ON THE FIRST ORDER ASYMPTOTICS OF PARTIAL BERGMAN KERNELS

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ABSTRACT. We show that under very general assumptions the partial Bergman kernel function of sections vanishing along an analytic hypersurface has exponential decay in a neighborhood of the vanishing locus. Considering an ample line bundle, we obtain a uniform estimate of the Bergman kernel function associated to a singular metric along the hypersurface. Finally, we study the asymptotics of the partial Bergman kernel function on a given compact set and near the vanishing locus.

CONTENTS

1. INTRODUCTION

Partial Bergman kernels were recently studied in different contexts, especially Kähler geometry [RS13, PS14, RWN14] or random polynomials [Ber07, SZ04].

Let us consider the following general setting.

(A) (X, ω) is a compact Hermitian manifold of dimension n, Σ is a smooth analytic hypersurface of X, and $t > 0$ is a fixed real number.

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(B) (L, h) is a singular Hermitian holomorphic line bundle on X with singular metric h which has locally bounded weights.

We define the space

$$H^0_0(X, L^p) := H^0\big(X, L^p \otimes \mathcal{O}\big(-\lfloor tp \rfloor \Sigma\big)\big)$$

of holomorphic sections of the p-th tensor power L^p vanishing to order at least $\lfloor tp \rfloor$ along Σ, where $\lfloor x \rfloor$ denotes the integral part of $x \in \mathbb{R}$. Set $d_p = \dim H^0(X, L^p)$ and $d_{0,p} = \dim H^0_0(X, L^p)$. We introduce on $H^0(X, L^p)$ the L^2 inner product $(\cdot\,,\cdot)_p$ induced by the metric $h_p = h^{\otimes p}$ and the volume form $\omega^n/n!$, see (9). This inner product is inherited by $H^0_0(X, L^p)$. The (full) Bergman kernel function is defined by taking an orthonormal basis $\{S^p_j : 1 \leq j \leq d_p\}$ of $(H^0(X, L^p), (\cdot\,,\cdot)_p)$ and setting

$$P_p(x) = \sum_{j=1}^{d_p} |S^p_j(x)|^2_{h_p}\,, \quad |S^p_j(x)|^2_{h_p} := \langle S^p_j(x), S^p_j(x)\rangle_{h_p},\ x \in X.$$

By considering an orthonormal basis $\{S^p_j : 1 \leq j \leq d_{0,p}\}$ of $(H^0_0(X, L^p), (\cdot\,,\cdot)_p)$, we define the *partial Bergman kernel function* $P_{0,p}$ by

$$P_{0,p}(x) = \sum_{j=1}^{d_{0,p}} |S^p_j(x)|^2_{h_p}\,,\ x \in X.$$

Note that this definition is independent of the choice of basis, cf. (10).

The asymptotics of the Bergman kernel function for a positive line bundle (L, h) [Cat99, Zel98], see also [MM07] for a comprehensive study, is very important in understanding the Yau-Tian-Donaldson conjecture. On the other hand, partial Bergman kernels are useful in connection to the slope semi-stability with respect to a submanifold [RT06]. On a toric variety X (and for a toric Σ) this study was carried out in [PS14]. In this context it is shown that the partial Bergman kernel has an asymptotic expansion, having rapid decay of order $p^{-\infty}$ in a neighborhood $U(\Sigma)$ of Σ, and giving the full Bergman kernel function to order $p^{-\infty}$ outside the closure of $U(\Sigma)$. Moreover [PS14] gives a complete distributional asymptotic expansion on X, whose leading term has an additional Dirac delta measure plus a dipole measure over $\partial U(\Sigma)$. These results were generalized in [RS13] to the case when the data in question are invariant under an S^1-action.

In general, if no symmetry is assumed, it was shown in [Ber07, Theorem 4.3] that if the bundle $L \otimes \mathscr{O}(-\Sigma)$ is ample, there exists a neighborhood $U(\Sigma)$ of Σ, such that $P_{0,p}(x)$ has exponential decay on $U(\Sigma)$ and $p^{-n}P_{0,p}(x)$ converges to $c_1(L, h)^n/\omega^n$ in L^1 outside the closure of $U(\Sigma)$.

Our first result is that under the very general hypotheses (A) and (B) above (in particular, without any positivity condition), the partial Bergman kernel function decays exponentially in a neighborhood of the divisor Σ.

Theorem 1.1. *Assume that conditions (A)-(B) are fulfilled. Then there exist a neighborhood* U_t *of* Σ *and a constant* $a \in (0,1)$ *such that* $P_{0,p} \le a^p$ *on* U_t *for* $p > 2t^{-1}$. *In particular* $P_{0,p} = O(p^{-\infty})$ *as* $p \to \infty$ *on* U_t.

For more precise statements see Theorem 3.1 and Corollary 3.3.

An object which is closely related to the partial Bergman kernel is the Bergman kernel for a singular metric. The full asymptotic expansion on compact subsets of the regular part of the metric was established in [HM14, Theorem 1.8]. We are here concerned with asymptotics at arbitrary points with dependence on the distance to the singular set. More precisely, we will consider the following situation.

Let $S_\Sigma \in H^0(X, \mathcal{O}(\Sigma))$ be a canonical holomorphic section of the line bundle $\mathcal{O}(\Sigma)$, vanishing to first order on Σ. We fix a smooth Hermitian metric h_Σ on $\mathcal{O}(\Sigma)$ such that

$$\varrho := \log \left|S_\Sigma\right|_{h_\Sigma} < 0 \ \text{ on } X. \tag{1}$$

We consider a function $\xi : X \to \mathbb{R} \cup \{-\infty\}$, smooth on $X \setminus \Sigma$, such that $\xi = t\rho$ in a neighborhood U of Σ. Let $\operatorname{dist}(\cdot\,,\cdot)$ be the distance on X induced by ω. Our main result is the following:

Theorem 1.2. *Let* $(X,\omega), (L,h), \Sigma$ *be as in (A)-(B), and assume* ω *is Kähler,* h *is smooth, and* $c_1(L,h) \ge \varepsilon\omega$ *for some constant* $\varepsilon > 0$. *Consider the singular Hermitian metric* $\widetilde{h} = he^{-2\xi}$ *on* L *and let* $\widetilde{P}_p$ *be the Bergman kernel function of* $H^0_{(2)}(X, L^p, \widetilde{h}_p, \omega^n/n!)$, *where* $\widetilde{h}_p := \widetilde{h}^{\otimes p}$. *There exists a constant* $C > 1$ *such that for every* $x \in X \setminus \Sigma$ *and every* $p \in \mathbb{N}$ *with* $p \operatorname{dist}(x,\Sigma)^{8/3} > C$ *we have*

$$\left| \frac{\widetilde{P}_p(x)}{p^n} \, \frac{\omega_x^n}{c_1(L,\widetilde{h})_x^n} - 1 \right| \le C p^{-1/8} \,. \tag{2}$$

Theorem 1.2 can be interpreted in two ways. First, if x runs in a compact set $K \subset X \setminus \Sigma$, we have a concrete bound $p_0 = C \operatorname{dist}(K,\Sigma)^{-8/3}$ such that for $p > p_0$ the estimate (2) holds. By [HM14, Theorem 1.8] we have $\widetilde{P}_p(x) = \sum_{r=0}^{\infty} \boldsymbol{b}_r(x) p^{n-r} + O(p^{-\infty})$ as $p \to \infty$ locally uniformly on $X \setminus \Sigma$. Hence, there exists $p_0(K) \in \mathbb{N}$ and C_K such that for $p > p_0(K)$ we have

$$\left| \frac{\widetilde{P}_p(x)}{p^n} \, \frac{\omega_x^n}{c_1(L,\widetilde{h})_x^n} - 1 \right| \le C_K p^{-1} \ \text{ on } K.$$

However, $p_0(K)$ is not easy to determine.

We can also recast Theorem 1.2 as a uniform estimate in p for the singular Bergman kernel on compact sets of $X \setminus \Sigma$ whose distance to Σ decreases as $p^{-3/8}$. Indeed, set $K_p = \{x \in X : \operatorname{dist}(x,\Sigma) \ge (C/p)^{3/8}\}$. Then (2) holds on K_p for every p.

We consider now the global behavior of the partial Bergman kernel. Given a compact set $K \subset X \setminus \Sigma$ we set

$$\begin{aligned} t_0(K) := \sup \Big\{ t > 0 : \exists\, \eta \in \mathscr{C}^\infty(X,[0,1]),\ \operatorname{supp}\eta \subset X \setminus K,\ \eta = 1 \text{ near } \Sigma, \\ \text{and } c_1(L,h) + t\, dd^c(\eta\varrho) \text{ is a Kähler current on } X \Big\}. \end{aligned} \tag{3}$$

A consequence of Theorems 1.1 and 1.2 is the following result about the asymptotics of the partial Bergman kernel:

Theorem 1.3. *Let $(X,\omega),(L,h),\Sigma$ be as in (A)-(B), and assume ω is Kähler, h is smooth, and $c_1(L,h) \geq \varepsilon\omega$ for some constant $\varepsilon > 0$. Let $K \subset X \setminus \Sigma$ be a compact set and let $t \in (0, t_0(K))$. Then there exist constants $C > 1$, $M > 1$ and a neighborhood U_t of Σ, all depending on t, such that for $x \in U_t$ we have*

$$Me^{t\varrho(x)} < 1 \text{ and } P_{0,p}(x) \leq (Me^{t\varrho(x)})^p \text{ for } p > 2/t, \tag{4}$$

$$P_{0,p}(x) \geq \frac{p^n}{C}\exp(2tp\varrho(x)) \text{ for } p\operatorname{dist}(x,\Sigma)^{8/3} > C, \tag{5}$$

where the function ϱ is defined in (1). *Moreover, we have uniformly on K,*

$$P_{0,p}(x) = P_p(x) + O(p^{-\infty}),\ \ p \to \infty, \tag{6}$$

and in particular,

$$P_{0,p}(x) = \boldsymbol{b}_0(x)p^n + \boldsymbol{b}_1(x)p^{n-1} + O(p^{n-2}),\ \ p \to \infty, \tag{7}$$

where

$$\boldsymbol{b}_0 = \frac{c_1(L,h)^n}{\omega^n},\ \ \boldsymbol{b}_1 = \frac{\boldsymbol{b}_0}{8\pi}\,(r^X - 2\Delta\log\boldsymbol{b}_0), \tag{8}$$

and r^X, Δ, are the scalar curvature, respectively the Laplacian, of the Riemannian metric associated to $c_1(L,h)$.

Hence, (4) and (5) show that on U_t the exponential decay estimate for the partial Bergman kernel function is sharp. Moreover, on K the partial Bergman kernel function has the same asymptotics as the full Bergman kernel function up to order $O(p^{-\infty})$. This was established in [RS13, Theorem 1.1] under the additional assumption that there is an S^1 action in a neighborhood of Σ. Our method is to estimate the partial Bergman kernel $P_{0,p}$ by above and below with the full Bergman kernel P_p and singular Bergman kernel $\widetilde{P}_p$. On the set where the singular metric $\widetilde{h}$ equals h, the kernels $\widetilde{P}_p$ and P_p differ by $O(p^{-\infty})$. This is shown in Theorem 5.1, which gives a general localization result for singular Bergman kernels. Theorem 5.1 is a straightforward consequence of [HM14].

However, in Theorem 1.3 we do not necessarily obtain a *partition* of the manifold X in two sets, one with exponential decay (4) and one with "full asymptotics" (6), since in general $U_t \cup K \neq X$. In [Ber07, RS13, PS14] a partition with two different regimes was exhibited under further hypotheses.

2. Preliminaries

2.1. **Bergman kernel function.** Let (L,h) be a singular Hermitian holomorphic line bundle over a compact Hermitian manifold (X,ω). We denote by $H^0(X,L^p)$ the space of holomorphic sections of $L^p := L^{\otimes p}$.

