Euler characteristic reciprocity for chromatic and order polynomials

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January 5, 2016

Abstract

The Euler characteristic of a semialgebraic set can be considered as a generalization of the cardinality of a finite set. An advantage of semialgebraic sets is that we can define "negative sets" to be the sets with negative Euler characteristics. Applying this idea to posets, we introduce the notion of semialgebraic posets. Using "negative posets", we establish Stanley's reciprocity theorems for chromatic and order polynomials at the level of Euler characteristics.

Contents

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1 Introduction

Let P be a finite poset. The *order polynomial* $\mathcal{O}^{\leq}(P,t) \in \mathbb{Q}[t]$ and the *strict order polynomial* $\mathcal{O}^{<}(P,t) \in \mathbb{Q}[t]$ are polynomials which satisfy

$$\begin{aligned} \mathcal{O}^{\leq}(P,n) &= \#\operatorname{Hom}^{\leq}(P,[n]), \\ \mathcal{O}^{<}(P,n) &= \#\operatorname{Hom}^{<}(P,[n]), \end{aligned} \tag{1}$$

where $[n] = \{1, \dots, n\}$ with normal ordering and

$$\operatorname{Hom}^{\leq(<)}(P,[n]) = \{f : P \longrightarrow [n] \mid x < y \Longrightarrow f(x) \leq (<) f(y)\}$$

is the set of increasing (resp. strictly increasing) maps.

These two polynomials are related to each other by the following reciprocity theorem proved by Stanley ([9, 10], see also [1, 3, 4] for a recent survey).

$$\mathcal{O}^{<}(P,t) = (-1)^{\#P} \cdot \mathcal{O}^{\leq}(P,-t). \tag{2}$$

By putting $t = n$, the formula (2) can be informally presented as follows.

$$\text{“ } \#\operatorname{Hom}^{<}(P,[n]) = (-1)^{\#P} \cdot \#\operatorname{Hom}^{\leq}(P,[-n]). \text{ ”} \tag{3}$$

It is a natural problem to extend the above reciprocity to homomorphisms between arbitrary (finite) posets P and Q. We may expect a formula of the following type.

$$\text{“ } \#\operatorname{Hom}^{<}(P,Q) = (-1)^{\#P} \cdot \#\operatorname{Hom}^{\leq}(P,-Q). \text{ ”} \tag{4}$$

Of course this is not a mathematically justified formula. In fact, we do not have the notion of a “negative poset $-Q$.” The negative poset $-Q$ in the right-hand side does not make sense.

In [8], Schanuel discussed what “negative sets” should be. A possible answer is that a negative set is nothing but a semialgebraic set which has a negative Euler characteristic (Table 1). For example, the open simplex

$$\overset{\circ}{\sigma}_d = \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid 0 < x_1 < \cdots < x_d < 1\}$$

Finite set	Semialgebraic set
Cardinality	Euler characteristic

Table 1: Negative sets

has the Euler characteristic $e(\overset{\circ}{\sigma}_d) = (-1)^d$, and the closed simplex

$$\sigma_d = \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid 0 \le x_1 \le \cdots \le x_d \le 1\}$$

has $e(\sigma_d) = 1$. Thus we have the following "reciprocity"

$$e(\overset{\circ}{\sigma}_d) = (-1)^d \cdot e(\sigma_d). \tag{5}$$

This formula looks alike Stanley's reciprocity (2). This analogy would indicate that (2) could be explained via the computations of Euler characteristics of certain semialgebraic sets.

In this paper, by introducing the notion of *semialgebraic posets*, we settle Euler characteristic reciprocity theorems for poset homomorphisms and chromatic functors, which imply Stanley's reciprocities as corollaries. Semialgebraic posets also provide a rigorous formulation for (4).

Briefly, a semialgebraic poset P is a semialgebraic set with poset structure such that the ordering is defined semialgebraically (see Definition 2.2). Finite posets and the open interval $(0, 1) \subset \mathbb{R}$ are examples of semialgebraic posets. A semialgebraic poset P has the Euler characteristic $e(P) \in \mathbb{Z}$ which is an invariant of semialgebraic structure of P (see §2.1). In particular, if P is a finite poset, then $e(P) = \#P$, and if P is the open interval $(0, 1)$, then $e((0, 1)) = -1$.

The philosophy presented in the literature [8] suggests to consider the "moduli space" $\mathrm{Hom}^{\le(<)}(P, Q)$ of poset homomorphisms from a finite poset P to a semialgebraic poset Q, and then computing the Euler characteristic of the moduli space instead of counting the number of maps. We can naturally expect that the product semialgebraic poset $Q \times (0, 1)$ could play the role of the negative poset "$-Q$" since $e(Q \times (0, 1)) = -e(Q)$. In fact, we have the following result.

