Layering ∂-Graphs and Networks: ∂-Graph Transformations, Discrete Harmonic Continuation, and a Generalized Electrical Inverse Problem

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January 2, 2016.

Abstract

We generalize electrical networks to include signed and complex linear networks, linear networks over arbitrary fields as well as infinite and nonlinear networks. We formalize the idea of "layering" graphs with boundary (∂-graphs) and networks using three related processes: (1) "Layer-stripping" ∂-graphs by contracting boundary spikes and boundary edges, (2) "elementary factorizations" in a category whose morphisms are graphs with input and output (as in Baez-Fong [1]), (3) structures called "scaffolds" involving partial orders on the set of edges. Layering theory provides a formal description of discrete harmonic continuation and the layer-stripping approach to the electrical inverse problem used by Curtis-Ingerman-Morrow [4] and Johnson [11].

We define a class of "solvable" ∂-graphs for which this approach works to solve the inverse boundary-value problem. Critical circular planar ∂-graphs and rectangular lattices are solvable, as are any ∂-subgraphs, covering ∂-graphs, and box products of solvable ∂-graphs. We thus reprove [11]'s result that critical circular planar networks with bijective zero-preserving nonlinear resistors can be recovered.

We generalize standard results about linear networks to arbitrary fields. Layering theory provides a useful language and motivation for the electrical linear group similar to the one described by Lam and Pylyavksyy [14], and helps us classify the possible boundary behaviors for linear networks over arbitrary fields. We show in particular that any feasible boundary behavior can be represented by a circular planar network.

Finally, we generalize theorems about ranks and connections proved by Curtis-Ingerman-Morrow [4] and de-Verdiere-Gitler-Vertigan to situations that include many non-planar and nonlinear networks.

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Preface

Approach and Prerequisites:

The main thrust of this paper is concrete and geometric–it is about cutting networks apart and gluing them together, stripping away a network layer by layer, and propagating potential and current information step-by-step through a network. The results are elementary and self-contained enough to be accessible to advanced undergraduates familiar with linear algebra, set theory, basic graph theory, and basic category theory.

But network theory is a multi-faceted subject reaching out to graph theory, physics, probability, algebraic topology, and symplectic Lie theory. I therefore make passing references to many branches of mathematics, yet none of the other results are essential to understand the main proofs here.

I devote more time than strictly necessary to exposition and motivation. The hope is that the reader will have to spend less time decrypting the technical details, and see that most of the insights are simple, and once they are known, it is only a matter of choosing the correct definitions to make the proofs work in the best generality.

Familiarity with the results of Curtis-Ingerman-Morrow [4] or de Verdiere-Gitler-Vertigan on electrical networks is very helpful in understanding the motivation. I include cursory explanations of the most important ideas.

Acknowledgements

James Morrow is the grandfather of this paper. He organized the University of Washington's math REU, and encouraged me to continue to think about electrical networks after the program in summer 2013 was finished. Both he and Ian Zemke spent a lot of time listening to and critiquing my ideas then. More recently, Avi Levy continues to inspire new ideas and find helpful results in the literature.

The tradition of the UW's REU, its library of techniques and examples invented by undergraduates, graduates, and professors together and preserved by Jim Morrow, has been the catalyst for my creative processes. This paper continues the work of Curtis-Ingerman-Morrow, Will Johnson, Konrad Schrøder, Ian Zemke, and others. The REU student papers are sometimes flawed, and it is hard to say for sure who came up with what ideas, but I will do my best to give credit where credit is due.

Several times, I found that some of my ideas had already been invented, in a different and more sophisticated form, by other mathematicians studying electrical networks (Lam and Pylyavsky, Baez and Fong), and their insights helped me sharpen my own results. This type of thing has happened many times in network theory: For instance, variants of the matrix-tree theorem were rediscovered many times. Curtis-Ingerman-Morrow and de Verdiere solved the inverse problem for circular planar networks simultaneously and independently. Further examples can easily be found by googling and citation-chasing. If I

have copied anyone else's results, be assured it is unintentional, and I will insert proper citations when I become aware.

The papers Curtis-Ingerman-Morrow [4] and de-Verdiere-Gitler-Vertigan [7] contain similar results. However, specific citations will be given from [4] since I am more familiar with that paper.

The pictures were produced using Till Tantau's package Tikz. I also used code posted on Stack Exchange by "Qrrbrbirlbels" and "Henri Menke" (http://tex.stackexchange.com/questions/163689/add-arrows-to-a-smooth-tikz-function).

Soli Deo Gloria.

Contents

1 Introduction

Although elementary, the first two sections of the introduction provide key intuition and motivation for the results. §1.3 gives the general definition of networks, and §1.4 describes the main results and outline of the paper.

1.1 Linear Resistor Networks

This paper is motivated by the study of the electrical inverse problem for linear resistor networks. I will sketch some basic ideas from [4] here, omitting proofs.

A *linear resistor network* consists of

- A finite connected graph G. V will denote the set of vertices and E the set of oriented edges.
- A designation of a certain vertices of the graph as "boundary vertices." The remaining vertices are called "interior." We call the set of boundary vertices ∂V and the set of interior vertices V°.
- An assignment of a positive number $\gamma_e = \gamma_{\overline{e}}$ for each edge e, called the conductance. The resistance is $1/\gamma_e$. The assignment is also viewed as function $\gamma : E \to (0, \infty)$.

An *electrical potential function* is a function $u : V \to \mathbb{R}$. If e is an oriented edge, then the *voltage* across e is given by $u(e_+) - u(e_-)$, where e_+ and e_- are the start and end points of e. The *current* across e is given by Ohm's Law as

$$c(e) = \gamma_e(u(e_+) - u(e_-)).$$

The *net current* at a vertex p is

$$\sum_{e:e_+=p} c(e) = \sum_{e:e_+=p} \gamma_e(u(p) - u(e_-)).$$

A potential u is called *harmonic* if it satisfies Kirchhoff's law that the net current at each interior vertex is zero. According to this convention, the net current at p is total current *going out* of p into the other vertices.

A potential $u : V \to \mathbb{R}$ can be viewed as a vector in $\mathbb{R}^V$, and so can the function $V \to \mathbb{R}$ that maps a vertex to its net current. The relations for harmonic potentials can thus be described in terms of the Kirchhoff matrix (a.k.a. weighted graph Laplacian) K given by

$$K_{p,q} = \begin{cases} -\displaystyle\sum_{\substack{e:e_+=p, \\ e_-=q}} \gamma_e, & p \neq q \\ \displaystyle\sum_{e:e_+=v} \gamma_e, & p = q. \end{cases}$$

We view the rows and columns of K as indexed by the set V. The Kirchhoff matrix K defines a linear transformation $\mathbb{R}^V \to \mathbb{R}^V$, and the component indexed by p is exactly the net current at p. K is symmetric and has row sums zero.

Based on analogy with continuous models of electricity in PDE, [4] considers the *Dirichlet problem:* Given $\phi \in \mathbb{R}^{\partial V}$, does there exist a harmonic potential u, with $u|_{\partial V} = \phi$? Is it unique? If so, what is the net current on the boundary?

Let us write K in block form putting the rows / columns indexed by ∂V first:

$$K = \begin{pmatrix} K_{\partial V, \partial V} & K_{\partial V, V^\circ} \\ K_{V^\circ, \partial V} & K_{V^\circ, V^\circ} \end{pmatrix} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

We will similarly write our potential u in block form as $(\phi, w)^T$. We then want to solve

$$\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \begin{pmatrix} \phi \\ w \end{pmatrix} = \begin{pmatrix} \psi \\ 0 \end{pmatrix},$$

where ϕ is given, w is the unknown, and ψ is the vector representing the net currents on the boundary vertices (we do not care what it is at the moment). We can find w from ϕ if and only if C is invertible. But it turns out C is positive-definite, so the Dirichlet problem has a unique solution.

The boundary current vector ψ is then given by

$$\psi = \Lambda \phi,$$

where Λ is the *Schur complement*

$$\Lambda = K / K_{V^\circ, V^\circ} = A - BC^{-1}B^T.$$

Λ is called the *response matrix* (a.k.a. the *Dirichlet-to-Neumann map*).

1.2 The Inverse Problem, Layer-Stripping

The *electrical inverse problem* is to determine γ from Λ and G. In other words, knowing the structure of the network and its boundary behavior, we want to figure out what type of resistors are in it. Algebraically, this amounts to checking injectivity of the multi-rational map

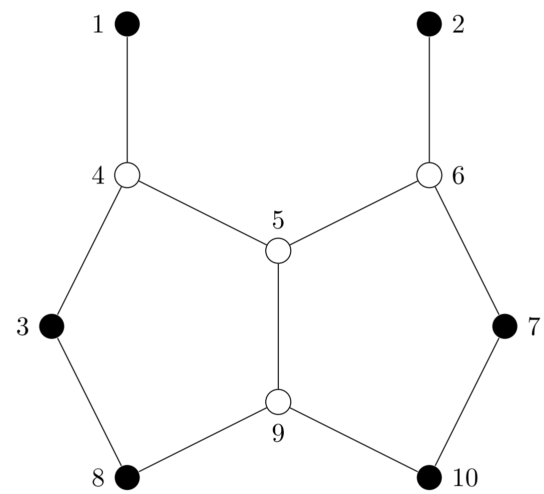
$$\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \mapsto A - BC^{-1}B^T$$

on the appropriate domain, but algebra alone is not too helpful.

The inverse problem cannot always be solved, nor has anyone discovered a strategy that works well in all situations. One fairly general approach for "recovering" the conductances used by [4] and [11] is known as "layer-stripping." We start by finding an edge that is close enough to the boundary to be "accessible." Using a cleverly chosen boundary value problem, we figure out the conductance of this edge. Then, armed with this new information, we proceed to find the conductances of edges further inside the network, away from the boundary.

As an illustration, consider the network Γ shown in Figure 1. The edge between vertices i and j will be called (i, j). An edge such as $(1, 4)$ for which

Figure 1: A network resembling the eyes and antennae of a fly. The boundary vertices are black and the interior vertices white. The vertices are indexed by $1, \ldots, 10$ as shown.



one of the endpoints is a boundary vertex of degree 1 is called a *boundary spike*. The conductances of boundary spikes are often the easiest to recover.

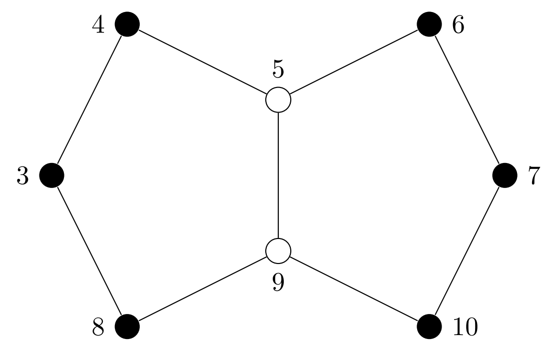
Recovering the Boundary Spike $\gamma_{1,4}$:

I claim that there exists a harmonic function u with potential zero at 8 and 3, net current zero at 3, and potential one at 1. To see why this is helpful, first observe that any such harmonic function will have net current $\gamma_{1,4}$ at vertex 1. Indeed, since $u(3) = u(8) = 0$, there is no current flowing on the edge $(3, 8)$. Since the net current on 3 is zero, there must also be no current on $(3, 4)$, hence $u(4) = 0$ as well. If $u(1) = 1$ and $u(4) = 0$, then the current on $(1, 4)$ is $\gamma_{1,4}$, and this is also the net current at 1.

So then why is $\gamma_{1,4}$ uniquely determined by Λ? Set-theoretically, we are claiming this: Search through all the pairs $(\phi, \Lambda\phi)$ and you will find one (possibly more than one) with $\phi(3) = \phi(8) = 0$ and $\phi(1) = 1$ and $(\Lambda\phi)(3) = 0$. Then pick some $(\phi, \Lambda\phi)$ satisfying these conditions, and magically $(\Lambda\phi)(1)$ will be $\gamma_{1,4}$. Hence, $\gamma_{1,4}$ is uniquely determined by Λ.

It only remains to prove that such a u exists. We will construct it step-by-step through "harmonic continuation." We start by setting $u(3) = u(8) = u(4) = 0$ and $u(1) = 1$. [For best results, please write on the picture.] The current flowing into vertex 4 from 3 and 1 has now been determined. Since 4 is an interior vertex, the net current flowing into it is zero, so the current flowing from 5 to 4 is uniquely determined, as is the potential $u(5)$. We have some freedom as to how to extend u further. We will somewhat arbitrarily set u to 0 at vertex 9. Then we choose $u(10)$ to make vertex 9 have net current zero.

Figure 2: The network Γ'



Choose the potential on 6 so as to make the net current at 5 zero. Arbitrarily set $u(7) = 0$, then choose $u(2)$ so that 6 will have net current zero.

Therefore, $\gamma_{1,4}$ is uniquely determined by Λ. We can similarly recover $\gamma_{2,6}$.

Removing the Boundary Spikes:

I claim that at this point, we can remove $(1, 4)$ and $(2, 6)$ from the network without losing any information. In a transformation known as "boundary spike contraction," we will collapse the edge $(1, 4)$ and merge vertices 1 and 4 into a single boundary vertex which we will still call 4. We do the same for $(2, 6)$. Call the new network Γ' (Figure 2).

I claim that the new response matrix Λ' can be determined from Λ, $\gamma_{1,4}$ and $\gamma_{2,6}$. Rather than giving an explicit formula for Λ' (which is done in [4] §8), I will instead work with the sets $\{(\phi, \Lambda\phi)\} \subset \mathbb{R}^{\partial V} \times \mathbb{R}^{\partial V}$ and $\{(\phi', \Lambda'\phi')\} \subset \mathbb{R}^{\partial V'} \times \mathbb{R}^{\partial V'}$.

Consider a pair $(\phi, \Lambda\phi)$ which represents the boundary data of a harmonic function u, and let $\psi = \Lambda\phi$. Let $u' = u|_{V'}$. Then u' is harmonic. Knowing the potential and net current at 1, we can deduce the potential at 4; it is $u(4) = u(1) + \psi(1)/\gamma_{1,4}$. The same holds for 2 and 6. When we restrict to the smaller network Γ', the net current on 4 is the same as the net current on 1 in Γ' since in Γ the net current on 4 was zero. Thus, $\psi'(4) = \psi(1)$. For the vertices in $\partial V'$ besides 4 and 6, the potential and net current of u' are easy to find from (ϕ, ψ). Thus, the boundary data (ϕ', ψ') is determined by the boundary data (ϕ, ψ) and given by a simple formula.

We thus have a "natural" map $(\phi, \psi) \to (\phi', \psi')$. Checking that this map is a bijection amounts to saying that any harmonic function on Γ' uniquely extends to a harmonic function on Γ, which the reader can easily check.

It follows that Λ' is determined by Λ, $\gamma_{1,4}$ and $\gamma_{2,6}$. Therefore, to recover Γ, it suffices to recover Γ'.

Recovering a Boundary Edge:

An edge such as $(3, 4)$ for which both endpoints are boundary vertices is called a *boundary edge*. These are also easy targets for recovering conductances.

To recover $(3,4)$, we only have to construct a harmonic function u such that $u(3) = u(8) = 0$ and $u(4) = 1$. Then the only current coming into 3 is from 4, and the net current at 3 is $-\gamma_{3,4}$. The construction of u is left as an exercise.

After recovering the conductance of $(3,4)$, we can *delete the boundary edge* by removing it from the graph while leaving the rest of the graph unchanged. Call the new network Γ''. Then any harmonic function on Γ'' extends uniquely to one on Γ' and the map $(\phi, \psi) \mapsto (\phi', \psi')$ can be explicitly computed (exercise). Thus, Λ'' is uniquely determined by Λ' and $\gamma_{(3,4)}$.

Continuing in this way, we can recover all the conductances in the network (exercise). In the process, we keep "stripping away layers," reducing to smaller and smaller networks until all the edges have been removed.

1.3 Generalizations of Electrical Networks

The ideas in the above example adapt to much more general situations; for instance,

- They were applied by Curtis-Ingerman-Morrow and de-Verdiere-Gitler-Vertigan to "critical circular planar networks" (e.g. [5]).
- Students at the University of Washington REU considered networks with signed conductances ([19] and [9]).[2] Also, [15] uses signed conductances in network transformations.
- Johnson [11] considers nonlinear electrical networks, where the current on an edge e is given as a nonlinear function of the voltage. Amusingly, the notation $\gamma_e(u(e_+) - u(e_-))$ is still used, but now γ_e is a function $\mathbb{R} \to \mathbb{R}$ rather than a real number. Dually, the voltage can be a nonlinear function of the current: $u(e_+) - u(e_-) = \rho_e(c(e))$, where $c(e)$ is the current. [11] shows that for critical circular planar networks where γ_e is bijective with $\gamma_e(0) = 0$, the inverse problem can be solved.
- Zemke [21] applies similar techniques to attack the inverse problem for infinite electrical networks.
- Harmonic continuation was used by various students of the UW REU for specific networks that are not circular planar.

Aside from the inverse problem, there are a variety of mathematical reasons to generalize electrical networks:

- Thinking of differential topology, Kenyon considers a vector bundle Laplacian where the potentials at each vertex take values in some vector space [13].

[2] I have not checked these papers thoroughly and [11] suggests they may contain errors. I don't use any results from these papers, only motivation.

- In cellular homology theory which includes graphs as a special case, one can consider weighted Laplacians on the chain groups over a given ring. So it could be worthwhile to consider a network where the conductances take values in some ring R and the potentials and currents take values in an R-module M. Then the $\mathbb{Z}$-module of harmonic functions on the network could be studied using homological algebra. Avi Levy and I will discuss this in an upcoming paper.

With these examples in mind, we make the following definitions/conventions:

A *graph* is given by a set V of vertices, a set E of oriented edges, an assignment of endpoints e_+ and e_- for each $e \in E$, a free involution $^- : E \to E$ mapping an edge to its reverse, such that $\overline{e}_+ = e_-$ and $\overline{e}_- = e_+$. We assume V and E are countable and that each vertex is incident to finitely many edges. We allow parallel and self-looping edges. We allow disconnected graphs. We denote by E' the set of edges (that is, pairs $\{e, \overline{e}\}$). We assume familiarity with basic graph terminology.

A *graph-with-boundary* or *∂-graph* is a graph together with a partition of V into a set of *boundary vertices ∂V* and a set of *interior vertices V°*. Unless otherwise specified, we will assume that every component of the graph has at least one boundary vertex.

Let M be an abelian group, written additively. A *network taking values in M* is a ∂-graph together with a map $\Theta : E \to \mathcal{P}(M \times M)$ which assigns to each edge a relation $\Theta_e \subset M \times M$, called its *voltage-current relation*, such that $\Theta_{\overline{e}} = -\Theta_e$.

A *harmonic function* on a network Γ is a pair of maps $u : V \to M$ and $c : E \to M$ such that

a. $c(\overline{e}) = -c(e)$ for each $e \in E$.

b. $(u(e_+) - u(e_-), c(e)) \in \Theta_e$ for each $e \in E$.

c. The net current $\sum_{e:e_+=p} c(e)$ is zero for each $p \in V^\circ$.

Note that the condition $\Theta_{\overline{e}} = -\Theta_e$ ensures that (a) and (b) are nicely consistent. In general, for any function $c : E \to M$ with $c(\overline{e}) = -c(e)$, we call $\sum_{e:e_+=p} c(e)$, the *net current at p*.

We say that Θ_e is given by a conductance function γ_e if

$$\Theta_e = \{(x, \gamma_e(x)) : x \in M\},$$

and it is given by a resistance function ρ_e if

$$\Theta_e = \{(\rho_e(y), y) : y \in M\}.$$

Thus, our definition includes both types of nonlinear networks considered in [11].

$\mathcal{H}(G, \Theta)$ denotes the space of harmonic functions. The *boundary behavior* $\mathcal{B} = \mathcal{B}(G, \Theta)$ of a network is the set of pairs $(\phi, \psi) \in M^{\partial V} \times M^{\partial V}$ such that there exists a harmonic function with potentials ϕ and net currents ψ on the boundary vertices.[3] There is a "natural" map $\Phi : \mathcal{H} \to \mathcal{B}$ that sends a harmonic function to its boundary data.

A BZ ("bijective zero-preserving") network is a network where each Θ_e is given by a bijective conductance function $\gamma_e : M \to M$ with $\gamma_e(0) = 0$ (or equivalently by a bijective zero-preserving resistance function). If M needs to be specified, then write BZ(M). For a harmonic pair (u, c) on a BZ network, c is given as a function of u, so we can work either with the harmonic function (u, c) or with the harmonic potential u.

BZ networks are the natural setting for the layer-stripping approach to the inverse problem (cf. [11]). Indeed, upon careful reflection, the main ingredients for recovering the network in the previous example were precisely that there was a bijective zero-preserving relation between the voltage and current on each edge. We will say that a ∂-graph G is *recoverable over BZ* if the Θ is uniquely determined by $\mathcal{B}$ for any BZ network (G, Θ) on G–that is, $\Theta \mapsto \mathcal{B}(G, \Theta)$ is injective on the set of Θ's where each Θ_e is a bijective zero-preserving relation.

1.4 Overview

It turns out the process of removing boundary spikes and boundary edges is structurally similar to step-by-step harmonic continuation, and both can be described by the idea of "layering." Formalizing this idea enables us to generalize the approach to the inverse problem given in [4], as well as some of their other results.

"Layering" has three distinct formulations, each with their advantages and disadvantages:

- A ∂-graph is *layerable* if there is a decreasing filtration of ∂-graphs $G_0, G_1, \dots$ such that G_{n+1} is obtained from G_n by a reduction operation–a combination of removing boundary spikes and boundary edges (§3).

- The process of concatenating graphs together is viewed as composition in a category of "IO-graphs" (as in Baez and Fong [1]). A graph with designated "input" and "output" vertices is viewed as a transformation from the inputs to the outputs, and the morphisms are composed by identifying the outputs of the first with the inputs of the second. In this framework, layering a graph G with inputs P and outputs Q corresponds to factorizing the morphism $P \to Q$ into elementary "layers" (§4).

- A *scaffold* is a certain type of partial order on the edges of G. A scaffold can describe in what order we remove edges from the graph in a layerable filtration, in what order we use the edges when constructing a harmonic function through harmonic continuation, or in what order they occur in an elementary factorization (§5). In this section, I define *solvable* and *totally*

[3] The idea of using the set of boundary data instead of a Dirichlet-to-Neumann map was used in [11]. The name "boundary behavior" is inspired by [1].

layerable ∂-graphs for which the inverse problem can be solved through layer-stripping, so that they are recoverable over BZ.

∂-subgraphs are fundamental to all three flavors of layering, so they are covered in §2. I define ∂-graph morphisms (essentially a combination of an inclusion and a covering map), and in §3 and §5 I show that the "layering structures" can be pulled back along ∂-graph morphisms by taking preimages. This means that if the layer-stripping process can solve the inverse problem on G, then it will also work for any ∂-subgraph or covering ∂-graph of G.

In §6, I specialize the theory to a graph embedded on the surface, use medial graphs to construct scaffolds, and prove that critical circular planar graphs are totally layerable (cf. [11]), as well as the "supercritical" half-planar graphs of [21]. §6 provides something that is conspicuously missing from the theory up through §5: general methods of constructing scaffolds without having any preexisting scaffolds.

In §7, I review standard results about linear networks and generalize to arbitrary fields. The layering theory is related to the action of the "electrical linear group" of Lam and Pylyavsky [14], which in turn is used to characterize the possible boundary behaviors of finite networks over a field $\mathbb{F}$. As a corollary, for any field other than $\mathbb{F}_2$, any feasible boundary behavior for n boundary vertices can be represented by a layerable network with $\leq n(n-1)/2+1$ edges, or by a layerable circular planar network.

In §8 I generalize [4]'s principle that the rank of the submatrix $\Lambda_{P,Q}$ is the maximum size connection between P and Q. For linear networks over arbirary fields, this principle holds generically as a consequence of the grove-determinant formula discussed in §7. But provided there is an IO-graph factorization from P to Q, this principle holds for *all* conductances and even generalizes to nonlinear networks. The converse holds for the special case of a unique connection between P and Q which also uses all the interior vertices.

§9 provides further methods of producing larger solvable ∂-graphs from smaller ones–box products and a weaker type of ∂-graph morphism. As a corollary, we prove that the an n-dimensional rectangular lattice is recoverable.

2 ∂-Graph Morphisms and Subnetworks

2.1 ∂-Graph Morphisms

If G_1 and G_2 are graphs, then a graph morphism $f : G_1 \to G_2$ consists of two maps $V(G_1) \to V(G_2)$ and $E(G_1) \to E(G_2)$ (which I will call f by abuse of notation) such that $\overline{f(e)} = f(\overline{e})$, $(f(e))_+ = f(e_+)$, and $f(e)_- = f(e_-)$.

A *∂-graph morphism* $f : G_1 \to G_2$ is a graph morphism such that

- f maps $V^\circ(G_1)$ into $V^\circ(G_2)$.
- If $p \in V^\circ(G_1)$, then f restricts to a bijection

$$\{e \in E(G_1) : e_+ = p\} \to \{e \in E(G_2) : e_+ = f(p)\}.$$

- If $p \in \partial V(G_1)$, then this induced map is an injection.

The ∂-graphs form a category. The above conditions mean that, roughly speaking, f is a graph isomorphism in a neighborhood of each interior vertex and a monomorphism in a neighborhood of each boundary vertex. Thus, f behaves like an immersion between smooth n-manifolds with boundary. Since such an immersion is an open map, interior points map to interior points, and local structure is preserved.

∂-Subgraphs: A *∂-subgraph* of a ∂-graph G is a subgraph G' such that the inclusion map $G' \to G$ is a ∂-graph morphism. Equivalently, G' is a ∂-subgraph if

- $V(G') \subset V(G)$, $E(G') \subset E(G)$, $V^\circ(G') \subset V^\circ(G)$.
- If $p \in V^\circ(G')$, then $p \in V^\circ(G)$ and all edges incident to p are in $E(G')$.

Pullbacks of ∂-subgraphs: If $f : G_1 \to G_2$ is a ∂-graph morphism, then any ∂-subgraph $S \subset G_2$ pulls back to a ∂-subgraph $f^{-1}(S) \subset G_1$ given by $V(f^{-1}(S)) = f^{-1}(V(S))$ and $E(f^{-1}(S)) = f^{-1}(E(S))$ and $V^\circ(f^{-1}(S)) = f^{-1}(V^\circ(S))$.

Categorical properties: The reader may verify that the category of ∂-graphs has pullbacks constructed in a fairly typical way, but not products. There is no terminal object. It has coproducts given by disjoint unions, but push-forwards are not well-behaved. If we remove the restriction that the sets of vertices and edges are countable, then the category has arbitrary limits. All this makes sense given the analogy with immersions of n-manifolds.

For better categorical structure, one might want to expand the definition of ∂-graph morphism, but for my purposes, the geometric properties of ∂-graph morphisms are more important. Intuitively, "layering" is analogous to creating a foliation of an n-manifold, and only regular enough maps can be used to pull back foliations.

Covering Maps: A ∂-graph morphism $f : G_1 \to G_2$ is a *covering map* if f is surjective on the vertices and edges of G_2, f maps boundary vertices to boundary vertices, and the induced map $\{e \in E(G_1) : e_+ = p\} \to \{e \in E(G_2) : e_+ = f(e)\}$ is a bijection for all vertices. Covering maps form a subcategory of ∂-graphs.

Covering maps are easy to construct explicity. Let G be a ∂-graph and $S = \{1, \ldots, n\}$ or $\mathbb{N}$. For each $e \in E(G)$, choose $\sigma_e \in \operatorname{Perm} S$ with $\sigma_{\overline{e}} = \sigma_e^{-1}$. Define a ∂-graph H by

- $V(H) = V(G) \times S$.
- $E(H) = E(G) \times S$.
- $V^\circ(H) = V^\circ(G) \times S$.
- $(e \times j)_+ = e_+ \times j$.
- $\overline{e \times j} = \overline{e} \times \sigma(j)$.

Then the map $H \to G$ is a covering map. As a fairly standard exercise, show that if G is connected, then up to isomorphism all covering maps are constructed this way. Also, any ∂-graph morphism $f : G_1 \to G_2$ can be factored as $f = g \circ h$ where h is an inclusion and g is a covering map.

Network Morphisms and Subnetworks: If $\Gamma_1 = (G_1, \Theta_1)$ and $\Gamma_2 = (G_2, \Theta_2)$ are networks, then a *network morphism* $f : \Gamma_1 \to \Gamma_2$ is ∂-graph morphism $G_1 \to G_2$ that preserves the voltage-current relations of the edges, that is, $(\Theta_2)_{f(e)} = (\Theta_1)_e$ for $e \in E(\Gamma_1)$. A *subnetwork* of Γ is a network on ∂-subgraph with the voltage-current relations inherited from Γ.

Pullbacks of Harmonic Functions: If (u, c) is a harmonic function on Γ_2, then $(u \circ f, c \circ f)$ is harmonic on Γ_1. Indeed, for $e \in E(\Gamma_1)$,

$$(u \circ f(e_+) - u \circ f(e_-), c \circ f(e)) = (u(f(e)_+) - u(f(e)_-), c(f(e))) \in (\Theta_2)_{f(e)} = (\Theta_1)_e,$$

and for each $p \in V^\circ(\Gamma_1)$,

$$\sum_{e : e_+ = p} c \circ f(e) = \sum_{e : e_+ = f(p)} c(e) = 0$$

since the map $\{e \in E(G_1) : e_+ = p\} \to \{e \in E(G_2) : e_+ = f(e)\}$ is a bijection and $f(p) \in V^\circ(\Gamma_2)$.

2.2 Subnetworks and Boundary Behavior

Overview: The way that subnetworks and boundary behavior interact is well-known and unsurprising. Roughly speaking,

- *Gluing:* If we glue together a collection of networks along boundary vertices, then the boundary behavior of the larger network depends only on the boundary behaviors of the smaller ones (see e.g. [2]).
- *Splicing:* If Γ' is obtained by replacing some part of Γ by another part with the same boundary behavior, then Γ and Γ' have the same boundary behavior (see e.g. [10]).[4]
- *Recoverability:* A subnetwork of a recoverable network is recoverable (see e.g. [2] Theorem 2.9, [16]).

In the case of gluing linear resistor networks, there is an explicit formula for the response matrix of the larger network based on the smaller ones ([2] §2.1). But these principles can be derived purely from set theory.

Subnetwork Partitions: A *subnetwork partition of* Γ is a collection of subnetworks $\{\Gamma_\alpha\}$ such that

- $V(\Gamma) = \bigcup_\alpha V(\Gamma_\alpha)$,
- $E(\Gamma)$ is the disjoint union of $E(\Gamma_\alpha)$.

[4]This principle is implicitly used when performing Y-Δ transformations–see §and [17].

- The $V^\circ(\Gamma_\alpha)$ is disjoint from $V(\Gamma_\beta)$ for any $\alpha \neq \beta$.

A ∂-subgraph partition is defined the same way except without the voltage-current relations.

Proposition 2.1. *Suppose $\{\Gamma_\alpha\}$ is subnetwork partition of Γ. Then the boundary behavior of Γ depends only on the boundary behavior of Γ_α.*

Proof. Let $S = \bigcup_\alpha \partial V(\Gamma_\alpha)$. Let $T \subset \prod_\alpha \mathcal{B}(\Gamma_\alpha)$ be the set of points $((\phi_\alpha, \psi_\alpha))$ where

a. If $p \in V(\Gamma_\alpha) \cap V(\Gamma_\beta)$, then $\phi_\alpha(p) = \phi_\beta(p)$.

b. If $p \in S \cap V^\circ(\Gamma)$, then
$$\sum_{\alpha : \alpha \in \partial V(\Gamma_\alpha)} \psi_\alpha(p) = 0.$$
Since p is an endpoint of only finitely many edges, and each edge is in only one subnetwork, the sum has only fintitely many nonzero terms.

Define $F : T \to M^{\partial V} \times M^{\partial V}$ by $\prod_\alpha (\phi_\alpha, \psi_\alpha) \mapsto (\phi, \psi)$, where

1. $\phi(p) = \phi_\alpha(p)$ for $p \in \partial V(\Gamma)$.
2. $\psi(p) = \sum_{\alpha : p \in \partial V(G_\alpha)} \psi_\alpha(p)$,

which is well-defined by definition of T. Then $\mathcal{B}(\Gamma) = F(T)$. Indeed, if $((\phi_\alpha, \psi_\alpha)) \in T$ and $(\phi_\alpha, \psi_\alpha)$ is the boundary data of a harmonic function (u_α, c_α), then (a) and (b) guarantee that they paste together to a harmonic function on Γ, and (1) and (2) describe how to find its boundary data. Conversely, given any harmonic function on Γ, the restrictions to Γ_α will be harmonic and their boundary data will be in T. Since we have described how to find $\mathcal{B}(\Gamma)$ from $\mathcal{B}(\Gamma_\alpha)$, we are done. □

Corollary 2.2. *Suppose that $\{\Gamma_\alpha\}$ and $\{\Gamma'_\alpha\}$ are subnetwork partitions of Γ and Γ' respectively such that $\partial V(\Gamma_\alpha) = \partial V(\Gamma_\alpha)$, $\partial V(\Gamma) = \partial V(\Gamma')$. If Γ_α has the same boundary behavior as Γ'_α, then Γ has the same boundary behavior as Γ'.*

Corollary 2.3. *If a ∂-graph G is recoverable over BZ(M), then so is any subgraph.*

Proof. Let S be a subgraph of G and define a subgraph S' by
$$V(S') = V(G) \setminus V^\circ(S), \qquad E(S') = E(G) \setminus E(S), \qquad V^\circ(S') = V^\circ(G) \setminus V(S).$$
Then S and S' form a ∂-subgraph partition of G. If S is not recoverable, then there are two networks Σ_1 and Σ_2 on S with different voltage-current relations and the same boundary behavior. Pick some BZ network Σ' on S'. Then the networks on G given by $\Sigma_1 \cup \Sigma'$ and $\Sigma_2 \cup \Sigma'$ have the same boundary behavior but different voltage-current relations, so G is not recoverable. □

Remark. One can replace "recoverability over BZ(M)" with "recoverability over" any given set of voltage-current relations (appropriately defined), for instance, linear voltage-current relations with positive real coefficients.

3 Layering I: ∂-Graph Reductions

Here I formalize the process of contracting boundary spikes and boundary edges:

∂-Graph Reductions: As explained in §1.2,

- A *boundary spike* is an edge or oriented edge with one endpoint that is a boundary vertex of degree 1. It is *non-degenerate* if the other endpoint is an interior vertex.
- If e is a non-degenerate spike of G with $e_+ \in \partial V$ and $e_- \in V^\circ$, then G' is obtained by *contracting the spike* if $V(G') = V(G) \setminus \{e_+\}$, $V^\circ(G') = V^\circ(G) \setminus \{e_-\}$, and $E(G') = E(G) \setminus \{e, \overline{e}\}$.
- A *boundary edge* is an edge or oriented edge such that both endpoints are boundary vertices.
- G' is obtained from G by *deleting the boundary edge* e if $V(G') = V(G)$, $V^\circ(G') = V^\circ(G)$, and $E(G') = E(G) \setminus \{e, \overline{e}\}$.
- A *disconnected boundary vertex* is a boundary vertex with no edges incident to it.
- G' is obtained from G by deleting the disconnected boundary vertex p if $V(G') = V(G) \setminus \{p\}$ and E and V° are the same for G' and G.
- We will refer to the reverse transformations as *adjoining* a boundary spike / boundary edge / disconnected boundary vertex.