Let $H^0_{(2)}(X,L^p) = H^0_{(2)}(X,L^p,h_p,\omega^n/n!)$ be the Bergman space of L^2-holomorphic sections of L^p relative to the metric $h_p := h^{\otimes p}$ induced by h and the volume form $\omega^n/n!$ on X, endowed with the inner product

$$(S,S')_p := \int_X \langle S,S'\rangle_{h_p}\frac{\omega^n}{n!}\,,\;\; S,S'\in H^0_{(2)}(X,L^p). \tag{9}$$

Set $\|S\|_p^2 = (S,S)_p$, $d_p = \dim H^0_{(2)}(X,L^p)$. If h has locally bounded weights (e. g. h is smooth) we have of course $H^0_{(2)}(X,L^p) = H^0(X,L^p)$. We have the following variational characterization of the partial Bergman kernel

$$P_{0,p}(x) = \max\Big\{|S(x)|^2_{h_p} : S\in H^0_0(X,L^p),\; \|S\|_p = 1\Big\}, \tag{10}$$

and similar characterizations hold for the full and singular Bergman kernel functions P_p and $\widetilde{P}_p$.

Throughout the paper we also use the following terminology. For a sequence of continuous functions f_p on a manifold M we write $f_p = O(p^{-\infty})$ if for every compact subset $K\subset M$ and any $\ell\in\mathbb{N}$ there exists $C_{K,\ell}>0$ such that for all $p\in\mathbb{N}$ we have $\|f_p\|_K \leq C_{K,\ell}\,p^{-\ell}$.

2.2. **Geometric set-up.** We prepare here the geometric set-up needed for the proofs of our results, by constructing a special neighborhood W of Σ.

Let (X,ω) be a compact Hermitian manifold of dimension n. Let (U,z), $z=(z_1,\dots,z_n)$, be local coordinates centered at a point $x\in X$. For $r>0$ and $y\in U$ we denote by

$$\Delta^n(y,r) = \{z\in U : |z_j - y_j|\leq r,\; j=1,\dots,n\}$$

the (closed) polydisk of polyradius $(r,\dots,r)$ centered at y. If ω is a Kähler form, the coordinates (U,z) are called Kähler at $y\in U$ if

$$\omega_z = \frac{i}{2}\sum_{j=1}^n dz_j\wedge d\overline{z}_j + O(|z-y|^2)\;\text{ on } U.$$

Since Σ is compact, we can find an open cover $\mathcal{W} = \{W_j\}_{1\leq j\leq N}$ of Σ, where W_j are Stein simply connected coordinate neighborhoods centered at points $y_j\in\Sigma$, such that

$$\begin{aligned}&\Delta^n(y_j,2)\subset W_j\,,\;\;\Sigma\subset W := \bigcup_{j=1}^N \Delta^n(y_j,1),\\ &\Sigma\cap W_j = \big\{z\in W_j : z_1 = 0\big\}\,,\text{ for } j = 1\dots,N,\end{aligned} \tag{11}$$

where $z = (z_1, \ldots, z_n)$ are the coordinates on W_j. Moreover, if ω is a Kähler form, we may also ensure that

(12) $\forall\, x \in \Delta^n(y_j, 1),\ \exists\, z = z(x)$ coordinates on $\Delta^n(y_j, 2)$ centered at x and Kähler at x.

As [CMM14, §2.5, Lemma 2.7] one can easily prove the following:

Lemma 2.1. *Let $(X, \omega), (L, h), \Sigma, \widetilde{h}$ be as in Theorem 1.2, and let $\mathcal{W} = \{W_j\}_{1\leq j\leq N}$ be an open cover of Σ verifying* (11) *and* (12)*. There exist constants $C_1 > 1$, $C_2 > 0$ and $r_1 > 0$ with the following property: if $j \in \{1, \ldots, N\}$, $x \in \Delta^n(y_j, 1)$ and $z = z(x)$ are the coordinates on $\Delta^n(y_j, 2)$ given by* (12)*, then:*

(i) $\Delta^n_z(x, r_1) \Subset \Delta^n(y_j, 2)$ and for $r \leq r_1$ we have

$$n!\, dm \leq (1 + C_1 r^2)\omega^n\,,\ \ \omega^n \leq (1 + C_1 r^2) n!\, dm \ \text{ on } \Delta^n_z(x, r), \tag{13}$$

where $dm = dm(z)$ is the Euclidean volume and $\Delta^n_z(x, \cdot)$ is the open polydisk relative to the coordinates z.

(ii) $(L, \widetilde{h})$ has a weight φ_x on W_j with

$$\begin{aligned} &\varphi_x = t \log|f| + \psi_x\,, \quad \psi_x \in \mathscr{C}^\infty(W_j), \\ &\psi_x(z) = \operatorname{Re} F_x(z) + \psi'_x(z) + \widetilde{\psi}_x(z) \ \text{ on } \Delta^n(y_j, 2), \end{aligned} \tag{14}$$

where f is a defining function for $\Sigma \cap W_j$, $F_x(z)$ is a holomorphic polynomial of degree ≤ 2 in z, $\psi'_x(z) = \sum_{\ell=1}^n \lambda_\ell |z_\ell|^2$, $\lambda_\ell = \lambda_\ell(x)$, and

$$|\widetilde{\psi}_x(z)| \leq C_2 |z|^3\,,\ \ z \in \Delta^n_z(x, r_1)\,. \tag{15}$$

3. Exponential decay

We prove here Theorem 1.1. Let $\mathcal{W} = \{W_j\}_{1\leq j\leq N}$ be the cover of Σ and $W \supset \Sigma$ be the neighborhood of Σ constructed in section 2.2 (see (11)). For a function $\varphi \in L^\infty_{loc}(W_j)$ set

$$\|\varphi\|_{\infty, W_j} = \sup\big\{|\varphi(w)| :\ w \in \Delta^n(y_j, 2)\big\}.$$

Let (L, h) be a singular Hermitian holomorphic line bundle on X, where the metric h has locally bounded weights. Since $L|_{W_j}$ is trivial, we fix a holomorphic frame e_j of $L|_{W_j}$, and denote by φ_j the corresponding weight of h on W_j, i.e. $|e_j|_h = e^{-\varphi_j}$. Set

$$\|h\|_\infty = \|h\|_{\infty, \mathcal{W}} := \max\big\{1, \|\varphi_j\|_{\infty, W_j} :\ 1 \leq j \leq N\big\}, \tag{16}$$

and let ϱ be the function defined in (1).