Theorem 1.1 (Proposition 2.6 and Theorems 3.1, 3.5). *Let P be a finite poset, and Q be a semialgebraic poset.*

(i) $\mathrm{Hom}^{\le}(P, Q)$ *and* $\mathrm{Hom}^{<}(P, Q)$ *possess structures of semialgebraic sets.*

(ii) The following reciprocity of Euler characteristics holds,

$$e(\mathrm{Hom}^{<}(P, Q)) = (-1)^{\#P} \cdot e(\mathrm{Hom}^{\le}(P, Q \times (0, 1))), \tag{6}$$

$$e(\mathrm{Hom}^{\le}(P, Q)) = (-1)^{\#P} \cdot e(\mathrm{Hom}^{<}(P, Q \times (0, 1))). \tag{7}$$

(iii) Let T be a semialgebraic totally ordered set. Then

$$e(\mathrm{Hom}^{\leq}(P,T)) = \mathcal{O}^{\leq}(P, e(T)), \tag{8}$$

$$e(\mathrm{Hom}^{<}(P,T)) = \mathcal{O}^{<}(P, e(T)). \tag{9}$$

The formula (6) may be considered as a rigorous formulation of (4). Furthermore, by putting $Q = [n]$ and $T = [n] \times (0,1)$, we can recover Stanley's reciprocity (2) from (6) and (8) (see §3.3).

Semialgebraic posets are also applicable to chromatic theory of finite graphs. Let $G = (V, E)$ be a finite graph. For any set X, we can associate the graph configuration space $\underline{\chi}(G, X)$ (Definition 4.1). If X is a semialgebraic set, then $\underline{\chi}(G, X)$ is also a semialgebraic set. The *chromatic polynomial* of G is a polynomial $\chi(G,t) \in \mathbb{Z}[t]$ determined by $\chi(G,n) = \#\underline{\chi}(G,[n])$. The next result generalizes a result in [7, Theorem 2], where X was chosen to be a complex projective space.

Theorem 1.2 (Theorem 4.2)**.** *Let $G = (V, E)$ be a finite graph and X be a semialgebraic set. Then*

$$e(\underline{\chi}(G, X)) = \chi(G, e(X)).$$

A "negative set" also appears in chromatic polynomials. Stanley established a reciprocity, showing that $(-1)^{\#V} \cdot \chi(G, -n)$ can be interpreted as the number of pairs of an acyclic orientation and a map $V \longrightarrow [n]$ compatible with the orientation ([11]). We can give a meaning of "$\#\underline{\chi}(G, [-n])$" using the above result, $e(\underline{\chi}(G, X)) = \chi(G, e(X))$, when $X = [n] \times (0,1)$. A precise setting and results are stated below.

Let T be a totally ordered set. We introduce the moduli space $\mathcal{AOC}^{\leq}(G,T)$ (resp. $\mathcal{AOC}^{<}(G,T)$) of the pairs consisting of an acyclic orientation on the edges E of G with a compatible (resp. strictly compatible) map $V \to T$ (Definition 4.4). When T is a semialgebraic totally ordered set, the moduli spaces $\mathcal{AOC}^{\leq}(G,T)$ and $\mathcal{AOC}^{<}(G,T)$ also admit structures of semialgebraic sets. Part of the Euler characteristic reciprocity can be formulated as follows.

Theorem 1.3 (Theorem 4.5, Corollary 4.7)**.** *Let $G = (V, E)$ be a finite graph and T be a semialgebraic totally ordered set. Then*

$$e(\mathcal{AOC}^{\leq}(G,T)) = (-1)^{\#V} \cdot e(\mathcal{AOC}^{<}(G, T \times (0,1))), \tag{10}$$

$$e(\mathcal{AOC}^{<}(G,T)) = \chi(G, e(T)). \tag{11}$$

By putting $T = [n]$ in (10) and $T = [n] \times (0,1)$ in (11), we obtain Stanley's reciprocity (Corollary 4.8)

$$\#\,\mathcal{AOC}^{\leq}(G, [n]) = (-1)^{\#V} \cdot \chi(G, -n).$$

2 Semialgebraic posets and Euler characteristics

2.1 Semialgebraic sets

A subset $X \subset \mathbb{R}^n$ is said to be a semialgebraic set if it is expressed as a Boolean connection (i.e. a set expressed by a finite combination of $\cup, \cap$ and complements) of subsets of the form

$$\{x \in \mathbb{R}^n \mid p(x) > 0\},$$

where $p(x) \in \mathbb{R}[x_1, \ldots, x_n]$ is a polynomial. Let $f : X \longrightarrow Y$ be a map between semialgebraic sets $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$. It is called semialgebraic if the graph

$$\Gamma(f) = \{(x, f(x)) \mid x \in X\} \subset \mathbb{R}^{m+n}$$

is a semialgebraic set. If f is semialgebraic then the pull-back $f^{-1}(Y)$ and the image $f(X)$ are also semialgebraic sets (see [2, 5] for details).