A *∂-graph reduction* is roughly speaking, some combination of contracting boundary spikes, deleting boundary edges, and deleting disconnected boundary vertices, such that the endpoints of edges removed do not overlap too much. Precisely, a reduction is a transformation of a ∂-graph G into a subgraph G' such that

1. The edges removed are all boundary spikes or boundary edges of G.
2. A boundary spike that is removed does not share any endpoints with any of the other edges that are removed.
3. The vertices removed are all boundary vertices of valence 0 or 1.
4. The only boundary vertices of G' that are interior in G are the endpoints of boundary spikes that were removed.

If there is exactly one boundary spike/ boundary edge / disconnected boundary vertex removed overall, then the reduction operation is called *simple*.

Reductions and ∂-Graph Morphisms If $f : G \to G'$ is a ∂-graph morphism, and S is obtained from G' by a reduction, then $f^{-1}(S)$ is obtained from G by a reduction (easy casework left to the reader). However, a boundary spike contraction in G' may produce a disconnected boundary vertex deletion in G

or some combination of boundary spike contraction and disconnected boundary vertex deletion in G. This is why the definition was phrased so as to allow mixing boundary spike contraction, boundary edge deletion, and disconnected boundary vertex deletion in one reduction operation.

Filtrations and Layerable ∂-graphs: A *(decreasing) filtration* of a graph G is a sequence of ∂-subgraphs $G = G_0 \supset G_1 \supset G_2 \supset \dots$ such that $\bigcap_{n=0}^{\infty} G_n = \varnothing$. If G_{n+1} is obtained from G_n by a reduction, then the filtration is called a *layerable filtration* and the ∂-graph is said to be *layerable.* A *partial filtration* is a sequence of subgraphs $G = G_0 \supset G_1 \supset \dots$, and it is a partial layerable filtration if each subgraph is obtained from the previous one by a reduction operation.

If $f : G \to G'$ is a ∂-graph morphism and $G_0', G_1', \dots$ is a layerable filtration of G', then $f^{-1}(G_0)$, $f^{-1}(G_1)$, …is a layerable filtration of G. Hence, layerability of G' implies layerability of G.

Electrical Properties: As exemplified in §1.2, if G' is obtained from G by a reduction, then the boundary behavior of G' is determined by $\mathcal{B}(G)$ and the voltage-current relations of the edges removed:

Lemma 3.1. *Suppose that G' is obtained from G by a reduction. Let Γ be a BZ network on G and let Γ' be the subnetwork on G'. Then*

- *Any harmonic function on Γ' extends to a harmonic function on Γ.*
- *In the case of contracting boundary spikes or deleting boundary edges, the extension is unique.*
- *$\mathcal{B}(\Gamma')$ is determined by $\mathcal{B}(\Gamma)$ and the voltage-current relation of the spike. The same holds switching Γ and Γ'.*

Proof. Any reduction can be expressed in three steps as a contraction of non-degenerate spikes, deletion of boundary edges, and deletion of disconnected boundary vertices, so it suffices to consider each of these operations individually.

Contracting boundary spikes: Let's consider the case of contracting one boundary spike. Let e be the oriented boundary spike, $\rho_e = \gamma_e^{-1}$ the resistance function. We want to show that any harmonic (u', c') on Γ' extends to a unique harmonic (u, c) on Γ. The only thing we need to decide is the current on e and the potential on e_+. Since e_- is boundary in Γ' but interior in Γ, there is only one possible choice for $c(e)$ that would yield net current zero on e_-. We then set $u(e_+) = u(e_-) + \rho_e(c(e))$.

Note that the boundary data of u is uniquely determined by ρ_e and the boundary data of u'. Indeed, the net current of u on $\iota(e)$ equals the net current of u' on $\tau(e)$ equals c_e, and $u(e_+) = u(e_-) + \rho_e(c_e)$. Also, $\partial V(\Gamma') \setminus \{e_-\} = \partial V(\Gamma) \setminus \{e_+\}$, and the potential / net current on these vertices is the same for u as it is for u'. Similarly, the boundary of u' is uniquely determined by ρ_e and the boundary data of u.

The same proof applies for contracting multiple boundary spikes, even infinitely many.

Deleting Boundary Edges: The argument is similar: Any harmonic function on Γ restricts to harmonic function on Γ'. To find the boundary data of (u, c) from (u', c') or (u', c') from (u, c), we keep the potentials the same, and adjust the net currents on the boundary vertices according to the boundary potentials together with conductance functions γ_e of the boundary edges removed.

Deleting Disconnected Boundary Vertices: Details left to reader. The disconnected boundary vertex can have whatever potential we want but must have net current zero since it does not interact with the rest of the network. ☐

4 Layering II: IO-Graphs and Elementary Factorization

4.1 The Category of Input-Output Graphs

Overview: John Baez and Brendan Fong describe "gluing networks together" in terms of a composition in a category [1]. I will use the same construction except that I assume less knowledge of category theory and work with infinite / nonlinear networks.

Roughly speaking, we label some of the boundary vertices of our graph as "input" and some as "output" (allowing a vertex to be both input and output), and think of our *network as a transformation (morphism)* from the input vertices to the output vertices. We compose such morphisms by identifying the output vertices of the first with the input vertices of the second. Next, we define a functor from this category to the category of relations, which sends a *network* with input and output to a *relation* describing how the boundary potential and current data of harmonic functions on the inputs and outputs are related.

IO-Graphs: A *graph with input and output* or *IO-graph* is a graph G together with two sets P and Q and injective "labelling" functions $i : P \to V(G)$ and $j : Q \to V(G)$. In this case, we say the triple (G, i, j) is an *IO-graph from P to Q*. If (G, i, j) and (G, i', j') are two IO-graphs from P to Q, then we say they are *isomorphic* if there is a graph isomorphism $f : G \to G'$ such that the following commutes:

$$\begin{array}{ccccc} & & V(G) & & \\ & \overset{j}{\nearrow} & & \overset{i}{\nwarrow} & \\ Q & & \Big\downarrow f & & P \\ & \underset{j'}{\searrow} & & \underset{i'}{\swarrow} & \\ & & V(G') & & \end{array}$$

Remark. [1] does not assume the labelling function is injective. I make this assumption for simplicity since I do not need the general case here.

The Category of IO-Graphs: We define the category of IO-graphs as follows:

- The objects are finite / countable sets.
- A morphism $\mathcal{G} : P \to Q$ is an isomorphism class of IO-graphs from P to Q.
- Composition is defined as follows: Suppose $\mathcal{G}_1 : P \to Q$ and $\mathcal{G}_2 : Q \to R$, and choose representatives (G_1, i_1, j_1) and (G_2, i_2, j_2) of the isomorphism classes. Let G be the graph obtained from the disjoint union of G_1 and G_2 by identifying $j_1(q)$ with $i_2(q)$ for each $q \in Q$. We define $i : P \to V(G)$ by composing i_1 with the obvious map $V(G_1) \to V(G)$ and $j : R \to V(G)$ is defined similarly. Then $\mathcal{G}_2 \circ \mathcal{G}_1 : P \to R$ is the isomorphism class represented by (G, i, j). (Check this is well-defined!)
- The identity morphism $P \to P$ is a represented by a graph with no edges and $V(G) = P$, and i and j are the identity $P \to P$.

The reason to use isomorphism classes is that the disjoint union of graphs is only well-defined up to canonical isomorphism.

IO-Graphs and ∂-Graphs: Any IO-graph can be made into a ∂-graph by defining $\partial V = i(P) \cup j(Q)$. Conversely, if we have a ∂-graph G and write ∂V as a union of two sets P and Q, then G represents an IO-graph morphism from P to Q.

The Category of IO-Networks: IO-networks are defined the same way but with voltage-current relations associated to the edges. An IO-network morphism will be denoted $\mathcal{G}$, and context will determine whether $\mathcal{G}$ represents an isomorphism class of IO-graphs or IO-networks.

The Category of Relations: The *category of relations* **Rel** is the category where the objects are sets and a morphism $U \to V$ is a *relation* $R \subset U \times V$. To emphasize that these relations are not necessarily given by functions $U \to V$, we shall write $R : U \rightsquigarrow V$. If $R_1 : U \rightsquigarrow V$ and $R_2 : V \rightsquigarrow W$, then $R_1 \circ R_2 : U \rightsquigarrow W$ is defined by $(u, w) \in R_1 \circ R_2$ if and only if there exists $v \in V$ such that $(u, v) \in R_1$ and $(v, w) \in R_2$.

The IO Boundary Behavior Functor: Recall our networks take values in a given abelian group M. We will define a functor $\mathcal{X}$ from the category of IO-networks over M to the category of relations. If P is a finite / countable set, define $\mathcal{X}(P) = M^P \times M^P$. By convention, $\mathcal{X}(\varnothing)$ is a one-element set. Next, suppose $\mathcal{G} : P \to Q$ is an IO-network morphism. Suppose $\mathcal{G}$ is represented by a network Γ and labellings $i : P \to V(\Gamma)$ and $j : Q \to V(\Gamma)$, and set $\partial V(\Gamma) = i(P) \cup j(Q)$. Let π_P be the projection $M^{\partial V} \to M^P$ and $\pi_Q : M^B \to M^Q$. Let ι_P and ι_Q be the canonical inclusions $M^P \to M^{\partial V}$ and $M^Q \to M^{\partial V}$. We define the relation

$$\mathcal{X}(\mathcal{G}) : (M^P \times M^P) \rightsquigarrow (M^Q \times M^Q)$$

as follows: If $x = (x_1, x_2) \in M^P \times M^P$ and $y = (y_1, y_2) \in M^Q \times M^Q$, then we say $(x, y) \in \mathcal{X}$ if and only if there exists a harmonic boundary data $(\phi, \psi) \in \mathcal{B}(\Gamma)$ such that

$$x_1 = \pi_P \phi, \qquad y_1 = \pi_Q \phi, \qquad \iota_P(x_2) - \iota_Q(y_2) = \psi.$$

More intuitively, $(x, y) \in X$ if there exists a harmonic function on Γ with boundary potentials consistent with x_1 and y_1 and boundary net currents consistent with x_2 and y_2. Here x_2 represents current flowing into the network at the input vertices, and y_2 represents current flowing out of the network at the output vertices. If a vertex is both input and output, then current can flow in at the input side and out at the output side.

If $\mathcal{G} : P \to Q$ is an IO-network morphism, then $\mathcal{X}(\mathcal{G})$ is independent of the choice of representation for the isomorphism class.

To see that $\mathcal{X}$ preserves composition, suppose $\mathcal{G}_1 : P \to Q$ and $\mathcal{G}_2 : Q \to R$. Let $\mathcal{G} = \mathcal{G}_2 \circ \mathcal{G}_1$ and let $\Gamma_1, \Gamma_2, \Gamma$ be specific networks representing the isomorphism classes. Without loss of generality Γ_1 and Γ_2 are subnetworks of Γ.

Composition in the category of relations gives us that $(x, z) \in X(\mathcal{G}_2) \circ X(\mathcal{G}_1)$ if and only if there exists some y with $(x, y) \in \mathcal{X}(\mathcal{G}_1)$ and $(y, z) \in \mathcal{X}(\mathcal{G}_2)$. In that case (x, y) and (y, z) represent the boundary data of harmonic functions on Γ_1 and Γ_2. By similar reasoning as in §2, these harmonic functions paste together to a harmonic function on Γ with boundary data consistent with (x, z). Conversely, for any (x, z) consistent with a harmonic function (u, c) on Γ, we can choose y such that (x, y) and (y, z) are consistent with the restricted harmonic functions on the subnetworks.

4.2 Elementary IO-graphs

The category of IO graphs enables us to express complicated networks as compositions of simpler ones. Our building blocks are networks on the following four types of *elementary IO-graphs*:

1. A graph in which every component consists of either (a) an isolated vertex which is both an input and an output or (b) one edge and two vertices, where one of the vertices is an input and the other is an output. See G_2, G_8, G_{10} in Figure 3.

2. A graph in which all the vertices are both inputs and outputs. See G_1, G_3, G_7, G_9 in Figure 3.

3. A graph with no edges in which every vertex is an input. We call the vertices which are not outputs *input stubs*. See G_6 in Figure 3.

4. A graph with no edges in which every vertex is an output. We call the vertices which are not inputs *output stubs*. See G_4 in Figure 3.

An *elementary factorization* of an IO-graph morphism $\mathcal{G} : P \to Q$ is a factorization $\mathcal{G} = \mathcal{G}_n \circ \dots \circ \mathcal{G}_1$ such that each $\mathcal{G}_j$ is represented by an elementary IO-graph and all the type 3 elementary IO-graphs come *before* (to the right of) the type 4 elementary IO-graphs. If $\mathcal{G}$ is the identity morphism $P \to P = Q$ then we make the convention that it has an elementary factorization of length zero. If we are given a graph G representing $\mathcal{G} : P \to Q$ and an elementary factorization of $\mathcal{G}$, then we can assume without loss of generality that $\mathcal{G}_j$ is represented by a subgraph G_j of G, and we will often make this simplification.

Remark. The usefulness of the stipulation that the type 3 IO-graphs come before the type 4 ones will become clear in the next section. For the moment, the reader can verify that, if we allowed type 3 IO-graphs to come after type 4 IO-graph, then *any* morphism $\mathcal{G} : P \to Q$ represented by a finite graph would admit a factorization. It is unreasonable to expect such cheap factorizations to provide useful information.

Factorization into Simple Elementary IO-Graphs: In the case of finite graphs, any type 1 IO-graph can be factorized into type 1 IO-graphs with only one edge. It is more convenient for writing out specific factorizations if we allow several edges, since it makes the list of IO-graphs shorter; but in proving general theorems we will often assume only one edge. The same considerations apply to the other types of elementary IO-graphs.

Elementary IO-networks and Their Boundary Behavior: We define an elementary IO-network to be a network on an elementary IO-graph. The behavior of $\mathcal{X}$ on such IO-networks is easy to describe in the cases we are interested in:

1. Suppose we have a type 1 network Γ with only one edge $\{e, \overline{e}\}$ with input vertex $p = e_+$ and output $q = e_-$. Suppose Θ_e is given by a resistance function $\rho_e : M \to M$. Then $(x, y) = ((x_1, x_2), (y_1, y_2))$ is in $\mathcal{X}(\Gamma)$ if and only if

$$\begin{aligned} &(y_1)_q = (x_1)_p - \rho_e((x_2)_p) \\ &(y_2)_q = (x_2)_p \\ &(x_1)_r = (y_1)_r \text{ and } (x_2)_r = (y_2)_r \text{ for } r \neq p, q. \end{aligned}$$

   As a result, $\mathcal{X}(\Gamma)$ defines a bijective function $M^P \times M^P \to M^Q \times M^Q$.

2. Suppose we have a type 2 network Γ with only one edge $\{e, \overline{e}\}$ with endpoints $p = e_+$ and $q = e_-$. Suppose Θ_e is given by a conductance function $\gamma_e : M \to M$. Then $(x, y) \in \mathcal{X}(\Gamma)$ if and only if

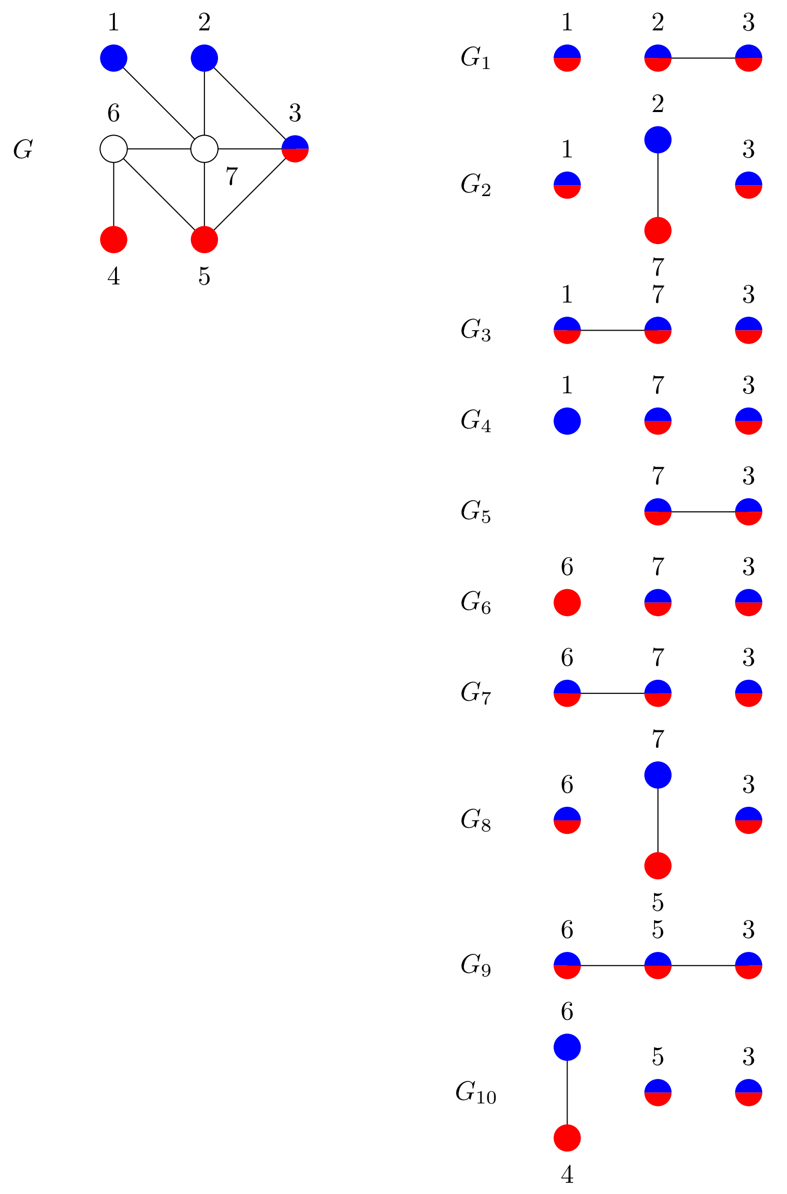
$$\begin{aligned} x_1 &= y_1 \\ (y_2)_p &= (x_2)_p - \gamma_e((x_1)_p - (x_1)_q) \\ (y_2)_q &= (x_2)_q + \gamma_e((x_1)_p - (x_1)_q) \\ (y_2)_r &= (x_2)_r \text{ for } r \neq p, q. \end{aligned}$$

   As a result, $\mathcal{X}(\Gamma)$ defines a bijective function $M^P \times M^P \to M^Q \times M^Q$.

3. Suppose we have a type 3 network Γ with only one input stub p. Then $(x, y) \in \mathcal{X}(\Gamma)$ if and only if $(x_2)_p = 0$ and for all $r \neq p$, $(x_1)_r = (y_1)_r$ and $(x_2)_r = (y_2)_r$.

4. The case of a type 4 network is symmetrical.

Figure 3: An elementary factorization. The inputs are shown in blue and the outputs in red.



4.3 Elementary Factorization and Layerability

Elementary IO-Graphs and Reduction Operations: Any ∂-graph represents an IO-graph morphism $\mathcal{G} : \varnothing \to \partial V$. Then

1. If $\mathcal{G}'$ is a type 1 IO-graph morphism, then $\mathcal{G}' \circ \mathcal{G}$ is obtained from $\mathcal{G}$ by adjoining a boundary spike. (Or to be precise, this holds for some pair of ∂-graphs representing $\mathcal{G}' \circ \mathcal{G}$ and $\mathcal{G}$.)

2. If $\mathcal{G}'$ is a type 2 IO-graph morphism, then $\mathcal{G}' \circ \mathcal{G}$ is obtained from $\mathcal{G}$ by adjoining a boundary edge.

3. Precomposing a type 3 IO-graph morphism corresponds to adjoining an isolated boundary vertex.

4. Precomposing a type 4 IO-graph morphism corresponds to changing a boundary vertex to interior.

The IO-graphs can thus be viewed as a geometric and categorical realization of reduction operations and other graph transformations. I invite the reader to reinterpret the proofs of §3 using elementary IO-networks.

If we consider $\mathcal{G} : \varnothing \to B$ instead and *post*compose the elementary IO-graphs, then the roles of type 3 and type 4 networks are reversed. These considerations lead to ...

Lemma 4.1. *Let G be a finite ∂-graph. The following are equivalent:*

a. G is layerable.

b. The IO-graph morphism $\varnothing \to \partial$ represented by G admits an elementary factorization into networks of types 1, 2, and 4.

c. For some $P, Q \subset \partial V$ with $P \cup Q = \partial V$, the morphism $P \to Q$ represented by G admits an elementary factorization.

Proof. (a) $\implies$ (b). A sequence of reductions on G can be interpreted as a factorization into elementary IO-graphs. Details left to the reader.

(b) $\implies$ (c) is trivial.

(c) $\implies$ (a). Let G represent an IO-graph morphism $\mathcal{G}$ with factorization $\mathcal{G} = \mathcal{G}_n \circ \cdots \circ \mathcal{G}_1$. We can choose some k such that $j \leq k$ for any type 3 network $\mathcal{G}_j$ and $j > k$ for any type 4 network $\mathcal{G}_j$. Then we define a layerable filtration of G using the subgraphs that represent

$$\begin{aligned}
&\mathcal{G}_n \circ \cdots \circ \mathcal{G}_1 \\
&\mathcal{G}_{n-1} \circ \cdots \circ \mathcal{G}_1 \\
&\ldots \\
&\mathcal{G}_{k+1} \circ \cdots \circ \mathcal{G}_1 \\
&\mathcal{G}_k \circ \cdots \circ \mathcal{G}_1 \\
&\mathcal{G}_k \circ \cdots \circ \mathcal{G}_2 \\
&\ldots \\
&\mathcal{G}_k \circ \mathcal{G}_{k-1} \\
&\mathcal{G}_k \\
&\varnothing
\end{aligned}$$

□

4.4 Parametrizing the Space of Harmonic Functions

For layerable networks, this idea provides an easy way to parametrize the space of harmonic functions and the boundary behavior. If G is layerable, it is not hard to show that we can express $\mathcal{G} : \varnothing \to \partial V$ in the form $\mathcal{G}_n \circ \cdots \circ \mathcal{G}_1 \circ \mathcal{G}_0$ where $\mathcal{G}_j$ is a type 1 or type 2 network for $j \geq 1$ and $\mathcal{G}_0$ is a type 4 network with no inputs. Suppose Γ is a network on G with bijective conductance functions $\gamma_e : M \to M$. Then $\mathcal{B}(\Gamma_0) = \{(\phi, 0)\} \subset M^{V(G_0)} \times M^{V(G_0)}$ since any potentials are possible but the net current at each vertex must be zero. Also, $\mathcal{X}(\mathcal{G}_0) = \mathcal{B}(\Gamma_0) \times \mathcal{X}(\varnothing)$, which we can identify with $\mathcal{B}(\Gamma_0)$ since $\mathcal{X}(\varnothing)$ is a one-element set. Then the boundary behavior of $\mathcal{G}_j \circ \cdots \circ \mathcal{G}_0$ is given by

$$\mathcal{B}_j = \mathcal{B}(\mathcal{G}_j \circ \cdots \circ \mathcal{G}_0) = \mathcal{X}(\Gamma_j) \circ \cdots \circ \mathcal{X}(\Gamma_1)(\mathcal{B}(\Gamma_0)).$$

Hence, we have a bijective parametrization of $\mathcal{B}_j$ by $M^{V(G_0)}$, and in particular $\mathcal{B}(\mathcal{G})$ has such a parametrization. In the process, we have also parametrized the space of harmonic functions $\mathcal{H}_\Gamma$ since all edges of the graph were included in one of the elementary factors. This yields the following corollary (the smoothness, linearity, etc. of the maps below follows from our explicit formula for $\mathcal{X}(\mathcal{G}_j)$):

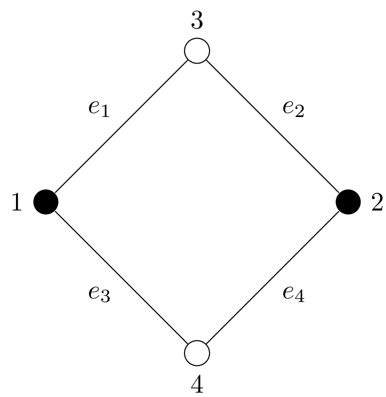
Proposition 4.2. *Suppose Γ is a finite layerable network with bijective conductance functions γ_e. Then $\Phi : \mathcal{H}(\Gamma) \to \mathcal{B}(\Gamma)$ is a bijection. Also,*

a. If M is a field $\mathbb{F}$ and γ_e is a linear isomorphism $\mathbb{F} \to \mathbb{F}$, then Φ is a linear isomorphism, and the space of harmonic functions has dimension $|\partial V|$.

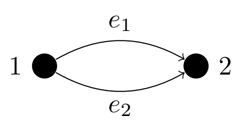
b. If M is a topological abelian group and γ_e is a homeomorphism $M \to M$, then Φ is a homeomorphism.

c. If $M = \mathbb{R}$ or $\mathbb{C}$ and γ_e is a diffeomorphism, then Φ is a diffeomorphism and $\mathcal{B}(\Gamma)$ is a smooth $|B|$-dimensional submanifold of $\mathbb{R}^{\partial V} \times \mathbb{R}^{\partial V}$.

This Fails in General: For an example where (a) fails, see 7.3. For an example where (c) fails, consider the following graph:



Define resistance functions $\mathbb{R} \to \mathbb{R}$ as follows: Let $\rho_{e_1}(t) = \rho_{e_3}(t) = t + \frac{1}{2}\sin t$ (the orientation of the edge does not matter since the function is odd), and let $\rho_{e_2}(t) = \rho_{e_3}(t) = -t$. These are bijective C^∞ resistance functions with a C^∞ inverse. The series with resistance functions ρ_{e_1} and ρ_{e_2} is equivalent to a single-edge with resistance $\rho_{e_1} + \rho_{e_2}$. Thus, the network is equivalent to a parallel connection



in which each edge has resistance function $\rho(t) = \frac{1}{2}\sin t$. Let e_1 and e_2 be the oriented edges shown in the picture. Thus, (u, c) is harmonic if and only if

$$u_1 - u_2 = \tfrac{1}{2}\sin c_{e_1} = \tfrac{1}{2}\sin c_{e_2}.$$

Now $\sin c_{e_1} = \sin c_{e_2}$ is equivalent to $c_{e_2} = c_{e_1} + 2\pi n$ or $c_{e_2} = \pi - c_{e_1} + 2\pi n$. If $c_{e_1} = c_{e_2} + 2\pi n$, then the net current $\psi_1 = c_{e_1} + c_{e_2} = 2c_{e_1} + 2\pi n$ and $\psi_2 = -\psi_1$ and $u_1 - u_2$ must be $\frac{1}{2}\sin\psi_1/2$. If $c_{e_2} = \pi - c_{e_1} + 2\pi n$, then $\psi_1 = (2n+1)\pi$ and $\psi_2 = -\psi_1$ and $u_1 - u_2$ could be any number in $[-1, 1]$. Thus,

$$\begin{aligned} L =& \{(\phi, \psi) : \phi_1 - \phi_2 = \tfrac{1}{2}\sin\psi_1/2,\ \psi_1 = -\psi_2\} \\ &\cup \{(\phi, \psi) : \phi_1 - \phi_2 \in [-1, 1],\ \psi_1 = (2n+1)\pi,\ \psi_2 = -\psi_1\}. \end{aligned}$$

This is not a smooth manifold because there is no Euclidean coordinate neighborhood of the points where $\phi_1 - \phi_2 = \pm 1$ and $\psi_1 = (2n+1)\pi$.

4.5 Parametrizing the IO Boundary Behavior

Motivation: In the last proposition, layerable filtrations correspond to only one specific type of IO factorization where either the input or the output set is trivial. If the input and output sets are nontrivial, the IO factorization gives us much more information about the boundary data on the input and output side are related: For instance, for what boundary data on the input side can we find a matching harmonic function on the network? How unique is it? What possible outputs are compatible with the given input?

This is exactly the sort of question we asked in the example of §1.2. We wanted to find a harmonic function that had certain potentials and net currents on some subset of ∂V. Specifically, referring to Figure 1, we wanted potential zero on 3 and 8, potential 1 on 1, and net current zero on 3. We could take the inputs to be $P = \{1, 3, 8\}$ and the outputs to be $Q = \{1, 8, 2, 7, 10\}$, and ask whether $x = ((1, 0, 0), (0, 0, 0))$ is a valid input. Here $(1, 0, 0)$ represents the potentials on vertices 1, 3, and 8 and $(0, 0, 0)$ represent their input net currents. Since 1 and 8 are both inputs and outputs, setting the input current to zero does not actually determine the net current of the harmonic function, but for vertex 3, we are asking for the overall net current to be zero.

Parametrizing the IO Boundary Behavior: To answer the above questions, assume that

- Γ is a network given by bijective functions $\gamma_e : M \to M$.
- Γ represents an IO-network morphism $\mathcal{G} : P \to Q$.
- There is an elementary factorization $\mathcal{G} = \mathcal{G}_n \circ \cdots \circ \mathcal{G}_1$, such that $\mathcal{G}_j$ has inputs P_{j-1} and outputs P_j, with $P_0 = P$ and $P_n = Q$.
- All the type 3 networks $\mathcal{G}_j$ satisfy $j \leq k$ and all the type 4 networks $\mathcal{G}_j$ satisfy $j > k$.
- Each elementary network has only one edge or one stub.
- The number of input stubs in N_i and the number of output stubs is N_o.

One could start with an input $x_0 \in M^{P_0} \times M^{P_0}$ and see what happens as one works toward the output. But things will be nicer if we start in the middle: Choose some $x_k \in M^{P_k} \times M^{P_k}$.

If $\mathcal{G}_k$ is type 1 or type 2, then there is a unique $x_{k-1} \in M^{P_{k-1}} \times M^{P_{k-1}}$ that is compatible with x_k. On the other hand, if $\mathcal{G}_k$ is type 3, then there are multiple compatible values of x_{k-1} since the potential of the input stub can be anything, but the input net current on the stub must be zero. However, in all cases, x_{k-1} uniquely determines x_k. Repeating this reasoning for $\mathcal{G}_{k-2}$, ..., $\mathcal{G}_1$, we can see that x_0 uniquely determines x_k, but if we start with x_k, then to find all possible values of x_0 we must add one arbitrary parameter on each input stub. So the set of possible x_0's which have a compatible x_k is in bijective correspondence with $M^{P_k} \times M^{P_k} \times M^{N_i}$.

Similarly, considering $\mathcal{G}_{k+1}$, ..., $\mathcal{G}_n$, for any given x_k, we can find a compatible x_n after choosing N_o arbitrary parameters for the output stubs. On the other hand, x_k is uniquely determined by x_n.

Let us call $r = |P_k|$ the *rank* of the factorization (check this is well-defined, independent of the choice of k so long as it is after all the type 3 networks and before the type 4 networks). Suppose that $\pi_P : \mathcal{X}(\mathcal{G}) \to M^P \times M^P$ and

$\pi_Q : \mathcal{X}(\mathcal{G}) \to M^Q \times M^Q$ are the obvious projections. Then the above reasoning gives us bijections:

$$\pi_P(\mathcal{X}(\mathcal{G})) \cong M^{2r+N_i}, \qquad \pi_Q(\mathcal{X}(\mathcal{G})) \cong M^{2r+N_o}.$$

These bijections are "nice" in that they preserve whatever extra structure our network has. For instance, for linear networks over a field, they are linear isomorphisms.

Moreover, if $x_0 \in \pi_P(\mathcal{X}(\mathcal{G}))$, then x_0 uniquely determines x_k, and from x_k, we have M^{N_o} choices for x_n. That is,

$$\begin{aligned} &\text{If } x_0 \in \pi_P(\mathcal{X}(\mathcal{G})), \text{ then } \pi_P^{-1}(x_0) \cong M^{N_o}. \\ &\text{If } x_n \in \pi_Q(\mathcal{X}(\mathcal{G})), \text{ then } \pi_Q^{-1}(x_n) \cong M^{N_i}. \end{aligned}$$

This tells us exactly how many values of x_0 are attainable and how many x_n's are compatible with a given x_0.

We have proven the following:

Proposition 4.3. *Suppose Γ is a finite BZ(M) network representing a morphism $\mathcal{G} : P \to Q$, and that it admits an elementary factorization of rank m with N_i input stubs and N_o output stubs. Then there are bijections:*

$$\begin{aligned} \pi_P(\mathcal{X}(\mathcal{G})) &\cong M^{2r+N_i}, \\ \pi_Q(\mathcal{X}(\mathcal{G})) &\cong M^{2r+N_o} \\ \pi_P^{-1}(x_P) &\cong M^{N_o} \\ \pi_Q^{-1}(x_Q) &\cong M^{N_i}, \end{aligned}$$

for any $x_P \in \pi_P(\mathcal{X}(\mathcal{G}))$ and $x_Q \in \pi_Q(\mathcal{X}(\mathcal{G}))$. Also,

- *If M is a topological space and γ_e is a homeomorphism, then these bijections are homeomorphisms.*
- *If M is a field and γ_e is a linear map, then these bijections are linear isomorphisms.*

The numbers r, N_i, and N_o are detectable from "dimension" of

$$\pi_P(\mathcal{X}(\mathcal{G})), \qquad \pi_P(\mathcal{X}(\mathcal{G})), \qquad \pi_P^{-1}(x_0), \qquad \pi_Q^{-1}(x_n),$$

provided x_0 and x_n are feasible input/output and the conductance functions γ_e have enough extra structure for the dimension to be well-defined. For instance, in the case of linear networks, we take $x_0 = 0$ and $x_n = 0$ and check the dimension of these sets as vector spaces. If M is a finite set, we check the cardinality. If $M = \mathbb{R}$ and γ_e is a homeomorphism $\mathbb{R} \to \mathbb{R}$, we check their dimension as topological manifolds.

Remark. The number r has meaning independent of the factorization: It is the maximum size connection between P and Q. See §8.

Exercises:

1. In the example of Figure 3, find the "dimensions" of $\pi_P(\mathcal{X}(\mathcal{G}))$, $\pi_Q(\mathcal{X}(\mathcal{G}))$, $\pi_P^{-1}(x_0)$, and $\pi_Q^{-1}(x_n)$.

2. Work out an elementary factorization of the example from §1.2 to show that the desired harmonic function exists. What degree of freedom do we have in choosing the extension?

Remark. The reason we required the type 3 IO-graphs come before the type 4 ones was precisely so that the above argument would work. If we had inputs stubs that came after output stubs, then if we started at the input side, we would need to insert extra parameters when we came to an output stubs, and then when we came to later input stubs to determine which parameters on the output stubs were actually valid. The solutions to such equations will not always yield the same answer for the final dimensions.

5 Layering III: Scaffolds

The uses of "scaffolds" are summarized in the following mantra: A scaffold describes

- The order the edges occur in an elementary factorization.
- The order the edges are removed in a layerable filtration.
- The order the edges are used in harmonic continuation.

I will start with the definition of scaffolds and some basic properties (§5.1), postponing the discussion of motivation. The next three sections will explain the three uses of scaffolds listed above, including motivation and examples as well as more technical details. In §5.5, the machinery is in place to define solvable graphs, for which the layer-stripping and harmonic-continuation approach works to solve the inverse problem, and to deduce that if $f : G \to H$ is a ∂-graph morphism, then solvability of H implies solvability of G.

5.1 Definition and Basic Properties

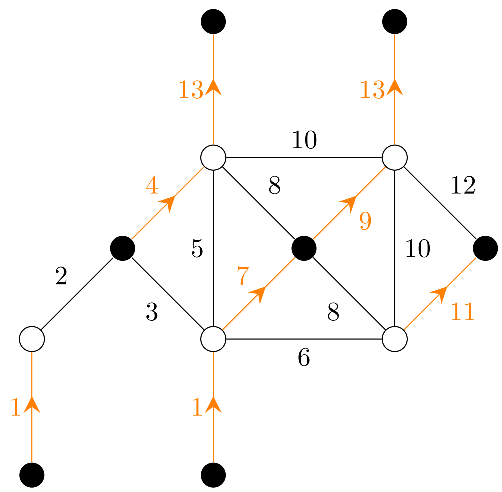
Scaffolds: A *scaffold* $\mathcal{S}$ on a ∂-graph G consists of

A. A strict partial order $\prec$ on the set of edges E'.

B. A partition of E' into two sets, $\mathrm{Lad}(\mathcal{S})$ and $\mathrm{Pl}(\mathcal{S})$, whose elements are called *ladders* and *planks* respectively.