Theorem 3.1. *In the setting of Theorem 1.1, there exists a constant $A \geq 1$ depending only on ρ and $\mathcal{W}$ such that for any $S \in H^0_0(X, L^p)$, $x \in W$, and $p \geq 1$, we have*

$$|S(x)|^2_{h_p} \leq (A e^{\rho(x)})^{2\lfloor tp\rfloor} e^{4p\|h\|_\infty} \|S\|^2_p\,.$$

Therefore, for every $x \in W$ and $p \geq 1$,

$$P_{0,p}(x) \leq (A e^{\rho(x)})^{2\lfloor tp\rfloor} e^{4p\|h\|_\infty}\,.$$

For the proof we need the following elementary lemma.

Lemma 3.2. *If $k \geq 0$ and $f \in \mathcal{O}(\Delta(0,2))$, where $\Delta(0,2) \subset \mathbb{C}$ is the closed disk centered at 0 and of radius 2, then*

$$\int_{\Delta(0,2)} |f(\zeta)|^2 \, dm(\zeta) \leq \frac{k+1}{2^{2k}} \int_{\Delta(0,2)} |\zeta|^{2k} |f(\zeta)|^2 \, dm(\zeta) \, .$$

Proof. Consider the power expansion $f(\zeta) = \sum_{j=0}^{\infty} a_j \zeta^j$ of f in $\Delta(0,2)$. Integrating in polar coordinates we obtain

$$\int_{\Delta(0,2)} |f(\zeta)|^2 \, dm(\zeta) = 2\pi \sum_{j=0}^{\infty} |a_j|^2 \int_0^2 r^{2j+1} \, dr = 2\pi \sum_{j=0}^{\infty} \frac{2^{2j+2}}{2j+2} |a_j|^2 \, .$$

On the other hand, $\zeta^k f(\zeta) = \sum_{j=k}^{\infty} a_{j-k} \zeta^j$, so

$$\begin{aligned} \int_{\Delta(0,2)} |f(\zeta)|^2 \, dm(\zeta) &= 2\pi \sum_{j=k}^{\infty} \frac{2^{2j+2}}{2j+2} |a_{j-k}|^2 = 2\pi \sum_{j=0}^{\infty} \frac{2^{2j+2+2k}}{2j+2+2k} |a_j|^2 \\ &\geq \frac{2^{2k}}{k+1} 2\pi \sum_{j=0}^{\infty} \frac{2^{2j+2}}{2j+2} |a_j|^2 = \frac{2^{2k}}{k+1} \int_{\Delta(0,2)} |f(\zeta)|^2 \, dm(\zeta) \, . \end{aligned}$$

□

Proof of Theorem 3.1. Let $x \in W$. Fix $j \in \{1, \ldots, N\}$ such that $x \in \Delta^n(y_j, 1)$ and let e_j be the local frame of $L|_{W_j}$ and φ_j be the corresponding weight of h as considered in (16). Let $S \in H_0^0(X, L^p)$. On W_j we write $S = s e_j^{\otimes p}$, with $s \in \mathcal{O}(W_j)$. Then we have $s(z) = z_1^{\lfloor tp \rfloor} \widetilde{s}(z)$, with $\widetilde{s} \in \mathcal{O}(W_j)$. Using the sub-averaging inequality we get

$$\begin{aligned} |S(x)|_{h_p}^2 &= |x_1|^{2\lfloor tp \rfloor} |\widetilde{s}(x)|^2 e^{-2p\varphi_j(x)} \leq |x_1|^{2\lfloor tp \rfloor} e^{-2p\varphi_j(x)} \frac{1}{\pi^n} \int_{\Delta^n(x,1)} |\widetilde{s}(z)|^2 \, dm(z) \\ &\leq |x_1|^{2\lfloor tp \rfloor} e^{-2p\varphi_j(x)} \int_{\Delta^n(0,2)} |\widetilde{s}(z)|^2 \, dm(z) \, . \end{aligned} \tag{17}$$

Applying Fubini's theorem for the splitting $z = (z_1, z')$ and Lemma 3.2 for the variable z_1, we obtain

$$\begin{aligned} \int_{\Delta^n(0,2)} |\widetilde{s}(z)|^2 \, dm(z) &= \int_{\Delta^{n-1}(0,2)} \int_{\Delta(0,2)} |\widetilde{s}(z_1, z')|^2 \, dm(z_1) dm(z') \\ &\leq \frac{\lfloor tp \rfloor + 1}{2^{\lfloor tp \rfloor}} \int_{\Delta^n(0,2)} |z_1|^{2\lfloor tp \rfloor} |\widetilde{s}(z)|^2 \, dm(z) \\ &\leq C \exp\left(2p \max_{\Delta^n(0,2)} \varphi_j\right) \int_{\Delta^n(0,2)} |s(z)|^2 e^{-2p\varphi_j(z)} \, \omega^n, \end{aligned} \tag{18}$$

where $C = C(\mathcal{W}) \geq 1$ is chosen such that $dm(z) \leq C\omega^n$ on each $\Delta^n(y_j, 2)$ in the local coordinates of W_j, for $j = 1, \ldots, N$. Combining (17) and (18) we get

$$|S(x)|^2_{h_p} \leq C\,|x_1|^{2\lfloor tp\rfloor} \exp\left(2p \max_{\Delta^n(0,2)} \varphi_j - 2p\varphi_j(x)\right) \|S\|^2_p \tag{19}$$

Note that there exists a constant $A' = A'(\rho, W) > 1$ such that

$$|x_1| \leq A' e^{\rho(x)}\,, \;\; x \in W\,. \tag{20}$$

Set $A = A'C$. The estimates (19) and (20) yield

$$|S(x)|^2_{h_p} \leq (C|x_1|)^{2\lfloor tp\rfloor} e^{4p\|h\|_\infty} \|S\|^2_p \leq (Ae^{\rho(x)})^{2\lfloor tp\rfloor} e^{4p\|h\|_\infty} \|S\|^2_p\,.$$

Taking into account (10) we immediately obtain the conclusion. □

Corollary 3.3. *In the setting of Theorem 3.1 we let*

$$U_t := \left\{x \in W\,:\, (Ae^{\rho(x)})^t\, e^{4\|h\|_\infty} < 1\right\}. \tag{21}$$

Then for any $x \in U_t$ and $p > 2t^{-1}$ we have

$$P_{0,p}(x) \leq \left[(Ae^{\rho(x)})^t\, e^{4\|h\|_\infty}\right]^p. \tag{22}$$

In particular $P_{0,p} = O(p^{-\infty})$ as $p \to \infty$ on U_t.