Any semialgebraic set X has a finite partition into Nash cells, namely, a partition $X = \bigsqcup_{\alpha=1}^k X_\alpha$ such that X_α is Nash diffeomorphic (that is a semialgebraic analytic diffeomorphism) to the open cell $(0,1)^{d_\alpha}$ for some $d_\alpha \geq 0$. Then the *Euler characteristic*

$$e(X) := \sum_{\alpha=1}^{k} (-1)^{d_\alpha} \tag{12}$$

is independent from the partition ([6]). Moreover, the Euler characteristic satisfies

$$e(X \sqcup Y) = e(X) + e(Y),$$
$$e(X \times Y) = e(X) \times e(Y).$$

Example 2.1. As mentioned in §1, the closed simplex σ_d and the open simplex $\mathring{\sigma}_d$ have $e(\sigma_d) = 1$ and $e(\mathring{\sigma}_d) = (-1)^d$.

2.2 Semialgebraic posets

Definition 2.2. $(P, \leq)$ is called a *semialgebraic poset* if

(1) $(P, \leq)$ is a partially ordered set, and

(2) there is an injection $i : P \hookrightarrow \mathbb{R}^n$ ($n \geq 0$) such that the image $i(P)$ is a semialgebraic set and the image of

$$\{(x, y) \in P \times P \mid x \leq y\},$$

by the map $i \times i : P \times P \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$, is also a semialgebraic subset of $\mathbb{R}^n \times \mathbb{R}^n$.

Let P and Q be semialgebraic posets. The set of homomorphisms (strict homomorphisms) of semialgebraic posets is defined by

$$\mathrm{Hom}^{\leq(<)}(P, Q) = \left\{ f : P \longrightarrow Q \;\middle|\; \begin{array}{l} f \text{ is a semialgebraic map s.t.} \\ x < y \Longrightarrow f(x) \leq (<) f(y) \end{array} \right\}. \quad (13)$$

Example 2.3. (1) A finite poset $(P, \leq)$ admits the structure of a semialgebraic poset, since any finite subset in $\mathbb{R}^n$ is a semialgebraic set. A finite poset has the Euler characteristic $e(P) = \#P$.

(2) The open interval $(0, 1)$ and the closed interval $[0, 1]$ are semialgebraic posets with respect to the usual ordering induced from $\mathbb{R}$. Their Euler characteristics are $e((0, 1)) = -1$ and $e([0, 1]) = 1$, respectively.

Let P and Q be posets. Recall that the product $P \times Q$ admits poset structure by the lexicographic ordering:

$$(p_1, q_1) \leq (p_2, q_2) \Longleftrightarrow \left\{ \begin{array}{l} p_1 < p_2, \text{ or,} \\ p_1 = p_2 \text{ and } q_1 \leq q_2, \end{array} \right.$$

for $(p_i, q_i) \in P \times Q$.

Proposition 2.4. Let P and Q be semialgebraic posets. Then the product poset $P \times Q$ (with lexicographic ordering) admits the structure of a semialgebraic poset.

Proof. Suppose $P \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$. Then

$$\begin{aligned} &\{((p_1, q_1), (p_2, q_2)) \in (P \times Q)^2 \mid (p_1, q_1) \leq (p_2, q_2)\} \\ &= \{(p_1, q_1, p_2, q_2) \in (P \times Q)^2 \mid (p_1 < p_2) \text{ or } (p_1 = p_2 \text{ and } q_1 \leq q_2)\} \\ &\simeq \left(\{(p_1, p_2) \in P^2 \mid p_1 < p_2\} \times Q^2\right) \sqcup \left(P \times \{(q_1, q_2) \in Q^2 \mid q_1 \leq q_2\}\right) \end{aligned}$$

is also semialgebraic since semialgebraicity is preserved by disjoint union, complement and Cartesian products. □

Proposition 2.5. Let P and Q be semialgebraic posets. Then the first projection $\pi : P \times Q \longrightarrow P$ is a homomorphism of semialgebraic posets.

