the arcs $I_\vartheta = (e^{-i\vartheta}, e^{i\vartheta})$, $\vartheta \in (0,\pi/2)$, and observing that

$$\sup_\vartheta \frac{\mu(S(I_\vartheta))}{|I_\vartheta|} = \infty.$$

If b is a μ-admissible function, then so are all of the reproducing kernels k^b_λ (along with finite linear combinations of them) and thus, with this admissibility assumption on b, $\mathscr{H}(b)_\mu$ is a dense linear manifold in $\mathscr{H}(b)$. This motivates our definition of a reverse Carleson measure for $\mathscr{H}(b)$.

Definition 7.8. For $\mu \in M_+(\mathbb{D}^-)$ and $b \in H^\infty_1$, we say that μ is a *reverse Carleson measure for* $\mathscr{H}(b)$ if $\mathscr{H}(b)_\mu$ is dense in $\mathscr{H}(b)$ and $\|f\|_b \lesssim \|f\|_\mu$ for all $f \in \mathscr{H}(b)_\mu$. In this definition, we allow the possibility for the integral $\|f\|_\mu$ to be infinite.

Here is a reverse Carleson measure result from [8] which focuses on the case when b is non-extreme.

Theorem 7.9. *Let $\mu \in M_+(\mathbb{D}^-)$ and let $b \in H^\infty_1$ be non-extreme and μ-admissible. If $h = d\mu|_{\mathbb{T}}/dm$, then the following assertions are equivalent:*

(i) μ is a reverse Carleson mesure for $\mathscr{H}(b)$;

(ii) $\|k^b_\lambda\|_b \lesssim \|k^b_\lambda\|_\mu$ for all $\lambda \in \mathbb{D}$;

(iii) $d\nu := (1-|b|)d\mu$ satisfies
$$\inf_I \frac{\nu(S_I)}{m(I)} > 0;$$

(iv) $\operatorname{ess\,inf}_{\mathbb{T}}(1-|b|)h > 0$.

The proof of this results is in the same spirit as Theorem 2.3. Also note that the condition (iv) implies that $(1-|b|)^{-1} \in L^1$. As a consequence of this observation, we see that if $b \in H^\infty_1$ is non-extreme and such that $(1-|b|)^{-1} \notin L^1$, then there are *no* reverse Carleson measures for $\mathscr{H}(b)$.

As was done with many of the other spaces discussed in this survey, one can say something about the equivalent measures for $\mathscr{H}(b)$ [8].

Theorem 7.10. *Let $b \in H^\infty_1$ be non-extreme and $\mu \in M_+(\mathbb{D}^-)$. Then the following are equivalent:*

(i) $\mathscr{H}(b)_\mu = \mathscr{H}(b)$ and $\|f\|_\mu \asymp \|f\|_b$ for all $f \in \mathscr{H}(b)$;

(ii) The following conditions hold:

(a) a is μ-admissible,

(b) (a,b) is a corona pair, i.e.,
$$\inf\{|a(z)| + |b(z)| : z \in \mathbb{D}\} > 0;$$

(c) $|a|^2$ satisfies the Muckenhoupt (A_2) condition, i.e.,
$$\sup_I \left(\frac{1}{m(I)}\int_I |a|^{-2}\,dm\right)\left(\frac{1}{m(I)}\int_I |a|^2\,dm\right) < \infty,$$
where I runs over all subarcs of $\mathbb{T}$;

(d) $d\nu := |a|^2\, d\mu$ *satisfies*

$$0 < \inf_I \frac{\nu\,(S_I)}{m(I)} \leqslant \sup_I \frac{\nu\,(S_I)}{m(I)} < \infty,$$

where the infimum and supremum above are taken over all open arcs I *of* $\mathbb{T}$.

One should note that if (a, b) is a corona pair and $|a|^2 \in (A_2)$, then $\mathscr{H}(b) = \mathscr{M}(a)$, where $\mathscr{M}(a) = aH^2$ equipped with the range norm, i.e., $\|ag\|_{\mathscr{M}(a)} = \|g\|_m$, for any $g \in H^2$ [49, IX-5]. Hence the above result says that it is possible to obtain an equivalent norm on $\mathscr{H}(b)$ expressed in terms of an integral only when $\mathscr{H}(b) = \mathscr{M}(a)$.