Proof. This follows immediately from Theorem 3.1, since $Ae^{\rho(x)} < 1$ for $x \in U_t$, and $2\lfloor tp\rfloor > 2tp - 2 > tp$ for $p > 2/t$. □

4. Singular Bergman kernel

In this section we prove Theorem 1.2 by using ideas of Berndtsson, who gave in [B03, Section 2] a simple proof for the first order asymptotics of the Bergman kernel function in the case of powers of an ample line bundle (see also [CMM14, Theorem 1.3]).

We start by recalling the following version of Demailly's estimates for the $\overline{\partial}$ operator [Dem82, Théorème 5.1] (see also [CMM14, Theorem 2.5]) which will be needed in our proofs.

Theorem 4.1. *Let (X, ω) be a compact Kähler manifold of dimension n, and let $B > 0$ be a constant such that $\mathrm{Ric}_\omega \geq -2\pi B\omega$ on X. Let (L, h) be singular Hermitian holomorphic line bundle on X such that $c_1(L, h) \geq \varepsilon\omega$, and fix p_0 such that $p_0\varepsilon \geq 2B$. Then for all $p > p_0$ and all $g \in L^2_{0,1}(X, L^p, loc)$ with $\overline{\partial} g = 0$ and $\int_X |g|^2_{h_p}\,\omega^n < \infty$ there exists $u \in L^2_{0,0}(X, L^p, loc)$ such that $\overline{\partial} u = g$ and $\int_X |u|^2_{h_p}\,\omega^n \leq \frac{2}{p\varepsilon} \int_X |g|^2_{h_p}\,\omega^n$.*

Proof of Theorem 1.2. Let $\mathcal{W} = \{W_j\}_{1\leq j\leq N}$ be an open cover of Σ verifying (11) and (12). If $j \in \{1, \ldots, N\}$ and $x \in \Delta^n(y_j, 1)$, let $z = z(x)$ be the coordinates on $\Delta^n(y_j, 2)$ given by (12), and let $e_{j,x}$ be a holomorphic frame of L on W_j such that $|e_{j,x}|_{\widetilde{h}} = e^{-\varphi_x}$, where φ_x is given by (14).

Assume now that $x \in \Delta^n(y_j, 1) \setminus \Sigma$ and define

$$r_x := \sup \Big\{ r \in (0, r_1] : \Delta^n_z(x, r) \subset \Delta^n(y_j, 2) \setminus \Sigma \Big\} .$$

We have

$$\begin{aligned} \omega_x &= \frac{i}{2} \sum_{\ell=1}^n dz_\ell \wedge d\bar{z}_\ell \,, \\ c_1(L, \widetilde{h})_x &= dd^c \varphi_x(0) = dd^c \psi_x(0) = dd^c \psi'_x(0) = \frac{i}{\pi} \sum_{\ell=1}^n \lambda_\ell \, dz_\ell \wedge d\bar{z}_\ell \,. \end{aligned} \tag{23}$$

Since $c_1(L, \widetilde{h})_x \geqslant \varepsilon \omega_x$ it follows that $\lambda_\ell \geq \varepsilon$, $\ell = 1, \ldots, n$. Moreover, there exists $H_x \in \mathcal{O}(\Delta^n_z(x, r_x))$ such that $\operatorname{Re} H_x = \operatorname{Re} F_x + t \log |f|$. We define a new frame for L over $\Delta^n_z(x, r_x)$ by $e_x = e^{H_x} e_{j,x}$. Hence

$$|e_x|_{\widetilde{h}} = \exp(\operatorname{Re} H_x) \exp(-\varphi_x) = \exp(-\psi'_x - \widetilde{\psi}_x) \,.$$

We fix now $j \in \{1, \ldots, N\}$ and $x \in \Delta^n(y_j, 1) \setminus \Sigma$ and we will estimate $\widetilde{P}_p(x)$. Let $r_p \in (0, r_x/2)$ be an arbitrary number which will be specified later. We start by estimating the norm of a section $S \in H^0_{(2)}(X, L^p, \widetilde{h}_p, \omega^n/n!)$ at x. Writing $S = s e_x^{\otimes p}$, where $s \in \mathcal{O}(\Delta^n_z(x, r_x))$, we obtain by the sub-averaging inequality for psh functions on $\Delta^n_z(x, r_p) = \Delta^n(0, r_p)$,

$$|S(x)|^2_{\widetilde{h}_p} = |s(0)|^2 \leq \frac{\int_{\Delta^n(0, r_p)} |s|^2 e^{-2p\psi'} \, dm}{\int_{\Delta^n(0, r_p)} e^{-2p\psi'} \, dm} \,.$$

We have further by (13), (15),

$$\begin{aligned} \int_{\Delta^n(0, r_p)} |s|^2 e^{-2p\psi'} \, dm &\leq (1 + C_1 r_p^2) \exp\big(2p \sup_{\Delta^n(0, r_p)} \widetilde{\psi}\big) \int_{\Delta^n(0, r_p)} |s|^2 e^{-2p(\psi'_x + \widetilde{\psi}_x)} \frac{\omega^n}{n!} \\ &\leq (1 + C_1 r_p^2) \exp\big(2C_2 p \, r_p^3\big) \|S\|^2_p \,. \end{aligned}$$

Set

$$E(r) := \int_{|\xi| \leq r} e^{-2|\xi|^2} \, dm(\xi) = \frac{\pi}{2} \left(1 - e^{-2r^2}\right) .$$