Proof. This is straightforward from the definition of the lexicographic ordering. □

The next result shows that the "moduli space" of homomorphisms from a finite poset to a semialgebraic poset has the structure of a semialgebraic set.

Proposition 2.6. Let P be a finite poset and Q be a semialgebraic poset. Then $\mathrm{Hom}^{\leq}(P, Q)$ and $\mathrm{Hom}^{<}(P, Q)$ have structures of semialgebraic sets.

Proof. Let us set $P = \{p_1, \dots, p_n\}$ and $\mathcal{L} = \{(i, j) \mid p_i < p_j\}$. Since each element $f \in \mathrm{Hom}^{\leq}(P, Q)$ can be identified with the tuple $(f(p_1), \dots, f(p_n)) \in Q^n$, we have the expression

$$\begin{aligned}\mathrm{Hom}^{\leq}(P, Q) &\simeq \{(q_1, \dots, q_n) \in Q^n \mid q_i \leq q_j \text{ for } (i, j) \in \mathcal{L}\} \\ &= \bigcap_{(i,j)\in\mathcal{L}} \{(q_1, \dots, q_n) \in Q^n \mid q_i \leq q_j\}.\end{aligned}$$

Clearly, the right-hand side is a semialgebraic set.

The semialgebraicity of $\mathrm{Hom}^{<}(P, Q)$ is similarly proved. □

3 Euler characteristic reciprocity

3.1 Main results

We can formulate a reciprocity theorem for semialgebraic posets.

Theorem 3.1. *Let P be a finite poset and Q be a semialgebraic poset. Then*

$$e(\mathrm{Hom}^{<}(P, Q)) = (-1)^{\#P} \cdot e(\mathrm{Hom}^{\leq}(P, Q \times (0, 1))), \tag{14}$$

and

$$e(\mathrm{Hom}^{<}(P, Q \times (0, 1))) = (-1)^{\#P} \cdot e(\mathrm{Hom}^{\leq}(P, Q)). \tag{15}$$

Note that $Q \times (0, 1)$ in the right-hand side of (14) is a semialgebraic poset with Euler characteristic

$$e(Q \times (0, 1)) = -e(Q). \tag{16}$$

In view of the relation (16), Theorem 3.1 may be considered as a generalization of Stanley's reciprocity (see Corollary 3.10).

Before the proof of Theorem 3.1, we present an example which illustrates the main idea of the proof.

Example 3.2. Let $P = Q = \{1, 2\}$ with $1 < 2$. Clearly we have

$$\mathrm{Hom}^{<}(P, Q) = \{\mathrm{id}\}.$$

Let us describe $\mathrm{Hom}^{\leq}(P, Q \times (0,1))$. Note that $Q \times (0,1)$ is isomorphic to the semialgebraic totally ordered set $(1, \frac{3}{2}) \sqcup (2, \frac{5}{2})$ by the isomorphism

$$\varphi : Q \times (0,1) \longrightarrow \left(1, \frac{3}{2}\right) \sqcup \left(2, \frac{5}{2}\right), (a, t) \longmapsto a + \frac{t}{2}.$$

A homomorphism $f \in \mathrm{Hom}^{\leq}(P, Q \times (0,1))$ is described by the two values $f(1) = (a_1, t_1)$ and $f(2) = (a_2, t_2) \in Q \times (0,1)$. The condition imposed on a_1, a_2, t_1 and t_2 (by the inequality $f(1) \leq f(2)$) is

$$(a_1 < a_2), \text{ or } (a_1 = a_2 \text{ and } t_1 \leq t_2),$$

which is equivalent to $a_1 + \frac{t_1}{2} \leq a_2 + \frac{t_2}{2}$. Therefore, the semialgebraic set $\mathrm{Hom}^{\leq}(P, Q \times (0,1))$ can be described as in Figure 1. Each diagonal triangle

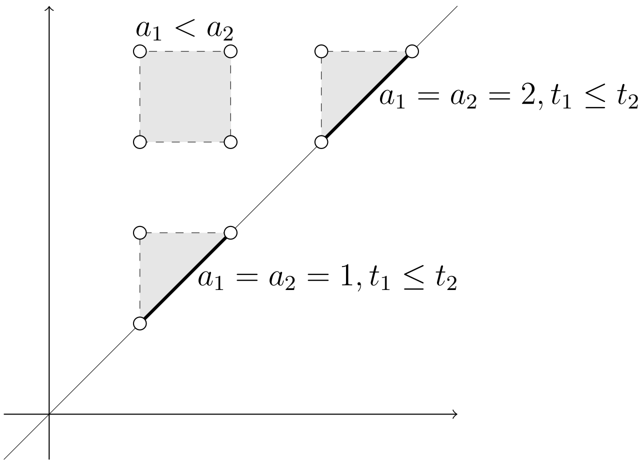

Figure 1: $f(1) \leq f(2)$.