C. For each ladder e, an assigned *head* endpoint $\mathrm{head}(e)$ and *foot* endpoint $\mathrm{foot}(e)$. The head and foot are required to be distinct, so in particular e is not a self-loop. A vertex will be called a *head* if it is the head of some ladder and a *non-head* otherwise, and the same for "foot."

Figure 4: A scaffold on a ∂-graph. The ladders are colored orange and depicted with an arrow pointing from the foot to the head.. The planks are simply black. The numbers of the edges indicate the partial order. That is, $e \prec e'$ if and only if the number on e is less than the number on e'.



(A), (B), (C) are required to satisfy the following conditions:

1. Every subset of E' has a minimal element with respect to $\prec$;

2. If an edge e' is incident to the head of a ladder e, then $e' \succeq e$;

3. If an edge e' is incident to the foot of a ladder e, then $e' \preceq e$;

4. If interior vertices p_1 and p_2 are incident to e_1 and e_2 respectively and if $e_1 \preceq e_2$, then either p_1 is a head or p_2 is a foot.

An example is shown in Figure 4. The ladders are depicted with an arrow pointing from the foot to the head. The ladders are unfortunately not always drawn vertically, so the reader will have to imagine the graph as floating in deep space where there is no global notion of "up." As an exercise, verify that the partial order and ladder assignment in the picture satisfies (1) - (4).

Top, Bottom, and Middle: Given a scaffold $\mathcal{S}$, we can partition the edges into three sets:

- The *top* $\operatorname{Top}\mathcal{S}$ is the set of e such that $e \succeq e'$ for some e' incident to an interior non-head.

- The *bottom* $\operatorname{Bot}\mathcal{S}$ is the set of e such that $e \succeq e'$ for some e' incident to an interior non-foot.

- The *middle* $\operatorname{Mid}\mathcal{S} = E' \setminus (\operatorname{Top}\mathcal{S} \cup \operatorname{Bot}\mathcal{S})$.

More intuitively, to find $\mathrm{Bot}\,\mathcal{S}$, locate all the interior non-feet, then form the set of edges incident to them, then form the set of everything $\preceq$ those. Condition (4) guarantees that $\mathrm{Bot}\,\mathcal{S}$ and $\mathrm{Top}\,\mathcal{S}$ are disjoint. Indeed, if there was an $e \in \mathrm{Bot}\,\mathcal{S} \cap \mathrm{Top}\,\mathcal{S}$, then by transitivity, there would be some $e_1 \preceq e_2$ where e_1 is incident to an interior non-head and e_2 is incident to an interior non-foot, contradicting (4).

Functoriality: Suppose $f : H \to G$ is a ∂-graph morphism and $\mathcal{S}$ is a scaffold on G. We define a scaffold $f^*\mathcal{S}$ on H as follows:

A. Set $e \prec e'$ if and only if $f(e) \prec f(e')$.

B. Let e be a ladder if and only if $f(e)$ is a ladder.

C. In that case, the two endpoints of e are distinct, and we can choose the head and foot so that $f(\mathrm{head}(e)) = \mathrm{head}(f(e))$ and $f(\mathrm{foot}(e)) = \mathrm{foot}(f(e))$ (and there is only one way to do this).

It's straightforward to see that this defines a partial order on $E'(H)$. Next, we check that (1) - (4) are satisfied:

1. To find a minimal element of $T \subset E'(H)$, pick any e such that $f(e)$ is minimal in $f(T)$.

2. Suppose p is the head of a ladder e and $e' \neq e$ is incident to p. Then $f(p)$ is the head of the ladder $f(e)$, and $f(e')$ is incident to it, so $f(e) \prec f(e')$, hence $e \prec e'$.

3. is symmetrical.

4. Suppose that $e_1 \preceq e_2$ in H, and p_1 and p_2 are interior endpoints of e_1 and e_2 respectively. Then $f(p_1)$ and $f(p_2)$ are interior in G and $f(e_1) \preceq f(e_2)$. Thus, either $f(p_1)$ is a head or $f(p_2)$ is a foot. If $f(p_1)$ is a head, then so is p_1; this is because by definition of ∂-graph morphism, f maps the edges incident to p_1 bijectively onto the edges incident to $f(p_1)$. Similarly, if $f(p_2)$ is a foot, then so is p_2.

Let $\mathrm{Scaf}\,G$ be the set of scaffolds on G. By the above construction, this defines a contravariant functor from ∂-graphs to sets. The reader may verify that $\mathrm{Top}\,f^*\mathcal{S} \subset f^{-1}(\mathrm{Top}\,\mathcal{S})$ and $\mathrm{Bot}\,f^*\mathcal{S} \subset f^{-1}(\mathrm{Bot}\,\mathcal{S})$, and hence $\mathrm{Mid}\,f^*\mathcal{S} \supset f^{-1}(\mathrm{Mid}\,\mathcal{S})$.

5.2 Scaffolds and Elementary Factorization

Scaffolds from Elementary Factorizations: Let G be a finite graph. The intuition for the scaffold definitions, especially (2), (3), and (4), should become clearer when we describe how to obtain a scaffold from an elementary factorization:

- The partial order is defined by $e \prec e'$ if the elementary IO-graph containing e comes earlier in the factorization than the one containing e'.

- The ladders are the edges in the type 1 elementary IO-graphs. The input is the foot and the output is the head.
- The planks are the edges in the type 2 elementary IO-graphs.
- (2) holds because if e is a ladder, then head(e) is the output of the elementary IO-graph containing e, and so any other edges incident to head(e) must have come in the later factors. (3) is symmetrical.
- An interior non-head must be an input stub and an interior non-foot must be an output stub. In an elementary factorization, an edge incident to an input stub must come strictly before an edge incident to an output stub. This implies that (4) holds by contrapositive.
- If we imagine a "flow of information" from inputs to outputs, then the ladders are parallel to the flow of information, and the planks are transverse to it.

As an exercise, draw the scaffold produced from the elementary factorization in Figure 3 from §4.2.

In general, scaffolds need not come from elementary factorizations. In a scaffold, a boundary vertex can be incident to two ladders, which does not happen when it comes from a factorization. Also, when a scaffold comes from an elementary factorization, a stronger version of (4) holds where $\preceq$ is replaced by $\not\succ$.

Motivation: Why Scaffolds? The reader has probably been wondering for a while why I introduce scaffolds rather than continuing to use elementary factorizations for everything. First, elementary factorizations are tricky to generalize to infinite graphs. One would have to compose infinitely many IO-graph morphisms. There might also be infinitely many type 3 and type 4 IO-graphs, so to put all the type 3 IO-graphs before the type 4 ones, one would need some indexing system more complicated than a sequence. (In fact, this is exactly the function of the scaffold's partial order.)

Second, elementary factorizations contain information that is redundant or unnecessary for harmonic continuation, and it is tedious to write out the input/output vertices for each elementary IO-graph when the most important thing is the order in which the edges are used, and which parts of the graph come after the input stubs and before the output stubs. A scaffold is much easier to define, both in general theorems and in specific examples.

Third, the scaffold axioms (2) - (4) are defined purely in terms of local conditions. All the global information is captured by the partial order. This is what makes scaffolds so easy to pull back through ∂-graph morphisms, which are also defined by local conditions.

Elementary factorizations on the other hand do not pull back nicely through ∂-graph morphisms. The problem comes from the inclusion maps of subgraphs. Suppose that we have a factorization of a ∂-graph G from P to Q and that H is a subgraph of G. The obvious way to try to get an elementary factorization of H is by intersecting H with each of the subgraphs representing the elementary

Figure 5: Example: Why one cannot induce factorizations on subgraphs.

factors. This gives some factorization in the category of IO-graphs, but not an elementary factorization.

A variety of things go wrong: The inputs for a given elementary factor might not be in H at all, and maybe some of the inputs are in H and some are not. Finally, there could be boundary vertices in the subgraph that are not inputs or outputs of the pulled-back factorization. Consider the following factorization of the graph G in the left of Figure 5:

- G_1 is a type 2 IO-graph using the first row of vertices and edges.
- G_2 is a type 1 using the first row of vertical edges.
- G_3 is type 2 using the middle row vertices and horizontal edges.
- G_4 is type 1 using the second row of vertical edges.
- G_5 is type 2 using the last row of vertices and edges.

In the right picture, there is a subgraph H obtained by changing one interior vertex to boundary. This does not give an elementary factorization of H in any obvious way. The middle vertex cannot be an input of the overall morphism since it is the output of one of the type 1 factors, nor can this vertex be an output for symmetrical reasons.

On the other hand, the scaffold on G derived from this elementary factorization does create a scaffold on H, precisely because scaffolds allow a boundary vertex to be incident to two ladders.

In sum, scaffolds are more flexible and easier to construct than elementary factorizations. However, elementary factorizations make results about the input-output relation easy to formulate precisely and prove, and in §8, it will be useful to have both constructions available.

Elementary Factorizations from Scaffolds: A scaffold can be used to produce a factorization as well, although it is more fussy and requires us to make some arbitrary choices along the way. Assume G is a finite graph. We will prove two claims:

a. If $\mathcal{S}$ is a scaffold with a total order and each boundary vertex is incident to at most one ladder, then $\mathcal{S}$ corresponds to an elementary factorization with P equal to the set of boundary non-heads and Q the set of boundary non-feet.

b. If $\mathcal{S}$ is any scaffold, then we can complete the partial order to a total order, and change some ladders to planks to produce a new scaffold where each boundary vertex is incident to at most one ladder.

(a) Let $e_1, \ldots, e_n$ be edges listed according to the total order. We can choose a k such that

$$\operatorname{Bot} \mathcal{S} \subset \{e_1, \ldots, e_k\} \text{ and } \operatorname{Top} \mathcal{S} \subset \{e_{k+1}, \ldots, e_n\}.$$

Let P be the set of boundary non-heads and Q the set of boundary non-feet. If $k \geq 1$, we claim that e_1 is either a boundary spike with the boundary endpoint in P or a boundary edge with both endpoints in P. Observe:

- If e_1 is a ladder, then there cannot be any other edges incident to the foot by condition (2) and minimality of e_1. The foot can't be interior–if it was, then it would be an interior non-head, and hence e_1 would be in $\operatorname{Top} \mathcal{S}$, contrary to our assumption. Thus, $h(e_1)$ is a boundary spike, and $\operatorname{head}(e_1) \in P$ by definition of P. So e_1 is a boundary spike with boundary endpoint in P.

- If e_1 is a plank, then neither of its endpoints can be a head by condition (3), and hence they must be boundary vertices because $e_1 \notin \operatorname{Top} \mathcal{S}$. The endpoints are also in P by definition. So e_1 is a boundary edge with endpoints in P.

Therefore, we have a factorization $\mathcal{G} = \mathcal{H}_1 \circ \mathcal{G}_1$ where $\mathcal{G}_1$ is a type 1 or type 2 IO-graph. We let P_1 be the set of outputs for $\mathcal{G}_1$. The reader may verify that P_1 contains all the boundary non-heads of $\mathcal{H}_1$. Thus, if $2 \leq k$, we can repeat the process using e_2 and $\mathcal{S}_1$ instead of e_1 and $\mathcal{G}$ and obtain a factorization $\mathcal{G} = \mathcal{S}_2 \circ \mathcal{G}_2 \circ \mathcal{G}_1$.

We repeat this process for the edges $e_2, \ldots, e_k$ to get a factorization

$$\mathcal{G} = \mathcal{H}_k \circ \mathcal{G}_k \circ \cdots \circ \mathcal{G}_1.$$

Next, in a symmetrical way, we start at e_n and work our way backwards to factorize

$$\mathcal{H}_k = \mathcal{G}_n \circ \cdots \circ \mathcal{G}_{k+1} \circ \mathcal{T}.$$

Then $\mathcal{T}$ cannot have any interior vertices, and so $\mathcal{T} = \mathcal{T}_1 \circ \mathcal{T}_2$ where $\mathcal{T}_1$ is an output-stub IO-graph and $\mathcal{T}_1$ is an input-stub IO-graph, and this completes the factorization.

(b) An easy induction which we leave to the reader allows us to complete the partial order of any given scaffold to a total order without changing the beginning and end of the scaffold.

Next, we will change some ladders to planks until each boundary vertex only has one ladder. Choose k such that $\operatorname{Bot} \mathcal{S} \subset \{e_j : j \leq k\}$ and $\operatorname{Top} \mathcal{S} \subset \{e_j : j > k\}$. Suppose that $e_i \prec e_j$ are ladders incident to a boundary vertex p, with $p = \tau(e_i) = \iota(e_j)$. Then either $i \leq k$ or $j > k$. In the first case, we can change e_i to a plank, and we will still have a scaffold. if the other endpoint $q = \operatorname{foot}(e_i)$

was interior, then we have created a new interior non-foot; but all the edges incident to q are $\preceq e_i$, so that the top and bottom are still disjoint. Similarly, in the case $j > k$, we change e_j to a plank. After repeating this for each boundary vertex that is incident to two ladders, we are done.

5.3 Scaffolds and Layerability

Let $G = G_0 \supset G_1 \supset \ldots$ be a layerable filtration. Then each edge e is in $E(G_{n_e}) \setminus E(G_{n_e+1})$ for some n_e. Define $\mathcal{S}$ as follows:

A. $e \prec e'$ if and only if $n_e < n_{e'}$.

B. e is a ladder if it is a boundary spike of G_{n_e} and it is a plank if it is a boundary edge of G_{n_e}.

C. If e is a boundary spike in G_{n_e}, then foot(e) is the boundary endpoint removed in the spike contraction and head(e) is the other endpoint.

We then check conditions (1) through (4):

1. If $T \subset E'$, then we can choose an $e \in T$ such that n_e is minimal, since n_e is always a positive integer.

2. Suppose that $e' \neq e$ is incident to the head of the ladder e. Then e is one of the boundary spikes removed from G_{n_e} and the head is the vertex that was not removed. By the definition of reduction operation, e' cannot have been removed in the reduction of G_{n_e} to G_{n_e+1}, so that $e' \succ e$.

3. Exercise.

4. This holds trivially because all interior vertices are heads. Indeed, every interior vertex must be removed at some point in the filtration, and before that, it must have been changed to a boundary vertex. That can only happen if it was the interior endpoint of some boundary spike which was contracted.

The precise relationship between layerability and scaffolds in the infinite case is described in the next technical lemma, which may be omitted on a first reading:

Lemma 5.1. *For a ∂-graph G, the following are equivalent:*

a. G admits a layerable filtration (that is, G is layerable).

b. There exists a scaffold $\mathcal{S}$ on G with $\operatorname{Top}\mathcal{S} = \varnothing$.

c. For any $e \in E'(G)$, there is a scaffold $\mathcal{S}$ on G with $e \notin \operatorname{Top}\mathcal{S}$.

d. For any $e \in E'(G)$, there is a finite partial layerable filtration $G = G_0 \supset \cdots \supset G_n$ with $e \notin E'(G_n)$.

Proof. (a) $\implies$ (b) $\implies$ (c) is immediate given the above discussion.

(c) $\implies$ (d). Define a new scaffold $\mathcal{S}'$ with the same ladders with the same heads and feet $\mathcal{S}$, but define the new partial order by taking the transitive closure of the relations defined by conditions (2) and (3) of the scaffold definition. (Thus, we are making as few edges comparable to each other as possible given our assignment of ladders.) Every subset of E has a minimal element with respect to $\mathcal{S}$, which will automatically be minimal with respect to $\mathcal{S}'$.

I claim that for any $e \in E'(G)$, there are only finitely many edges $e \preceq e_0$ in $\mathcal{S}'$. If we suppose not, then there is a minimal edge e_0 for which the claim does not hold. There are only finitely many edges $e_1, \dots, e_n$ which incident to and less than e_0, and $\{e \preceq e_0\} = \bigcup_{j=1}^{n} \{e \preceq e_j\} \cup \{e_0\}$ since the relations (2) and (3) used to define our partial order only compare edges which are incident to each other. By minimality of e_0, $\{e \preceq e_j\}$ is finite, which implies $\{e \preceq e_0\}$ is finite, which is a contradiction.

Now choose e. Let $e_1, \dots, e_k = e$ be the edges $\preceq e$ in $\mathcal{S}'$. We can assume they are listed in some nondecreasing order. Let $G_0 = G$. Then e_1 is a minimal edge in G_0. The conditions in the definition of a scaffold force e_1 to be a boundary spike if it is vertical and a boundary edge if it is a plank. Let G_1 be the graph formed by deleting/contracting this edge as appropriate. Then e_2 is a minimal edge in G_1, hence a boundary spike or boundary edge. So (e) follows by induction.

(d) $\implies$ (a). We assumed in §1 that our graphs have countably many edges, so we can write them in a sequence $e_1, e_2, \dots$. For each e_n, choose a k_n and a sequence of subgraphs $G = G_{n,1} \supset \cdots \supset G_{n,k_n}$ as in (d). Then consider the following filtration:

$$\begin{array}{llll} G = G_{1,1}, & G_{1,2}, & \dots & G_{1,k_1}, \\ G_{1,k_1} \cap G_{2,1}, & G_{1,k_1} \cap G_{2,2}, & \dots & G_{1,k_1} \cap G_{2,k_2} \\ G_{1,k_1} \cap G_{2,k_2} \cap G_{3,1}, & \dots & G_{1,k_1} \cap G_{2,k_2} \cap G_{3,k_3} & \\ \dots\dots & & & \end{array}$$

The consecutive elements of this sequence, if they are not equal, are obtained by removing a boundary spike or boundary edge as a result of our earlier observation. Thus, we have a partial layerable filtration which removes all the edges in the graph. We can obtain a new filtration by replacing each reduction with two reductions–first remove the edges in the original reduction, then remove any isolated boundary vertices. The new filtration will remove all the vertices in the graph as well as all the edges. □

5.4 Scaffolds and Harmonic Continuation

Introduction: On finite graphs, we have already described the relationship between scaffolds and elementary factorization, as well as elementary factorization and harmonic continuation. Now we will perform harmonic continuation directly from a scaffold. The process is roughly what one would expect based on §4.5:

- If e is a ladder, then at some point we will know the potential and net current at the foot of e and use that information to find the potential at the head of e.
- If e is a plank, then at some point we will know the potentials at both the endpoints of e and use that information to find the current on e.
- If p is an interior non-head (similar to an output stub), then at some point we will have to make an arbitrary decision about what potential to put on p.
- If p is an interior non-foot, then the continuation process cannot always guarantee that p has net current zero unless we choose the right initial conditions. (In fact, when we are proving the existence of harmonic functions, we will cheat by starting with a harmonic function that is defined on all of $\operatorname{Bot} \mathcal{S}$ and zero on most of it, so the step-by-step continuation will not be used at this point.)

Example: Let us illustrate this by revisiting the example from §1.2; refer to that section as needed.

Suppose we have a BZ(M) network on Γ. Our goal is to recover the voltage-current relation on the edge 3, which is given by a bijective zero-preserving function $\gamma_{1,4} : M \to M$ (we assume the edge is oriented from the vertex 1 to the vertex 4). Given $x \in M$, we want to create a harmonic function that has potential zero on vertex 4 and potential x at vertex 1. Then the net current at 1 will be $\gamma_1(x)$; if we can do this for any x, then we will know $\gamma_{1,4}$.

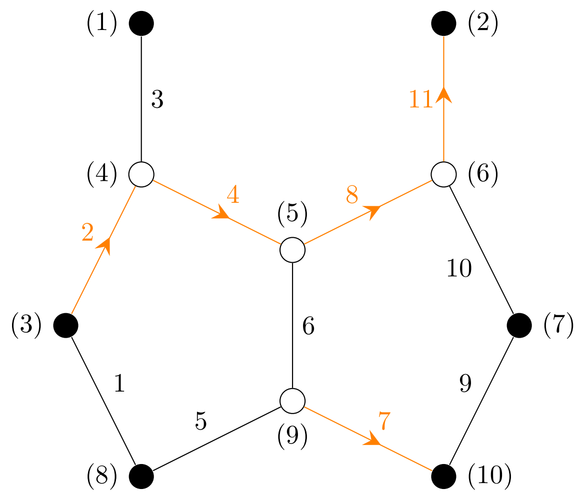
We will start by defining (u, c) on the edges 1, 2, 3 and their endpoints, and then show that this extends to a harmonic function on the whole network. We set potential zero on vertices 3, 4, and 8, and potential x at vertex 1. Now we know the potential at the foot of edge 4 but not the head, and we know the current on edge 4 based on the currents of edges 3 and 2. So there is a unique potential on vertex 5 which yields net current zero at vertex 4.

We next have to decide what happens on edge 5, but first we must decide the potential on vertex 9. This can be anything since 9 is not the head of any ladder. Once that is chosen, then the current on the planks 5 and 6 is determined, hence so is the current on the ladder 7. So the potential at the foot (9) determines the potential at the head (10). Similarly, we use the ladder 8 to find the potential on vertex 6.

After assigning an arbitrary potential to vertex 7, the current on the planks 9 and 10 is determined. From that, the current on the ladder 11 is determined, and hence the potential on its head, vertex 2. At each interior vertex p, the last edge we used was a ladder with p as its foot, and the current on the ladder was chosen to make the function harmonic on at p.

Thus, we have proven the "existence" claim that there exists a harmonic function with potential zero at vertices 3 and 8 and net current zero at 3, and potential x at vertex 1. The second ("uniqueness") claim we have to prove is that any such harmonic function must have net current $\gamma_{1,4}(x)$ at vertex 1. In

Figure 6: A scaffold on the network from Figure 1. The numbers on the vertices are simply their indices, while the numbers on the edges indicate the partial order, which in this case is a total order.



this case, it is easy, but to motivate the general case, let us use the scaffold to prove it. The edge 1 is a plank where the potential at both endpoints is zero. This implies the current on the plank is zero. The next edge 2 is a ladder. We know the net current and potential at the foot, and the current at the other edge incident to the foot, so we know the current on the ladder, hence the potential at its head, vertex 4. Since the potential at both endpoints of the plank 3 is known, so is the current on 3.

Strategy for the General Case: The general use of scaffolds and harmonic continuation to recover the voltage-current relation on a boundary spike (or boundary edge) is the same in spirit. However, to prove the existence of a harmonic extension in the infinite case, we must use Zorn's lemma and phrase things as a proof by contradiction. Both the existence and uniqueness proofs crucially rely on the fact that in a scaffold, every subset of E' has a minimal element. We also need a general way of deciding what vertices we will use for our starting point in both the existence and uniqueness part of the proof; there are many reasonable choices, and I picked one that seemed to be the least work for me writing the proof.

Statements and Proofs:

We need two more pieces of terminology: Let $T \subset E'$. The *∂-subgraph G_T induced by T* is defined as follows:

- $E'(G_T) = T$.
- $V(G_T)$ is the set of vertices incident to edges in T.

- A vertex is interior in G_T if and only if it is interior in G and all the edges incident to it are in T.

A ∂-subgraph $G' \subset G$ is induced if and only if any vertex $p \in V(G') \cap I(G)$ with all edges incident to it contained in $E(G')$ must be interior in G'.

Let $\mathcal{S}$ be a scaffold on G. We say that $G' \subset G$ is a *lower ∂-subgraph* if $e \prec e' \in E(G')$ implies $e \in E(G')$. We say that $G' \subset G$ is an *upper ∂-subgraph* if $e \succ e' \in E(G')$ implies $e \in E(G')$.

Lemma 5.2. *Suppose $\mathcal{S}$ is a scaffold on G and G' is an induced lower subgraph. Let e_0 be a minimal edge not in G' and suppose that $e_0 \notin \operatorname{Bot}\mathcal{S}$. Let G'' be the subgraph induced by $E'(G') \cup \{e_0\}$. Let Γ be a BZ(M) network on G. Then any harmonic function on Γ' has some extension to Γ''.*

Proof. First, suppose e_0 is a ladder. Since G' is an induced lower subgraph, the head of e_0 is not in G'. Since $e_0 \notin \operatorname{Bot}\mathcal{S}$, the head of e_0 must either be a boundary vertex or a foot in G. Hence, not all the edges incident to head(e_0) are in G'', so that head(e_0) must be a boundary vertex of G''. So e_0 is a boundary spike of G'', and thus, we can extend any harmonic function on Γ' to a harmonic function on Γ''.

Next, suppose e_0 is a plank. Then each of its endpoints is either a boundary vertex or a foot in G, and hence a boundary vertex of G''. Thus, e_0 is a boundary edge of G'', so any harmonic function on Γ' extends to a harmonic function on Γ''. □

Lemma 5.3. *Suppose $\mathcal{S}$ is a scaffold on G and G' is an induced lower subgraph with $\operatorname{Bot}\mathcal{S} \subset E'(G')$. Let Γ be a BZ(M) network on G. Then any harmonic function on Γ' extends to a harmonic function on Γ.*

Proof. Let u' be a harmonic potential on Γ'.[5] Consider the set $\mathcal{Z}$ of pairs (Σ, v), where Σ is an induced lower subnetwork of Γ, $\Gamma' \subset \Sigma$, and v is a harmonic potential on Σ which equals u' on Γ'. Let $\mathcal{Z}$ be partially ordered by setting $(\Sigma_1, v_1) \leq (\Sigma_2, v_2)$ if $\Sigma_1 \subset \Sigma_2$ and $v_2|_{V(\Sigma_1)} = v_1$. Note that $(\Gamma_1, u_1) \in \mathcal{Z}$. To apply Zorn's lemma, note that every totally ordered subset $\mathcal{C}$ of $\mathcal{Z}$ has an upper bound. Indeed, for two networks (Σ, v) and $(\Sigma', v) \in \mathcal{C}$, the corresponding harmonic functions agree on the overlap, and hence they produce a well-defined harmonic function v^* on $\Sigma^* = \bigcup_{(\Sigma,v)\in\mathcal{C}} \Sigma$, and (Σ^*, v^*) is an upper bound for $\mathcal{C}$. Hence, by Zorn's lemma $\mathcal{Z}$ has a maximal element (Σ^*, v^*).

If Σ^* is not all of Γ, then there is a minimal edge not in Σ^*. Then by the previous lemma, we can extend v^* to a larger induced lower subnetwork, which contradicts maximality of (Σ^*, v^*). So we are done. □

Lemma 5.4. *Suppose $\mathcal{S}$ is a scaffold on G, and Γ is a BZ(M) network on G. Let Γ' be an induced lower subnetwork of Γ and suppose that $E(\Gamma') \subset \operatorname{Bot}\mathcal{S} \cap \operatorname{Mid}\mathcal{S}$. If (u, c) is any harmonic function on Γ, then $(u, c)|_{\Gamma'}$ is uniquely determined by*

[5] Since the potentials on an edge uniquely determines the current, we can work with u rather than (u, c).

- *The potentials on the vertices of $\partial V(\Gamma) \cap V(\Gamma')$.*

- *The net current on the boundary vertices of Γ that are feet of ladders in Γ'.*

Proof. Suppose that u and v are two harmonic potentials on Γ' with the same potential on $\partial V(\Gamma) \cap V(\Gamma')$ and net current on the boundary vertices of Γ that are feet of ladders in Γ'. Suppose for contradiction that u and v do not agree on all of Γ'. Let T be the set of edges e in Γ' such that u and v disagree on one or both endpoints of e. Then T has a minimal element e_0. Then

- Suppose e_0 is a ladder. If the foot is a boundary vertex of Γ, then by assumption u and v have the same potential and net current there. Also, by minimality of e_0, u and v agree on all edges less than e_0, and in particular all other edges incident to $\mathrm{foot}(e_0)$. Thus, u and v must have the same current on e_0, and hence the same potential at $\mathrm{head}(e_0)$, which contradicts our choice of e_0.

- In the case where e_0 is vertical and $b(e_0)$ is interior in Γ, we know from $e_0 \notin \mathrm{Top}\,\mathcal{S}$ that $\mathrm{foot}(e_0)$ has some edges incident to it at $\mathrm{foot}(e_0)$, and hence u and v have the same potential and net current on $\mathrm{foot}(e_0)$. The same argument yields a contradiction.

- Suppose e_0 is a plank. Since $e_0 \notin \mathrm{Top}\,\mathcal{S}$, we conclude that each endpoint is either a boundary vertex of Γ or incident to edges less than e_0. Thus, u and v have potentials which agree on both endpoints of e_0, which contradicts our choice of e_0.

Therefore, the only possibility is T is empty and $u = v$ on Γ'. ☐

Lemma 5.5. *Let G be a ∂-graph. Suppose that either*

1. e_0 is a boundary spike and there is a scaffold $\mathcal{S}$ with $e_0 \in \mathrm{Pl}\,\mathcal{S} \cap \mathrm{Mid}\,\mathcal{S}$, or

2. e_0 is a boundary edge and there is a scaffold $\mathcal{S}$ with $e_0 \in \mathrm{Lad}\,\mathcal{S} \cap \mathrm{Mid}\,\mathcal{S}$.

For any BZ(M) network Γ on G, Θ_{e_0} is uniquely determined by $\mathcal{B}(\Gamma)$ over BZ(M).

Proof. Consider the case of a boundary spike first. Let p be the valence-one boundary vertex of the spike, q the other vertex. Choose $t \in M$. Let

- Γ_0 be the subnetwork induced by $\{e \prec e_0\}$.

- Γ_1 be the subnetwork induced by $\{e \not\succeq e_0\}$.

- Γ_2 be the subnetwork induced by $\{e \not\succ e_0\}$.

Note $\Gamma_0 \subset \Gamma_1 \subset \Gamma_2 \subset \Gamma$.

I claim that the potential u_2 on Γ_2 which is t at p and 0 everywhere else is harmonic. Because $e_0 \in \mathrm{Mid}\,\mathcal{S}$, we know q is the bottom vertex of some vertical edge, hence not all edges of Γ incident to q are in Γ_2, so that q is a boundary vertex of Γ_1. Hence, p is not adjacent to any interior vertices of Γ_2, so u_2 is harmonic.

By Lemma 5.3, u_2 extends to a harmonic function u on Γ satisfying:

a. $u(p) = t$.

b. $u = 0$ on $\partial V(\Gamma) \cap V(\Gamma_0)$.

c. u has net current zero on the boundary vertices of Γ that are feet of ladders in Γ_0.

By Lemma 5.4, any harmonic function satisfying properties (a) - (c) must be identically zero on Γ_0, and in particular it must have potential zero at q. Since q is the only neighbor of p, this implies the net current on p is $-\gamma_{e_0}(t)$ (if e_0 is oriented with $(e_0)_+ = p$). Thus, by imposing the boundary conditions of (a) - (c), we obtain a unique net current on p which is $-\gamma_{e_0}(t)$. Since this holds for all BZ(M) networks, $\gamma_{e_0}(t)$ is uniquely determined by $\mathcal{B}(\Gamma)$ over BZ(M). Since t is arbitrary, γ_{e_0} is determined.

In the case of a boundary edge, the argument is the same with the following changes:

- Let $q = \text{foot}(e_0)$, $p = \text{head}(e_0)$.
- To define u_1 on Γ_2, note $p, q \in B(\Gamma_2)$ and e_0 is the only edge incident to p in Γ_2. Define u_1 to be zero on Γ_2 and t at vertex p.
- We recover $\gamma_{e_0}(t)$ by noting that it is the net current on q. This is because by (3) all edges incident to q except e_0 are in Γ_0 and hence have current zero. □

5.5 Solvable and Totally Layerable ∂-Graphs

Let $G_0, G_1, \ldots$ be a layerable filtration of a ∂-graph G. We say that it is a *solvable filtration* if it satisfies the following:

- For each spike e removed from G_n, there is a scaffold on G_n in which e is a plank in Mid $\mathcal{S}$.
- For each boundary edge e removed from G_n, there is a scaffold on G_n in which e is a ladder in Mid $\mathcal{S}$.

A ∂-graph which admits a solvable filtration is called *solvable*. This name is appropriate because these are precisely the graphs for which the inverse problem can be solved through layer-stripping with repeated application of harmonic continuation:

Theorem 5.6. *Any solvable ∂-graph is recoverable over BZ(M).*

Proof. Let Γ be a BZ network on G. Let $G_0, G_1, \ldots$ be a solvable filtration, and let $\Gamma_0, \Gamma_1, \ldots$ be the corresponding subnetworks. By Lemma 5.5, the conductance functions of the edges removed from G_n are uniquely determined by $\mathcal{B}(\Gamma_n)$ over BZ(M). By Lemma 3.1, $\mathcal{B}(\Gamma_{n+1})$ is determined by $\mathcal{B}(\Gamma_n)$ and these conductance functions. Hence, by induction each conductance function and each $\mathcal{B}(\Gamma_n)$ is uniquely determined by $\mathcal{B}(\Gamma)$ over BZ(M). $\square$

Proposition 5.7. *Let $f : H \to G$ be a ∂-graph morphism. If $G_0, G_1, \ldots$ is a solvable filtration of G, then $f^{-1}(G_0), f^{-1}(G_1), \ldots$ is a solvable filtration of H. Hence, if G is solvable, then so is H.*

Proof. We already know that a layerable filtration pulls back to a layerable filtration. To see that $f^{-1}(G_0), f^{-1}(G_1), \ldots$ is a solvable filtration, we just pull back the scaffolds used for each edge as in §5.1. $\square$

A more symmetrical (and it turns out stronger) condition than solvability is total layerability. We say that a ∂-graph G is *totally layerable* if for any edge e, there exists a scaffold $\mathcal{S}$ with $e \in \operatorname{Pl}\mathcal{S} \cap \operatorname{Mid}\mathcal{S}$ and a scaffold $\mathcal{S}'$ with $e \in \operatorname{Lad}\mathcal{S}' \cap \operatorname{Mid}\mathcal{S}'$.

Proposition 5.8.

1. *If $f : H \to G$ is a ∂-graph morphism and G is totally layerable, then so is H.*

2. *Any totally layerable ∂-graph is layerable.*

3. *If G is a totally layerable, then it is solvable. In fact, any layerable filtration of G is a solvable filtration.*

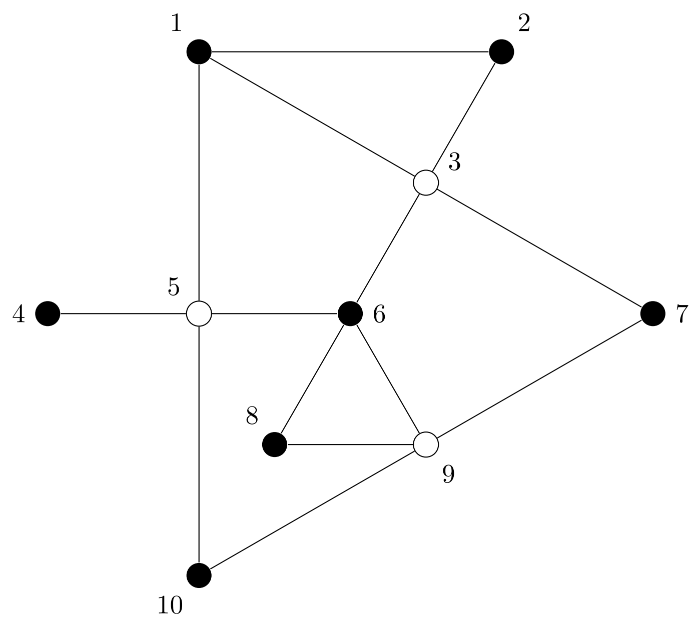
Proof. The first claim follows immediately from properties of the Scaf functor in §5.1. (2) follows from Lemma 5.1. To prove (3), let $G_0, G_1, \ldots$ be a layerable filtration. If e is a boundary spike / boundary edge of G_{n-1}, then there exists a scaffold on G in which e is a horizontal / vertical middle edge, and this induces a scaffold on G_n as well. $\square$

Solvability and total layerability are not equivalent. Figure 7 shows a ∂-graph which is solvable but not totally layerable. A solvable filtration is shown in Figure 8. To check the solvability condition for the first three reduction operations, we use the scaffold depicted in Figure 9. Constructing the scaffolds for the remaining steps is left as an exercise for the reader.[6]

However, this ∂-graph is not totally layerable. Let's index the vertices as in Figure 7 and denote by (i, j) the edge between vertices i and j. I claim that there does not exist a scaffold in which $(1, 2)$ is a ladder in $\operatorname{Mid}\mathcal{S}$.