Surely an example is important here: Let $a(z) := c_\alpha(1 - z)^\alpha$, where $\alpha \in (0, 1/2)$ and c_α is suitable chosen so that $a \in H^\infty_1$. When $0 < \alpha < 1/2$, one can show that $|a|^2$ satisfies the (A_2) condition. Choose b to be the outer function in H^∞_1 satisfying $|a|^2 + |b|^2 = 1$ on $\mathbb{T}$. Standard theory [24], using the fact that a is Hölder continuous on $\mathbb{D}^-$, will show that b is continuous on $\mathbb{D}^-$. From here it follows that (a, b) is a corona pair. If $\sigma \in M_+(\mathbb{D}^-)$ is any Carleson measure for H^2, then one can show that $d\mu := |a|^{-2}dm + d\sigma$ satisfies the conditions of Theorem 7.10.

For $\mathscr{H}(b)$ spaces when b non-extreme, the isometric measures: $\|f\|_\mu = \|f\|_b$ for all $f \in \mathscr{H}(b)$, are not worth discussing as illustrated by the following result.

Theorem 7.11. *When* b *is non-constant and non-extreme, there are no positive isometric measures for* $\mathscr{H}(b)$.

Also not worth discussing for general $\mathscr{H}(b)$ spaces is the notion of dominating sets [8]: $E \subset \mathbb{T}$, $0 < m(E) < 1$, for which

$$\|f\|^2_b \lesssim \int_E |f|^2 dm \quad \forall f \in \mathscr{H}(b).$$

Indeed, we have the following:

Theorem 7.12. *Let* $b \in H^\infty_1$ *such that* $\mathscr{H}(b)$ *has a dominating set. Then either* b *is an inner function or* $\|b\|_\infty < 1$.

As one can see, the case for extreme b seems to be very much open. When b is inner, much has been said about the Carleson and reverse Carleson measures for $\mathscr{H}(b) = \mathcal{K}_b$. When b is extreme but not inner, there are a few things one can say [8] but there is much work to be done to complete the picture.

8. Harmonically weighted Dirichlet spaces

For $\mu \in M_+(\mathbb{T})$ let

$$\varphi_\mu(z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\xi - z|^2} d\mu(\xi), \quad z \in \mathbb{D},$$

denote the Poisson integral of μ. The *harmonically weighted Dirichlet space* $\mathscr{D}(\mu)$ [19, 45] is the set of all analytic functions f on $\mathbb{D}$ for which

$$\int_{\mathbb{D}} |f'|^2 \varphi_\mu dA < \infty,$$

where $dA = dxdy/\pi$ is normalized planar measure on $\mathbb{D}$. Notice that when $\mu = m$, we have $\varphi_\mu \equiv 1$ and $\mathscr{D}(\mu)$ becomes the classical Dirichlet space [19]. One can show that $\mathscr{D}(\mu) \subset H^2$ [45, Lemma 3.1] and the norm $\|\cdot\|_{\mathscr{D}(\mu)}$ given by

$$\|f\|^2_{\mathscr{D}(\mu)} := \int_{\mathbb{T}} |f|^2 dm + \int_{\mathbb{D}} |f'|^2 \varphi_\mu dA$$

makes $\mathscr{D}(\mu)$ into a reproducing kernel Hilbert space of analytic functions on $\mathbb{D}$. It is known that both the polynomials as well as the linear span of the Cauchy kernels form dense subsets of $\mathscr{D}(\mu)$ [45, Corollary 3.8].