in Figure 1 has a stratification $\mathring{\sigma}_2 \sqcup \mathring{\sigma}_1$. Therefore the Euler characteristic is $e(\mathring{\sigma}_2 \sqcup \mathring{\sigma}_1) = e(\mathring{\sigma}_2) + e(\mathring{\sigma}_1) = (-1)^2 + (-1)^1 = 0$. On the other hand, the square region corresponding to $a_1 < a_2$ has the Euler characteristic $(-1)^2 = 1$. Hence we have

$$e(\mathrm{Hom}^{\leq}(P, Q \times (0,1))) = 1 = e(\mathrm{Hom}^{<}(P, Q)).$$

The following lemma will be used in the proof of Theorem 3.1.

Lemma 3.3. *Let $P \subset \mathbb{R}^n$ be a d-dimensional polytope (i.e., a convex hull of a finite set). Fix a hyperplane description*

$$P = \{\alpha_1 \geq 0\} \cap \cdots \cap \{\alpha_N \geq 0\}$$

of P where α_i are affine maps from $\mathbb{R}^n$ to $\mathbb{R}$. For a given $x_0 \in P$, define the associated locally closed subset P_{x_0} of P (see Figure 2) by

$$P_{x_0} = \bigcap_{\alpha_i(x_0)=0} \{\alpha_i \geq 0\} \cap \bigcap_{\alpha_i(x_0)>0} \{\alpha_i > 0\}.$$

Then the Euler characteristic is

$$e(P_{x_0}) = \begin{cases} (-1)^d, & \text{if } x_0 \in \overset{\circ}{P} \\ 0, & \text{otherwise } (x_0 \in \partial P), \end{cases}$$

where $\overset{\circ}{P}$ is the relative interior of P and $\partial P = P \setminus \overset{\circ}{P}$.

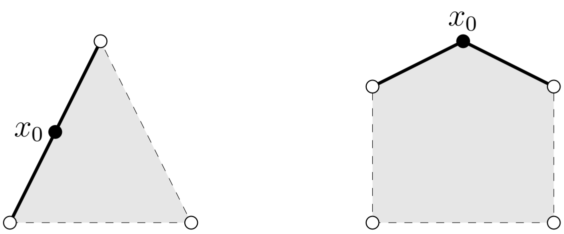

Figure 2: P_{x_0}.

Proof. If $x_0 \in \overset{\circ}{P}$, then $P_{x_0} = \overset{\circ}{P}$. The Euler characteristic is $e(\overset{\circ}{P}) = (-1)^d$.

Suppose $x_0 \in \partial P$. Then P_{x_0} can be expressed as

$$P_{x_0} = \bigsqcup_{F \ni x_0} \overset{\circ}{F}, \tag{17}$$

where F runs over the faces of P containing x_0 and $\overset{\circ}{F}$ denotes its relative interior. Then we obtain the decomposition

$$P_{x_0} = \overset{\circ}{P} \sqcup \bigsqcup_{F \ni x_0, F \subset \partial P} \overset{\circ}{F}.$$

We look at the structure of the second component $Z := \bigsqcup_{F \ni x_0, F \subset \partial P} \overset{\circ}{F}$. For any point $y \in Z$, the segment $[x_0, y]$ is contained in Z. Hence Z is contractible open subset of ∂P, which is homeomorphic to the $(d-1)$-dimensional open disk. The Euler characteristic is computed as

$$\begin{aligned} e(P_{x_0}) &= e(\overset{\circ}{P}) + e(Z) \\ &= (-1)^d + (-1)^{d-1} \\ &= 0. \end{aligned}$$

□

3.2 Proof of the main results

Now we prove (14) of Theorem 3.1. Let $\varphi \in \mathrm{Hom}^{<}(P, Q \times (0,1))$. Then φ is a pair of maps

$$\varphi = (f, g),$$

where $f : P \longrightarrow Q$ and $g : P \longrightarrow (0,1)$. Let $\pi_1 : Q \times (0,1) \longrightarrow Q$ be the first projection. Since π_1 is order-preserving (Proposition 2.5), so is $f = \pi_1 \circ \varphi$, and hence $f \in \mathrm{Hom}^{\leq}(P, Q)$.