Since $\lambda_\ell \geq \varepsilon$ we obtain

$$\frac{E(r_p \sqrt{p\varepsilon}\,)^n}{p^n \lambda_1 \ldots \lambda_n} \leq \int_{\Delta^n(0, r_p)} e^{-2p\psi'} \, dm \leq \int_{\mathbb{C}^n} e^{-2p\psi'} \, dm = \frac{(\pi/2)^n}{p^n \lambda_1 \ldots \lambda_n} \,.$$

Combining these estimates it follows that

$$|S(x)|^2_{\widetilde{h}_p} \leq \frac{(1 + C_1 r_p^2) \exp\big(2C_2 p \, r_p^3\big)}{E(r_p \sqrt{p\varepsilon})^n} \, p^n \lambda_1 \ldots \lambda_n \, \|S\|^2_p \,. \tag{24}$$

The singular Bergman kernel also satisfies a variational formula,

$$\widetilde{P}_p(x) = \max \Big\{ |S(x)|^2_{\widetilde{h}_p} \, : \, S \in H^0_{(2)}(X, L^p, \widetilde{h}_p, \omega^n/n!), \ \|S\|_p = 1 \Big\}.$$

Hence (24) implies the following upper estimate for the singular Bergman kernel,

$$\frac{\widetilde{P}_p(x)}{p^n\lambda_1\ldots\lambda_n} \leq \frac{(1+C_1r_p^2)\exp\big(2C_2p\,r_p^3\big)}{E(r_p\sqrt{p\varepsilon})^n}\,, \quad \forall\, r_p\in(0,r_x/2). \tag{25}$$

For the lower estimate of $\widetilde{P}_p$, let $0\leq\chi\leq 1$ be a smooth cut-off function on $\mathbb{C}^n$ with support in $\Delta^n(0,2)$ such that $\chi\equiv 1$ on $\Delta^n(0,1)$, and set $\chi_p(z)=\chi(z/r_p)$. Then $F=\chi_p e_x^{\otimes p}$ is a section of L^p and $|F(x)|_{\widetilde{h}_p}=|e_x^{\otimes p}(x)|_{\widetilde{h}_p}=1$. We have

$$\begin{aligned} \|F\|_p^2 &\leq \int_{\Delta^n(0,2r_p)} e^{-2p(\psi'_x+\widetilde{\psi}_x)}\,\frac{\omega^n}{n!} \\ &\leq (1+4C_1r_p^2)\exp\big(16C_2p\,r_p^3\big)\int_{\Delta^n(0,2r_p)} e^{-2p\psi'_x}\,dm \\ &\leq \left(\frac{\pi}{2}\right)^n\frac{(1+4C_1r_p^2)\exp\big(16C_2p\,r_p^3\big)}{p^n\lambda_1\ldots\lambda_n}\,. \end{aligned} \tag{26}$$

Set $\alpha=\overline{\partial}F$. Since $\|\overline{\partial}\chi_p\|^2=\|\overline{\partial}\chi\|^2/r_p^2$, where $\|\overline{\partial}\chi\|$ denotes the maximum of $|\overline{\partial}\chi|$, we obtain as above

$$\|\alpha\|_p^2=\int_{\Delta^n(0,2r_p)}|\overline{\partial}\chi_p|^2e^{-2p(\psi'_x+\widetilde{\psi}_x)}\,\frac{\omega^n}{n!}\leq\frac{\|\overline{\partial}\chi\|^2}{r_p^2}\left(\frac{\pi}{2}\right)^n\frac{(1+4C_1r_p^2)\exp\big(16C_2p\,r_p^3\big)}{p^n\lambda_1\ldots\lambda_n}\,.$$

There exists $p_0\in\mathbb{N}$ such that for $p>p_0$ we can solve the $\overline{\partial}$–equation by Theorem 4.1. We get a smooth section G of L^p with $\overline{\partial}G=\alpha=\overline{\partial}F$ and

$$\|G\|_p^2\leq\frac{2}{p\varepsilon}\|\alpha\|_p^2\leq\frac{2\|\overline{\partial}\chi\|^2}{p\varepsilon r_p^2}\left(\frac{\pi}{2}\right)^n\frac{(1+4C_1r_p^2)\exp\big(16C_2p\,r_p^3\big)}{p^n\lambda_1\ldots\lambda_n}\,. \tag{27}$$

Note that G is holomorphic on $\Delta^n(0,r_p)$ since $\overline{\partial}G=\overline{\partial}F=0$ there. So the estimate (24) applies to G on $\Delta^n(0,r_p)$ and gives

$$\begin{aligned} |G(x)|^2_{\widetilde{h}_p} &\leq \frac{(1+C_1r_p^2)\exp\big(2C_2p\,r_p^3\big)}{E(r_p\sqrt{p\varepsilon})^n}\,p^n\lambda_1\ldots\lambda_n\|G\|_p^2 \\ &\leq \frac{2\|\overline{\partial}\chi\|^2}{p\varepsilon r_p^2E(r_p\sqrt{p\varepsilon})^n}\left(\frac{\pi}{2}\right)^n(1+4C_1r_p^2)^2\exp\big(18C_2p\,r_p^3\big). \end{aligned}$$

Let $S=F-G\in H^0_{(2)}(X,L^p,\widetilde{h}_p,\omega^n/n!)$. Then

$$\begin{aligned} |S(x)|^2_{\widetilde{h}_p} &\geq (|F(x)|_{\widetilde{h}_p}-|G(x)|_{\widetilde{h}_p})^2=(1-|G(x)|_{\widetilde{h}_p})^2 \\ &\geq \left[1-\left(\frac{\pi}{2}\right)^{n/2}\frac{\sqrt{2}\,\|\overline{\partial}\chi\|(1+4C_1r_p^2)}{r_p\sqrt{p\varepsilon}\,E(r_p\sqrt{p\varepsilon})^{n/2}}\,\exp\big(9C_2p\,r_p^3\big)\right]^2=:K_1(r_p)\,. \end{aligned}$$