[6]Alternatively, the theory of critical circular planar graphs developed in the next chapter can be used to show that the third graph in the filtration is totally layerable, hence solvable.

Figure 7: A ∂-graph which is solvable but not totally layerable.



Suppose for contradiction such a scaffold $\mathcal{S}$ exists.[7] As a result of §5.2, we can assume without loss of generality that each boundary vertex is incident to at most one ladder. We can also assume that 1 is the foot of $(1,2)$ and 2 is the head. Now we have

$$(1,3) \prec (1,2) \prec (2,3).$$

Since $(1,2)$ is assumed to be in the middle of $\mathcal{S}$, we know 3 must be both a head and a foot. Hence, $(3,6)$ and $(3,7)$ are both ladders, and one must be oriented going into 3 and one going out. Since each boundary vertex (in particular, vertex 6 or 7) is incident to at most one ladder, we conclude that $(9,7)$ and $(9,6)$ are planks.

We now treat two cases:

- Suppose $(6,3)$ is oriented from 6 to 3 and $(3,7)$ is oriented from 3 to 7. Then

  $$(9,7) \succ (3,7) \succ (2,3) \succ (1,2),$$

  and so $(9,7) \notin \operatorname{Bot}\mathcal{S}$. Hence, 9 must be a foot. We already know $(9,7)$ and $(9,6)$ are planks, so the ladder with 9 as its foot must be either $(9,8)$

[7]For best results when reading this proof, the reader should keep referring to Figure 7 and mark in pencil the ladders and planks in each of the scenarios considered. Time and space constraints prevent me from including figures for each case.

Figure 8: A solvable filtration of the ∂-graph in Figure 7.

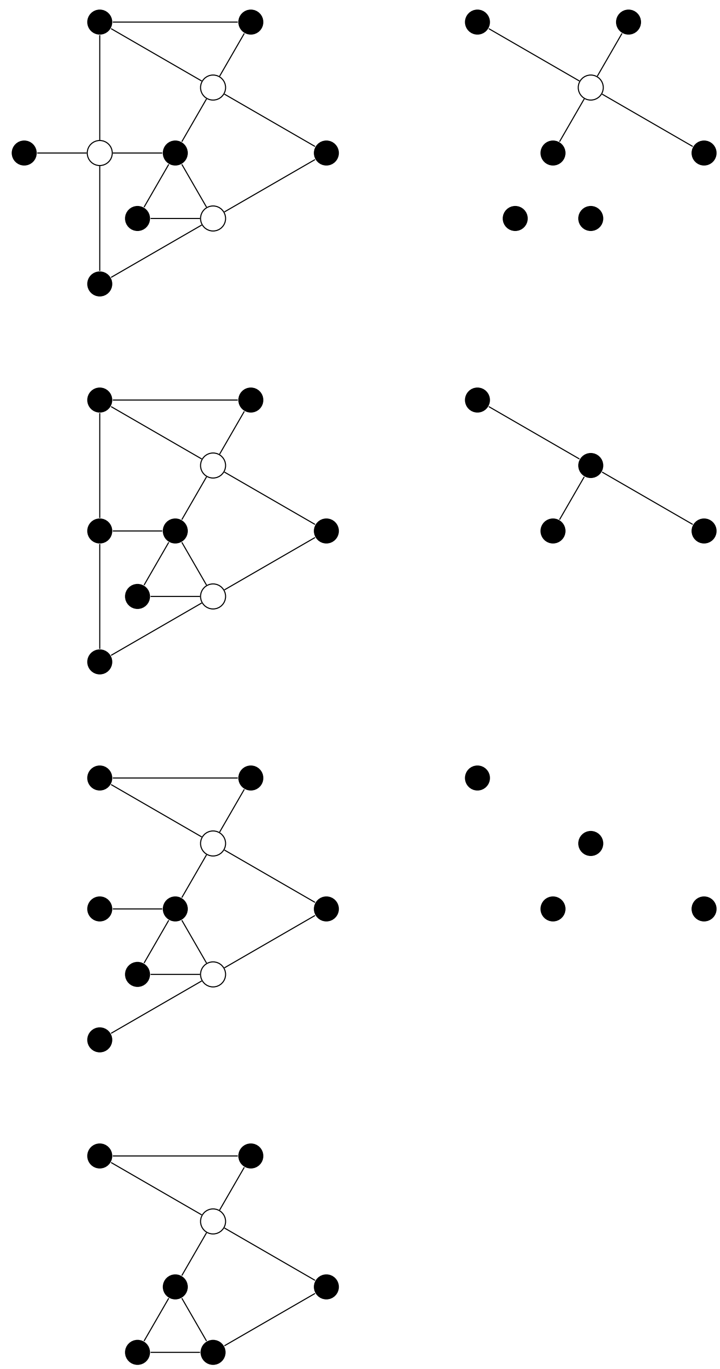
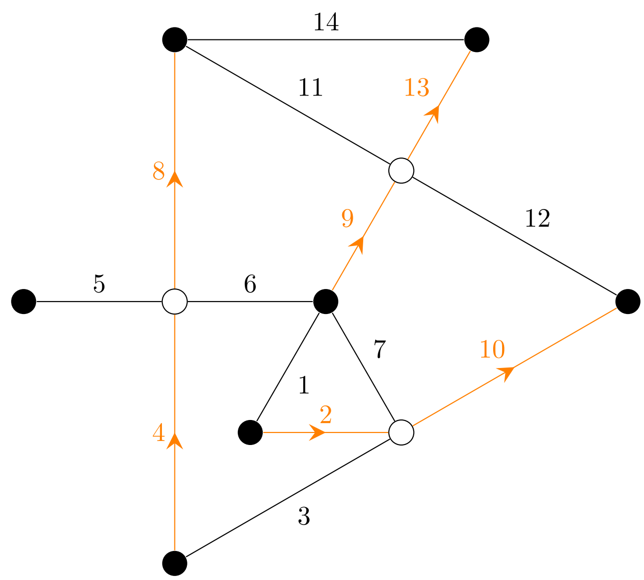


Figure 9: A scaffold on the graph from Figure 7. The vertical edges are orange. The numbering indicates the order of the edges, which in this case is a total order.



or $(9,10)$. It cannot be $(9,8)$ because in that case

$$(9,8) \prec (8,6) \prec (6,3) \prec (3,7) \prec (7,9) \prec (9,8),$$

a contradiction. So suppose $(9,10)$ is a ladder with 9 as its foot. In that case, 5 is incident to only one ladder $(4,5)$, since all the other boundary vertices adjacent to 5 already have other ladder incident to them. Since $(5,1) \prec (1,2) \in \operatorname{Mid}\mathcal{S}$, vertex 5 must be a head. But since

$$(5,10) \succ (9,10) \succ (9,7) \succ (3,7) \succ (2,3) \succ (1,2),$$

vertex 5 must be a foot as well. This contradicts the fact that 5 can only be incident to one ladder.

- Suppose $(7,3)$ is oriented from 7 to 3 and $(3,6)$ is oriented from 3 to 6. By similar reasoning as before, since $(9,7) \prec (1,2)$, we have $(9,7) \notin \operatorname{Top}\mathcal{S}$, hence 9 must be a head. We know $(8,9)$ cannot be a ladder with 9 as its head, since then we would have

$$(8,9) \prec (9,7) \prec (7,3) \prec (3,6) \prec (6,8) \prec (8,9).$$

On the other hand, if $(10,9)$ is a ladder with its head at 9, then similar to before, 5 can only be incident to one ladder. Since

$$(5,10) \prec (10,9) \prec (9,7) \prec (7,3) \prec (3,1) \prec (1,2) \in \operatorname{Mid}\mathcal{S},$$

we know 5 must be a head. But since

$$(5,6) \succ (6,3) \succ (2,3) \succ (1,2) \in \operatorname{Mid}\mathcal{S},$$

we know 5 must be a foot. Thus, we have another contradiction.

For further intuition about why solvability is a weaker condition than total layerability, see §9.

6 Layering Graphs on Surfaces

6.1 Medial Strand Arrangements

The study of graphs on surfaces, especially the disk, has made heavy use of graph embeddings and the medial graph. Before we detail how to construct scaffolds and elementary factorizations using a medial graph, we need some technical definitions. Our goal here is to define "embedding" and "medial graph" in enough generality to cover a lot of degenerate cases that are not usually considered, so that we can safely pass to any subgraph without having to modify our definitions ad hoc. In achieving this goal, we do not care whether "the" medial graph is well-defined, only that *some* medial graph is there for us to use. This approach was essentially adopted in [11] for the disk.

Those who are unfamiliar with medial graphs should consult [6], [11], or Figure 10 for intuition before diving in.

For any graph G, there is a corresponding topological space, the quotient space obtained from $E \times [0,1]$ by identifying (e,t) with $(\overline{e}, 1-t)$ and identifying $(e,0)$ and $(e',0)$ if $e_+ = (e')_+$. We will call this topological space G as well since no confusion will result. An *embedding* of a graph on a surface with boundary S is a function $f : G \to S$ which is a homeomorphism onto its image, such that $f(x) \in \partial S$ if and only if x corresponds to a boundary vertex. Let's identify $f(G)$ with G.

The embedding is *non-degenerate* if each component of $S \setminus G$ is homeomorphic to an open disk. Unfortunately, a non-degenerate embedding can easily become degenerate when we pass to a subgraph, or even delete a boundary edge.

It will be helpful to have a generalization of a "chord diagram" or "pseudoline arrangement," which we will call a "strand arrangement." A *strand arrangement* on a surface with boundary S is a collection of curves on S called *strands* such that

- Each strand s admits a continuous parametrization f_s by $[0,1]$, S^1, $\mathbb{R}$, or $[0,\infty)$ which is a closed map, and is locally a homeomorphism onto its image.
- The endpoints of any strand parametrized by $[0,1]$ or $[0,\infty)$ must be on ∂S. No other points of the strands are allowed to be on ∂S.
- For each $x \in \overline{S}$ there are at most two strand segments which intersect there. That is, $\bigcup_s f_s^{-1}(x)$ contains at most two points.
- Call a point where two strands intersect or one strand intersects itself a *vertex*. We assume the vertices form a discrete set, and none of these points are on ∂S.
- For any point $x \in S$, there is a neighborhood that intersects at most two strands. This prevents infinitely many strands from accumulating near a point.

If $\overline{S}$ is compact, then any strand arrangement will form a ∂-graph embedded on S where the vertices of the strand arrangement are interior vertices of the graph and the endpoints of the strands are boundary vertices (by some tedious topological argument). This fails in the non-compact case because some strands may run off to ∞, but this is perfectly allowable for my purposes.

A *lens* in a strand arrangement is a loop formed by one or two arcs of strands. If an arrangement has no lenses, it is called *lensless*.

The components of S minus the union of the strands are called *cells*. Two cells $\mathcal{A}$ and $\mathcal{B}$ are *adjacent* if $\partial\mathcal{A} \cap \partial\mathcal{B}$ contains some strand segment. A *two-coloring* of the cells is an assignment of a "white" or "black" color to each cell such that no adjacent cells are the same color. Depending on the surface, not all strand arrangements may admit a two-coloring of the cells.

For a graph G embedded on S, a *compatible medial strand arrangement* is a strand arrangement with a two-coloring of the cells such that

- Each black cell is homeomorphic to the disk (though the closure might not be homeomorphic to the closed disk).
- There is a bijective correspondence between vertices of G and black cells such that each black cell contains the corresponding vertex.
- If $\mathcal{A}$ is a black cell, then $\partial\mathcal{A}$ intersects ∂S if and only if the corresponding vertex of G is a boundary vertex.
- There is a bijective correspondence between the edges of G and vertices of the strand arrangement ("medial vertices") such that each edge of G contains the corresponding medial vertex and no other points of any strand.
- At each medial vertex, the two strands "cross" the edge e of G. That is, there is some neighborhood N of the vertex and homeomorphism $h : N \to \mathbb{D}$ such that $h(N \cap e) = \mathbb{D} \cap \mathbb{R}$ and the image of each parametrized strand starts on side of the x-axis moves the other side, intersecting it exactly once.

Depending on how degenerate the embedding is, there may be many different compatible medial strand arrangements.

6.2 ∂-Subgraph Partitions and Elementary Factorizations for Embedded Graphs

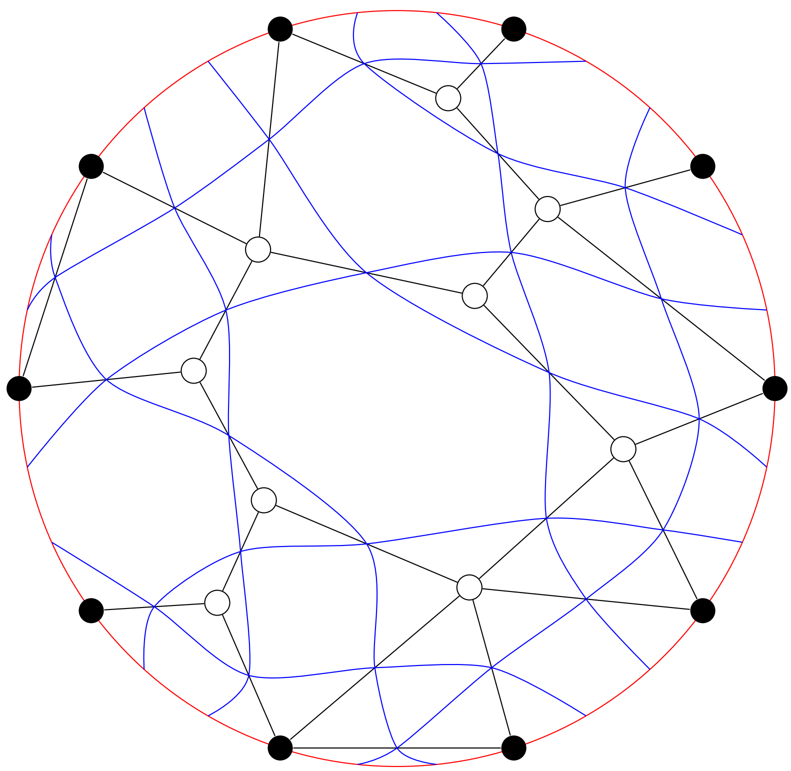
Embeddings and medial strand arrangements provide a way of constructing ∂-subgraph partitions of a graph. Suppose G is a ∂-graph embedded on S with medial strand arrangement $\mathcal{M}$. Let $\mathcal{C}$ be another strand arrangement. Let $\{S_\alpha\}$ be the components of $S \setminus \mathcal{C}$. Assume that

- For each medial cell $\mathcal{A}$, $\mathcal{A} \cap S_\alpha$ is homeomorphic to a disk.
- Each strand of $\mathcal{C}$ intersects the boundary of a medial cell in finitely many points.
- $\mathcal{C}$ doesn't contain any medial vertices of G.

We can define a subgraph G_α of G as follows:

- The vertices of G_α are the vertices of G whose medial cells intersect S_α.
- The edges of G_α are the edges of G whose medial vertices are contained in S_α°.
- A vertex of G_α is interior if and only if its medial cell is contained in S_α°.

Figure 10: A lensless strand arrangement for a ∂-graph embedded on the disk. The medial strands are blue. As an exercise, color in all the cells which have vertices of G.



Then the G_α's form a ∂-subgraph partition of G (exercise). Also, G_α is embedded in S_α with medial strand arrangement $\mathcal{M} \cap S_\alpha$. If a subgraph partition is constructed in this way, we say it is *compatible with the embedding (and medial strand arrangment)*.

The embedding also provides a way to assign input and output vertices to make G into an IO-graph. Take a partition of ∂S into two sets D_1 and D_2 (for instance, two arcs of the boundary of a disk). Then declare $p \in \partial V(G)$ to be input if the closure of its medial cell intersects D_1 and output if the closure of its medial cell intersects D_2.

If D_1 and D_2 are above, an elementary factorization of G into IO-graph morphsims $G_1, \dots, G_n$ is *compatible with the embedding* if

- $G_1, \dots, G_n$ form a ∂-subgraph partition compatible with the embedding, where G_j corresponds to the subsurface S_j.
- The input vertices of G_1 are the ones whose medial cells in S_1 touch D_1, and the output vertices of G_n are the one whose medial cells in S_n touch D_2.
- The input vertices of G_{j+1} are the ones whose medial cells touch $\partial S_j \cap \partial S_{j+1}$ and the output vertices of G_i are the ones whose medial cells touch $\partial S_j \cap \partial S_{j+1}$.

Roughly speaking, the factorization is produced by cutting S into thin slices, along with the medial cells.

6.3 Producing Scaffolds from the Medial Strands

An *orientation* of a strand arrangement is a choice of orientation for each strand. It is *acyclic* if there is a no loop formed by oriented strand segments.

An orientation $\mathcal{O}$ of the medial strands naturally produces a relation on E' and an assignment of vertical and horizontal edges (which may or may not form a scaffold). We can define a relation $\prec$ on the medial vertices by setting $a \prec b$ if there is an increasing path from a to b along medial edges. Define $\prec$ on the edges $E'(G)$ by the relation on the corresponding medial vertices.

Suppose $e \in E'(G)$ corresponds to a medial vertex a. Define e to be a ladder if and only if the ingoing medial edges at a are on the boundary of one black cell, and the outgoing edges are on the boundary of the other black cell. In this case, foot(e) is the G vertex corresponding to the black cell bounded by the ingoing edges, and head(e) is the black cell bounded by the outgoing edges. Equivalently, e is a ladder if the oriented strands cross e in opposite directions. Otherwise, e is a plank. See Figure 11.

For this to be a bona fide scaffold, we need to guarantee several things:

- $\prec$ defines a partial order; this is a equivalent to saying that the orientation of the medial strands is acyclic.
- Every subset has a minimal element (for infinite graphs).

- Conditions (2), (3), and (4) in the scaffold definition are satisfied.

Desired Behavior of Medial Cells: One way to achieve (2), (3), and (4) is to arrange that for any interior medial black cell $\mathcal{A}$, $\partial\mathcal{A}$ can be partitioned into two arcs, the first arc when oriented according to $\mathcal{O}$ moves counterclockwise around $\partial\mathcal{A}$, and the second arc moves clockwise (see Figure 12). That is, all the strand segments in the first arc are oriented counterclockwise around $\partial\mathcal{A}$ and the strand segments in the second are oriented clockwise. For boundary medial black cells, we want the same behavior, except that either arc of $\partial\mathcal{A}$ is allowed to contain portions of the boundary of the surface. If an interior or boundary cell is non-compact, then either one of the two arcs is allowed to contain the point at ∞. Unable to think of a better name, we refer to these conditions as the *Desired Behavior* for the strands that bound a medial cell.

This will guarantee that there are at most two ladders incident to any vertex of G, and that scaffold conditions (2) and (3) are satisfied. It also guarantees that each interior vertex is both a head and a foot, so that scaffold condition (4) is trivially satisfied, and every edge is in $\mathrm{Mid}\,\mathcal{S}$.

To produce a scaffold, the orientation must be chosen judiciously. We will explain how to do this on the disk and half-plane since other surfaces are more complicated and not well understood.

Remark. There is no reason that we could not divide the strands into segments and give a different orientation to each segment, so long as the segment divisions do not fall on medial vertices. This approach is potentially more flexible and adaptable to general surfaces.

6.4 ∂-Graphs in the Disk

Overview: A ∂-graph that can be embedded in the disk is called *circular planar*. A circular planar ∂-graph is called *critical* if it has a lensless medial strand arrangement. [4] and de Verdiere showed that critical circular planar ∂-graphs are recoverable over positive linear real conductances, and [11] extended that result to the nonlinear case. Here I will use scaffolds produced from the medial strands to show that critical ∂-graphs are totally layerable, as well to produce a host of elementary factorizations for them.

The Orientation $\mathcal{O}_\theta$: Suppose we are given a ∂-graph embedded in the disk with a lensless medial strand arrangement. If $e^{i\theta} \in \partial\mathbb{D}$ is not the endpoint of any strand, then we can define an $\mathcal{O}_\theta$ of the strands as follows: If a strand s has endpoints $e^{i\alpha}$ and $e^{i\beta}$ with $\theta < \alpha < \beta < \theta + 2\pi$, then the positive direction moves from $e^{i\alpha}$ to $e^{i\beta}$. I claim $\mathcal{O}_\theta$ produces a scaffold. First, we show it is acyclic:

Lemma 6.1. *For any lensless strand arrangement in $\mathbb{D}$, $\mathcal{O}_\theta$ defines an acyclic orientation.*

Proof. The proof is by induction on the number of strands. It clearly holds for one strand. Suppose it holds for $n-1$ strands and consider n strands $s_1, \ldots, s_n$

Figure 11: Scaffold produced by orienting medial strands in the disk. Medial strands are blue. Ladders are orange.

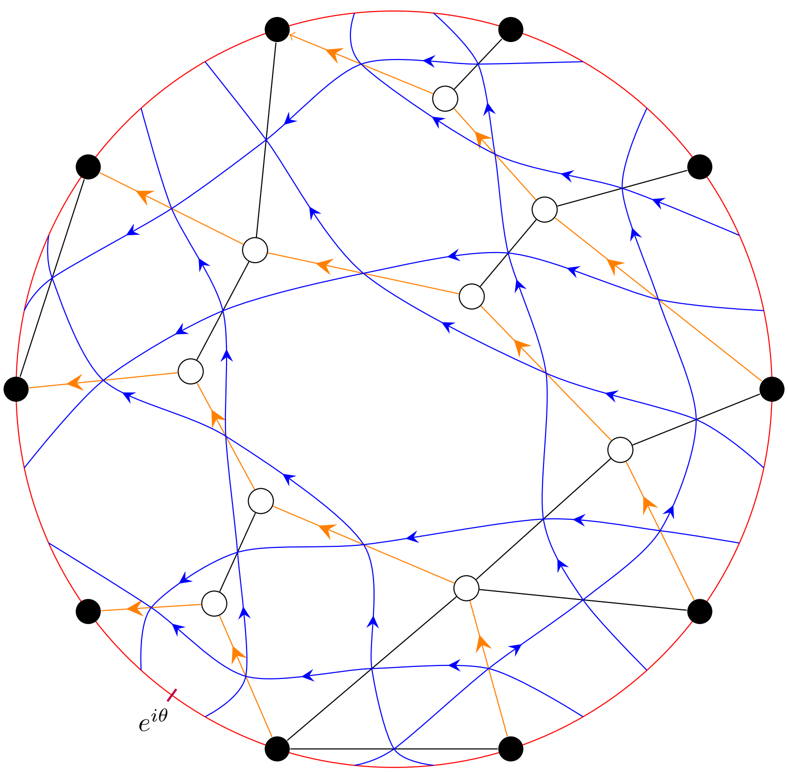
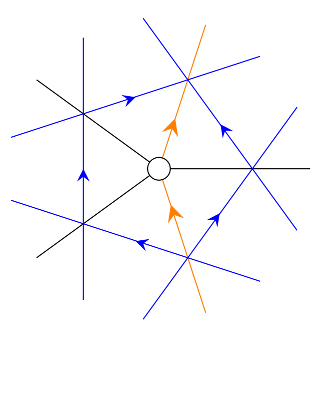


Figure 12: Desired Behavior of oriented medial strands on the boundary of medial black cell containing an interior vertex.



with endpoints $e^{i\alpha_j}$ and $e^{i\beta_j}$ with $\theta < \alpha_j < \beta_j < \theta + 2\pi$. Without loss of generality, $\alpha_n = \min(\alpha_j)$.

From the Jordan curve theorem, we know that $\mathbb{D} \setminus s_j$ has two components, one on the left of s_j and one on the right of s_j. Since the strand arrangement is lensless, s_j can only cross s_k in one direction and the direction can be detected from the positions of the start and endpoints of s_j and s_k on $\partial\mathbb{D}$. For any $j \neq n$, we have $\theta < \alpha_n < \alpha_j$, and this implies that s_j either does not cross s_n or s_j crosses s_n from right to left. Thus, there is no strand that crosses s_n from left to right.

From the induction hypothesis, $s_1, \dots, s_{n-1}$ do not form any oriented loops. Thus, if a loop exists it must contain some segment of s_n and clearly it cannot be entirely contained in s_n. Thus, the loop must exit s_n at some point. After that, it must move into the left component of $\mathbb{D} \setminus s_n$ because no strand crosses s_n from left to right. But then at some point the loop must return to (or cross) s_n from the left component of $\mathbb{D} \setminus s_n$, which implies there is some strand which crosses s_n from left to right, causing a contradiction. So there is no loop. □

Next, we describe the behavior of $\mathcal{O}_\theta$ on the boundary of a medial cell. Note that for a *lensless* strand arrangement on the disk, each medial cell is bounded by a Jordan curve formed by segments of the strands (as can be proved using the Jordan curve theorem and induction on the number of strands). Hence, the boundary of the cell has a well-defined counterclockwise orientation.

Lemma 6.2. *Let $\mathcal{A}$ be a cell of a lensless strand arrangement on $\mathbb{D}$. Let $s_1, \dots, s_n$ be the strands that intersect $\partial\mathcal{A}$, listed in CCW order around $\partial\mathcal{A}$ and oriented in the same direction as the CCW orientation of $\partial\mathcal{A}$ (with $\mathcal{A}$ on the left of each s_j). Let x_j and y_j be respectively the start and end of s_j. Then $x_1, \dots, x_n$ occur in CCW order around $\partial\mathbb{D}$, and so do $y_1, \dots, y_n$.*

Remark. We do not assume in the hypothesis that $s_1, \dots, s_n$ are distinct, although that turns out to be true.

Proof. Suppose $\mathcal{A}$ is an interior cell. Let z be the vertex of $\partial\mathcal{A}$ where s_1 and s_2 intersect. Let C be the counterclockwise arc of ∂D from x_1 to x_2. Let h_1 and h_2 be the arcs of s_1 and s_2 from x_1 and x_2 to z, so that C, h_1, and h_2 bound a triangle T.

Suppose for contradiction that there is some other $x_j \in C$. Let w be the first point where s_j hits ∂T. If $w \in h_2$, then s_j crosses s_2 there from left to right. It cannot intersect s_2 again since $\mathcal{M}$ is lensless, but that implies it cannot intersect $\partial\mathcal{A}$ because $\mathcal{A}$ is on the left side of s_2. So suppose $w \in h_1$. Then at w, s_j crosses from the left to the right side of s_2, and this occurs before the point z along s_2, which implies $z \in \partial\mathcal{A}$ is on the right side of s_j. This also is impossible because $\mathcal{A}$ is supposed to be on the left side of s_j.

This contradiction proves that there is no x_j between x_1 and x_2, and the same argument applies to x_k and x_{k+1} for all k, hence $x_1, \dots, x_n$ occur in counterclockwise order. By a symmetrical argument, $y_1, \dots, y_n$ occur in counterclockwise order. In the case of a boundary cell, similar reasoning applies except that arcs of $\partial\mathbb{D}$ may intervene between the strand segments; details left to the reader. $\square$

Now consider a medial cell $\mathcal{A}$. Let s_j and x_j and y_j as above. Suppose that $x_j = e^{ia_j}$ and $y_j = e^{ib_j}$. We can assume without loss of generality that $\theta < a_1 < a_2 < \cdots < a_n < \theta + 2\pi$, that $b_1 < \cdots < b_n < b_1 + 2\pi$, and that $a_j < b_j < a_j + 2\pi$. Then whenever $b_j < \theta + 2\pi$, the orientation of s_j given by $\mathcal{O}_\theta$ matches the orientation of $\partial\mathcal{A}$, and whenever $\theta + 2\pi < b_j$, the orientations are opposite. Let k be the last index with $b_k < \theta + 2\pi$. Then $\partial\mathcal{A}$ can be divided into two arcs

$$\partial\mathcal{A} \cap (s_1 \cup \cdots \cup s_k), \qquad \partial\mathcal{A} \cap (s_{k+1} \cup \cdots \cup s_n),$$

such that $\mathcal{O}_\theta$ orients the first arc CCW around $\mathcal{A}$ and the second CW. This shows that the strands that bound a medial cell have the Desired Behavior, and thus

Lemma 6.3. *For a ∂-graph in the disk with a lensless medial strand arrangement $\mathcal{M}$, the orientation $\mathcal{O}_\theta$ defines a scaffold $\mathcal{S}_\theta$.*

Total Layerability:

Theorem 6.4 (cf. [4] Theorem 2 and [11] Theorem 6.7)**.** *A critical circular planar ∂-graph is totally layerable, hence solvable and recoverable over BZ(M) for any M.*

Proof. Let e be any edge and let a be the corresponding medial vertex, and s_1 and s_2 the strands that meet there. Note s_1 and s_2 divide $\mathbb{D}$ into four components, and e is contained in two opposite components. If $e^{i\theta}$ is on the boundary of one of the components that contains e, then e is a plank in the scaffold $\mathcal{S}_\theta$, and if $e^{i\theta}$ is on the boundary of one of the other components, then e is a ladder. In either case, $e \in \operatorname{Mid}\mathcal{S}_\theta$ since all edges are in $\operatorname{Mid}\mathcal{S}_\theta$. $\square$

Elementary Factorizations between Circular Pairs: The scaffold $\mathcal{S}_\theta$ not only allows us to prove total layerability, but it can also be used to construct elementary factorizations compatible with the medial strand arrangement.

Any two "cut-points" $e^{i\theta}$ and $e^{i\phi}$ divide $\partial\mathbb{D}$ into two arcs; let C_1 be the CCW arc from $e^{i\theta}$ to $e^{i\phi}$ and let C_2 be the other arc. Let P and Q be the sets of vertices of G whose medial cells touch C_1 and C_2 respectively. Then P and Q are called a *circular pair.* $P \cap Q$ contains at most two vertices. The strands fall into three types:

- A strand with both endpoints on C_1 is called *reentrant on* C_1.
- A strand with both endpoints on C_2 is called *reentrant on* C_2.
- A strand with one endpoint on C_1 and one on C_2 is called *transverse*.

Theorem 6.5 (cf. [6])**.** *Let G be a ∂-graph on $\mathbb{D}$ with a lensless $\mathcal{M}$. Assume the boundary of each medial cell intersects $\partial\mathbb{D}$ in at most one arc. Suppose P and Q are a circular pair produced by cut-points $e^{i\theta}$ and $e^{i\phi}$. Then*

1. *The IO-graph morphism $\mathcal{G} : P \to Q$ represented by G admits an elementary factorization compatible with the medial strand arrangement.*

2. *If r is the rank of the elementary factorization, then*

$$2r = \#(\text{transverse strands}) + |P \cap Q|.$$

Proof. We produce a factorization from the scaffold $\mathcal{S}_\theta$ and medial strands by a similar method to §5.2, also adapting techniques from [6]. Our first goal is to find one of the following:

a. A C_1-reentrant medial strand s with no medial vertices on it. In this case, there is a black cell on one side of s. Because the closure of a medial cell only intersects C_1 in one arc, not two, the black medial cell must be one component of $\mathbb{D} \setminus s$ and must represent an isolated boundary vertex of G on C_1.

b. A triangular medial cell formed by two medial strand segments and an arc of C_1. The two strand segments meet at some medial vertex a. If the cell is black, then a represents a boundary spike of G and the black cell is the boundary vertex of the spike and is in P and not Q. If the cell is white, then a represents a boundary edge of G between two vertices in P.

The transverse strands are all oriented to start at C_1 and end at C_2. Using the Jordan curve theorem, the reentrant strands on C_1 do not intersect those on C_2. Thus, the medial vertices on the C_1-reentrant strands come before those on the C_2-reentrant geodesics in our partial order, when they are comparable. Let W be the set of medial vertices a such that $a \not\succeq b$ for some b on a C_2-reentrant geodesic.

Assume (a) does not occur and that W is nonempty, and we will prove (b) occurs. Let a_1 be a minimal element of W. Then two medial strands s_1 and t_1 meet at a_1, and s and t have no medial vertices between C_1 and a_1. Let T_1 be the triangle formed by C_1 and the segments of s_1 and t_2 from C_1 to a_1. Now T_1 is either a medial cell satisfying (b), or else T_1 contains some whole medial strands, which are necessarily C_1-reentrant. In this case, let $\mathcal{M}_1$ be the union of the medial strands contained in T_1. Let a_2 be a minimal medial vertex in $\mathcal{M}_1$. Then a_2 is the vertex of a medial triangle T_2 by the same reasoning as before. T_2 either satisfies (b) or contains some $\mathcal{M}_2$. This process must terminate after finitely many steps since $\mathcal{M}_{j+1}$ contains strictly fewer strands than $\mathcal{M}_j$. Hence, there is a triangle satisfying (b).

Therefore, either (a) or (b) occurs or else there are no C_1-reentrant strands and W is empty. If (a) or (b) occurs, we can write $\mathcal{G} = \mathcal{G}' \circ \mathcal{G}_1$ where G_1 is an elementary IO-graph of type 1, 2, or 3 and the factorization can be represented by cutting $\mathbb{D}$ into two components with a curve g_1 from $e^{i\theta}$ to $e^{i\phi}$.

Let U_1 be the component of $\mathbb{D} \setminus g_1$ containing G'. Then U_1 is homeomorphic to $\mathbb{D}$ (by standard results from topology) and the scaffold satisfies all the same properties as before. (The cutting may produce medial cells which intersect ∂U_1

in two arcs, but not we cannot produce any which intersect g_1 in two arcs.) We can repeat this process with U_1 instead of $\mathbb{D}$ and g_1 instead of C_1, and we can continue to repeat it until the smaller ∂-graph has no reentrant strands on its version of C_1 and its version of W is empty. We thus obtain a factorization $\mathcal{G} = \mathcal{G}'' \circ \mathcal{G}_n \circ \cdots \circ \mathcal{G}_1$, where the $\mathcal{G}_j$'s are type 1, 2, or 3, and $\mathcal{G}''$ has no reentrant medial strands on its "input" boundary g_n, and in fact all medial vertices are $\succeq$ to a medial vertex on a C_2-reentrant strand.

Next, we repeat this process starting at C_2 with a C_2-reentrant strands with no medial vertices or a maximal medial vertex in our partial order. So we produce elementary IO-graph morphisms $\mathcal{G}_j'$ of type 1, 2, or 4. In the end, we have $\mathcal{G} = \mathcal{G}_1' \circ \cdots \circ \mathcal{G}_m' \circ \mathcal{G}^* \circ \mathcal{G}_n \circ \cdots \circ \mathcal{G}_1$. This $\mathcal{G}^*$ has no reentrant strands and no edges in the graph; it represents the identity IO-graph morphism. Thus, the factorization is complete, proving (1).