In order to compute the Euler characteristics, we consider the map

$$\pi_{1*} : \mathrm{Hom}^{\leq}(P, Q \times (0,1)) \longrightarrow \mathrm{Hom}^{\leq}(P, Q),\ \varphi \longmapsto \pi_1 \circ \varphi = f. \tag{18}$$

Let us set

$$\begin{aligned} M :=& \mathrm{Hom}^{\leq}(P, Q) \smallsetminus \mathrm{Hom}^{<}(P, Q) \\ =& \{ f \in \mathrm{Hom}^{\leq}(P, Q) \mid \exists x < y \in P \text{ s.t. } f(x) = f(y) \}. \end{aligned} \tag{19}$$

Then obviously, we have

$$\mathrm{Hom}^{\leq}(P, Q) = \mathrm{Hom}^{<}(P, Q) \sqcup M. \tag{20}$$

This decomposition induces that of $\mathrm{Hom}^{\leq}(P, Q \times (0,1))$,

$$\mathrm{Hom}^{\leq}(P, Q \times (0,1)) = \pi_{1*}^{-1}\left(\mathrm{Hom}^{<}(P, Q)\right) \sqcup \pi_{1*}^{-1}(M). \tag{21}$$

By the additivity of the Euler characteristics, we obtain

$$e\left(\mathrm{Hom}^{\leq}(P, Q \times (0,1))\right) = e\left(\pi_{1*}^{-1}\left(\mathrm{Hom}^{<}(P, Q)\right)\right) + e(\pi_{1*}^{-1}(M)). \tag{22}$$

We claim the following two equalities which are sufficient for the proof of (14).

$$e\left(\pi_{1*}^{-1}\left(\mathrm{Hom}^{<}(P, Q)\right)\right) = (-1)^{\#P} \cdot e\left(\mathrm{Hom}^{<}(P, Q)\right) \tag{23}$$

$$e(\pi_{1*}^{-1}(M)) = 0 \tag{24}$$

We first prove (23). Let $\varphi \in \pi_{1*}^{-1}(\mathrm{Hom}^{<}(P,Q))$, that is $\varphi = (f,g)$ with $f \in \mathrm{Hom}^{<}(P,Q)$. By the definition of the ordering of $Q \times (0,1)$, (f,g) is contained in $\pi_{1*}^{-1}(\mathrm{Hom}^{<}(P,Q))$ for arbitrary map $g : P \longrightarrow (0,1)$. This implies

$$\pi_{1*}^{-1}(\mathrm{Hom}^{<}(P,Q)) \simeq \mathrm{Hom}^{<}(P,Q) \times (0,1)^{\#P}, \tag{25}$$

which yields (23).

The proof of (24) requires further stratification of M. Let

$$\mathcal{L}(P) := \{(p_1,p_2) \in P \times P \mid p_1 < p_2\}.$$

For given $f \in M$, consider the set of collapsing pairs,

$$K(f) := \{(p_1,p_2) \in \mathcal{L}(P) \mid f(p_1) = f(p_2)\}.$$

Note that $f \in M$ if and only if $K(f) \neq \emptyset$. We decompose M according to $K(f)$. Namely, for any nonempty subset $X \subset \mathcal{L}(P)$, define the subset $M_X \subset M$ by

$$M_X := \{f \in M \mid K(f) = X\}.$$

Since $\mathcal{L}(P)$ is a finite set,

$$M = \bigsqcup_{\substack{X \subset \mathcal{L}(P)\\ X \neq \emptyset}} M_X \tag{26}$$

is a decomposition of M into finitely many semialgebraic sets. Therefore, we obtain

$$e(\pi_{1*}^{-1}(M)) = \sum_{\substack{X \subset \mathcal{L}(P)\\ X \neq \emptyset}} e(\pi_{1*}^{-1}(M_X)).$$

Thus it is enough to show $e(\pi_{1*}^{-1}(M_X)) = 0$ for all $X \subset \mathcal{L}(P)$ as long as $\pi_{1*}^{-1}(M_X) \neq \emptyset$ (note that $\pi_{1*}^{-1}(M_X) = \emptyset$ can occur for a nonempty X e.g. when $\#Q = 1$).