Moreover, by (26) and (27)

$$\|S\|_p^2\leq(\|F\|_p+\|G\|_p)^2\leq\left(\frac{\pi}{2}\right)^n\frac{K_2(r_p)}{p^n\lambda_1\ldots\lambda_n}\,,$$

where

$$K_2(r_p) = (1+4C_1r_p^2)\exp\big(16C_2p\,r_p^3\big)\left(1+\frac{\sqrt{2}\,\|\overline{\partial}\chi\|}{r_p\sqrt{p\varepsilon}}\right)^2.$$

Therefore

$$\widetilde{P}_p(x) \geq \frac{|S(x)|^2_{\widetilde{h}_p}}{\|S\|_p^2} \geq \left(\frac{2}{\pi}\right)^n p^n\lambda_1\dots\lambda_n\,\frac{K_1(r_p)}{K_2(r_p)}\,. \tag{28}$$

Using now (23), (25) and (28) we deduce that for every $x \in \bigcup_{j=1}^N \Delta^n(y_j,1)\setminus\Sigma$, $r_p < r_x/2$ and $p > p_0$,

$$\frac{K_1(r_p)}{K_2(r_p)} \leq \widetilde{P}_p(x)\,\frac{\omega_x^n}{p^n c_1(L,\widetilde{h})_x^n} \leq K_3(r_p)\,, \tag{29}$$

where

$$K_3(r_p) = \left(\frac{\pi/2}{E(r_p\sqrt{p\varepsilon})}\right)^n (1+C_1r_p^2)\exp\big(2C_2p\,r_p^3\big)\,.$$

We take now $r_p = p^{-3/8}$, so $p\,r_p^3 = p^{-1/8} \to 0$ and $p\,r_p^2 = p^{1/4} \to \infty$ as $p \to \infty$. Note that there exists a constant $C_3 > 0$ such that

$$K_1(p^{-3/8}) \geq 1 - C_3p^{-1/8}\,,\;\; K_2(p^{-3/8}) \leq 1 + C_3p^{-1/8}\,,\;\; K_3(p^{-3/8}) \leq 1 + C_3p^{-1/8}\,.$$

It follows by (29) that there exists a constant $C_4 > 0$ such that

$$1 - C_4\,p^{-1/8} \leq \widetilde{P}_p(x)\,\frac{\omega_x^n}{p^n c_1(L,\widetilde{h})_x^n} \leq 1 + C_4\,p^{-1/8}\,, \tag{30}$$

holds for every $x \in \bigcup_{j=1}^N \Delta^n(y_j,1)\setminus\Sigma$, $p^{-3/8} < r_x/2$ and $p > p_0$. Now $r_x > c\,\mathrm{dist}(x,\Sigma)$, for some constant $c > 0$, so there exists a constant $C_5 > 0$ such that (30) holds for $p > C_5\,\mathrm{dist}(x,\Sigma)^{-8/3}$. This concludes the proof of (2) for $x \in \bigcup_{j=1}^N \Delta^n(y_j,1)\setminus\Sigma$.

By [HM14, Theorem 1.8] there exist $C_6 > 0$ and $p_0' \in \mathbb{N}$ such that

$$\left|\widetilde{P}_p(x)\,\frac{\omega_x^n}{p^n c_1(L,\widetilde{h})_x^n} - 1\right| \leq \frac{C_6}{p}\,,$$

for $x \in X\setminus\bigcup_{j=1}^N \Delta^n(y_j,1)$ and $p > p_0'$. The proof of Theorem 1.2 is complete. □

5. Estimates for the partial Bergman kernel

In this section we prove Theorem 1.3. Let $t < t_0(K)$. By the definition (3) of $t_0(K)$, there exist $\eta \in \mathscr{C}^\infty(X,[0,1])$ and $\delta > 0$ such that $\mathrm{supp}\,\eta \subset X\setminus K$, $\eta = 1$ near Σ and $c_1(L,h) + tdd^c(\eta\varrho) \geq \delta\omega$ in the sense of currents on X. Define

$$\widetilde{h}_t = h\exp(-2t\eta\varrho)\,,\;\; \widetilde{h}_{t,p} = \widetilde{h}_t^{\otimes p}\,.$$

Note that $\widetilde{h}_t = h$ in a neighborhood of K and $\widetilde{h}_t \geq h$ on X. Since Σ is smooth, it follows by (1) that $H^0_0(X, L^p) = H^0_{(2)}(X, L^p, \widetilde{h}_{t,p}, \omega^n/n!)$. We denote the norm on $H^0_{(2)}(X, L^p, \widetilde{h}_{t,p}, \omega^n/n!)$ by

$$\|S\|^2_{t,p} = \int_X |S|^2_{\widetilde{h}_{t,p}} \frac{\omega^n}{n!} = \int_X |S|^2_{h_p} \exp(-2tp\eta\varrho) \frac{\omega^n}{n!} \cdot$$

Let $\widetilde{P}_{t,p}$ be the Bergman kernel function of $H^0_{(2)}(X, L^p, \widetilde{h}_{t,p}, \omega^n/n!)$. Recall that $\|S\|_p$ is the norm given by the scalar product (9) on $H^0_0(X, L^p)$. Since $\varrho < 0$ we have $\|S\|^2_{t,p} \geq \|S\|^2_p$ for any $S \in H^0_0(X, L^p)$. Let $S \in H^0_0(X, L^p)$ with $\|S\|^2_{t,p} \leq 1$. Then $\|S\|^2_p \leq 1$, too, hence

$$|S|^2_{\widetilde{h}_{t,p}} = |S|^2_{h_p} \exp(-2tp\eta\varrho) \leq P_{0,p} \exp(-2tp\eta\varrho)\,,$$

and thus

$$\widetilde{P}_{t,p} \leq P_{0,p} \exp(-2tp\eta\varrho)\,.$$

Denote now by P_p the Bergman kernel function of $H^0(X, L^p)$ endowed with the scalar product (9). Since $H^0_0(X, L^p)$ is isometrically embedded in $H^0(X, L^p)$ we have $P_{0,p} \leq P_p$. Consequently we have shown:

$$\begin{aligned} &\widetilde{P}_{t,p} \exp(2tp\eta\varrho) \leq P_{0,p} \leq P_p \ \text{ on } X, \\ &\widetilde{P}_{t,p} \leq P_{0,p} \leq P_p \ \text{ near } K. \end{aligned} \tag{31}$$