In $\mathcal{G}^*$, all the medial cells touch both boundary arcs, and so all the vertices of $\mathcal{G}^*$ are both inputs and outputs. Since $\mathcal{G}^*$ is after to the type 3 IO-graphs and before the type 4 IO-graphs, the maximum connection between the two boundary arcs is the same for $\mathcal{G}^*$ as for $\mathcal{G}$, that is, the number of vertices of $\mathcal{G}^*$. When we passed to a subgraph at each stage of the proof, we did not change the number of transverse strands or the number of medial black cells whose closures contain a cut-point (the cells corresponding to $P \cap Q$). Thus, $\mathcal{G}^*$ and $\mathcal{G}$ have the same number of transverse medial strands. Since (2) holds for $\mathcal{G}^*$, it holds for $\mathcal{G}$ as well. □

6.5 ∂-Graphs in the Half-Plane

In [21], a ∂-graph G embedded in the upper half-plane $\mathbb{H} \subset \mathbb{C}$ is called *supercritical* if it a compatible medial strand arrangement such that

- Each medial strand begins and ends on $\mathbb{R}$ rather than going off to ∞.
- The medial graph is lensless.

[21] adapts the techniques of [11] to prove recoverabiliy of supercritical half-planar ∂-graphs. i shall prove

Theorem 6.6. *Any supercritical half-planar ∂-graph G is totally layerable. More precisely, for each edge e_0 there is a scaffold $\mathcal{S}$ such that e_0 is a ladder (resp. plank) and* $\operatorname{Mid} \mathcal{S} = E$*, and $\mathcal{S}$ can be chosen so that $\{e : e \succ e_0\}$ is finite.*

Finiteness of $\{e : e \succ e_0\}$ implies that the harmonic functions constructed for solving the inverse problem in §5.4 are *finitely supported*. Here's why this could be useful: For positive linear networks or nonlinear networks with "physically meaningful" increasing conductance functions $\gamma_e : \mathbb{R} \to \mathbb{R}$, one might only want to consider bounded or finite-power harmonic functions (as in [21]). However, harmonic continuation might a priori produce unbounded or infinite-power harmonic functions. On the other hand, finitely supported functions automatically satisfy whatever growth conditions one wants to impose at infinity.

Without the finiteness condition, one can prove that a supercritical half-planar ∂-graph is totally layerable using an orientation of the medial strands, similar to the method for the disk, but slightly more complicated, because we must make sure every subset has a minimal element. However, the finiteness condition makes the proof more tricky. The basic plan is as follows:

Let $t_0 < t_1$ be two points on the real line that are not the endpoints of medial strands. The goal is to construct a scaffold such that the harmonic continuation process will "begin" on $(-\infty, t_0] \cup [t_1, \infty)$ and "end" on $[t_0, t_1]$. This cannot be accomplished by simply orienting each medial strand. Instead, we will divide $\mathbb{H}$ into three pieces, produce a scaffold on each piece, and then patch the scaffolds together. The most annoying part of the proof is finding the correct way of cutting up $\mathbb{H}$.[8]

Division of $\mathbb{H}$ into Three Regions: Each strand divides $\mathbb{H}$ into two components–one is bounded, and we will call it the "inside," and the other is unbounded, and we will call it the "outside." Each strand has an endpoint which is further left on the real axis and one which is further right, and hence there is a left-to-right orientation of each strand. In the left-to-right orientation of the strand, the inside is on the right of the strand and the outside is on the left.

Let U be the union of all the following regions:

- The inside of a $[t_0, t_1]$-reentrant strand.
- Any triangle bounded by a segment of a strand with one endpoint on $(-\infty, t_0]$ and one endpoint on $[t_0, t_1]$, a strand with one endpoint on $[t_0, t_1]$ and one endpoint on $[t_1, \infty)$, and a segment of $[t_0, t_1]$.

Claim. *U is the region to the right of some oriented Jordan arc C_0 formed by strand segments and segments of $[t_0, t_1]$ such that*

- *The path starts at t_0 and ends at t_1.*
- *Each strand used in the path has at least one endpoint on $[t_0, t_1]$.*
- *For each strand segment in the path, the orientation of the path matches the left-to-right orientation of the strand.*
- *For each segment of $[t_0, t_1]$ in the path, the orientation in the path matches the increasing orientation of $[t_0, t_1]$.*

Proof. Let $\mathcal{O}$ be the orientation of $\mathcal{M}$ formed by orienting each strand from left to right. Then $\mathcal{O}$ is acyclic. Indeed, any cycle would be formed by only finitely many strands $s_1, \dots, s_n$. If F is a conformal map of $\mathbb{H}$ onto $\mathbb{D}$ and $e^{i\theta} = F(\infty)$, then the orienation $\mathcal{O}$ of $s_1, \dots, s_n$ corresponds to $\mathcal{O}_\theta$ on the disk. But we already showed this is acyclic. Thus, $\mathcal{O}$ defines a partial order on the medial vertices. This can be extended to a partial order on the medial vertices *and*

[8] On a first reading of the paper, one might skip this proof. When reading the proof, start by reading the claims and drawing a picture.

endpoints of strands such that if two endpoints x, y are on the real line with $x < y$ in $\mathbb{R}$, then $x \prec y$.

We say that a region satisfies $(*)$ if it is the region to the right of some path satisfying the conditions of the Claim. Note that U is defined as the union of finitely many regions which satisfy $(*)$. Thus, it suffices to show that if U_1 and U_2 satisfy $(*)$, then so does $U_1 \cup U_2$. Let g_1 and g_2 be the corresponding paths. The intersection points / intervals of g_1 and g_2 must occur in increasing order along g_1 and in increasing order along g_2, and hence they occur in the same order for g_1 and g_2. Thus, we can form a path g_3 as follows: Between any two consecutive intersection points / intervals, follow either g_1 or g_2, whichever one is farther to the left. Then $U_1 \cup U_2$ is the region to the right of g_3, hence satisfies $(*)$. ☐

Claim. *There exists an oriented Jordan arc C_0' such that*

- *C_0' does not contain any medial vertices.*
- *If s is a medial strand with one endpoint on $[t_0, t_1]$ and one endpoint on $(-\infty, t_0] \cup [t_1, +\infty)$, then C_0' intersects s exactly once.*
- *The region to the left of C_0' contains $\overline{U}$ and does not contain any medial vertices not in $\overline{U}$.*
- *The start point t_0' of C_0' is to the left of t_0 with no endpoints of strands in between them. The end point t_1' of C_0' is to the right of t_1 with no endpoints of strands in between.*

Proof. Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be the medial cells outside U whose closures intersect C_0, listed in order along C_0. Construct C_1 inductively starting on $\mathbb{R} \cap \partial \mathcal{A}_1$, then going into $\mathcal{A}_2$, and so forth.

The hardest condition to verify is the second one: Suppose s is a medial strand with one endpoint on $(-\infty, t_0]$ and one endpoint on $[t_0, t_1]$. If s crosses C_0', then it must enter $\overline{U}$ immediately afterward. At the point where it enters U, it must either cross a $[t_0, t_1]$-reentrant strand or enter a triangle formed by strands s_1 and s_2, where s_1 has enpoints on $(-\infty, t_0]$ to $[t_0, t_1]$ and s_2 has endpoints on $[t_0, t_1]$ and $[t_1, +\infty)$. Move along s starting at the endpoint on $(-\infty, t_0$. If s crosses a $[t_0, t_1]$-reentrant strand, then it cannot cross it again, and hence is trapped inside $\overline{U}$ and cannot cross C_0' again. If it enters a triangle formed by s_1 and s_2, then it must have crossed s_1 at some point. Then the triangle formed by s_1 and s is inside U, so the rest of s must also be inside $\overline{U}$. A symmetrical argument works if s has one endpoint on $[t_0, t_1]$ and one on $[t_1, \infty)$. ☐

Claim. *There is a point z on C_0' such that*

- *Any strand starting on $(-\infty, t_0]$ and ending on $[t_0, t_1]$ must intersect C_0' before z ("before" along C_0').*

- *Any strand starting on* $[t_1, +\infty)$ *and ending on* $[t_0, t_1]$ *must intersect* C_0' *after* z*.*

Proof. Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be as above. Since C_0' ends on the outside of all strands with endpoints on $(-\infty, t_0]$ and $[t_0, t_1]$, there must be a first $\mathcal{A}_j$ that is on the the outside of all such strands. Let z be a point of C_0' inside $\mathcal{A}_j$, and let s_1 be the last strand with endpoints on $(-\infty, t_0]$ before $\mathcal{A}_j$.

Suppose for contradiction s is a strand with endpoints on $[t_1, \infty)$ and $[t_0, t_1]$ that intersects C_0' before z. Since s only intersects C_0' once, the only way it can do this is by crossing s_1 outside of C_0', which contradicts the definition of U. ☐

Claim. *There exists an oriented curve* C_1 *injectively parametrized by* $[0, +\infty)$ *such that*

- C_1 *starts at* z *and goes to complex* ∞*.*
- C_1 *does not contain any medial vertices.*
- C_1 *intersects each strand at most once.*
- C_1 *only crosses strands from inside to outside.*
- C_1 *never intersects* C_0' *again.*

Proof. For a given medial cell $\mathcal{A}$ bounded by strands $s_1, \ldots, s_n$, there are two possibilities:

1. $\mathcal{A}$ is on the inside of some s_j.
2. $\mathcal{A}$ is on the outside of each s_j. In this case, by a connectedness argument, $\mathcal{A}$ is exactly the intersection of the outsides of the s_j's, and hence is unbounded.

We construct C_1 inductively cell by cell, starting at z. As long as we are in a cell where (1) holds, we can continue into another cell by crossing a strand from inside to outside. If we ever reach a cell where (2) holds, we can stay inside the cell and go to ∞. Because we only ever cross strands from inside to outside, we never cross the same strand twice or enter the same medial cell twice.

We never enter $\overline{U}$ because to do that, we would have to cross from the outside to the inside of some strand (by previous Claims about U). Thus, we can arrange that we never cross C_0' (since C_0' was defined to "skirt the outside of" $\overline{U}$).

Now we prove the path goes to ∞. This is trivial if (2) ever occurs.

If (1) occurs infinitely many times, then I claim the path is eventually outside any given strand s. The path crosses infinitely many strands from inside to outside. However, the inside of s only intersects finitely many strands, so the path cannot stay inside s forever, and once it goes outside of s it cannot come back inside.

Suppose $K \subset \overline{\mathbb{H}}$ is compact, and we will show that the path is eventually outside of K. Then only finitely many medial cells intersect K. Let K' be the union of the medial cells that intersect K and satisfy (1). Since we assumed (2) never occurs, the path never enters any unbounded cells, so it suffices to show the path is eventually outside K'. But any cell of K' is on the inside of some strand, and we just proved that the path is eventually outside every strand. □

Claim. *Let C_2 be the arc of C_0' before z and let C_3 be the arc of C_0' after z. Then C_1, C_2, and C_3 divide $\mathbb{H}$ into three simply connected regions homeomorphic to the disk:*

- *R_1 is the region outside C_0' and to the left of C_1. It is bounded by $(-\infty, t_0']$, C_1, and C_2.*
- *R_2 is the region outside C_0' and to the right of C_2. It is bounded by $[t_1', +\infty)$, C_1, and C_3.*
- *R_3 is the region inside C_0'. It is bounded by $[t_0', t_1']$, C_2 and C_3.*

Proof. Use the Jordan curve theorem and conformal equivalence of the half-plane and disk. □

Claim. *Let G_1, G_2, G_3 be the ∂-subgraph partition of G induced by the division of $\mathbb{H}$ into R_1, R_2, and R_3, and let $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ be the corresponding medial strand arrangements. Then*

- *Any strand of $\mathcal{M}_1$ either has both endpoints on $(-\infty, t_0']$ or one endpoint on $(-\infty, t_0']$ and one on $C_2 \cup C_1$.*
- *Any strand of $\mathcal{M}_2$ either has both endpoints on $[t_1', +\infty)$ or one endpoint on $[t_1', +\infty)$ and one on $C_3 \cup C_1$.*

Proof. Consider a medial strand s from the original medial strand arrangement $\mathcal{M}$.

- If s is $[t_0, t_1]$-reentrant since then it would is entirely contained in $\overline{U} \subset R_3$, so there is nothing to prove.
- Suppose s has one endpoint on $(-\infty, t_0] \cup [t_1, +\infty)$ and one on $[t_0, t_1]$. Then it crosses C_0' exactly once. Since C_1 only crosses strands from inside to outside and it starts outside s, we know s never crosses C_1, so we are done.
- Suppose s has one endpoint on $(-\infty, t_0]$ and one on $[t_1, +\infty)$, and that it never crosses C_0'. Then we are done since C_1 intersects each strand at most once.
- Suppose s has one endpoint on $(-\infty, t_0]$ and one on $[t_1, +\infty)$, and that it crosses C_0' at some time. Orient s to start on $(-\infty, t_0]$ and end on $[t_1, +\infty)$. Note s cannot intersect a $[t_0, t_1]$-reentrant strand. Thus, once s enters $\overline{U}$, it must be inside one of the triangles in the definition of U, hence it has gone to the inside of a strand s' with one endpoint on $[t_0, t_1]$ and

one endpoint on $[t_1, +\infty)$. Since C_1 is outside of s', s can never intersect C_1 after this point. But by a symmetrical argument, s can never intersect C_1 *before exiting* $\overline{U}$. Thus, it can never intersect C_1 at all.

Furthermore, if s crosses C_2 and hence enters $\overline{U}$, it is inside s' and hence remains outside R_1 and never crosses C_2 again. Thus, s can must cross C_2 exactly once and C_3 exactly once by symmetry.

□

Construction of Scaffold:

The scaffold will be defined so that the direction of harmonic continuation is roughly as follows:

- In G_1, it will go from $(-\infty, t_0']$ to $C_2 \cup C_1$.
- In G_2, it will go from $[t_1', +\infty)$ to $C_3 \cup C_1$.
- In G_3, it will go from $C_2 \cup C_3$ to $[t_0', t_1']$.

Claim. *Let $\mathcal{O}_1$ be the orientation of $\mathcal{M}_1$ defined as follows:*

- *A $(-\infty, t_0']$-reentrant strand is oriented from right to left.*
- *A strand with one endpoint on $(-\infty, t_0']$ and one on $[t_0', t_1']$ is oriented to start on $(-\infty, t_0']$.*

Then $\mathcal{O}_1$ defines a scaffold $\mathcal{S}_1$ on G_1.

Proof. Let $F : \mathbb{H} \to \mathbb{D}$ be a conformal map and let $e^{i\theta} = F(t_0')$. The orientation $\mathcal{O}_1$ matches $\mathcal{O}_\theta$ on the disk, and hence is acyclic. The same argument shows that the medial cells have the Desired Behavior.

To show that every subset has a minimal element, it suffices to show that any descreasing path of medial strand segments must terminate. Let C be any such path, and let Z be the set of strands used in the path. Let s_0 be the strand with the endpoint closest to t_0' on the real line. Then no strand can cross from the right (outside) of s to the left (inside) of s. Hence, once the decreasing path reaches s, it remains trapped in the closure of the region inside s, which contains only finitely many medial vertices. Hence, the path must terminate. □

Claim. *Symmetrically, Let $\mathcal{O}_2$ be the orientation of $\mathcal{M}_2$ defined as follows:*

- *A $[t_1', +\infty)$-reentrant strand is oriented from right to left.*
- *A strand with one endpoint on $[t_1', +\infty)$ and one on $[t_0', t_1']$ is oriented to start on $[t_1', +\infty)$.*

Then $\mathcal{O}_2$ defines a scaffold $\mathcal{S}_2$ on G_2.

To contruct a scaffold on G_3, note there is a homeomorphism $F : R_3 \to \mathbb{D}$ (by corollaries of the Jordan curve theorem), and the homeomorphism extends to the closures. In particular, G_3 is circular planar with no lenses in the medial strands. Let $\mathcal{S}_3$ be the scaffold obtained by pulling back $\mathcal{S}_\theta$ through F, where θ is chosen with $e^{i\theta} = F(t_1')$. This is chosen so that all the strands with one endpoint on $C_2 \cup C_3$ and one on $[t_0', t_1']$ are oriented from $C_2 \cup C_3$ to $[t_0', t_1']$.

The following observations are useful for patching the three scaffolds together:

- Any interior vertex of G_1 or G_2 is both a head and a foot.
- Observe that a vertex of G_1 whose medial cell touches $C_1 \cup C_2$ is a head, but not a foot, because there are no oriented strands that start at $C_1 \cup C_2$. A symmetrical claim holds for G_2.

To define the scaffold $\mathcal{S}$ on G, we splice the partial orders for $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$, and declare that the edges of G_1 are $\prec$ the edges of G_2, and the edges of G_2 are $\prec$ the edges of G_3. The above observations with some casework imply that all the conditions in the scaffold definition are satisfied, except that the conditions (2) and (3) might not be satisfied at a vertex of G whose medial cell is split by C_1, C_2, and C_3. If the medial cell is split be C_1, then the corresponding vertex p will be the head of a ladder in G_1 and a ladder in G_2. We change the ladder from G_2 to a plank, and then because edges in G_1 are $\prec$ edges in G_2, conditions (2) and (3) are now satisfied at p.

After doing this for all the vertices whose medial cells are split by C_1, we do the same thing for all the vertices whose medial cells are split by $C_2 \cup C_3$. Such a vertex must be a head in $\mathcal{S}_1$ or $\mathcal{S}_2$. It may or may not be a head in $\mathcal{S}_3$, but by changing some ladder to a plank if necessary, we can arrange that it is not.

These changes may reduce the number of interior vertices which are feet. But everything which was a head before is still a head, since we only changed a ladder to a plank in cases where the head of the ladder was also the head of a different ladder. Therefore, condition (4) is satisfied, and we indeed have a scaffold $\mathcal{S}$ on G. Moreover, $\operatorname{Top} \mathcal{S} = \varnothing$, since every interior vertex is a head.

Properties of the Scaffold:

Claim. *Let $D \subset R_3$ be the union of the regions enclosed by a $s \cup C_2 \cup C_3$ for each strand s in $\mathcal{M}_3$ that is reentrant on $C_2 \cup C_3$. Then*

a. Any edge whose medial vertex is in $R_3 \setminus \overline{D}$ is in $\operatorname{Mid} \mathcal{S}$.

b. Any edge whose medial vertex is on a $[t_0, t_1]$-reentrant strand is in $\operatorname{Mid} \mathcal{S}$.

c. If $e \in E'(G_3)$, there are only finitely many edges $\succ e$ in $\mathcal{S}$.

Proof. (a) Suppose p is a vertex in P was the head of a ladder e in $\mathcal{S}_3$. Let s_1 and s_2 be the strands which cross at the medial vertex of e. If we assume that s_1 and s_2 both have their ending points on $[t_0', t_1']$, then $\mathcal{A}$ is contained in the triangle bounded by s_1, s_2, and $[t_0', t_1']$. Since $\mathcal{A}$ touches $C_2 \cup C_3$ this is impossible. This implies the medial vertex of e must be in $\overline{D}$.

Since $\mathrm{Mid}\,\mathcal{S}_3$ is all of $E'(G_3)$ by construction of $\mathcal{S}_\theta$, the only way that an edge $e_0 \in E'(G_3)$ can be in $\mathrm{Bot}\,\mathcal{S}$ is if $e_0 \preceq e_1$ where e_1 is incident to a vertex which was changed from a foot in $\mathcal{S}_3$ to a non-foot in $\mathcal{S}$. This vertex is the foot in $\mathcal{S}_3$ of some ladder e_2 such that $p = \mathrm{head}(e_2)$ is a vertex in P. The previous paragraph implies that the medial vertex of e_2 is in $\overline{D}$. But since $e_0 \preceq e_1 \prec e_2$, and there are no oriented strands which ever *enter* $\overline{D}$ from outside, this implies that the vertex of e_0 is also in $\overline{D}$.

By contrapositive, if the medial vertex of e_0 is not in $\overline{D}$, then $e_0 \in \mathrm{Mid}\,\mathcal{S}$. This proves the first claim.

(b). By construction of R_3, any $[t_0, t_1]$-reentrant strand is fully contained in R_3 and is also $[t'_0, t'_1]$-reentrant. Such a strand cannot ever enter D because $\mathcal{M}_3$ is lensless. Thus, all the medial vertices on such a strand are in $R_3 \setminus \overline{D}$.

(c). If $e \in E'(G_3)$, then the only edges $\succ e$ are edges in G_3, and G_3 is finite since it is contained in a compact region. □

Proof of Theorem 6.6: Choose an edge e_0. By choosing t_0 and t_1 correctly and constructing a scaffold as above, we can arrange that the medial vertex of e_0 is on a $[t_0, t_1]$-reentrant strand, and e_0 is either a plank or a ladder as desired. Thus, the Theorem follows from the previous claim.

7 Linear Networks

This chapter has several purposes:

- To collect known results about linear resistor networks, and explain them in a manner consistent with this paper's notation.
- To generalize known results to networks over any field rather than just $\mathbb{R}$, or indicate how they fail to generalize.
- To detail the consequences of layering theory for linear networks.

7.1 Basic Notions

Linear Networks: Let $\mathbb{F}$ be a field. Then a *linear network* over $M = \mathbb{F}$ is a network on a ∂-graph G such that each $\Theta_e = \{(t, a_e t)\}$ for some nonzero $a_e = a_{\overline{e}} \in \mathbb{F}$. Linear networks are automatically BZ($\mathbb{F}$). We shall assume in this chapter that all ∂-graphs are finite, and there are no self-looping edges.

Remark. Some of the results hold for rings as well as fields, but we shall work with fields for simplicity. Many of them also hold in the "projective" case where Θ_e is allowed to be any line in $\mathbb{F} \times \mathbb{F}$, but this will add too many annoying exceptions, so I will leave it to others.

Harmonic Potentials: Let $\mathcal{H}(\Gamma)$ be the set of $\mathbb{F}$-valued harmonic functions (u, c) on Γ, with $c_e = a_e \cdot (u_{\iota(e)} - u_{\tau(e)})$. Then $\mathcal{H}(\Gamma)$ is a linear subspace of $\mathbb{F}^V \times \mathbb{F}^E$. Since c is determined by u, $\mathcal{H}(\Gamma)$ is linearly isomorphic to the space $\mathcal{U}(\Gamma)$ of harmonic potentials, so we will work with $\mathcal{U}(\Gamma)$ instead of $\mathcal{H}(\Gamma)$.

The Kirchhoff Matrix: Recall $\mathcal{U}(\Gamma)$ is the set of all $u \in F^V$ such that

$$\sum_{e \in \iota^{-1}(v)} a_e(u(e_+) - u(e_-)) = 0 \text{ for each } v \in I.$$

These equations can be compactly expressed in terms of the Kirchhoff matrix $K \in M_V(\mathbb{F})$, which is defined exactly the same way as in §1.1, to wit:

$$K_{p,q} = \begin{cases} \sum_{\substack{e:e_+=p,\\ e_-=q}} a_e, & p \neq q \\ -\sum_{e:e_+=p} a_e, & p = q. \end{cases}$$

The Kirchhoff matrix is symmetric. K defines a linear transformation $\mathbb{F}^V \to \mathbb{F}^V$, and the component indexed by p is

$$(Ku)_p = -\sum_{e \in \iota^{-1}(p)} a_e(u_{\iota(e)} - u_{\tau(e)}),$$

which is the net current at p produced by u. If $\pi_{V^\circ} : \mathbb{F}^V \to \mathbb{F}^{V^\circ}$ and $\pi_{\partial V} : \mathbb{F}^V \to \mathbb{F}^{\partial V}$ are the obvious projection maps, then u is harmonic if and only if $\pi_{V^\circ} Ku = 0$, hence

$$\mathcal{U}(\Gamma) = \ker(\pi_{V^\circ} K) \subset \mathbb{F}^V.$$

The Dirichlet Problem: As discussed in §1.1, the Dirichlet problem is this: Given $\phi \in \mathbb{F}^{\partial V}$, does there exists a harmonic potential u such that $\pi_{\partial V}(u) = \phi$? The answer is yes for all ϕ if and only if the submatrix K_{V°,V° is invertible. In this case, we say that the network is *Dirichlet-nonsingular*, and otherwise, it is *Dirichlet-singular*. If Γ is Dirichlet-nonsingular, then the net current for a harmonic function u with $\pi_{\partial V}(u) = \phi$ is given by $\Lambda\phi$, where Λ is the Schur complement $K/K_{V^\circ,V^\circ}$.

The Neumann Problem: While the Dirichlet problem specifies the potentials on the boundary vertices, the Neumann problem specifies the net currents. Of course, if $\psi \in \mathbb{F}^{\partial V}$ is to represent the net current of some harmonic function, then the coordinates of ψ must sum to zero, because the sum of the entries in any column of the Kirchhoff matrix is zero. In fact, the net currents of the boundary vertices in any connected component of G must sum to zero as well.

Assuming that G is connected[9] and has some boundary vertices, the Neumann problem is this: Given $\psi \in \mathbb{F}^{\partial V}$ such that $\sum_{p \in \partial V} \psi_p = 0$, is there some harmonic potential u such that the net current of u on the boundary vertices is given by ψ? In other words, is there some u such that $\pi_{\partial V}(Ku) = \psi$? Is this u unique up to adding a constant? If we let $A \subset \mathbb{F}^V$ be the subspace on which the coordinates sum to zero, then having a unique solution to the Neumann problem is equivalent to $K|_A : A \to A$ being invertible, or equivalent to K having rank $|V| - 1$.

[9] If G is not connected, then we can simply treat the components separately.

7.2 A Grove-Determinant Formula

Overview: To attack the Dirichlet and Neumann problems, and for other purposes described in §8, we want to determine when certain submatrices of K are invertible. Our main tool is Proposition 7.4 below, which is a consequence of the grove-determinant formula Theorem 7.1.

The grove-determinant formula is a generalization of the matrix-tree theorem attributed to Kirchhoff. The version presented here is a special case of Robin Forman's [8], often used in the probabilistic study of spanning forests by Richard Kenyon and David Wilson (see e.g. [13]), and it is similar in spirit to the "determinant-connection formula" ([4] Lemma 4.1).

Forests and Groves: Let G be a graph. A *spanning tree* T is a subgraph (without boundary) such that T is connected, every vertex is in T, and T has no cycles. A *forest* F is a subgraph F with no cycles; the components of F have no cycles, and are therefore *trees*. A *grove* is a forest such that each component contains a boundary vertex.

Let P and Q be disjoint subsets of ∂V with $|P| = |Q| = n$. Let $\mathcal{F}(P, Q)$ be the set of groves F such that each connected component either

- contains exactly one vertex from P and one from Q and no other boundary vertices, or
- contains exactly one vertex from $\partial V \setminus (P \cup Q)$ and no other boundary vertices.

Choose a fixed indexing of V by the integers $1, \ldots, |V|$ with the boundary vertices written first. Let $K_{P\cup V^\circ, Q\cup V^\circ}$ be the submatrix of K with rows indexed by $P \cup V^\circ$ and columns by $Q \cup V^\circ$, ordered according to our given indexing. Let $p_1, \ldots, p_n$ be the vertices of P and $q_1, \ldots, q_n$ the vertices of Q ordered according to the same indexing. For any $F \in \mathcal{F}(P, Q)$, there is a permutation $\tau \in S_n$ such that p_j and $q_{\tau(j)}$ are in the same component of F; call this permutation τ_F.

Theorem 7.1 (Grove-determinant Formula, [8], [13]). *Let Γ be a finite linear network over $\mathbb{F}$. Let P and Q be disjoint subsets of B with $|P| = |Q| = n$. Then*

$$\det K_{P\cup V^\circ, Q\cup V^\circ} = (-1)^n \sum_{F\in\mathcal{F}(P,Q)} \operatorname{sgn} \tau_F \prod_{e\in E'(F)} a_e.$$

The proof consists of intensive bookkeeping and will not be used in the rest of this paper. It is included here in my notation for the sake of completeness.

Proof. Let $m = |V^\circ|$. Let $p_1, \ldots, p_{n+m}$ be the vertices of $P \cup I$ and $q_1, \ldots, q_{n+m}$ be the vertices of $Q \cup I$, so that $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_n\}$ and $p_j = q_j \in V^\circ$ for $j > n$. For $\sigma \in S_{n+m}$, let m_σ be the number of indices j with

$p_j = q_{\sigma(j)}$, or equivalently the number of fixed points $j > n$ of σ. Then

$$\det K_{P\cup V^\circ, Q\cup V^\circ} = \sum_{\sigma\in S_{n+m}} \operatorname{sgn}\sigma \prod_{j=1}^{n+m} \kappa_{p_j, q_{\sigma(j)}}$$

$$= \sum_{\sigma\in S_{n+m}} \operatorname{sgn}\sigma \left(\prod_{p_j\neq q_{\sigma(j)}} \sum_{\substack{e: e_+ = p_j \\ e_- = q_{\sigma(j)}}} (-a_e) \right) \left(\prod_{p_j = q_{\sigma(j)}} \sum_{e: e_+ = p_j} a_e \right)$$

$$= \sum_{\sigma\in S_{n+m}} (-1)^{n+m-m_\sigma} \operatorname{sgn}\sigma \left(\prod_{p_j\neq q_{\sigma(j)}} \sum_{\substack{e: e_+ = p_j \\ e_- = q_{\sigma(j)}}} a_e \right) \left(\prod_{p_j = q_{\sigma(j)}} \sum_{e: e_+ = p_j} a_e \right)$$

Our goal is to expand each of the sums inside the product. Fix σ; choosing one term from each of the inner sums amounts to choosing for each j an edge e_j such that (1) $(e_j)_+ = p_j$ and (2) if $p_j \neq q_{\sigma(j)}$, then $(e_j)_- = q_{\sigma(j)}$. Let $\mathcal{Y}$ be the collection of all sequences $Y = \{e_1, \ldots, e_{n+m}\}$ such that $(e_j)_+ = p_j$. We will say that $\sigma \in S_{n+m}$ and $Y \in \mathcal{Y}$ are *compatible* if (2) is satisfied for every $e_j \in Y$. Then

$$\det K_{P\cup I, Q\cup I} = \sum_{\sigma\in S_{n+m}} (-1)^{n+m-m_\sigma} \operatorname{sgn}\sigma \sum_{\substack{\text{compatible} \\ Y\in\mathcal{Y}}} \prod_{e\in Y} a_e$$

$$= \sum_{Y\in\mathcal{Y}} \sum_{\substack{\text{compatible} \\ \sigma\in S_{n+m}}} (-1)^{n+m-m_\sigma} \operatorname{sgn}\sigma \prod_{e\in Y} a_e$$

$$= \sum_{Y\in\mathcal{Y}} \left(\prod_{e\in Y} a_e \right) \sum_{\substack{\text{compatible} \\ \sigma\in S_{n+m}}} (-1)^{n+m-m_\sigma} \operatorname{sgn}\sigma.$$

Our outer sum is now indexed by $\mathcal{Y}$ rather than S_{n+m}. Our next goal is to show that the inner sum will be zero for any Y which contains a cycle of the graph or a pair $\{e, \overline{e}\}$. Suppose that Y contains a sequence of edges $e_{j(1)}, \ldots, e_{j(k)}$ with $(e_{j(\ell)})_- = (e_{j(\ell+1)})_+$ for $\ell = 1, \ldots, k-1$ and $(e_{j(k)})_- = (e_{j(1)})_+$. If σ is compatible with Y, there are two possibilities: Either

(A) $\sigma(j(\ell)) = j(\ell)$ for all ℓ or

(B) $j(1) \mapsto j(2) \mapsto \ldots \mapsto j(k) \mapsto j(1)$ is a cycle of σ.

In fact, there is a one-to-one correspondence between compatible permutations satisfying (A) and those satisfying (B). Let $\xi \in S_{n+m}$ be the cycle $j(1) \mapsto j(2) \mapsto \ldots \mapsto j(k) \mapsto j(1)$. Then the permutations compatible with Y can be grouped into pairs $\{\sigma, \xi\sigma\}$, where σ satisfies (A) and $\xi\sigma$ satisfies (B). Then $m_{\xi\sigma} = m_\sigma - k$ and $\operatorname{sgn}\xi = (-1)^{k+1}$, so

$$(-1)^{n+m-m_{\xi\sigma}} \operatorname{sgn}(\xi\sigma) = (-1)^{n+m-m_\sigma-k}(-1)^{k+1} \operatorname{sgn}\sigma = -(-1)^{n+m-m_\sigma} \operatorname{sgn}\sigma,$$

and the two terms have opposite signs. Thus,

$$\sum_{\substack{\text{compatible} \\ \sigma \in S_{n+m}}} (-1)^{n+m-m_\sigma} \operatorname{sgn} \sigma = 0$$

because the terms for σ and $\xi\sigma$ cancel.

Therefore, it suffices to consider elements $Y \in \mathcal{Y}$ which do not contain cycles or pairs $\{e, \overline{e}\}$, and which have at least one compatible σ. Let $\mathcal{Z}$ be the set of all such Y, so that

$$\det K_{P \cup V^\circ, Q \cup V^\circ} = \sum_{Y \in \mathcal{Z}} \left(\prod_{e \in Y} a_e \right) \sum_{\substack{\text{compatible} \\ \sigma \in S_{n+m}}} (-1)^{n+m-m_\sigma} \operatorname{sgn} \sigma.$$

We now claim there is a one-to-one correspondence between $\mathcal{Z}$ and $\mathcal{F}(P, Q)$, and that for each $Y \in \mathcal{Z}$, there is exactly one compatible permutation σ. We begin by assigning a forest $F(Y)$ to each Y. If $Y \in \mathcal{Z}$, then there is a unique forest F with $E(F) = Y \cup \overline{Y}$. Call it $F(Y)$. In order to show $F(Y) \in \mathcal{F}(P, Q)$, observe:

a. Any vertex $p \in P \cup V^\circ$ has a unique outgoing oriented edge in Y. Therefore, starting at a given $p \in V$, we can form a path of oriented edges in Y that reaches a vertex in $V \setminus (P \cup V^\circ) = \partial V \setminus P$. Therefore, *every component of $F(Y)$ contains an element of $\partial V \setminus P$.*

b. Assume for contradiction that two elements of $\partial V \setminus P$ are in the same component of $F(Y)$. Then there is a path of oriented edges $e_1, \dots, e_k$ in $Y \cup \overline{Y}$ with $(e_1)_+ \in \partial V \setminus P$ and $(e_k)_- \in \partial V \setminus P$. Since Y has no edges exiting vertices in $\partial V \setminus P$, we have $e_1 \in \overline{Y}$ and $e_k \in Y$. Since each e_j is either in Y or $\overline{Y}$, there is some j with $e_j \in \overline{Y}$ and $e_{j+1} \in Y$, which contradicts the fact that Y has at most one edge exiting a given vertex. This contradiction shows that *every component of $F(Y)$ contains only one element of $\partial V \setminus P$.*

c. Suppose $p_j \in P$. Let σ be the permutation compatible with Y. For some k, $\sigma^k(j) = j \leq n$, so there must be a smallest value of $k \geq 1$ such that $\sigma^k(j) < n$. Then for $1 \leq \ell < k$, $p_{\sigma^\ell(j)} = q_{\sigma^\ell(j)}$ is an interior vertex, and $q_{\sigma^k(j)} \in Q$. This implies that there is a path in Y from p_j to some vertex of Q. Thus, we have *every component of $F(Y)$ that has a vertex in P also has a vertex in Q.*

d. Combining (a), (b), and (c) together with the fact that $|Q| = |P|$, we conclude that each component of $F(Y)$ contains at most one vertex of P, and therefore $F(Y) \in \mathcal{F}(P, Q)$.

Next, we want to prove that $Y \mapsto F(Y)$ defines a *bijection* $\mathcal{Z} \to \mathcal{F}(P, Q)$. *Injectivity:* Note $F(Y)$ uniquely determines $Y \cup \overline{Y}$, so we only need to show $F(Y)$ determines the orientation of each edge in Y. Since any component of $F(Y)$ contains a vertex of $\partial V \setminus P$, for any given edge e in $F(Y)$, we can choose a path in $F(Y)$ from some vertex $p \in V$ to some vertex $q \in \partial V \setminus P$ which uses e, and we can assume the path is an *embedded path*, that is, it has no repeated vertices or edges. Since $F(Y)$ is a forest, there is only one such path from p to q, so this must be the same as the one constructed in (a). Thus, the orientation of e in Y must match its orientation in the path from p to q. So Y is uniquely determined by $F(Y)$.