Now we fix $X \subset \mathcal{L}(P)$ such that $\pi_{1*}^{-1}(M_X) \neq \emptyset$. Then we can show that $\pi_{1*}^{-1}(M_X) \longrightarrow M_X$ is a trivial fibration. Indeed, for any $f \in M_X$, the condition imposed on g by $(f,g) \in \mathrm{Hom}^{\leq}(P, Q \times (0,1))$ is

$$(p_1,p_2) \in X \Longrightarrow g(p_1) \leq g(p_2).$$

Hence the fiber $\pi_{1*}^{-1}(f)$ is independent of $f \in M_X$ and isomorphic to

$$F_X := \{(t_p)_{p \in P} \in (0,1)^P \mid (p_1,p_2) \in X \Longrightarrow t_{p_1} \leq t_{p_2}\}, \tag{27}$$

and we have

$$\pi_{1*}^{-1}(M_X) \simeq M_X \times F_X. \tag{28}$$

The fiber F_X is a locally closed polytope defined by the following inequalities.

$$0 < t_p < 1, t_{p_1} \leq t_{p_2} \text{ for } (p_1, p_2) \in X.$$

The closure $\overline{F_X}$ is defined by

$$\overline{F_X} = \{(t_p)_{p \in P} \in [0,1]^P \mid t_{p_1} \leq t_{p_2} \text{ for } (p_1, p_2) \in X\}.$$

Then F_X is equal to the locally closed polytope $(\overline{F_X})_{x_0}$ associated to the point $x_0 = (\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}) \in \partial\overline{F_X}$. Since $X \neq \emptyset$, x_0 is not contained in the interior of $\overline{F_X}$. By Lemma 3.3, $e(F_X) = 0$. Together with (28), we conclude $e(\pi_{1*}^{-1}(M_X)) = 0$. This completes the proof of (14) of Theorem 3.1.

The proof of the other formula (15) is similar to and actually simpler than that of (14) since we do not need Lemma 3.3. Again the first projection $\pi_1 : Q \times (0,1) \longmapsto Q$ induces a map

$$\pi_{1*} : \mathrm{Hom}^{<}(P, Q \times (0,1)) \longrightarrow \mathrm{Hom}^{\leq}(P, Q).$$

We can prove that this map is surjective and each fiber of $\pi_{1*}^{-1}(M_X)$ (now $X = \emptyset$ is allowed) is isomorphic to

$$\mathring{F}_X = \{(t_p)_{p \in P} \in (0,1)^P \mid t_{p_1} < t_{p_2} \text{ for all } (p_1, p_2) \in X\}.$$

This fiber is an open polytope of dimension $\#P$ and hence is isomorphic to $(0,1)^{\#P}$ whose Euler characteristic is $(-1)^{\#P}$. Thus we obtain

$$\begin{aligned} e(\mathrm{Hom}^{<}(P, Q \times (0,1))) &= \sum_{X \subset \mathcal{L}(P)} e(\pi_{1*}^{-1}(M_X)) = \sum_{X \subset \mathcal{L}(P)} e(M_X \times \mathring{F}_X) \\ &= \sum_{X \subset \mathcal{L}(P)} e(M_X) \cdot (-1)^{\#P} = (-1)^{\#P} \cdot e\left(\bigsqcup_{X \subset \mathcal{L}(P)} M_X\right) \\ &= (-1)^{\#P} \cdot e(\mathrm{Hom}^{\leq}(P, Q)). \end{aligned}$$

3.3 Stanley's reciprocity for order polynomials

In this section, we deduce Stanley's reciprocity (2) from Theorem 3.1. The idea is to apply the theorem for semialgebraic totally ordered sets.

Example 3.4. Any semialgebraic set $X \subset \mathbb{R}$ with induced ordering is a semialgebraic totally ordered set. Furthermore, since $\mathbb{R}^n$ is totally ordered by the lexicographic ordering, any semialgebraic set $X \subset \mathbb{R}^n$ admits the structure of a semialgebraic totally ordered set.

The Euler characteristic of $\mathrm{Hom}^{\leq}(P,T)$, with T a semialgebraic totally ordered set, can be computed by using the order polynomial $\mathcal{O}^{\leq(<)}(P,t)$.

Theorem 3.5. *Let P be a finite poset and T be a semialgebraic totally ordered set. Then*

$$e(\mathrm{Hom}^{\leq}(P,T)) = \mathcal{O}^{\leq}(P, e(T)), \tag{29}$$
$$e(\mathrm{Hom}^{<}(P,T)) = \mathcal{O}^{<}(P, e(T)). \tag{30}$$

Before proving Theorem 3.5, we need several lemmas on the Euler characteristics of configuration spaces.

Definition 3.6. Let X be a semialgebraic set. The ordered configuration space of n-points on X, denoted by $C_n(X)$, is defined by

$$C_n(X) = \{(x_1,\ldots,x_n) \in X^n \mid x_i \neq x_j \text{ if } i \neq j\}.$$

Lemma 3.7. $e(C_n(X)) = e(X)\cdot(e(X)-1)\cdots(e(X)-n+1)$.