Let now W be the neighborhood of Σ defined in (11) and let U_t be defined as in (21), so that the exponential estimate (22) holds on U_t for $p > 2t^{-1}$. By shrinking U_t we can assume that $\eta = 1$ on U_t. Setting $M := e^{4\|h\|_\infty} A^t$ we obtain(4). By Theorem 1.2 we have

$$\widetilde{P}_{t,p}(x) \geq (1 - Cp^{-1/8}) p^n \frac{c_1(L, \widetilde{h}_t)^n_x}{\omega^n_x}$$

for every $p \in \mathbb{N}$ with $p \operatorname{dist}(x, \Sigma)^{8/3} > C$. Note that $c_1(L, \widetilde{h}_t) \geq \delta\omega$ in the sense of currents on X. Since $c_1(L, \widetilde{h}_t)$ is smooth on $X \setminus \Sigma$ we have $\dfrac{c_1(L, \widetilde{h}_t)^n}{\omega^n} \geq \delta^n$ on $X \setminus \Sigma$. By increasing C if necessary, it follows that

$$\widetilde{P}_{t,p}(x) \geq \frac{p^n}{C} \quad \text{for } p > C \operatorname{dist}(x, \Sigma)^{-8/3}.$$

Hence

$$P_{0,p}(x) \geq \frac{p^n}{C} \exp(2tp\varrho(x)) \quad \text{for } x \in U_t \text{ and } p > C \operatorname{dist}(x, \Sigma)^{-8/3}.$$

This proves (5).

In order to prove (6) we need the following localization theorem for the Bergman kernel.

Theorem 5.1. *Let (X, ω) be a compact Hermitian manifold and $L \to X$ be a holomorphic line bundle. Consider two singular Hermitian metrics h_1 and h_2 on L, which are smooth outside a proper analytic set $\Sigma \subset X$ and such that $c_1(L, h_1)$, $c_1(L, h_2)$ are Kähler currents.*

Let $P_p^{(j)}$ be the Bergman projection on $H^0(X, L^p, h_j^p, \omega^n/n!)$, $j = 1, 2$. We assume that there exists an open set $U \Subset X \setminus \Sigma$ such that $h_1 = h_2$ on U. Then the Bergman kernels satisfy $P_p^{(1)}(z,w) - P_p^{(2)}(z,w) = O(p^{-\infty})$ on U in any $\mathscr{C}^\ell$-topology, $\ell \in \mathbb{N}$, as $p \to \infty$.

Proof. The proof follows essentially from the analysis in [HM14] (see also [HM16]). Let h_0 be any singular Hermitian metric on L, smooth on $X \setminus \Sigma$ and satisfying $c_1(L, h_0) \geq \varepsilon\omega$ in the sense of currents on X, for some $\varepsilon > 0$. Let $P_p^{(0)}$ be the Bergman projection on $H^0(X, L^p, h_0^p, \omega^n/n!)$.

Consider an open set $D \subset U$ such that $L|_D$ is trivial. Let $s : D \to L$ be a holomorphic frame and let $\varphi \in \mathscr{C}^\infty(D)$ be the weight of h_0 corresponding to s, that is, $|s|_{h_0} = e^{-\varphi}$. Let us denote by $\mathscr{E}'(D)$ the space of distributions with compact support on D and by $L^2(D)$ the space of square-integrable functions with respect to the volume form $\omega^n/n!$. The localized Bergman projection with respect to s is the operator $P_{p,s}^{(0)} : L^2(D) \cap \mathscr{E}'(D) \to L^2(D)$, defined by $P_p^{(0)}(ue^{p\varphi}s^{\otimes p}) = P_{p,s}^{(0)}(u)e^{p\varphi}s^{\otimes p}$. It is easy to see that

$$P_p^{(0)}(z,w) = P_{p,s}^{(0)}(z,w)e^{p(\varphi(z)-\varphi(w))}s^{\otimes p}(z) \otimes (s^{\otimes p})^*(w) \in L_z^p \otimes (L_w^p)^*\,, \;\; z, w \in D. \tag{32}$$

By [HM14, Theorem 9.2] the kernel of $P_{p,s}^{(0)}$ satisfies

$$P_{p,s}^{(0)}(z,w) = \mathcal{S}_p(z,w) + O(p^{-\infty}) \text{ on } D, \tag{33}$$

where $\mathcal{S}_p$ is the localized approximate Szegő kernel defined in [HM14, (3.43)]. Note that by [HM14, Theorem 3.12] we have

$$\mathcal{S}_p(z,w) = e^{ip\Psi(z,w)}b(z,w,p) + O(p^{-\infty}) \text{ on } D, \tag{34}$$

where $\Psi : D \times D \to \mathbb{C}$ is a phase function depending on the eigenvalues of $c_1(L, h_0)$ with respect to ω and described precisely in [HM14, Theorem 3.8]. Moreover, $b(\cdot,\cdot,p) : D \times D \to \mathbb{C}$ is a semi-classical symbol of order $n = \dim X$, depending only on the restriction of h and ω to D.

We apply now these results for $h_0 = h_1$ and $h_0 = h_2$. Since $h_1|_D = h_2|_D$ we deduce that the weight φ, the phase Ψ and the symbol $b(\cdot,\cdot,p)$ above are the same for h_1 and h_2. We infer from (33) and (34) that $P_{p,s}^{(1)}(z,w) - P_{p,s}^{(2)}(z,w) = O(p^{-\infty})$ on D. Finally, (32) yields $P_p^{(1)}(z,w) - P_p^{(2)}(z,w) = O(p^{-\infty})$ on D. The proof of Theorem 5.1 is complete. □

We apply now Theorem 5.1 to the metrics $\widetilde{h}_t$ and h, which are equal on a neigborhood V of K and infer that

$$\widetilde{P}_{t,p} - P_p = O(p^{-\infty}) \text{ locally uniformly on } V. \tag{35}$$

Combined with (31), (35) yields (6). Finally, (7) and (8) follow from the expansion of the Bergman kernel P_p (see [MM07, Theorems 4.1.1–3]) or of the singular Bergman kernel (see [HM14, Theorem 1.8]).

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