Surjectivity: Given some $F \in \mathcal{F}(P, Q)$, each component contains exactly one vertex of $\partial V \setminus P$. One can easily check that procedure used to prove injectivity produces a well-defined orientation for each edge in F, hence defines some $Y \in \mathcal{Y}$ with no cycles or pairs $\{e, \overline{e}\}$. To check $Y \in \mathcal{Z}$, we must also produce some σ compatible with Y.

Decompose τ_F into disjoint cycles $\eta_1, \dots, \eta_K$. For each η_k, we define a cycle $\sigma_k \in S_{n+m}$ as follows: Let η_k be given by $i_1 \mapsto i_2 \mapsto i_R \mapsto i_1$ (the dependence on k has been suppressed in the notation). There is a unique embedded path in F from p_{i_r} to $q_{i_{r+1}}$ and the other vertices in the path are interior, so the vertices in all the paths have the form

$$
\begin{aligned}
&p_{i_1},\, p_{j_{1,1}} = q_{j_{1,1}},\, p_{j_{1,2}} = q_{j_{1,2}},\, \dots,\, p_{j_{1,k_1}} = q_{j_{1,n_1}},\, q_{i_2} \\
&p_{i_2},\, p_{j_{2,1}} = q_{j_{2,1}},\, p_{j_{2,2}} = q_{j_{2,2}},\, \dots,\, p_{j_{2,k_2}} = q_{j_{2,n_2}},\, q_{i_3} \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&p_{i_R},\, p_{j_{R,1}} = q_{j_{R,1}},\, p_{j_{R,2}} = q_{j_{R,2}},\, \dots,\, p_{j_{R,k_2}} = q_{j_{R,n_R}},\, q_{i_1}.
\end{aligned}
$$

We define the cycle ξ_k by

$$
\begin{aligned}
&i_1 \mapsto j_{1,1} \mapsto j_{1,1} \mapsto j_{1,2} \mapsto \dots \mapsto j_{1,n_1} \mapsto i_2 \\
&i_2 \mapsto j_{2,1} \mapsto j_{2,1} \mapsto j_{2,2} \mapsto \dots \mapsto j_{2,n_2} \mapsto i_3 \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&i_R \mapsto j_{R,1} \mapsto j_{R,1} \mapsto j_{R,2} \mapsto \dots \mapsto j_{R,n_R} \mapsto i_1.
\end{aligned}
$$

Then let $\sigma = \xi_1 \xi_2 \dots \xi_n$. This implies that $Y \mapsto F(Y)$ is a bijection $\mathcal{Z} \to \mathcal{F}(P, Q)$.

I claim that in fact σ is uniquely determined by Y (or equivalently by F). Suppose σ is compatible with Y. Write σ as a product of cycles $\xi_1, \dots, \xi_\ell$. Suppose ξ_k is given by $j_1 \mapsto j_2 \mapsto \dots \mapsto j_L \mapsto j_1$. If each j_ℓ was greater than n (corresponding to an interior vertex), then we would have $p_{j_\ell} = q_{j_\ell} \in I$ and the edges $e_{j_1}, \dots, e_{j_L} \in Y$ would form a cycle or pair $\{e, \overline{e}\}$, contradicting our assumptions about Y. Thus, some of the indices in the cycle are $\leq n$; it follows that $\xi_1, \dots, \xi_k$ must represent boundary-to-boundary paths just as in our original construction, and the paths are uniquely determined by F.

We now know there is a bijection between $\mathcal{Z}$ and $\mathcal{F}(P, Q)$, and there is exactly one permutation σ_Y compatible with Y. It only remains to relate $\operatorname{sgn} \sigma_Y$ and $\operatorname{sgn} \tau_{F(Y)}$. Consider a cycle η_k which maps $i_1 \mapsto i_2 \mapsto i_R \mapsto i_1$, and let $j_{i,1}, \dots, j_{i,n_r}$ be as above. Let $z_k = \sum_{r=1}^R n_r$, which is the number of interior vertices in the paths corresponding to η_k. Then $\operatorname{sgn} \xi_k = (-1)^{z_k} \operatorname{sgn} \eta_k$. The total number of interior vertices in the paths is $\sum_{k=1}^{K} z_k$. The interior vertices not in the paths are exactly the vertices p_j for which $\sigma(j) = j$. Hence, $\sum_{k=1}^{K} z_k = m - m_\sigma$. Therefore,

$$\operatorname{sgn}\sigma = \operatorname{sgn}(\xi_1 \dots \xi_n) = (-1)^{\sum_k z_k} \operatorname{sgn}(\eta_1 \dots \eta_n) = (-1)^{m-m_\sigma} \operatorname{sgn}\tau_F.$$

Thus, $(-1)^{n+m-m_\sigma} \operatorname{sgn}\sigma = (-1)^n \operatorname{sgn}\tau_F$. Therefore,

$$\sum_{Y\in\mathcal{Z}} (-1)^{n+m-m_{\sigma_Y}} \operatorname{sgn}\sigma_Y \prod_{e\in Y} a_e = (-1)^n \sum_{F\in\mathcal{F}(P,Q)} \operatorname{sgn}\tau_F \prod_{e\in E'(F)} a_e. \qquad \square$$

Corollary 7.2. *Let* $\mathcal{F} = \mathcal{F}(\varnothing, \varnothing)$. *Then* $\det K_{V^\circ,V^\circ} = \sum_{F\in\mathcal{F}} \prod_{e\in E'(F)} a_e$.

Corollary 7.3 (Matrix-Tree Theorem)**.** *Let G be a connected graph (without boundary). Let K be the Kirchhoff matrix of the electrical network where each edge has conductance $a_e = 1$. For $p, q \in V$, $(-1)^{p-q} \det K_{V\setminus\{p\},V\setminus\{q\}}$ is the number of spanning trees of G.*

Proof. If $p = q$, then make G into a graph with boundary by setting $\partial V = \{p\}$. Reindex the vertices so that p occurs first; this does not change the determinant. Then by the previous theorem,

$$\det K_{V\setminus\{p\},V\setminus\{p\}} = \det K_{V^\circ,V^\circ} = \sum_{F\in\mathcal{F}} \operatorname{sgn}\tau_F.$$

Since p is the only boundary vertex, each grove is a spanning tree, so the result is the number of spanning trees. If $p \neq q$, set $\partial V = \{p, q\}$. Reindex the vertices so that p and q occur first; this does not change the determinant, but it does change $(-1)^{p-q}$ to -1. Compute

$$\det K_{V\setminus\{p\},V\setminus\{q\}} = \det K_{\{q\}\cup V^\circ,\{p\}\cup V^\circ} = -\sum_{F\in\mathcal{F}(\{q\},\{p\})} \operatorname{sgn}\tau_F.$$

Again, since p and q are the only boundary vertices, each grove is a spanning tree, and τ_F is the identity. $\square$

The grove-determinant formula allows us to test when $K_{P\cup V^\circ,Q\cup V^\circ}$ is invertible for networks over various fields. In particular:

Proposition 7.4. *Let G be a finite linear network over a field $\mathbb{F}$. Let $P, Q \subset \partial V$ be disjoint with $|P| = |Q| = n$. Then*

a. If $\mathcal{F}(P,Q) = \varnothing$, then $\det K_{P\cup V^\circ,Q\cup V^\circ} = 0$.

b. If $\mathcal{F}(P,Q)$ has exactly one element, then $\det K_{P\cup V^\circ,Q\cup V^\circ} \neq 0$.

c. If $\mathcal{F}(P,Q)$ has more than one element, then there exist linear networks over $\mathbb{R}$ for which $\det K_{P\cup V^\circ,Q\cup V^\circ}$ *is positive, negative, and zero. The determinant is nonzero for some positive numbers.*

Proof. In case (a), the grove-determinant formula expresses the term as a sum over an empty index set, which is zero. In case (b), there is exactly one term in the sum, which is a product of nonzero numbers, hence nonzero. Now consider case (c). Let F_1 and F_2 be two distinct groves. All the groves must have the same number of edges, as is clear from the proof of the grove-determinant formula. Thus, there is some $e_0 \in E'(F_1) \setminus E'(F_2)$ and $e_1 \in E'(F_2) \setminus E'(F_1)$. Choose a sign $\operatorname{sgn} e = \pm 1$ for each edge in E as follows: Set $\operatorname{sgn} e = 1$ for $e \neq e_0, e_1$ and choose $\operatorname{sgn} e_0$ such that $\operatorname{sgn} e_0 \operatorname{sgn} \tau_{F_1} = 1$ and $\operatorname{sgn} e_1 \operatorname{sgn} \tau_{F_2} = -1$. Choose $\epsilon < 1/|\mathcal{F}(P,Q)|$ and set

$$a_e = \begin{cases} \operatorname{sgn} e, & e \in E'(F_1) \\ \epsilon \operatorname{sgn} e, & \notin E'(F_1). \end{cases}$$

Then in the grove expansion of $\det K_{P\cup I, Q\cup I}$, the term for F_1 dominates, making the determinant positive. In the other hand, if

$$b_e = \begin{cases} \operatorname{sgn} e, & e \in E'(F_2) \\ \epsilon \operatorname{sgn} e, & \notin E'(F_2), \end{cases}$$

then the determinant is negative. Applying the intermediate value theorem to the connected region $\{c \in \mathbb{R}^{E'} : \operatorname{sgn} c_e = \operatorname{sgn} e\}$ shows that there are nonzero numbers c_e which will make the determinant zero.

The same argument shows that whatever sign we choose for the edges, we can make $\det K_{P\cup I, Q\cup I}$ nonzero; in particular, this holds if we want the weights to be positive. □

7.3 Dirichlet-Singular and Neumann-Singular Networks over $\mathbb{R}$

Proposition 7.4 enables us to determine immediately when the Dirichlet and Neumann problems have a unique solution.

Dirichlet-Singular Networks: Assume that G is connected and has some boundary vertices. As the reader can verify, this implies that there is at least one grove in $\mathcal{F}$. Hence, if $a_e > 0$,

$$\det K_{V^\circ, V^\circ} = \sum_{F \in \mathcal{F}} \prod_{e \in E'(F)} a_e > 0.$$

Therefore, the Dirichlet problem has a unique solution (see §7.1). However, for most graphs there will be real values of $a_e \neq 0$ for which K_{V°, V° is not invertible. In fact, the only ∂-graphs for which the Dirichlet problem always a unique solution for any linear network over $\mathbb{R}$ are so trivial that the Dirichlet problem will also have a unique solution in the nonlinear case:

Proposition 7.5. *Let G be a finite ∂-graph. The following are equivalent:*

a. *If G' is the graph obtained from G by deleting all boundary edges, then G' is a forest with one boundary vertex in each component.*

b. *The Dirichlet problem has a unique solution for any BZ(M) network on G.*

c. *The Dirichlet problem has a unique solution for any linear network over $\mathbb{R}$ on G.*

Proof. (c) $\implies$ (a). By Proposition 7.4, (c) implies that $\mathcal{F}$ has only one element, and this implies (a) as the reader may verify.

(a) $\implies$ (b). Given any $\phi \in M^{\partial V}$, we define u to be constant on each component of G'. Details left to the reader.

(b) $\implies$ (c) trivially. □

A more delicate question is, for linear networks, what are the possible values of $\dim\ker K_{V^\circ,V^\circ}$? This depends on the graph, but in some cases, it is easy to find a lower bound: Suppose $G_1, \dots, G_N$ form a subgraph partition of G and $\partial V(G_k) \subset \partial V(G)$ for all k. Suppose there are Dirichlet-singular conductances for each G_k, and let the conductances on G be the same as the conductances on the G_k's. Since $\ker K_{I,I}$ is nontrivial for each G_k, there is a nonzero harmonic potential u_k on G_k, and we can extend it to G by setting it to zero on the other vertices. The potentials thus defined are linearly independent because u_k is nonzero on G_k, but u_j for $j \neq k$ is zero on G_k. Thus, $\dim\ker K_{I,I} \geq N$.

Neumann-Singular Networks: If $a_e > 0$, the Neumann problem has a unique solution. By similar reasoning as in Corollary 7.3, for any p, q,

$$(-1)^{p-q} \det K_{V\setminus\{p\},V\setminus\{q\}} = \sum_{\substack{\text{spanning} \\ \text{trees } T}} \prod_{e \in E'(T)} a_e.$$

Since G is connected, it has a spanning tree, so the right hand side is positive if $a_e > 0$. So K has rank $|V|-1$ and the Neumann problem has a unique solution. This also shows that the determinant of any $|V|-1$ by $|V|-1$ submatrix of K is the same up to sign, so to see whether the Neumann problem has a unique solution, it suffices to check one of them. Similar to the Dirichlet problem, we have

Proposition 7.6. *Let G be a finite connected ∂-graph with at least one boundary vertex. The following are equivalent:*

a. *G is a tree.*

b. *The Neumann problem has a unique solution (up to adding a constant to the potential) for any BZ(M) network on G.*

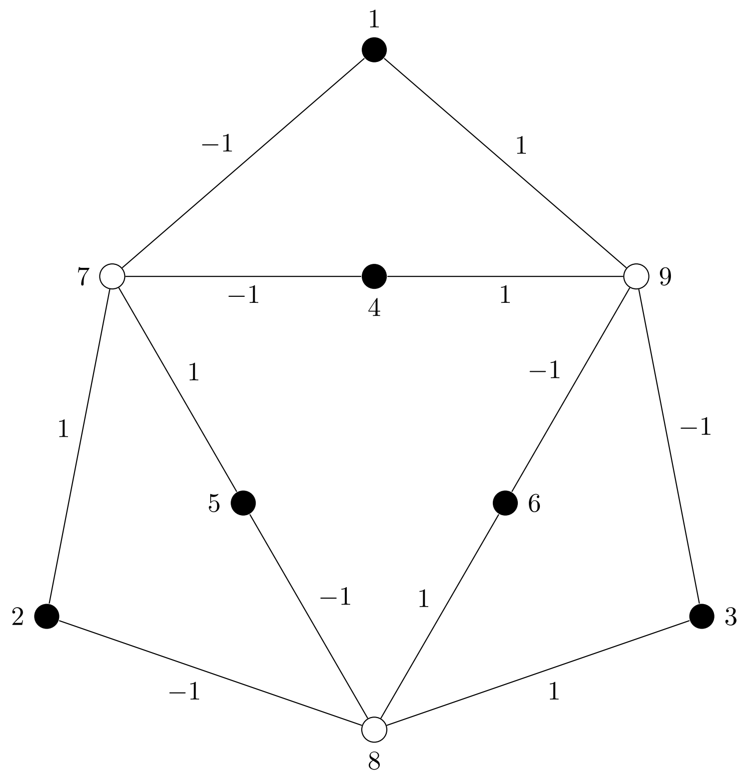
c. *The Neumann problem has a unique solution (up to adding a constant to the potential) for any linear network over $\mathbb{R}$ on G.*

Proof. (c) $\implies$ (a) because if G is not a tree, then there is more than one spanning tree for G, so by Proposition 7.4, there exist real weights for the edges of G that will produce a Neumann-singular network.

(a) $\implies$ (b) is left to the reader.

(b) $\implies$ (c) trivially. □

Figure 13: Singular conductances on the triangle-in-triangle network. Boundary vertices are colored in. Vertices are labelled with their index. Edges are labelled with their conductance.



For linear networks, what are the possible values of $\dim\ker K$? It must be ≥ 1. Now suppose $G_1, \ldots, G_N$ form a subgraph partition of G, such that each G_k is connected and any cycle of G is contained in some G_k. Suppose there exist Neumann-singular conductances on each G_k, and use them to define conductances on G. Then for each G_k, there exists a non-constant harmonic potential u_k on G_k with net current zero on every vertex. We can extend u_k to G by defining it to be constant on each G_k; this will be consistent because every cycle is contained in some G_k. Then the u_k's are linearly independent, so $\dim\ker K \geq N + 1$.

Harmonic Functions with Potential and Net Current Zero on the Boundary: For some networks, it is possible for a nonzero harmonic function to have potential and current zero on the boundary, even if there are no components without boundary vertices. Consider the "triangle-in-triangle" network with boundary vertices $\{1, \ldots, 6\}$ and interior vertices $\{7, 8, 9\}$ and edges with coefficients a_e shown in the figure. The Kirchhoff matrix is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

Let e_p be the vector with 1 on vertex p and zero elsewhere. Then $e_7 + e_8 + e_9$ is a harmonic potential which is zero on the boundary and also has net current zero at each boundary vertex.

However, if G is a finite layerable graph, then this behavior is impossible: Any harmonic function with potential zero and net current zero on each boundary vertex must be identically zero by Proposition 4.2.

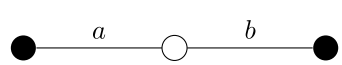
7.4 Local Network Equivalences

Overview: In §2.2, we stated the principle of "subnetwork splicing": If Γ' is obtained by replacing some subnetwork of Γ by another subnetwork with the same boundary behavior, then Γ and Γ' have the same boundary behavior. For linear networks with positive weights, there are several well-known replacements we can make ("local network equivalences"):

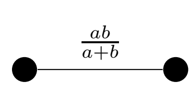
- A "series" can be replaced by a single edge with the appropriately chosen weight.
- A pair of parallel edges can be replaced by a single edge.
- A "Y" and a "Δ" are interchangeable.
- A "star" can be replaced by a network on a complete graph.

However, over arbitrary fields, it is not always possible to find weights on the new graph that will produce the same boundary behavior. I will explain each of these transformations, carefully noting when they do and do not generalize to arbitrary fields, and finish by summarizing two applications from the literature.

Series Reduction: A *series* is the following configuration:



If $a + b \neq 0$, then this has the same boundary behavior as



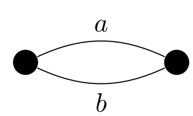
In other words, a series can be reduced to a single edge, and the resistances add: The original resistances were $1/a$ and $1/b$, and the new resistance is $1/a + 1/b$. This shows that the series is not recoverable; in fact, there is a one-parameter family of conductances on the series graph which produce the same boundary behavior.

If $a + b = 0$, then the series is Dirichlet-singular. The two boundary vertices must have the same potential. The potential of the interior vertex is independent of the boundary potentials, but depends on the current flowing from one boundary vertex to the other. In this case, changing the conductances to ca and cb for some $c \neq 0$ will produce an electrically equivalent network.

Any ∂-graph which contains a series is not recoverable for any field except $\mathbb{F}_2$ If a and b are any weights on the series with $a + b \neq 0$, we can produce a network with the same boundary behavior by replacing the series subnetwork with a single-edge subnetwork. This transformation is called a *series reduction.*

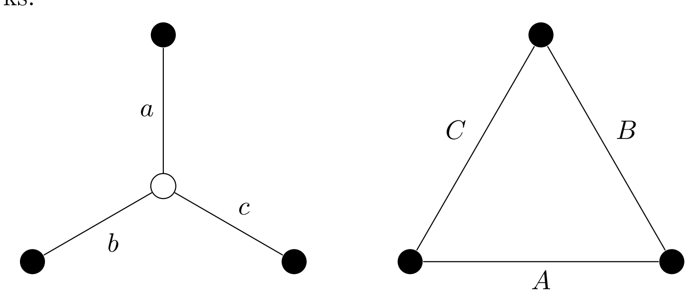
Suppose $a + b = 0$ and p and q are the endpoints of the series, and r is the middle vertex. If the series is a subnetwork of a larger network in which p is an interior vertex, then we can produce an electrically equivalent network by "collapsing" the series–identifying p and q and removing r and the edges in the series. This is because any harmonic function must have the same potential on p and q, and the amount of current flowing from p to q is independent of the potentials.

Parellel Reduction: A parallel circuit is the following configuration:



If $a + b \neq 0$, then this is equivalent to a single edge with conductance $a + b$. If $a + b = 0$, then it is equivalent to a network with no edges. Substituting a parallel edge for a single edge or no edge is another local electrical equivalence.

Y-Δ Transformation: A Y (left) and a Δ (right) are the following types of networks:[10]



For any Y with $a + b + c \neq 0$, there is a unique equivalent Δ with

$$A = \frac{bc}{a+b+c}, \quad B = \frac{ac}{a+b+c}, \quad C = \frac{ab}{a+b+c}.$$

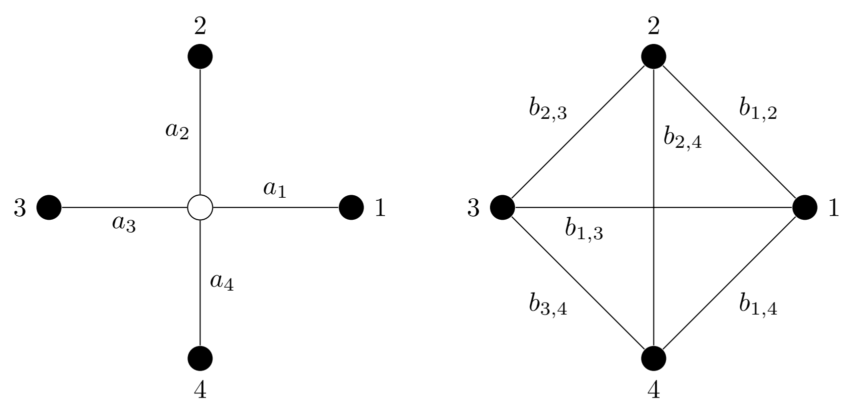
[10]The Y-Δ transformation has been known to electrical engineers since at least 1899 [12].

This can be proved by computing the response matrix Λ for each network. If $a+b+c=0$, then in the Y the Dirichlet problem does not always have a solution; however, this is impossible in a Δ, so there is no equivalent Δ. For any Δ with $1/A+1/B+1/C \neq 0$, there is a unique equivalent Y with

$$a = \frac{AB+BC+CA}{A}, \quad b = \frac{AB+BC+CA}{B}, \quad c = \frac{AB+BC+CA}{C}.$$

However, if $1/A+1/B+1/C=0$, then the Δ is Neumann-singular because it is a tree, so there is no equivalent Y. A *Y-Δ transformation* is the transformation that replaces a Y subnetwork with an equivalent Δ subnetwork or vice versa.

★-$\mathcal{K}$ Transformation: The final type of local electrical equivalence is the *★-$\mathcal{K}$ transformation* (also known as the "star-mesh" or "star-polygon transformation").[11] An n-star is a graph with n boundary vertices and one interior vertex, and one edge from the interior vertex to each boundary vertex. The *complete graph* $\mathcal{K}_n$ is a graph with n boundary vertices and one edge between each pair of distinct boundary vertices. For example, here are networks on 4-star and $\mathcal{K}_4$ graphs:



Index the vertices of the n-star and $\mathcal{K}_n$ by $1,\dots,n$. Let a_j be the conductance of the star edge incident to j and $b_{i,j}$ the conductance of the edge in the $\mathcal{K}_n$ between vertices i and j. Let $\sigma = a_1 + \cdots + a_n$. For any star with $\sigma \neq 0$, there is an equivalent $\mathcal{K}_n$ with conductances $b_{i,j} = a_i a_j/\sigma$. If $\sigma = 0$, then the star has Dirichlet-singular boundary behavior and hence is not equivalent to a $\mathcal{K}_n$. If $n \geq 4$, most $\mathcal{K}_n$'s are not equivalent to a star, unlike the $n=3$ case of Y-Δ transformations:

Lemma 7.7 (modification of [17])**.** *Let $n \geq 4$. A network on a $\mathcal{K}_n$ has the same boundary behavior as some n-star if and only if*

- *It satisfies the quadrilateral rule: $b_{i,j}b_{k,\ell} = b_{i,k}b_{j,\ell}$ for distinct i,j,k,ℓ.*
- *It is Neumann-nonsingular.*

[11] This has also been known to electrical engineers for a long time. See e.g. [20]. The terminology and applications described here are from [17] and [10].

Proof. If the network has the same boundary behavior as some n-star, then for distinct i, j, k, ℓ,

$$b_{i,j}b_{k,\ell} = \frac{a_i a_j a_k a_\ell}{\sigma^2} = b_{i,k}b_{j,\ell}.$$

A star is a tree and is therefore not Neumann-singular. Thus, the Neumann problem always has *some* solution on the $\mathcal{K}_n$ network. Since this network has only boundary vertices, this implies that $\operatorname{rank} K = |V| - 1$, so the $\mathcal{K}_n$ network is Neumann-nonsingular.

Suppose conversely that a $\mathcal{K}_n$ network satisfies these two conditions. Fix i and choose distinct $k, \ell \neq i$, and let

$$a_i = \sum_{j \neq i} b_{i,j} + \frac{b_{i,k}b_{i,\ell}}{b_{k,\ell}}.$$

The quadrilateral rule guarantees that the right hand side is independent of k and ℓ. But for a fixed k and ℓ, this is the current on vertex i of the potential $\chi_i - (b_{i,\ell}/b_{k,\ell})e_k$ on the $\mathcal{K}_n$ network. This function has net current 0 on vertex ℓ, but since $b_{i,\ell}/b_{k,\ell}$ is independent of the choice of ℓ, it has current 0 on all vertices other than k and i. Since the potential is not constant and the network is not Neumann-singular, there must be nonzero net current on i and k, so a_i must be nonzero.

Now we must verify that $\sigma = \sum a_i \neq 0$ and that $a_i a_j/\sigma = b_{ij}$. By extending $\mathbb{F}$ to a larger field if necessary, we can assume that there exists c_i with

$$c_i^2 = b_{i,k}b_{i,\ell}/b_{k,\ell} \text{ for distinct } k, \ell \neq i,$$

and again this is independent of k, ℓ. Then

$$c_i^2 c_j^2 = \frac{b_{i,k}b_{i,j}}{b_{j,k}} \frac{b_{j,k}b_{i,j}}{b_{i,k}} = b_{i,j}^2$$

so that $c_i c_j = \pm b_{i,j}$. By choosing c_1 first and then modifying c_j for $j \neq 1$ if necessary, we can guarantee $c_1 c_j = b_{1,j}$ for $j \neq 1$. Then for $i \neq 1$ we have

$$c_i c_j = b_{1,i}b_{1,j}/c_1^2 = b_{i,j}$$

as well. Then

$$a_i = \sum_{j \neq i} b_{i,j} + \frac{b_{i,k}b_{i,\ell}}{b_{k,\ell}} = \sum_{j \neq i} c_i c_j + c_i^2 = c_i \sum_{j=1}^{n} c_j.$$

Since $a_i \neq 0$, the sum is nonzero; hence,

$$\sigma = \sum_{i=1}^{n} c_i \sum_{j=1}^{n} c_j = \left(\sum_{i=1}^{n} c_i \right)^2 \neq 0.$$

The $\mathcal{K}_n$ is equivalent to the star because

$$\frac{a_i a_j}{\sigma} = \frac{\left(c_i \sum_{k=1}^n c_k\right)\left(c_j \sum_{k=1}^n c_k\right)}{\left(\sum_{k=1}^n c_k\right)^2} = c_i c_j = b_{i,j}. \qquad \square$$

Y-Δ Transformations and Recoverability: [4] §6 and §13 applies Y-Δ transformations as follows: Y-Δ transformations preserve recoverability over the positive linear conductances. For suppose G' is obtained from G by a Y-Δ transformation and G' is recoverable over the positive linear conductances. For any positive linear conductances on G, we can find equivalent conductances on G'. These conductances are uniquely determined by L over the positive linear conductances. In particular, the conductances on the Y or Δ in G' are determined, but then we can find the conductances on the corresponding Δ or Y in G, so G is also recoverable.

We say two graphs are Y-Δ *equivalent* if there is a sequence of Y-Δ transformations which will change one into the other. This is an equivalence relation. If G is Y-Δ equivalent to G' and G' has a series or parallel configuration, then G' is not recoverable, and hence G is not recoverable over the positive linear conductances. This is one of the best methods for showing a graph is not recoverable over $\mathbb{R}$.

[4] shows that if a ∂-graph in the disk has a medial strand arrangement with lenses, then there is a sequence of Y-Δ moves that will produce a parallel or series configuration, and hence G is not recoverable over the positive linear conductances. Combined with Theorem 6.4, this yields

Theorem 7.8 (cf. [4] Lemmas 13.1 and 13.2, [11] Theorem 6.7)**.** *Let G be a ∂-graph embedded in the disk with a medial strand arrangement $\mathcal{M}$. Then the following are equivalent:*

- *$\mathcal{M}$ is lensless.*
- *G is recoverable over positive linear conductances over $\mathbb{R}$.*
- *G is recoverable over BZ(M) for any abelian group M.*

Using $\bigstar$-$\mathcal{K}$ Transformations to Compute the Boundary Behavior: [17] used $\bigstar$-$\mathcal{K}$ transformations as follows: For any finite graph G, there is a sequence of $\bigstar$-$\mathcal{K}$ moves and parallel circuit reductions that will transform it into a graph with no interior vertices. Let Γ be a signed linear network on G, and suppose that at each step, the star is non-singular, so an equivalent $\mathcal{K}$ can be found. After the final step, the response matrix is exactly the Kirchhoff matrix because there are no interior vertices. So the $\bigstar$-$\mathcal{K}$ transformation provides a way to compute the response matrix from the Kirchhoff matrix in small steps, and in some cases, this is a useful technique for determining recoverability over positive (real) linear conductances.

7.5 The Electrical Linear Group

Overview: It is well-known that there is a relationship between electrical networks and symplectic vector spaces (e.g. [14], [1], [18]). In particular, Lam and Pylyavskyy [14] describe an "electrical linear group" whose positive part acts on positive linear circular planar networks with n boundary vertices by adjoining boundary spikes and boundary edges. They prove it is isomorphic to the symplectic Lie group $\mathrm{Sp}_{2n}(\mathbb{R})$.

I will construct an electrical linear group $EL_n(\mathbb{F})$ which (in the final analysis) differs only slightly from Lam and Pylyavskyy's. However, the approach will be different–it will be defined over arbitrary fields, using the language/motivation of elementary IO-network morphisms, and with no a priori restrictions on network planarity.[12]

Type 1 and 2 Elementary IO-Networks over $\mathbb{F}$: Let **IO-net**$(\mathbb{F})$ be the category of IO-networks given by linear networks over $\mathbb{F}$. Let $[n] = \{1, \ldots, n\}$. For $t \in \mathbb{F} \setminus \{0\}$, let $U_j(t) : [n] \to [n]$ be the type 1 IO-network morphism given by a network described as follows:

- The vertices are $p_1, \ldots, p_n$ and p'_j. The labelling $[n] \to V$ for the inputs is given by $k \mapsto p_k$, and the labelling $[n] \to V$ of the outputs is given by $j \mapsto p'_j$ and $k \mapsto p_k$ for $k \neq j$.
- There is only one edge. It is between p_j and p'_j, and the conductance is $1/t$.

For $i \neq j$, let $U_{i,j}(t) : [n] \to [n]$ be the type 2 IO-network morphism described as follows:

- The vertices are $p_1, \ldots, p_n$. The labellings $[n] \to V$ for the inputs and the outputs are both given by $k \mapsto p_k$.
- There is only one edge. It is between p_i and p_j, and the conductance is t.

We define $U_j(0) = \mathrm{id}$ and $U_{i,j}(0) = \mathrm{id}$.

In §4.2 we gave an explicit description of the IO boundary behavior of type 1 and type 2 elementary IO-networks. From this, we can see that the relation

$$\mathcal{X}(U_j(t)) : \mathbb{F}^{[n]} \times \mathbb{F}^{[n]} \rightsquigarrow \mathbb{F}^{[n]} \times \mathbb{F}^{[n]}$$

defines a bijective function (which is clearly linear). We will identify $\mathbb{F}^{[n]} \times \mathbb{F}^{[n]}$ with $\mathbb{F}^{2n}$, so that the first n coordinates represent potentials and the last n coordinates represent input/output net current. From the formulas in §4.2 we easily deduce that the matrix of $\mathcal{X}(U_j(t)) : \mathbb{F}^{2n} \to \mathbb{F}^{2n}$ is

$$\Xi_j(t) = \begin{pmatrix} I & -tE_{j,j} \\ 0 & I \end{pmatrix},$$

where $E_{i,j}$ is the $n \times n$ matrix with a 1 in the (i,j) position and zeroes elsewhere. (The conductance was chosen to be $-1/t$ so that we would get a t in the final formula.) Similarly, $\mathcal{X}(U_{i,j}(t))$ defines a linear isomorphism $\mathbb{F}^{2n} \to \mathbb{F}^{2n}$ whose matrix is

$$\Xi_{i,j}(t) = \begin{pmatrix} I & 0 \\ -t(E_{i,i} + E_{j,j} - E_{i,j} - E_{j,i}) & I \end{pmatrix}.$$

[12]But unfortunately I will not address the generalizations of electrical Lie groups suggested in their paper.

From the parallel and series reduction transformations, we can conclude that

$$\Xi_j(s)\Xi_j(t) = \Xi_j(s+t) \text{ and } \Xi_{i,j}(s)\Xi_{i,j}(t) = \Xi_{i,j}(s+t).$$

Definition of $EL_n(\mathbb{F})$: The *electrical linear group* $EL_n(\mathbb{F})$ is the subgroup of $GL_{2n}(\mathbb{F})$ generated by $\Xi_j(t)$ and $\Xi_{i,j}(t)$ for $t \in \mathbb{F}$. Equivalently, $EL_n(\mathbb{F})$ is the set of linear isomorphisms of $\mathbb{F}^{2n}$ representing $\mathcal{X}(\mathcal{G})$ for some $\mathcal{G} : [n] \to [n]$ in **IO-net**($\mathbb{F}$) which admits an elementary factorization into type 1 and type 2 elementary IO-networks. (We know that the latter is a subgroup because $\Xi_j(t)^{-1} = \Xi_j(-t)$ and $\Xi_{i,j}(t)^{-1} = \Xi_{i,j}(-t)$.)

Remark. The matrices $\Xi_j(t)$ and $\Xi_{i,j}(t)$ were written down in [11] based on the Lie algebra relations in [14]. The **IO-net**($\mathbb{F}$) perspective gives these matrices a "natural" and concrete electrical meaning.

The Electrical Grassmannian: Define the *electrical Grassmannian* $EG_n(\mathbb{F})$ as the set of boundary behaviors $\mathcal{B}(\Gamma) \subset \mathbb{F}^{2n}$ for linear networks over $\mathbb{F}$ with n boundary vertices.

Action of EL_n on EG_n: As in §4.4, if $\mathcal{G}$ is an **IO-net**($\mathbb{F}$) morphism $\varnothing \to [n]$, then $U_j(t) \circ \mathcal{G}$ is a morphism formed by adjoining a spike of conductance $1/t$ to the network, and moreover, the boundary behavior

$$\mathcal{B}(U_j(t) \circ \mathcal{G}) = \mathcal{X}(U_j(t)) \circ \mathcal{B}(\mathcal{G}) = \Xi_j(\mathcal{B}(\mathcal{G})).$$

That is, the boundary behavior of the new equivalence class of networks is found by taking the image of the original boundary behavior under the map multiplication by Ξ_j. The same applies for $U_{i,j}(t)$, $\Xi_{i,j}(t)$, and adjoining a boundary edge of conductance t. This implies that $EL_n(\mathbb{F})$ acts on $EG_n(\mathbb{F})$ by taking images under linear transformations.