When $\zeta \in \mathbb{T}$ and $d\mu = \delta_\zeta$, a result from [50] shows that

$$\mathscr{D}(\delta_\zeta) = \mathscr{H}(b),$$

where $w_0 = (3 - \sqrt{5})/2$ and

$$b(z) = \frac{(1 - w_0)\overline{\zeta} z}{1 - w_0 \overline{\zeta} z}. \tag{8.1}$$

Furthermore, the norms on these spaces are the same. In fact, these are the only harmonically weighted Dirichlet spaces which are equal to an $\mathscr{H}(b)$ space with equal norm [11]. In [14] it was shown that if

$$\mu = \sum_{j=1}^{n} c_j \delta_{\zeta_j}, \quad c_j > 0, \zeta_j \in \mathbb{T} \tag{8.2}$$

is a finite linear combination of point masses on $\mathbb{T}$ and a is the unique polynomial with $a(0) > 0$ and with simple zeros at ζ_j (and no other zeros) and b is the Pythagorean mate for a (which must also be a polynomial), then $\mathscr{H}(b) = \mathscr{D}(\mu)$ with equivalent norms. In this case we can use Theorem 7.7 to obtain a characterization of the Carleson measures for $\mathscr{D}(\mu)$:

Theorem 8.3. *For μ as in* (8.2) *and $\nu \in M_+(\mathbb{D}^-)$, the following assertions are equivalent:*

(i) ν is a Carleson measure for $\mathscr{D}(\mu)$;

(ii) $\prod_{i=1}^n |z - \zeta_i|^2 \, d\nu$ is a Carleson measure for H^2.

This result appeared in [9] (see also [10]). In fact, Theorem 6.1 from [9] shows that the above conditions are equivalent to

$$\|k_\lambda^{\mathscr{D}(\mu)}\|_\nu \lesssim \|k_\lambda^{\mathscr{D}(\mu)}\|_{\mathscr{D}(\mu)} \quad \forall \lambda \in \mathbb{D}.$$

In other words, at least when μ is a linear combination of point masses, the reproducing kernel thesis characterizes the Carleson measures for $\mathscr{D}(\mu)$.

The discussion of reverse Carleson measures for $\mathscr{D}(\mu)$ is dramatically simpler since they do not exist! Indeed, suppose that $\nu \in M_+(\mathbb{D}^-)$ and $\|f\|_\mu \lesssim \|f\|_\nu$ for all $f \in \mathscr{D}(\mu)$. In particular, this is true for the monomials z^n, $n \geqslant 0$. But $\|z^n\|_\nu \lesssim 1$ and $\|z^n\|_\mu^2 = 1 + n\mu(\mathbb{T})$, which gives a contradiction when n tends to ∞.

We point out some related results from [10] which discuss a type of reverse Carleson measure for $\mathscr{D}(\mu)$ spaces except that the definitions of "reverse Carleson measures" and "sets of domination" (dominating sets) are quite different, and not equivalent, to ours.

References

[1] A. B. Aleksandrov. Isometric embeddings of co-invariant subspaces of the shift operator. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 232(Issled. po Linein. Oper. i Teor. Funktsii. 24):5–15, 213, 1996.

[2] A. B. Aleksandrov. Embedding theorems for coinvariant subspaces of the shift operator. II. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 262(Issled. po Linein. Oper. i Teor. Funkts. 27):5–48, 231, 1999.

[3] A. D. Baranov. Bernstein-type inequalities for shift-coinvariant subspaces and their applications to Carleson embeddings. *J. Funct. Anal.*, 223(1):116–146, 2005.

[4] A. D. Baranov. Stability of bases and frames of reproducing kernels in model spaces. *Ann. Inst. Fourier (Grenoble)*, 55(7):2399–2422, 2005.

[5] A. D. Baranov, R. Bessonov, and V. Kapustin. Symbols of truncated Toeplitz operators. *J. Funct. Anal.*, 261(12):3437–3456, 2011.

[6] A.D. Baranov, E. Fricain, and J. Mashreghi. Weighted norm inequalities for de Branges-Rovnyak spaces and their applications. *Amer. J. Math.*, 132(1):125–155, 2010.

[7] A. Blandignères, E. Fricain, F. Gaunard, A. Hartmann, and W. Ross. Reverse Carleson embeddings for model spaces. *J. Lond. Math. Soc. (2)*, 88(2):437–464, 2013.