7.6 Characterization of Linear Boundary Behavior

EL_n and EG_n have a simple description in terms of symplectic vector spaces. Let Ω be the $2n \times 2n$ matrix given in block form by

$$\Omega = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Recall that

- The *standard symplectic form* on $\mathbb{F}^{2n}$ is $\omega(x, y) = x^T \Omega y$.
- The *symplectic group* $\mathrm{Sp}_{2n}(\mathbb{F})$ is the group of matrices in $GL_{2n}(\mathbb{F})$ satisfying $A^T \Omega A = \Omega$, or equivalently, $\omega(Ax, Ay) = \omega(x, y)$ for all x, y.
- A *Lagrangian subspace* of $\mathbb{F}^{2n}$ is a subspace W of dimension n such that $\omega(x, y) = 0$ for all $x, y \in W$.

Theorem 7.9 (cf. [14] Theorem 3.1, [1])**.** *Let $c_0 = (\vec{1}, \vec{0}) \in \mathbb{F}^{2n}$ be the vector with n ones and n zeroes. Then*

- *If $\mathbb{F} \neq \mathbb{F}_2$, then $EL_n(\mathbb{F})$ is the group of symplectic matrices $A \in \mathrm{Sp}_{2n}(\mathbb{F})$ with $Ac_0 = c_0$.*
- *$EG_n(\mathbb{F})$ is the set of Lagrangian subspaces containing c_0.*

The theorem will be broken into several lemmas, and exposition mingled with the proof. We will first prove that any element of EG_n is Lagrangian and any element of EL_n is symplectic, then tackle the harder converse.

Lemma 7.10. *If Γ is a finite linear network over $\mathbb{F}$ with $\partial V = [n]$, then $\mathcal{B}(\Gamma)$ is n-dimensional.*

Proof. This is clearly the case for Dirichlet-nonsingular networks. However, the statement is not as obvious as it initially appears, since the space of *harmonic potentials* can have dimension *strictly greater* than n; for instance, see the triangle-in-triangle example of §7.3.

Recall $\mathcal{U}(\Gamma)$ is the kernel of $K_{V^\circ,V}$, the submatrix of the Kirchhoff matrix containing the rows indexed by the interior vertices. Let $\Phi : \mathcal{U}(\Gamma) \to \mathcal{B}(\Gamma)$ be the map that sends a harmonic function to its boundary potential / net current data, which is surjective by definition of $\mathcal{B}(\Gamma)$. Then $\ker\Phi$ is the space of harmonic potentials that have potential and net current zero on the boundary. That is,

$$\ker\Phi = \left\{ \begin{pmatrix} 0 \\ w \end{pmatrix} \in \mathbb{F}^V : w \in \mathbb{F}^{V^\circ}, K\begin{pmatrix} 0 \\ w \end{pmatrix} = 0 \right\},$$

which is isomorphic to $\ker K_{V,V^\circ}$. Since K is symmetric, $K_{V,V^\circ} = K_{V^\circ,V}^T$. Then after three applications of the rank-nullity theorem, we have

$$\begin{aligned} \dim\mathcal{B}(\Gamma) + \dim\ker\Phi &= \operatorname{rank}\Phi + \dim\ker\Phi \\ &= \dim\mathcal{U}(\Gamma) = \dim\ker K_{V^\circ,V} \\ &= |V| - \operatorname{rank} K_{V^\circ,V} \\ &= |\partial V| + |V^\circ| - \operatorname{rank} K_{V,V^\circ} \\ &= |\partial V| + \dim\ker K_{V,V^\circ} \\ &= |\partial V| + \dim\ker\Phi, \end{aligned}$$

and hence $\dim\mathcal{B}(\Gamma) = |\partial V|$. □

Lemma 7.11. *If Γ is as above, then $\mathcal{B}(\Gamma)$ is a Lagrangian subspace of $\mathbb{F}^{2n}$.*

Proof. Heuristically, this is a generalization of the fact that the response matrix of a network is symmetric. Indeed, if $\langle\cdot,\cdot\rangle$ is the standard "inner product" on $\mathbb{F}^n$, then

$$\omega(x, y) = 0 \text{ for } x, y \in \mathcal{B}(\Gamma)$$

is equivalent to

$$\langle\phi_1, \psi_2\rangle = \langle\phi_2, \psi_1\rangle \text{ for } (\phi_1, \psi_1), (\phi_2, \psi_2) \in \mathcal{B}(\Gamma).$$

This clearly holds if $\psi_j = \Lambda\phi_j$ and Λ is symmetric.

For the general case, suppose that (ϕ_1, ψ_1) and (ϕ_2, ψ_2) are the boundary data for harmonic potentials (ϕ_1, w_1) and (ϕ_2, w_2), with $w_j \in \mathbb{F}^{V^\circ}$. Then by harmonicity, $K_{V^\circ,\partial V}\phi_j + K_{V^\circ,V^\circ}w_j = 0$, and also $\psi_j = K_{\partial V,\partial V}\phi_j + K_{\partial V,V^\circ}w_j$. Applying symmetry of the Kirchhoff matrix,

$$\begin{aligned}\langle \phi_1, \psi_2\rangle &= \langle \phi_1, K_{\partial V,\partial V}\phi_2\rangle + \langle \phi_1, K_{\partial V,V^\circ}w_2\rangle \\ &= \langle \phi_1, K_{\partial V,\partial V}\phi_2\rangle + \langle K_{V^\circ,\partial V}\phi_1, w_2\rangle \\ &= \langle \phi_1, K_{\partial V,\partial V}\phi_2\rangle - \langle K_{V^\circ,V^\circ}w_1, w_2\rangle\,,\end{aligned}$$

which is unchanged when we switch ϕ_1 and ϕ_2 and w_1 and w_2. □

Lemma 7.12. *If Γ is as above, then $\mathcal{B}(\Gamma)$ contains c_0.*

Proof. The constant potential function 1 is harmonic. □

Lemma 7.13. *Any $A \in EL_n(\mathbb{F})$ is symplectic and fixes c_0.*

Proof. By direct computation, the generators $\Xi_j(t)$ and $\Xi_{i,j}(t)$ are in $\mathrm{Sp}_{2n}(\mathbb{F})$ and fix c_0.

Alternatively, whenever we have a IO-network morphism $\mathcal{G} : [n] \to [n]$ of linear networks such that $\mathcal{X}(\mathcal{G})$ is a bijective function, it must be a linear isomorphism represented by a symplectic matrix. To verify this, take a network Γ representing $\mathcal{G}$. If we forget the labelling of inputs and outputs, $\mathcal{B}(\Gamma)$ is a Lagrangian subspace of $\mathbb{F}^{\partial V} \times \mathbb{F}^{\partial V}$, and with a little casework this implies $\mathcal{X}(\mathcal{G})$ is represented by a symplectic matrix. Since the constant potential is harmonic, this matrix must also fix c_0. □

Our next goal is to show that any Lagrangian subspace of $\mathbb{F}^{2n}$ is the boundary behavior of some network, for which this lemma turns out to be useful:

Lemma 7.14. *Suppose V is a Lagrangian subspace of $\mathbb{F}^{2n}$. For $S \subset [2n]$, let $\pi_S : \mathbb{F}^{2n} \to \mathbb{F}^S$ be the coordinate projection. Then there is a partition of $[n]$ into two sets S and T such that*

- *$\pi_S(x) = 0$ implies $\pi_{[n]}(x) = 0$ for $x \in V$.*
- *$\pi_{S\cup(n+T)}$ defines an isomorphism $V \to \mathbb{F}^{S\cup(n+T)}$.*

Proof. Let $W = \{w \in \mathbb{F}^n : (0, w) \in V\}$. If $(x, y) \in V$ and $w \in W$, then

$$0 = \omega((x, y), (0, w)) = -\langle x, w\rangle\,,$$

and hence

$$W \subset \pi_{[n]}(V)^\perp = \{w \in \mathbb{F}^n : \langle w, x\rangle = 0 \text{ for } x \in \pi_{[n]}(V)\}.$$

However, note that $W \cong \ker(\pi_{[n]}|_V)$, hence by the rank-nullity theorem $\dim W + \dim \pi_{[n]}(V) = \dim V = n$. We also know by the rank-nullity theorem that $\dim \pi_{[n]}(V) + \dim \pi_{[n]}(V)^\perp = n$ for any field. Therefore, $W = \pi_{[n]}(V)^\perp$.

From basic linear algebra, we can choose $S \subset [n]$ such that $\pi_S : \mathbb{F}^n \to \mathbb{F}^S$ restricts to an isomorphism $\pi_{[n]}(V) \to \mathbb{F}^S$. Let $T = [n] \backslash S$. Since $W = \pi_{[n]}(V)^\perp$, this implies that $\pi_T : \mathbb{F}^n \to \mathbb{F}^T$ defines an isomorphism $W \to \mathbb{F}^T$ (details[13]). This implies that $\pi_{S \cup (n+T)} : \mathbb{F}^{2n} \to \mathbb{F}^{S\cup(n+T)}$ defines an isomorphism $V \to \mathbb{F}^{S\cup(n+T)}$. One way to see this is to by applying the five-lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W & \longrightarrow & V & \longrightarrow & \pi_{[n]}(W) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{F}^T & \longrightarrow & \mathbb{F}^{S\cup(n+T)} & \longrightarrow & \mathbb{F}^S & \longrightarrow & 0. \end{array}$$

□

Corollary 7.15. *If Γ is a linear network over $\mathbb{F}$, then there is a partition of ∂V into two sets P and Q such that potentials on P and net currents on Q uniquely determine the other boundary data.*

Lemma 7.16. *Let V be a Lagrangian subspace of $\mathbb{F}^{2n}$ containing c_0. Then V is the boundary behavior of some linear network over $\mathbb{F}$.*

Proof. Choose a partition of $[n]$ into two sets S and T as in the previous lemma. By reindexing the coordinates, assume that $S = [\ell]$ for some $\ell \leq n$. Let $m = n - \ell$. Then we can choose a basis $x_1, \dots, x_n$ of V such that

$$\begin{pmatrix} | & \dots & | \\ x_1 & \dots & x_n \\ | & \dots & | \end{pmatrix} \text{ is of the form } \begin{pmatrix} I & 0 \\ * & 0 \\ * & * \\ 0 & I \end{pmatrix},$$

where the sizes of the blocks are

$$\begin{pmatrix} \ell \times \ell & \ell \times m \\ m \times \ell & m \times m \\ \ell \times \ell & \ell \times m \\ m \times \ell & m \times m \end{pmatrix}.$$

Then define $V' := \Xi_{\ell+1}(-1) \dots \Xi_n(-1)(V)$ and note that

$$V' = \operatorname{im} \begin{pmatrix} I & E_{\ell+1,\ell+1} \\ 0 & I \end{pmatrix} \dots \begin{pmatrix} I & E_{n,n} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ * & 0 \\ * & * \\ 0 & I \end{pmatrix} = \operatorname{im} \begin{pmatrix} I & 0 \\ * & I \\ * & * \\ 0 & I \end{pmatrix} = \operatorname{im} \begin{pmatrix} I & 0 \\ 0 & I \\ * & * \\ * & I \end{pmatrix},$$

which can be written as

$$V' = \operatorname{im} \begin{pmatrix} I \\ \Lambda \end{pmatrix}$$

[13] $\pi_{[n]}(V) \cap (\mathbb{F}^T \times 0^S) = 0$ in $\mathbb{F}^n$, which implies $\pi_{[n]}(V) + (\mathbb{F}^T \times 0^S) = \mathbb{F}^n$ since $\dim \pi_{[n]}(V) = |S| = n - |T|$. Hence taking orthogonal complements $\pi_{[n]}(V)^\perp \cap (\mathbb{F}^S \times 0^T) = 0$

with $n \times n$ blocks. Since $\Xi_j(t)$ is symplectic and fixes c_0, V' is a Lagrangian subspace that contains c_0. This implies Λ is symmetric and has row sums zero. Thus, Λ has the form

$$\Lambda = \sum_{i<j} -\lambda_{i,j}(E_{i,i} - E_{i,j} - E_{j,i} + E_{j,j}),$$

and this implies that

$$\begin{pmatrix} I \\ \Lambda \end{pmatrix} = \prod_{i<j} \begin{pmatrix} I & 0 \\ -\lambda_{i,j}(E_{i,i} - E_{i,j} - E_{j,i} + E_{j,j}) & I \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix},$$

or in other words,

$$V' = \prod_{i<j} \Xi_{i,j}(\lambda_{i,j})(\mathbb{F}^n \times 0^n),$$

so that

$$V = \prod_{k=\ell+1}^{n} \Xi_k(1) \prod_{i<j} \Xi_{i,j}(\lambda_{i,j})(\mathbb{F}^n \times 0^n).$$

Now $\mathbb{F}^n \times 0^n$ is the boundary behavior of a network with n boundary vertices with no edges. Hence, if we adjoin boundary edges of conductances $\lambda_{i,j}$ between vertices i and j whenever $\lambda_{i,j} \neq 0$, and then adjoin boundary spikes of conductance -1 to the vertices $\ell+1, \ldots, n$, then we obtain a network whose boundary behavior is V. □

Corollary 7.17. *Any $V \in EG_n(\mathbb{F})$ can be expressed as the boundary behavior of a layerable network with $\leq \frac{1}{2}n(n+1)+1$ edges.*

Proof. In the previous proof, the number of boundary edges added was the number of nonzero entries of Λ above the diagonal. Since

$$\Lambda = \begin{pmatrix} * & * \\ * & I \end{pmatrix},$$

with the last block being $m \times m$, the number of nonzero entries is at most $\frac{1}{2}\ell(\ell-1) + \ell m$. The number of boundary spikes adjoined was m, so recalling $\ell + m = n$, the total number of edges is at most

$$m + \frac{1}{2}(n-m)(n-m-1) + (n-m)m = \frac{1}{2}n(n-1) - \frac{1}{2}m(m-3) \leq \frac{1}{2}n(n-1)+1.$$

□

Corollary 7.18. *Let $Y = \{(i,j) \in [n] \times [n] : i < j\}$. For $S \subset n$, define $\mathcal{F}_S : \mathbb{F}^Y \to EG_n(\mathbb{F})$ by*

$$\mathcal{F}_S((t_{i,j})) = \prod_{k \in S} \Xi_k(-1) \prod_{i<j} \Xi_{i,j}(t_{i,j})(\mathbb{F}^n \times 0^n).$$

Then the images $U_S = \mathcal{F}_S(\mathbb{F}^Y)$ cover $EG_n(\mathbb{F})$ and the transition maps are rational functions. Hence, for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, EG_n is a smooth real/complex manifold of dimension $n(n-1)/2$.

Proof. The fact that the U_S's cover EG_n follows from the previous proofs, and the reader may explicitly compute the transition functions. $\square$

The proof of Theorem 7.9 is completed by the next lemma:

Lemma 7.19. *Suppose $\mathbb{F} \neq \mathbb{F}_2$. If $A \in \mathrm{Sp}_{2n}(\mathbb{F})$ and $Ac_0 = c_0$, then $A \in EL_n(\mathbb{F})$.*

Proof. The proof is elementary but a bit tedious: It amounts to constructing explicit factorizations in terms of the generators for $EL_n(\mathbb{F})$. We proceed by induction on n.

For $n = 1$, any symplectic matrix A that fixes c_0 must be of the form

$$A = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \Xi_1(t).$$

For the induction step, it suffices to find $A_1, \dots, A_\ell \in EL_n(\mathbb{F})$ such that

$$A_1 \dots A_\ell A = \begin{pmatrix} * & 0 & * & 0 \\ 0 & 1 & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where each "$*$" is $(n-1) \times (n-1)$. (Heuristically, $A_1 \dots A_N A$ is the behavior of IO-network where the nth input vertex equals the nth output, and this vertex is isolated; we are thus reducing to a case of networks with $n-1$ inputs and outputs.) The matrix A' formed by deleting the nth and $2n$th row and column of $A_1 \dots A_N A$ must be symplectic and fix $c_0 \in \mathbb{F}^{2(n-1)}$. So by the induction hypothesis $A' \in EL_{n-1}(\mathbb{F})$, which implies $A \in EL_n(\mathbb{F})$.

Our first goal is to find $A_1, \dots, A_m$ generators of $EL_n(\mathbb{F})$ such that $A_m \dots A_1 A$ fixes e_{2n} (the last column is e_{2n}). (Heuristically, $A_m \dots A_1 A$ corresponds to an IO-network where the nth input vertex is the same as the nth output, but is not necessarily an isolated vertex.) Let $x = Ae_{2n}$; it suffices to show that by multiplying by elements of EL_n we can map x to e_{2n}. There are several cases:

1. Suppose that the "potential" $x_n \neq 0$ and that the "net currents" $x_{n+1}, \dots, x_{2n-1} \neq 0$. Let
$$y = \left(\prod_{k=1}^{n-1} \Xi_k(x_k / x_{n+k}) \right) x.$$
Then $y_1, \dots, y_{n-1} = 0$, $y_n = x_n \neq 0$. Next, let
$$z = \left(\prod_{k=1}^{n-1} \Xi_{k,n}(-y_{n+k}/y_n) \right) y.$$
Then $z_1, \dots, z_{n-1} = 0$ and $z_{n+1}, \dots, z_{2n-1} = 0$. But $\omega(c_0, z) = \omega(c_0, x) = 1$, so $z_{2n} = 1$. Thus, multiplying by $\Xi_n(-z_n)$ will make the nth entry zero, yielding e_{2n}.

2. If $x_n = 0$ but $x_{n+1}, \dots, x_{2n-1}, x_{2n} \neq 0$, then we can multiply by $\Xi_n(1)$ to make $x_n \neq 0$, then proceed to Case 1.

3. Suppose that some of "currents" $x_{n+1}, \dots, x_{n+k}$ are zero, but the "potentials" $x_1, \dots, x_n$ are not all equal. For each j with $x_{n+j} = 0$, we can find a k with $x_j \neq x_k$. Then multiply by some $\Xi_{j,k}(t)$ to make it nonzero. In order to guarantee that the "net current" at k is still nonzero, we choose $t \neq 0$ and $t \neq -x_{n+k}/(x_k - x_j)$. This is possible because $\mathbb{F}$ has at least three elements. Once we have done this for every j, proceed to Case 2.

4. Suppose that $x_1, \dots, x_n$ are all equal to some constant t. Since the vector c_0 is fixed by A and all matrices in EL_n, it is not possible that $x_{n+1}, \dots, x_{2n}$ are all zero. Hence, there is some $x_{n+k} \neq 0$, and we can multiply by some $\Xi_k(1)$ to make the new $x_k \neq t$. Then proceed to Case 3.

Thus, if we let $A_1, \dots, A_m$ be the matrices used in the above operations and $B = A_m \dots A_1 A$, then $Be_{2n} = e_{2n}$.

Our next task is find $A_{m+1} \dots A_\ell$ such that $A_\ell \dots A_{m+1} B$ fixes both e_{2n} and e_n. Let $x = Be_n$, and consider the following cases:

1. Suppose that the "net currents" $x_{n+1}, \dots, x_{2n}$ are all nonzero. Observe
$$x_n = \omega(e_{2n}, x) = \omega(Be_{2n}, Be_n) = \omega(e_{2n}, e_n) = 1.$$
Let
$$y = \left(\prod_{k=1}^{n-1} \Xi_k(x_k/x_{n+k}) \right) x,$$
so that $y_1, \dots, y_{n-1} = 0$ and $y_n = 1$. Then let
$$z = \left(\prod_{k=1}^{n-1} \Xi_{k,n}(-y_{n+k}) \right) y.$$
Then $z_1 = y_1, \dots, z_n = y_n$, and $z_{n+1}, \dots, z_{2n-1} = 0$. But $\omega(c_0, z) = \omega(c_0, e_n) = 0$, so $z_{2n} = 0$ as well. Hence, $z = e_n$.

2. If some of "currents" $x_{n+1}, \dots, x_{n+k}$ are zero, but the "potentials" $x_1, \dots, x_n$ are not all equal, we can multiply by $\Xi_{j,k}(t)$'s to make all the "currents" nonzero (as in the previous part of the proof). Then proceed to Case 1.

3. Suppose that $x_1, \dots, x_n$ are all equal to 1. One of the "net currents" must be nonzero; so in fact, at least two of them are nonzero. Hence, we can multiply by $\Xi_k(1)$ for some $k \neq n$ to make the new $x_k \neq 1$. Then proceed to Case 2.

In all these cases, we never multiplied by a $\Xi_n(t)$ matrix. Thus, if we let A_{m+1}, $\dots$, A_ℓ be the matrices used in the above operations, then each one fixes e_{2n}, and thus
$$C = A_\ell \dots A_{m+1} B = A_\ell \dots A_1 A$$

also fixes e_{2n}, besides fixing e_n.

Because $C^T \Omega C = \Omega$, we know $C^T = \Omega C^{-1} \Omega^{-1}$. Since C^{-1} fixes e_n and e_{2n}, we know C^T fixes $\Omega e_n = e_{2n}$ and $\Omega e_{2n} = -e_n$. Thus, the nth and $2n$th rows of C are e_n and e_{2n}, and so are the nth and $2n$th columns. Thus, C has the desired form and the induction step is complete. □

Closing Remarks: This lemma fails in the case of $\mathbb{F}_2$. For instance, for $n = 2$,

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \notin EL_2(\mathbb{F}_2)$$

despite being symplectic and fixing c_0. An easy way to see this is to compute the orbit of e_4 under the action of $EL_2(\mathbb{F}_2)$ on $\mathbb{F}_2^4$; the orbit has only four elements and does not contain $e_1 + e_2 + e_4$, which is the last column of the matrix of above.[14]

As with $EG_n(\mathbb{F})$, the construction in Lemma 7.19 provides parametrizations of $EL_n(\mathbb{F})$ for which the transition functions are rational. For a given A, we parametrize a "neighborhood" using the parameters for Case 1 of each step, keeping the parameters in the other steps fixed. From this, we work out that the "dimension" of $EL_n(\mathbb{F})$ is $n(2n-1)$, which is the same as fo $EG_{2n}(\mathbb{F})$.

The action of $EL_n(\mathbb{F})$ on $EG_n(\mathbb{F})$ is transitive; indeed, the proof of Lemma 7.16 showed that every element of $EG_n(\mathbb{F})$ is in the orbit of $\mathbb{F}^n \times 0^n$. However, the action is not faithful: There exist nontrivial elements of EL_n which fix every element of EG_n. These elements are the kernel of the homomorphism Υ from EL_n to the group of bijections $EG_n \to EG_n$ given by $\Xi \mapsto F_\Xi$, where $F_\Xi : EG_n \to EG_n : L \mapsto \Xi(L)$. The reader can verify that (for $\mathbb{F} \neq \mathbb{F}_2$) the kernel consists of matrices of the form

$$\begin{pmatrix} I + \mathbf{1}\alpha^T & \mathbf{1}\beta^T + \beta\mathbf{1}^T \\ 0 & I - \alpha\mathbf{1}^T \end{pmatrix},$$

where $\mathbf{1}$ is the vector with every entry 1 and $\alpha, \beta \in \mathbb{R}^n$ with $\sum_{k=1}^n \alpha_k = 0$.

7.7 Generators of EL_n and Circular Planarity

The Quest for Planarity: Network planarization (given a network, find a circular planar network with the same boundary behavior) has long been a goal of electrical engineers, who desired to print out flat circuit components with certain behavior. For instance, [20] suggests using the ★-$\mathcal{K}$ transformation to find planar equivalents. Thanks to [4] Theorem 4 (and related results), we now know exactly what response matrices can occur for circular planar networks with positive linear conductances, which ought to be the end of the matter as far as engineering is concerned.

[14] I have not worked out precisely what happens for $\mathbb{F}_2$, but might do it later. This would be a good problem for REU students.

Many non-planar networks with positive real conductances cannot have the same boundary behavior as a circular planar network with positive conductances. However, if we allow negative conductances, it is much easier to "planarize" a network. The REU paper [19] conjectured that any real response matrix could be represented by a circular planar network with signed real conductances, and [9] and [11] suggest using the ★-$\mathcal{K}$ transformation with signed conductances.

This turns out to be true for all fields other $\mathbb{F}_2$:

Theorem 7.20. *Let $\mathbb{F} \neq \mathbb{F}_2$. Every element of $EG_n(\mathbb{F})$ can be represented by a layerable circular planar network.*

To prove this, we will use the electrical linear group and ★-$\mathcal{K}$ transformation:

Theorem 7.21. *Let $\mathbb{F} \neq \mathbb{F}_2$. The electrical linear group is generated by $\Xi_j(t)$ for $j = 1, \ldots, n$ and $\Xi_{j,j+1}(t)$ for $j = 1, \ldots, n-1$ and $t \in \mathbb{F} \setminus \{0\}$.*

Recall that we defined $EL_n(\mathbb{F})$ using $\Xi_j(t)$ together with $\Xi_{j,k}(t)$ for all $j \neq k$. The smaller set of generators in Theorem 7.21 more closely resembles [14]'s definition of the electrical linear group. If we view $EL_n(\mathbb{F})$ as acting on $EG_n(\mathbb{F})$ by adjoining boundary spikes and boundary edges to networks, the theorem says that it suffices to consider adjoining boundary edges between consecutively-indexed boundary vertices, rather than between any pair of boundary vertices.

Theorem 7.20 will follow easily from Theorem 7.21. Indeed, by Corollary 7.17, any element of $EG_n(\mathbb{F})$ can be represented by a layerable network, and hence has the form

$$A(\mathbb{F}^n \times 0^n) \text{ for some } A \in EL_n(\mathbb{F}).$$

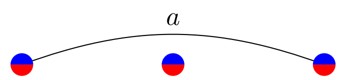
But A can be represented as a product of the generators in Theorem 7.21, which implies that $A(\mathbb{F}^n \times 0^n)$ is the boundary behavior of a network obtained from a network of isolated boundary vertices by adjoining boundary spikes, and adjoining boundary edges between consecutively-indexed boundary vertices. If we embed the original trivial network in the disk with the boundary vertices indexed in CCW order, then at each step the modified network can still be embedded in the disk with the boundary vertices indexed in CCW order, so Theorem 7.20 follows.

To prove Theorem 7.21, it suffices to show that each $\Xi_{j,k}$ for $j < k-1$ can be written in terms of the desired generators. By induction on $k - j$, it suffices to show $\Xi_{j,k}(t)$ can be expressed in terms of Ξ_{k-1}'s, $\Xi_{j,k-1}$'s and $\Xi_{k-1,k}$'s, which follows from the next lemma:

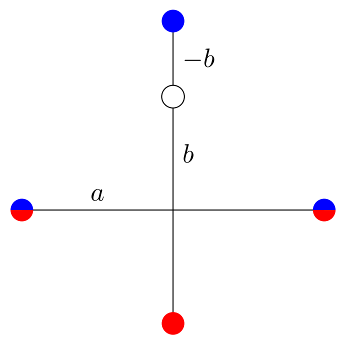
Lemma 7.22. *Let $\mathbb{F} \neq \mathbb{F}_2$. For any distinct indices i, j, k, $\Xi_{i,k}(t)$ can be expressed in terms of $\Xi_{i,j}$'s, Ξ_j's, and $\Xi_{j,k}$'s.*

Proof. For simplicity in drawing pictures, we will assume $i = 1$, $j = 2$, and $k = 3$. We can also assume $n = 3$, since for general n, one simply has to add more rows/columns to all the matrices, filling the new spaces with ones on the diagonal and zeroes elsewhere (this corresponds to adding isolated input/output boundary vertices to an IO-network for the indices larger than 3).

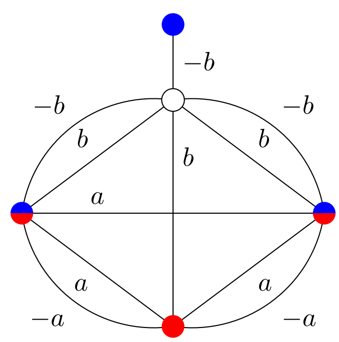
We begin with an IO-network representing $\Xi_{1,3}(a)$ for given $a \neq 0$. Here the inputs are blue and the outputs red, and the inputs/outputs 1, 2, and 3 are in order from left to right:



We then transform this network by local network equivalences. For some parameter b to be chosen later, add in a series with conductances b and $-b$ (representing $\Xi_2(1/b)$ and $\Xi_2(-1/b) = \Xi_2(1/b)^{-1}$):



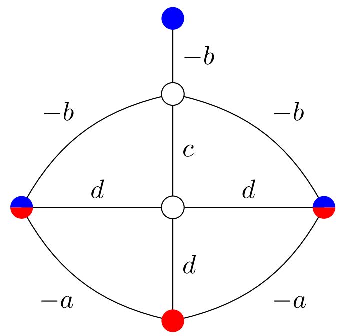
Next, add some cancelling parallel edges:



We want to choose b so that the $\mathcal{K}_4$ subnetwork in the middle will be equivalent to a star. Examining the formulas in Lemma 7.7, we choose $b \neq 0$ so that $a + 3b \neq 0$, which is possible because $\mathbb{F}$ has at least three elements. Set

$$c = 3a + a^2/b = (a + 3b)(a/b) \neq 0, \qquad d = a + 3b,$$

and then the $\mathcal{K}_4$ is equivalent to a 4-star with conductances c, d, d, d, and hence our network becomes



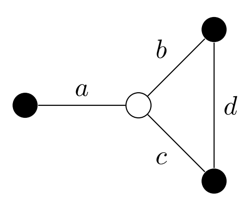
This represents the product of[15]

$$\begin{array}{l}\Xi_2(-1/b)\\ \Xi_{1,2}(-b)\Xi_{2,3}(-b)\\ \Xi_2(1/c)\\ \Xi_{1,2}(d)\Xi_{2,3}(d)\\ \Xi_2(1/d)\\ \Xi_{1,2}(-a)\Xi_{2,3}(-a),\end{array}$$

which completes the proof. □

Inefficiency of This Construction: Though Theorem 7.20 is a good result, the proof given here is a terribly inefficient algorithm for constructing a circular planar network representing a given boundary behavior, in the sense that it adds too many unnecessary edges. I urge future researchers to find a better method–perhaps by giving a circular planar version of the proofs of Lemmas 7.16 and 7.19.

Impossibility of Using *Critical* Circular Planar: One might hope to show that any boundary behavior can be represented by a *critical* circular planar network, but this is overly optimistic. Consider the following network:



Suppose that $a+b+c=0$ and $1/b+1/c+1/d=0$ (which can happen for most fields). Then the network is both Dirichlet-singular and Neumann-singular. However, there does not exist a critical circular planar network, or indeed any network recoverable over positive linear conductances, which has three boundary vertices and is both Dirichlet- and Neumann-singular. To be Dirichlet-singular,

[15]The matrix at the top of the list is applied first, which means that it goes on the *right* when we write the product out.

it must have an interior vertex, and the interior vertex must have degree ≥ 3 for the network to be critical circular planar. Since any such network cannot have more than 3 edges, the only possibility is a Y. However, a Y cannot be Neumann-singular.

This example also shows that not every network is equivalent to a network with $\leq \frac{1}{2}n(n-1)$ edges, as we might hope, so the bound in Corollary 7.17 is sharp in this case.

8 Rank and Connections

[4] and others have noted the relationship between connections through a ∂-graph and the rank of certain submatrices of the response matrix. Known generic results for linear networks can be derived from the grove-determinant formula, but in some cases, elementary factorizations provide results that hold in all cases generalize to nonlinear networks.

All networks in this chapter are assumed to be finite.

8.1 Connections and the Grove-Determinant Formula

Connections: Let P and Q be sets of boundary vertices. A *connection from P to Q* is a collection of disjoint boundary-to-boundary paths such that each path starts in P and ends in Q. There may be a vertex $p \in P \cap Q$; in this case, any connection from P to Q may include the length-0 path from p to itself. But because the paths are disjoint, no other paths can use a vertex in $P \cap Q$.

A connection between P and Q is *full* if every vertex in P and every vertex in Q is in one of the paths. If there is a full connection from P to Q, then P and Q must have the same cardinality. Because of our convention for the case where $P \cap Q \neq \varnothing$, there is a one-to-one correspondence between full connections from P to Q and full connections from $P \setminus Q$ to $Q \setminus P$.

Maximum Connection: Let $m(P,Q)$ be the maximum size connection between P and Q (the maximum number of paths in a connection). Equivalently, $m(P,Q)$ is the largest size of a full connection from some subset of P to some subset of Q.

Ranks and Connections for Dirichlet-Nonsingular Linear Networks:

Assume Γ is a inear network over $\mathbb{F}$ and that K_{V°,V° is invertible. Suppose P and Q are disjoint subsets of B with $|P| = |Q|$. Then the submatrix $\Lambda_{P,Q}$ is equal to the Schur complement $K_{P\cup V^\circ, Q\cup V^\circ}/K_{I,I}$ by elementary computation. Schur's formula for Schur complements tells us that[16]

$$\det \Lambda_{P,Q} = \det K_{P\cup V^\circ, Q\cup V^\circ} / \det K_{V^\circ,V^\circ},$$

and hence $\Lambda_{P,Q}$ is invertible if and only if $K_{P\cup V^\circ, Q\cup V^\circ}$ is invertible.

If there exists a full connection from P to Q, then edges in the paths can be completed to a grove in $\mathcal{F}(P,Q)$, and conversely, any grove in $\mathcal{F}(P,Q)$ contains

[16] This is easily proved by block row reduction, see e.g. [3].

a full connection. If there is no connection from P to Q, then Proposition 7.4 tells us that $\det \Lambda_{P,Q} = 0$. If there is a connection, then we can choose positive numbers such that $K_{P\cup V^\circ, Q\cup V^\circ}$ is invertible (and we already know K_{V°,V° is invertible for positive conductances), and hence Λ is defined and $\det \Lambda_{P,Q} \neq 0$. Thus, we have

Lemma 8.1. *Suppose Γ is a finite linear network over $\mathbb{F}$, and $\det K_{V^\circ,V^\circ} \neq 0$.*

- *If $\Lambda_{P,Q}$ is invertible, then there is a full connection between P and Q.*
- *If there is a full connection between P and Q and $\mathbb{F} = \mathbb{R}$, there exist some positive conductances that will make $\Lambda_{P,Q}$ invertible.*

Now suppose P and Q do not necessarily have the same cardinality, but are still disjoint sets of boundary vertices. Then by considering all subsets of P and Q we see that

Proposition 8.2. *For finite linear networks with $\det K_{V^\circ,V^\circ} \neq 0$, we have $\operatorname{rank} \Lambda_{P,Q} \leq m(P,Q)$ always, and equality holds for some positive real conductances.*

8.2 Rank, Connections, and Elementary Factorization

Elementary factorizations enable us to describe conditions on G that will guarantee that $\operatorname{rank} \Lambda_{P,Q} = m(P,Q)$ for all Dirichlet-nonsingular networks on G. In fact, we will find a substitute for $\operatorname{rank} \Lambda_{P,Q}$ that makes sense even for Dirichlet-singular networks.

Rather than assuming P and Q are disjoint as we did before, we will assume $P \cup Q = \partial V$. Suppose Γ is linear network on G. Then Γ represents an IO-network morphism $\mathcal{G} : P \to Q$, and

$$\mathcal{X}(\mathcal{G}) : \mathbb{F}^P \times \mathbb{F}^P \rightsquigarrow \mathbb{F}^Q \times \mathbb{F}^Q.$$

is a linear relation, that is, a linear subspace of $(\mathbb{F}^P \times \mathbb{F}^P) \times (\mathbb{F}^Q \times \mathbb{F}^Q)$.

Rank of a Linear Relation: In general, for finite-dimensional vector spaces W_1 and W_2 and a linear relation $R : W_1 \rightsquigarrow W_2$, we define $\operatorname{rank} R$ to be the maximal rank of a linear map T from some subspace of W_1 to a subspace of W_2 such that $(w, Tw) \in R$ for all w. Let π_1 and π_2 be the projections $R \to W_1$ and $R \to W_2$. Then R defines a linear isomorphism

$$\pi_1(R)/\pi_1 \circ \pi_2^{-1}(0) \to \pi_2(R)/\pi_2 \circ \pi_1^{-1}(0).$$

From this (with some linear algebra) we can see

$$\begin{aligned} \operatorname{rank} R &= \dim \pi_1(R)/\pi_1 \circ \pi_2^{-1}(0) = \dim \pi_2(R)/\pi_2 \circ \pi_1^{-1}(0) \\ &= \dim \pi_1(R) - \dim \pi_1^{-1}(0) = \dim \pi_2(R) - \dim \pi_1^{-1}(0) \end{aligned}$$

(since $\dim \pi_2 \circ \pi_1^{-1}(0) = \dim \pi_1^{-1}(0)$ and the same holds with π_1 and π_2 switched).