[8] A. Blandignères, E. Fricain, F. Gaunard, A. Hartmann, and W.T. Ross. Direct and reverse Carleson measures for $\mathscr{H}(b)$ spaces. *Indiana Univ. Math. J.*, 64:1027–1057, 2015.

[9] G. R. Chacón, E. Fricain, and M. Shabankhah. Carleson measures and reproducing kernel thesis in Dirichlet-type spaces. *Algebra i Analiz*, 24(6):1–20, 2012.

[10] G. Chacon P. *Carleson-type inequalitites in harmonically weighted Dirichlet spaces*. PhD thesis, University of Tennessee, 2010.

[11] N. Chevrot, D. Guillot, and T. Ransford. De Branges-Rovnyak spaces and Dirichlet spaces. *J. Funct. Anal.*, 259(9):2366–2383, 2010.

[12] J. Cima, A. Matheson, and W. Ross. *The Cauchy transform*, volume 125 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2006.

[13] B. Cohn. Carleson measures for functions orthogonal to invariant subspaces. *Pacific J. Math.*, 103(2):347–364, 1982.

[14] C. Costara and T. Ransford. Which de Branges-Rovnyak spaces are Dirichlet spaces (and vice versa)? *J. Funct. Anal.*, 265(12):3204–3218, 2013.

[15] L. de Branges. *Hilbert spaces of entire functions*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1968.

[16] P. L. Duren. *Theory of H^p spaces*. Academic Press, New York, 1970.

[17] Peter Duren and Alexander Schuster. *Bergman spaces*, volume 100 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2004.

[18] K. M. D′yakonov. Entire functions of exponential type and model subspaces in H^p. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 190(Issled. po Linein. Oper. i Teor. Funktsii. 19):81–100, 186, 1991.

[19] O. El-Fallah, K. Kellay, J. Mashreghi, and T. Ransford. *A primer on the Dirichlet space*, volume 203 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2014.

[20] J. Garnett. *Bounded analytic functions*, volume 236 of *Graduate Texts in Mathematics*. Springer, New York, first edition, 2007.

[21] M. L. Gorbachuk and V. I. Gorbachuk. *M. G. Krein's lectures on entire operators*, volume 97 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1997.

[22] A. Hartmann, X. Massaneda, A. Nicolau, and J. Ortega-Cerdà. Reverse Carleson measures in Hardy spaces. *Collect. Math.*, 65:357–365, 2014.

[23] W. W. Hastings. A Carleson measure theorem for Bergman spaces. *Proc. Amer. Math. Soc.*, 52:237–241, 1975.

[24] V. P. Havin and F. A. Šamojan. Analytic functions with a Lipschitzian modulus of the boundary values. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 19:237–239, 1970.

[25] H. Hedenmalm, B. Korenblum, and K. Zhu. *Theory of Bergman spaces*, volume 199 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[26] S. Janson, J. Peetre, and R. Rochberg. Hankel forms and the Fock space. *Rev. Mat. Iberoamericana*, 3(1):61–138, 1987.

[27] V. È. Kacnel′son. Equivalent norms in spaces of entire functions. *Mat. Sb. (N.S.)*, 92(134):34–54, 165, 1973.

[28] P. Koosis. *Introduction to H_p spaces*, volume 115 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, second edition, 1998. With two appendices by V. P. Havin [Viktor Petrovich Khavin].

[29] M. T. Lacey, E. T. Sawyer, Ch.-Y. Shen, I. Uriarte-Tuero, and B. D. Wick. Two weight inequalities for the Cauchy transform from $\mathbb{R}$ to $\mathbb{C}_+$. ArXiv:1310.4820v2.

[30] P. Lefèvre, D. Li, H. Queffélec, and L. Rodríguez-Piazza. Some revisited results about composition operators on Hardy spaces. *to appear, Revista Mat. Iberoamericana*.

[31] V. Ja. Lin. On equivalent norms in the space of square integrable entire functions of exponential type. *Mat. Sb. (N.S.)*, 67 (109):586–608, 1965.