Lemma 8.3. *Suppose Γ is a Dirichlet-nonsingular linear network. Suppose P and Q are a partition of ∂V, so that Γ represents a morphism $\mathcal{G} : P \to Q$. Then*

$$\operatorname{rank} \mathcal{X}(\mathcal{G}) = 2 \operatorname{rank} \Lambda_{P,Q}.$$

Proof. Let π_P, π_Q be the projection of $\mathcal{X}(\mathcal{G})$ onto $\mathbb{F}^P \times \mathbb{F}^P$ and $\mathbb{F}^Q \times \mathbb{F}^Q$. Note that $\pi_Q^{-1}(0)$ is isomorphic to the space of harmonic potentials which have zero potential and net current on Q. Since the Dirichlet problem has a unique solution, these functions are parametrized by their potentials on P. Thus, the space is isomorphic to $\ker \Lambda_{Q,P}$, so that $\dim \pi_Q^{-1}(0) = \dim \ker \Lambda_{Q,P}$. To compute $\pi_P(\mathcal{X}(\mathcal{G}))$, we apply Λ to $\mathbb{F}^{\partial V}$, then record the potential and current data on P. This means the matrix

$$\begin{pmatrix} I_P & 0 \\ \Lambda_{P,P} & \Lambda_{P,Q} \end{pmatrix}$$

maps $\mathbb{F}^B = \mathbb{F}^P \times \mathbb{F}^Q$ onto $\pi_P(\mathcal{X}(\mathcal{G})) \subset \mathbb{F}^P \times \mathbb{F}^P$. Row reducing the left half will make the matrix block diagonal, and then we can see its rank is $|P| + \operatorname{rank} \Lambda_{P,Q}$. Hence,

$$\begin{aligned} \operatorname{rank} \mathcal{X}(\mathcal{G}) &= \dim \pi_P(\mathcal{X}(\mathcal{G})) - \dim \pi_Q^{-1}(0) \\ &= |P| + \operatorname{rank} \Lambda_{P,Q} - \dim \ker \Lambda_{Q,P} \\ &= \operatorname{rank} \Lambda_{P,Q} + \operatorname{rank} \Lambda_{Q,P} = 2 \operatorname{rank} \Lambda_{P,Q}. \end{aligned}$$

□

Exercise. *Suppose P and Q are not disjoint but $\partial V = P \cup Q$. Let $P' = P \setminus Q$ and $Q' = Q \setminus P$. Find* $\operatorname{rank} \mathcal{X}(\mathcal{G})$ *in terms of* $\operatorname{rank} \Lambda_{P',Q'}$.

Theorem 8.4. *Let G represent a morphism $\mathcal{G} : P \to Q$ which admits an elementary factorization of rank r. Then*

1. *$r = m(P, Q)$.*
2. *For any linear network on G, we have* $\operatorname{rank} \mathcal{X}(\mathcal{G}) = 2r = 2m(P, Q)$.
3. *For any Dirichlet-nonsingular network on G, we have* $\operatorname{rank} \Lambda_{P,Q} = m(P, Q)$.

Proof. (1). Write the elementary factorization as $\mathcal{G} = \mathcal{G}_n \circ \cdots \circ \mathcal{G}_1$, and let k be an index that is before all the type 4 networks and after or equal to the type 3 networks. Let $\mathcal{G}_j : P_{j-1} \to P_j$. To create a connection of size m, we start from the middle of the factorization. We want each path to contain exactly one element of P_k. If $\mathcal{G}_k$ is type 2 or 3, then our paths are length zero, and if it is type 1, we use the edges in the network for our paths and hence have a connection from P_k to P_{k-1}. We continue to extend the paths inductively. Once we have a connection from P_k to some subset R_j of P_j through $\mathcal{G}_k \circ \cdots \circ \mathcal{G}_j$, we extend the paths into $\mathcal{G}_{j-1}$–if it is type 2 or 3, there is nothing to do, and if it is type 1 we use the edges that have endpoints in R_j and thus obtain a connection to some $R_{j-1} \subset P_{j-1}$. Hence, we have a connection from P_k to some subset of P_0. In the same way, we can extend our paths from P_k through $\mathcal{G}_{k+1}, \ldots, \mathcal{G}_n$.

Therefore, we have a connection of size $|P_k| = r$ from P to Q, with the paths formed by edges from type 1 networks.

On the other hand, it is easy to verify (by induction on the number of elementary IO-graphs) that any path from a vertex in P to a vertex in Q must contain a vertex of every P_j. In particular, every path in a connection from a subset of P to a subset of Q must use a vertex from P_k, so there can be at most $|P_k| = m$ paths.

(2). From §4.5, we see that

$$\operatorname{rank} \mathcal{X}(\mathcal{G}) = \dim \pi_P(\mathcal{X}(\mathcal{G})) - \dim \pi_Q^{-1}(0) = 2r + N_i - N_i = 2r,$$

where N_i is the number of input stubs. (3) follows from the previous lemma. ☐

Generalization to the Nonlinear Case: As remarked in §4.5, the rank r of the factorization can be detected from the "dimensions" of $\pi_P(\mathcal{X}(\mathcal{G}))$ and $\pi_Q^{-1}(0)$ for any class of nonlinear networks for which a suitable notion of dimension exists–for instance, if $M = \mathbb{R}$ and $\gamma_e : \mathbb{R} \to \mathbb{R}$ is a homeomorphism, or if M is finite. Hence, the maximum size connection $m(P, Q)$ is also detectable. Informally, the size of connections in the ∂-graph is visible in the dimensions of projections and slices of the boundary behavior $\mathcal{B}(\Gamma)$. This is a consequence of topological-combinatorial structure of the ∂-graph that does not rely upon linearity.

8.3 Application to Circular Planar ∂-Graphs: The Cut-Point Lemma

Theorem 8.4 is especially useful in the case of critical ∂-graphs on the disk, since we know a priori that elementary factorizations exist between any circular pair P and Q by Theorem 6.5, and moreover the rank of the elementary factorization is related to the number of transverse medial strands. This yields the following corollary, in which the last equality was proved by [4] under the name of the "Cut-Point Lemma:"

Corollary 8.5 (cf. [4] Theorem 4.2, [6] "Cut-Point Lemma"). *Let Γ be a linear network on a critical ∂-graph on the disk, and let P and Q be a circular pair. There is an elementary factorization of G from P to Q. If r is the rank of the factorization, then*

$$\operatorname{rank} \mathcal{X}(\mathcal{G}) = 2r = 2m(P, Q) = \#(\textit{transvserse strands}) + |P \cup Q|.$$

The original argument of [4] that $\operatorname{rank} \Lambda_{P,Q} = m(P, Q)$ worked for non-critical ∂-graphs on the disk, but relied upon having positive linear conductances: For positive linear conductances, if there is a full connection between $P' \subset P$ and $Q' \subset Q$, then by circular planarity, there is only one possible permutation τ_F for $F \in \mathcal{F}(P, Q)$ in the grove-determinant formula and hence $\operatorname{sgn} \det \Lambda_{P',Q'} = \operatorname{sgn} \tau_F$ and $\det \Lambda_{P',Q'} \neq 0$. Thus, $\operatorname{rank} \Lambda(P, Q) \geq m(P, Q)$ by virtue of having a nonzero minor of size $m(P, Q)$. The *same* minor is nonzero for all positive conductances. By contrast, for linear conductances over an arbitrary field, any given minor of $\Lambda_{P,Q}$ of size $m(P,Q)$ might (a priori) be zero. Our corollary says that the size $m(P,Q)$ minors *cannot all be zero at the same time*.

Curtis-Morrow in [6] also showed that for a critical network on the disk with positive linear conductances, the ∂-graph is determined up to Y-Δ equivalence by the response matrix. In a nutshell, the response matrix tells us the maximum size connections between all circular pairs, and from this we can deduce how many reentrant medial strands there are on a given arc of the boundary circle. Next, we determine what order the endpoints of the medial strands occur on the boundary circle (the "Z-sequence"), and then what the ∂-graph could possibly be. Thanks to Corollary **??**, the same process will work to determine the Z-sequence for linear networks over arbitrary fields, and for appropriate nonlinear networks; however, the notion of Y-Δ equivalence is more problematic and requires further study.

8.4 Unique Full Connections Using All Interior Vertices

The relationship between connections, factorizations, and mixed-data boundary value problems is particularly strong in the case of a unique full connection which uses all the interior vertices.

Theorem 8.6. *Let G be a finite ∂-graph and assume each interior vertex has valence at least 2. Suppose $\partial V = P \cup Q$ and $P' = P \setminus Q$ and $Q' = Q \setminus P$. The following are equivalent:*

a. There is a unique full connection between P' and Q', and this connection uses all the interior vertices.

b. There exists a scaffold $\mathcal{S}$ in which the heads are $V \setminus P$ and the feet are $V \setminus Q$.

c. The IO-graph morphism $P \to Q$ represented by G admits a factorization into type 1 and type 2 networks.

d. For any M and any BZ(M) network on G, potentials on P and net currents on P' determine a unique harmonic function on Γ.

e. For any signed linear network Γ over $\mathbb{R}$, potentials on P and net currents on P' determine a unique harmonic function on the network.

Remark. Let $(*)$ be the condition that potentials on P and net currents on P' determine a unique harmonic function. In (2) and (3) it is important that $(*)$ holds for *all* conductances. Even if it holds for *most* signed linear conductances, the elementary factorization may not exist.

Proof. (b) $\implies$ (c) $\implies$ (d) follows from the general theory developed so far, and (d) $\implies$ (e) is trivial.

To prove (e) $\implies$ (a), note that tor signed linear conductances $\{a_e\}$, $(*)$ is equivalent to the submatrix $K_{P' \cup V^\circ, Q' \cup V^\circ}$ being invertible. If this holds for all signed linear conductances, then $\mathcal{F}(P,Q)$ has exactly one element by Proposition 7.4. Let F be this element. Each component contains either one vertex in $P \cap Q$, or it contains one vertex in P' and one in Q'. Each component is a tree, but I claim that each component is actually a path. Otherwise, there would be an interior vertex p with only one edge e in F incident to it. By assumption, there is another edge e' incident to p. The other endpoint of e' is in some component of F, so $F \setminus \{e\} \cup \{e'\}$ is another grove in $\mathcal{F}(P,Q)$. The components of F thus provide a full connection from P to Q. The full connection is unique because if there were another full connection, then we could add edges to complete it to a different grove.

(a) $\implies$ (b). There is a unique full connection between P' and Q' if and only if there is a unique full connection between P and Q, as a simple consequence of our definition of connection. Suppose there is a unique full connection between P and Q that uses all the interior vertices. We define a scaffold $\mathcal{S}$ as follows:

i. The ladders are the edges in the paths of the connection.

ii. The foot-to-head orientation of each ladder is the same as its orientation in the path.

iii. We define $\prec$ by setting $e \prec e'$ if $e \in \operatorname{Lad}\mathcal{S}$ and e' are incident at foot(e) and $e \succ e'$ if $e \in \operatorname{Lad}\mathcal{S}$ and e' are incident at head(e). Then we take the transitive closure.

Assuming that $\prec$ actually defines a partial order, the conditions (2) and (3) of the scaffold definition are satisfied by construction, (4) is trivial since any interior vertex is both a head and a foot, and (1) is trivial since the ∂-graph is finite.

To prove that $\prec$ defines a partial order, it suffices to show that there is no "precedence loop"

$$e_1 \prec e_2 \prec \cdots \prec e_K \prec e_1,$$

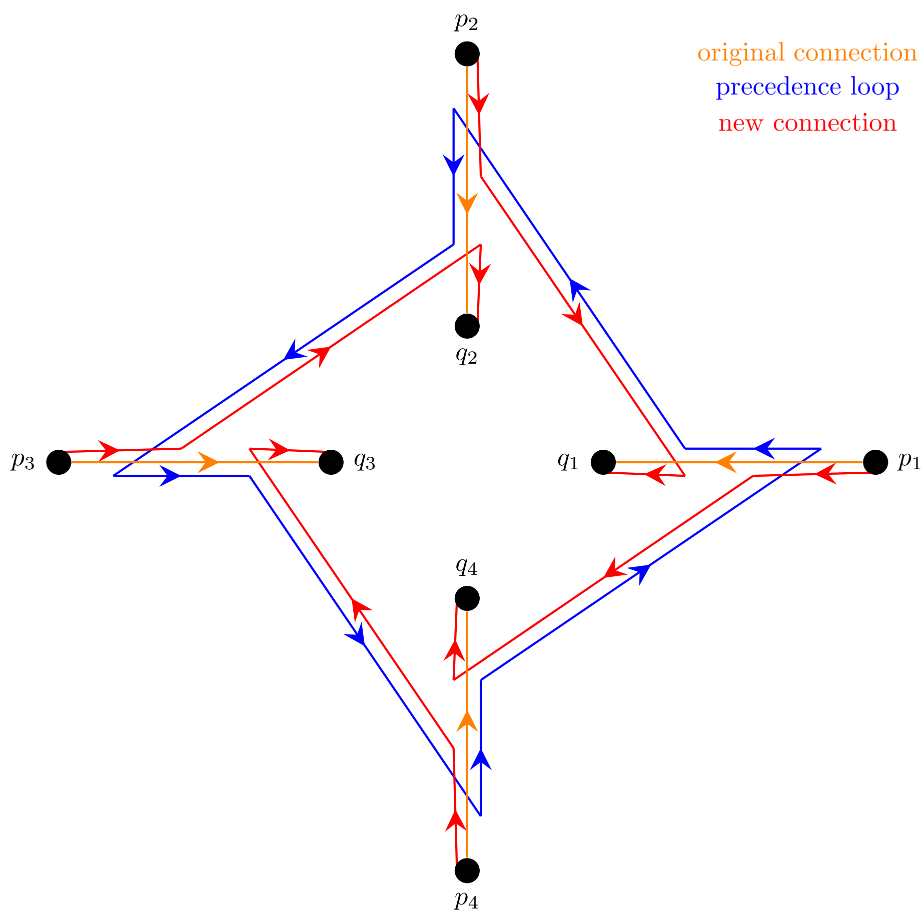
in which each pair of edges is comparable by the primitive relations given in (iii). The basic idea is that if we had such a loop, then we could construct a different connection between P and Q as indicated in Figure 14.

To make this rigorous, consider the precedence loops with the minimal number of planks, and from those, choose one with the minimal number of edges in all. Let $\alpha_1, \ldots, \alpha_n$ be the paths in the connection. Then observe:

- Any precedence loop must contain some planks, since otherwise it would have to be contained in one of the α_m's, which is impossible. We also cannot have two planks in a row since the primitive relations do not compare planks with planks.

- In the loop which we chose, $e_1, \ldots, e_K$ must be distinct, since otherwise we could find a loop with either fewer planks or the same number of planks and fewer ladders.

Figure 14: Proof of Theorem 8.6: (a) $\implies$ (b)



- Suppose there are some $i < j < k$ where e_j a plank and e_i and e_k are ladders in the same path α_m, and that e_i comes before e_k in the path α_m. If we replace the segment $e_{i+1} \dots e_{k-1}$ of the loop with the segment of α_m from e_i to e_k, then we get a precedence loop with fewer planks. Thus, this cannot happen in our chosen loop. The same reasoning holds for any cyclic permutation of the indices $1, \dots, K$. Thus, the loop must intersect each path in an "interval"; that is, $I_m = \{k : e_k \in \alpha_m\}$ is of the form $\{1, \dots, k\}$ after some cyclic permutation of the indices.

Hence, our loop has the following form: It moves forward along some path of the connection (which we will call α_1 after reindexing), then crosses by a plank to some other path α_2, and it continues in the same way until it crosses from some α_ℓ back to α_1. The paths $\alpha_1,\dots,\alpha_\ell$ are distinct. It follows that the vertices in our loop must be distinct and the loop looks essentially like the one portrayed in the Figure except that it might not visit every path of the connection. If the remaining paths are $\alpha_{\ell+1}, \dots, \alpha_n$, then we construct our new full connection as follows: $\alpha'_j = \alpha_j$ for $j = \ell + 1, \dots, n$. For $j = 1, \dots, \ell$, α'_j follows α_j until it meets an endpoint of a plank from the loop, then it crosses the plank to α_{j-1}, and it continues along α_{j-1} until it reaches Q (indices written mod ℓ). So $\prec$ does actually define a partial order and (b) is proved. $\square$

Some Remarks:

- Note the similarity of Theorem 8.6 to Propositions 7.5 and 7.6 regarding the Dirichlet and Neumann problems. These theorems say that geometric conditions characterize all the situations when a certain mixed-data boundary value problem has a solution for any linear network over $\mathbb{R}$. Surprisingly, the conditions are the same for the linear case and the non-linear case.
- (c) $\implies$ (a) is easy to prove directly, but (a) $\implies$ (c) is rather surprising. One corollary is the following: Suppose G is a ∂-graph with no boundary spikes or boundary edges, and suppose there is a full connection between P and Q that uses the all the interior vertices. Then there is another full connection between P and Q. Test this out on a few examples.
- For G a critical ∂-graph on the disk and P, Q a circular pair, we can add a sixth equivalent condition: All the medial strands are transverse.

9 Box Products and Weak ∂-Graph Morphisms

A standard construction in graph theory is the *box product*; for ∂-graphs G and H, we define $G \square H$ as follows:

- $V(G \square H) = V(G) \times V(H)$.
- $V^\circ(G \square H) = V^\circ(G) \times V^\circ(H)$.

- $E(G\Box H) = E(G) \times V(H) \coprod V(G) \times E(H)$.
- $\overline{e \times p} = \overline{e} \times p$ and $\overline{p \times e} = p \times \overline{e}$.
- $(e \times p)_+ = e_+ \times p$ and $(p \times e)_+ = p \times e_+$.

A natural question is whether the box product of solvable or totally layerable ∂-graphs is solvable or totally layerable. This would for instance provide an easy to way to show that variants of a rectangular lattice are recoverable over $\mathrm{BZ}(M)$.

We want to pull back scaffolds on G_1 and G_2 to scaffolds on $G_1 \Box G_2$ via the projection maps π_1, π_2 from $G_1 \Box G_2$ to G_1 and G_2 that send an element of $V(G_1 \Box G_2) \coprod E(G_1 \Box G_2)$ to the first or second coordinate. However, these maps are not a ∂-graph morphisms since they do not even map edges to edges– by construction, the first or second coordinate of an edge in $E(G_1 \Box G_2)$ could be a vertex in G_1 or G_2.

Thus, we extend the definition of ∂-graph morphism as follows: A *weak ∂-graph morphism* $f : G \to H$ is a function $V(G) \coprod E(G) \to V(H) \coprod E(H)$ such that

- A vertex maps to a vertex.
- An interior vertex maps to an interior vertex.
- If $e \in E(G)$ and $f(e)$ is an oriented edge, then $f(\overline{e}) = \overline{f(e)}$ and $(f(e))_+ = f(e_+)$.
- If $e \in E(G)$ and $f(e)$ is a vertex, then $f(\overline{e}) = f(e)$ and $f(e_+) = f(e)$.
- If p is any vertex, then the map $\{e : e_+ = p, f(e) \neq p\} \to \{e : e_+ = f(p)\}$ is injective, and if p is interior then it is bijective.

Exercise. *The projections $G_1 \Box G_2 \to G_1$ and $G_1 \Box G_2 \to G_2$ are weak B-graph morphisms.*

Exercise. *∂-graphs with weak ∂-graph morphisms form a category.*

Exercise. *Define a weak network morphism. Suppose $f : \Gamma_1 \to \Gamma_2$ is a weak network morphism, and that $(0,0) \in \Theta_e$ for each edge in Γ_1. If (u, c) is harmonic on Γ_2, show that $(f^* u, f^* c) = (u \circ f, c \circ f)$ is harmonic on Γ_1, where we define $f^* c_e = 0$ if $f(e)$ is a vertex.*

If $f : G \to H$ is a weak ∂-graph morphism and H' is a subgraph of H, then we define $f^{-1}(H')$ as follows:

$$\begin{aligned} V(f^{-1}(H')) &= V(G) \cap f^{-1}(V(H')), \\ E(f^{-1}(H')) &= f^{-1}(V(H') \cup E(H')), \\ V^\circ(f^{-1}(H')) &= f^{-1}(V^\circ(H')) \cap V^\circ(G). \end{aligned}$$

Now that we allow edges to map to vertices, we must modify the definition of scaffold to make the partial order include the vertices. An *extended scaffold* on a ∂-graph G consists of

- A partial order $\preceq$ on $V(G) \coprod E'(G)$.
- A partition of $E'(G)$ into ladders and planks.
- An assignment of a head and foot for each plank.

such that

1. Every subset has a minimal element.
2. If e is a ladder then $\text{foot}(e) \prec e \prec \text{head}(e)$.
3. If e' is incident to the head of a ladder e, then $e' \succ e$.
4. If e' is incident to the foot of a ladder e, then $e' \prec e$.
5. If p_1 and p_2 are interior vertices incident to e_1 and e_2 respectively, with $e_1 \preceq e_2$, then either p_1 is a head or p_2 is a foot. The same holds if $p_1 \preceq p_2$ or $p_1 \prec e_2$ or $e_1 \prec p_2$.

Any extended scaffold defines a scaffold when $\prec$ is restricted to the edges. Conversely, any scaffold can be completed to an extended scaffold by adding the relations $\text{foot}(e) \prec e \prec \text{head}(e)$ for each ladder to the partial order, then taking the transitive closure. To show this is a partial order, we only have to show there is a no loop $x_1 \prec \cdots \prec x_n \prec x_1$ for $x_j \in V(G) \coprod E'(G)$, where each of the comparisons is one of the relations in our original scaffold or one of the relations $\text{foot}(e) \prec e \prec \text{head}(e)$. If a sequence of the form $e \prec p \prec e'$ occurs in the loop, then e and e' must be vertical and $\text{head}(e) = p = \text{foot}(e')$. Hence, $e \prec e'$ and we can delete p from the loop. Thus, any loop in the new order could be shortened to a loop in the original order, which shows there cannot be a loop.

To show every subset has a minimal element, consider $S \subset V(G) \coprod E'(G)$. Let S' be the set of edges which are in S or incident to vertices in S. Because S' has a minimal element by assumption, we can deduce by some casework that S has a minimal element. (2) and (3) follow from the corresponding conditions for scaffolds and (5) is easy to verify by casework.

Suppose that $f : G \to H$ is a weak ∂-graph morphism and $\mathcal{S}$ is an extended scaffold on H, then we define $f^*\mathcal{S}$ on G as follows:

- e is a ladder if and only if $f(e)$ is a ladder.
- In that case, we choose $\text{head}(e)$ and $\text{foot}(e)$ such that $f(\text{head}(e)) = \text{head}(f(e))$ and $f(\text{foot}(e)) = \text{foot}(f(e))$.
- $x \prec y$ if and only if $f(x) \prec f(y)$.

The reader may verify that this defines a scaffold and is functorial. Then we have

Theorem 9.1.

a. If G and H are totally layerable, then so is $G\Box H$*.*

b. If $f : G \to H$ *is a weak* ∂*-graph morphism,* H *is solvable, and* G *has no self-loops or parallel edges, then* G *is solvable.*

Proof. For (a), choose an edge $e \times p \in E'(G\Box H)$. There is an extended scaffold on G where e is a ladder / plank in $\mathrm{Mid}\,\mathcal{S}$, and this induces a scaffold on $E'(G\Box H)$. The case for $p \times e \in E'(G\Box H)$ is symmetrical.

For (b), let $H = H_0, H_1, \ldots$ be a solvable filtration of H and assume without loss of generality that each step only includes one type of reduction operation (contracting non-degenerate spikes, deleting boundary edges, deleting isolated boundary vertices). Then consider three cases:

1. Suppose H_n is obtained from H_{n-1} by deleting boundary edges. Then $f^{-1}(H_n)$ is obtained from $f^{-1}(H_{n-1})$ by deleting boundary edges. For any boundary edge e that is removed from H_{n-1}, we have an extended scaffold in which it is a ladder in the middle of the scaffold. This pulls back to an extended scaffold where the edges in $f^{-1}(e)$ are ladders in the middle of the scaffold.

2. Suppose H_n is obtained from H_{n-1} by contracting non-degenerate boundary spikes. Then $f^{-1}(H_n)$ is obtained from $f^{-1}(H_{n-1})$ in two steps:

   A. Delete the edges in $f^{-1}(p)$ for any boundary vertex p at the end of a spike; these are necessarily boundary edges.

   B. Contract the edges in $f^{-1}(e)$ for each boundary spike e contracted in H_{n-1}; the edges $f^{-1}(e)$ are now boundary spikes.

   To create the extended scaffolds for step (A), choose a spike e with boundary vertex p, and let $\mathcal{S}$ be an extended scaffold on H_{n-1} where e is a plank in the middle of the scaffold. We can assume $\mathcal{S}$ is obtained from an ordinary scaffold in the manner described above, and so p is not comparable to anything in the partial order. Then in $f^*\mathcal{S}$, the edges in $f^{-1}(p)$ are planks and not comparable to anything else. Pick an edge $\epsilon \in f^{-1}(p)$. We modify the scaffold as follows:

   - Change ϵ to a ladder, and choose a distinct head and foot (it does not matter which one is which). This is possible because G has no self-looping edges.
   - If $\epsilon' \in f^{-1}(e)$ is incident to ϵ at $\mathrm{head}(\epsilon)$, set $\epsilon' \succ \epsilon$ and do the symmetrical thing at the foot. We assume G has no parallel edges, and hence we will not have $\epsilon' \prec \epsilon \prec \epsilon'$.
   - Let η be the edge in $f^{-1}(e)$ incident to $\mathrm{head}(\epsilon)$. Set $\epsilon \prec \eta$ and everything which is greater than η, and do the symmetrical thing at the foot of ϵ. Since ϵ was not comparable to anything originally, we still have a partial order, and since η was in the middle of the orginal scaffold, ϵ is in the middle of the new one.

For step (B), for each spike e removed, we can choose an exctended scaffold $\mathcal{S}$ on H in which e is a ladder in the middle of the scaffold. Then $f^*\mathcal{S}$ is an extended scaffold where the edges in $f^{-1}(e)$ are ladders in the middle of the scaffold.

3. Suppose H_n is obtained from H_{n-1} by deleting isolated boundary vertices. Let p be such a vertex. Since H_n is solvable, it has some extended scaffold $\mathcal{S}$ on it (in the case where H_n has no edges, it has a scaffold trivially). The extended scaffold $f^*\mathcal{S}$ on $f^{-1}(H_n)$ can be extended to an extended scaffold on $f^{-1}(H_{n-1})$ since it is the disjoint union of $f^{-1}(H_n)$ and some components with only boundary vertices, and no loops or parallel edges. Similarly to case (2), we can arrange that any given edge in $f^{-1}(p)$ is vertical. □

Example: Rectangular Lattices:

A (finite) rectangular lattice is the box product of ∂-graphs $G_1 \square G_2 \square \dots \square G_n$, where each G_j is a path. Various lattices can be constructed by different assignments of boundary vertices for each path. It seems most natural that for a given path, the boundary vertices should be nothing, one of the endpoints, or two of the endpoints.

Proposition 9.2. *Let $n \geq 2$, and suppose G is the box product of paths G_1, $\dots$, G_n, and that it in each G_j, both endpoints are boundary vertices. Then G is totally layerable.*

Proof. Let $f_j : G \to G_j$ be the projection. For each path G_j, we can form a scaffold $\mathcal{S}_j$ where every edge is a ladder and $\operatorname{Mid}\mathcal{S}_j$ is everything. Given $e \in E(G_1)$ and $p_2, \dots, p_n$ in $V(G_2)$, $\dots$, $V(G_n)$ respectively, the edge $e \times p_2 \times \dots \times p_n$ is a ladder in $\operatorname{Mid} f_1^*\mathcal{S}_1$ and a plank in $\operatorname{Mid} f_2^*\mathcal{S}_2$. The same argument applies to any edge in G after permuting the coordinates. □

Proposition 9.3. *Let $n \geq 2$. Let G_1 be a path ∂-graph where both endpoints are boundary vertices, let G_2 be a path ∂-graph where one endpoint is a boundary vertex, and let $G_3, \dots, G_n$ be path ∂-graphs with no boundary vertices. Then the box product of G_1, $\dots$, G_n is solvable.*

Proof. $G_1 \square G_2$ is circular-planar, and has a lensless medial strand arrangement (exercise). Thus, it is totally layerable, hence solvable, so by the Theorem, $G_1 \square \dots \square G_n$ is solvable. □

The total layerability or solvability of other types of rectangular lattices can be deduced from these two Propositions: Indeed, if G' is obtained from G by changing some interior vertices to boundary, then G' is a ∂-subgraph of G, hence total layerability or solvability of G implies total layerability or solvability of G'.

10 Problems for Further Research

Beyond Linear Algebra: [11] and this paper show that much of what [4] initially proved using linear algebra actually holds for any BZ(M) network. As developed here, network theory is not fundamentally about linear algebra–it is about the topological-combinatorial structure of ∂-graphs. On the other hand, linear algebra was still used for some of the results.

Beyond Circular Planar ∂-Graphs: Various attempts have been made to move beyond circular planar ∂-graphs (see e.g. [13]). In particular, Lam and Pylyavskyy [15] study the inverse problem for positive linear networks on the cylinder $S^1 \times [0, 1]$ (or equivalently the annulus). Rather than the response matrix as typically defined, they consider the "universal response matrix"–an appropriately defined infinite response matrix for a periodic network on an infinite strip corresponding to the given cylindrical network. They conjecture that if the medial graph of the network on the strip is lensless, then the conductances are uniquely determined by the universal response matrix up to an action of the symmetric group S_n, where n is the number of self-looping medial strands on the cylinder.

Using the standard response matrix rather than the universal response matrix, the theory of networks on the cylinder is much more dicey. Based on my earlier work at the REU:

- There exists a network on the annulus which is recoverable for positive linear conductances; however, its dual (with respect to the annular embedding) is not recoverable. However, these two networks have the same medial strands, so there is no simplistic medial-strand test for recoverability.

- There exists a network on the annulus which is recoverable for positive linear conductances and most real conductances, but fails to be recoverable for all signed real conductances. Incidentally, it is recoverable for bijective zero preserving conductance functions $\gamma_e : \mathbb{R} \to \mathbb{R}$ which are increasing. Sign conditions are crucial to the recovery process.

- The theory of layering adapts to the annulus better than the methods of [11].[17] and medial strands can be used to create scaffolds and recover networks in certain cases, but the criteria for solvability are not yet known and are probably ugly.

- It is unclear whether linear recoverability over $\mathbb{R}$ implies nonlinear recoverability.

For more complicated surfaces, things will become even messier, and the universal cover itself will be less tractable.

Beyond ∂-Graphs on Surfaces: Given the complexity of annular networks, I decided to phrase layering theory in purely ∂-graph-theoretic terms

[17] Indeed, I arrived at the idea of elementary factorizations through annular networks

without embeddings. Embeddings and medial graphs are sometimes extremely useful, but sometimes another approach is better. For instance, embedding a ten-dimensional rectangular lattice on a surface is a terrible way to see whether it is recoverable. There may yet be other combinatorial/∂-graph-theoretic criteria for solvability, total layerability, or recoverability over $\mathrm{BZ}(M)$ that do not assume an embedding on a surface.

Strengths and Weaknesses of Layering Theory:

- Layering theory is good for using geometric conditions to prove results about boundary behavior for a lot of nonlinear networks. However, to start with properties of boundary behavior and deduce geometric results, we often resorted to linear algebra.
- Solvability and total layerability make sense for infinite and non-planar ∂-graphs; however, they are limited to layerable graphs, which are a very small portion of all ∂-graphs.
- It is easy to create scaffolds from other scaffolds (or from medial graphs), but hard to create them from scratch.
- Layering theory provides many tools for proving ∂-graphs are recoverable, but none for proving they are not recoverable.
- Layering theory provides many ways of constructing complicated solvable graphs, but not an efficient way to decide whether a given complicated graph is solvable.

Problems for Further Research:

1. Find interesting examples of M and subsets of $\mathrm{BZ}(M)$.[18]
2. Find universal ∂-graph-theoretic conditions that characterize recoverability over $\mathrm{BZ}(M)$ for all M.
3. Are the conditions the same for general $\mathrm{BZ}(M)$ as for signed linear networks over $\mathbb{R}$?
4. Find nicer conditions that are equivalent to solvability.
5. For a given surface, find elegant conditions that will guarantee total layerability or solvability.
6. Given a ∂-graph which is not solvable, is there some covering ∂-graph of it, which is solvable?[19]

[18] More generally, one can choose a different M for each vertex, and/or a different M for the potential and currents, and formulate a definition of bijective zero-preserving networks for which the layering theory mostly still applies. One application would be to situations as in [13], where there have a different vector space for each vertex, and a notion of "parallel transport" along an edge that would be useful for a graph on a smooth surface.

[19] For instance, for a circular planar ∂-graph G where the medial graph has only one lens which is a two-pole lens and contains 0, taking the preimage of G and $\mathcal{M}$ under $z \mapsto z^2$ will create a circular planar ∂-graph with a lensless strand arrangement.

7. Generalize the theory of harmonic continuation to non-layerable networks.

8. The idea of discrete harmonic continuation ("information propagation") and partial orders may be useful for things other than ∂-graphs.[20]

9. Study the nonlinear inverse problem on non-layerable networks. It is conceivable that some networks are recoverable for "most" smooth nonlinear bijective zero-preserving $\gamma_e : \mathbb{R} \to \mathbb{R}$, and that the linear conductances represent a "small" pathological set of conductances.[21]

10. Associate algebraic, topological, or some other type of objects to a ∂-graph that will test layerability, solvability, or total layerability. If possible, make them functorial on the category of ∂-graphs.[22]

11. Find an efficient algorithm to transform a non-planar linear network into a circular planar network with the same boundary behavior. (Efficient means smallest number of steps or smallest number of edges in the final result.)

12. Characterize $EL_n(\mathbb{F}_2)$ and $EG_n(\mathbb{F}_2)$.

13. For a given ∂-graph and field $\mathbb{F}$, characterize the set of possible boundary behaviors.

14. Let $\mathbb{F}$ be an algebraically closed field. Study $EL_n(\mathbb{F})$ and $EG_n(\mathbb{F})$ from the viewpoint of algebraic geometry. "Projectivize" by letting the voltage-current relation on an edge be any linear relation $\mathbb{F} \rightsquigarrow \mathbb{F}$ rather than a bijective one.

[20] Harmonicity is given by a set of relations between different quantities called potential and current. The quantities and relations are indexed by the vertices/edges/interior vertices/incidences of the ∂-graph. The boundary behavior is a relation between certain quantities which are "visible" while other quantities remain "hidden." One could replace the ∂-graph by some other combinatorial structure, and still use an auxiliary partial order like a scaffold as an aid to harmonic continuation for a more general type of inverse problem. However, such a theory would be completely opaque without other motivating examples.

[21] For instance, suppose G does not have a series or a parallel circuit, but it Y-Δ equivalent to a network that does. Y-Δ transformations do not work for smooth nonlinear conductances. The more complicated nonlinear boundary behavior might capture the properties of the conductance functions better than in the linear case.

[22] Avi Levy and I are developing a "harmonic cohomology" for ∂-graphs that tests something like the failure of layerability.

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