[32] V. N. Logvinenko and Ju. F. Sereda. Equivalent norms in spaces of entire functions of exponential type. *Teor. Funkciĭ Funkcional. Anal. i Priložen.*, 175(Vyp. 20):102–111, 1974.

[33] D. H. Luecking. Inequalities on Bergman spaces. *Illinois J. Math.*, 25(1):1–11, 1981.

[34] D. H. Luecking. Equivalent norms on L^p spaces of harmonic functions. *Monatsh. Math.*, 96(2):133–141, 1983.

[35] D. H. Luecking. Forward and reverse Carleson inequalities for functions in Bergman spaces and their derivatives. *Amer. J. Math.*, 107(1):85–111, 1985.

[36] D. H. Luecking. Dominating measures for spaces of analytic functions. *Illinois J. Math.*, 32(1):23–39, 1988.

[37] J. W. Moeller. On the spectra of some translation invariant spaces. *J. Math. Anal. Appl.*, 4:276–296, 1962.

[38] F. Nazarov and A. L. Vol′berg. The Bellman function, the two-weight Hilbert transform and embeddings of the model spaces K_Θ. *J. Anal. Math.*, 87:385–414, 2002.

[39] N. K. Nikolski. *Operators, functions, and systems: an easy reading. Vol. 1*, volume 92 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002. Hardy, Hankel, and Toeplitz, Translated from the French by Andreas Hartmann.

[40] J. Ortega-Cerdà. Sampling measures. *Publ. Mat.*, 42(2):559–566, 1998.

[41] B. P. Panejah. On some problems in harmonic analysis. *Dokl. Akad. Nauk SSSR*, 142:1026–1029, 1962.

[42] B. P. Panejah. Certain inequalities for functions of exponential type and a priori estimates for general differential operators. *Uspehi Mat. Nauk*, 21(3 (129)):75–114, 1966.

[43] V. Paulsen and M. Raghupathi. *An Introduction to the theory of reproducing kernel Hilbert spaces*. Cambridge University Press, 2016.

[44] A. Poltoratski. Boundary behavior of pseudocontinuable functions. *Algebra i Analiz*, 5(2):189–210, 1993.

[45] S. Richter. A representation theorem for cyclic analytic two-isometries. *Trans. Amer. Math. Soc.*, 328(1):325–349, 1991.

[46] W. Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.

[47] D. Sarason. Invariant subspaces. In *Topics in operator theory*, pages 1–47. Math. Surveys, No. 13. Amer. Math. Soc., Providence, R.I., 1974.

[48] D. Sarason. *Sub-Hardy Hilbert spaces in the unit disk*. University of Arkansas Lecture Notes in the Mathematical Sciences, 10. John Wiley & Sons Inc., New York, 1994. A Wiley-Interscience Publication.

[49] D. Sarason. *Sub-Hardy Hilbert spaces in the unit disk*. University of Arkansas Lecture Notes in the Mathematical Sciences, 10. John Wiley & Sons Inc., New York, 1994. A Wiley-Interscience Publication.

[50] D. Sarason. Local Dirichlet spaces as de Branges-Rovnyak spaces. *Proc. Amer. Math. Soc.*, 125(7):2133–2139, 1997.

[51] D. Sarason. Algebraic properties of truncated Toeplitz operators. *Oper. Matrices*, 1(4):491–526, 2007.

[52] K. Seip. *Interpolation and sampling in spaces of analytic functions*, volume 33 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2004.

[53] A. L. Vol′berg. Thin and thick families of rational fractions. In *Complex analysis and spectral theory (Leningrad, 1979/1980)*, volume 864 of *Lecture Notes in Math.*, pages 440–480. Springer, Berlin, 1981.

[54] A. L. Vol′berg and S. R. Treil′. Embedding theorems for invariant subspaces of the inverse shift operator. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 149(Issled. Linein. Teor. Funktsii. XV):38–51, 186–187, 1986.

[55] K. Zhu. *Analysis on Fock spaces*, volume 263 of *Graduate Texts in Mathematics*. Springer, New York, 2012.

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