Proof. It is proved by induction. When $n=1$, it is obvious from $C_1(X)=X$. Suppose $n>1$. Consider the projection

$$\pi : C_n(X) \longrightarrow C_{n-1}(X), (x_1,\ldots,x_n) \longmapsto (x_1,\ldots,x_{n-1}).$$

Then the fiber of π at the point $(x_1,\ldots,x_{n-1}) \in C_{n-1}(X)$ is

$$X \smallsetminus \{x_1,\ldots,x_{n-1}\},$$

which has the Euler characteristic

$$e(X \smallsetminus \{x_1,\ldots,x_{n-1}\}) = e(X) - (n-1).$$

Therefore, from the inductive assumption, we have

$$\begin{aligned} e(C_n(X)) &= e(C_{n-1}(X))\cdot(e(X)-n+1) \\ &= e(X)\cdot(e(X)-1)\cdots(e(X)-n+1). \end{aligned}$$

□

Remark 3.8. We will give a stronger result later (Theorem 4.2 and Corollary 4.3).

Lemma 3.9. *Let T be a semialgebraic totally ordered set. Then*

$$e(\mathrm{Hom}^{<}([n],T)) = \frac{e(T)\cdot(e(T)-1)\cdots(e(T)-n+1)}{n!}. \tag{31}$$

Proof. The set

$$\mathrm{Hom}^{<}([n], T) = \{(x_1, \dots, x_n) \in T^n \mid x_1 < \cdots < x_n\}.$$

This is obviously a subset of the configuration space $C_n(T)$. Moreover, using the natural action of the symmetric group $\mathfrak{S}_n$ on $C_n(T)$ and the fact that T is totally ordered, we have

$$C_n(T) = \bigsqcup_{\sigma \in \mathfrak{S}_n} \sigma(\mathrm{Hom}^{<}([n], T)).$$

Since the group action preserves the Euler characteristic, we obtain the following.

$$e(C_n(T)) = n! \cdot e(\mathrm{Hom}^{<}([n], T)).$$

□

Proof of Theorem 3.5. We fix $\varepsilon \in \{\leq, <\}$. Let $f \in \mathrm{Hom}^{\varepsilon}(P, T)$. Since P is a finite poset, the image $f(P) \subset T$ is a finite totally ordered set. Suppose $\# f(P) = k$. Then the map f is decomposed as $f = \beta \circ \alpha$, where

$$P \stackrel{\alpha}{\longrightarrow} [k] \stackrel{\beta}{\longrightarrow} T$$

$\alpha : P \longrightarrow [k]$ is surjective, while $\beta : [k] \longrightarrow T$ is injective. Hence β can be considered as an element of $\mathrm{Hom}^{<}([k], T)$, and we have the following decomposition,

$$\mathrm{Hom}^{\varepsilon}(P, T) = \bigsqcup_{k \geq 1} \mathrm{Hom}^{\varepsilon, \mathrm{surj}}(P, [k]) \times \mathrm{Hom}^{<}([k], T), \tag{32}$$

where $\mathrm{Hom}^{\varepsilon, \mathrm{surj}}(P, [k])$ is the set of surjective maps in $\mathrm{Hom}^{\varepsilon}(P, [k])$. By putting $T = [n]$ and then extending n to real numbers t, we obtain the expression for the (strict) order polynomial,

$$\mathcal{O}^{\varepsilon}(P, t) = \sum_{k \geq 1} \# \mathrm{Hom}^{\varepsilon, \mathrm{surj}}(P, [k]) \cdot \frac{t(t-1) \cdots (t-k+1)}{k!}, \tag{33}$$

which was already obtained by Stanley [9, Theorem 1]. Using (32), Lemma 3.9 and (33), we have

$$\begin{aligned} e(\mathrm{Hom}^{\varepsilon}(P, T)) &= \sum_{k \geq 1} e(\mathrm{Hom}^{\varepsilon, \mathrm{surj}}(P, [k])) \cdot e(\mathrm{Hom}^{<}([k], T)) \\ &= \sum_{k \geq 1} \# \mathrm{Hom}^{\varepsilon, \mathrm{surj}}(P, [k]) \cdot \frac{e(T)(e(T)-1) \cdots (e(T)-k+1)}{k!} \\ &= \mathcal{O}^{\varepsilon}(P, e(T)). \end{aligned}$$

This completes the proof of Theorem 3.5. □

Corollary 3.10 (Stanley's reciprocity [9]). *Let P be a finite poset and $n \in \mathbb{N}$. Then*

$$\#\operatorname{Hom}^{<}(P,[n]) = (-1)^{\#P} \cdot \mathcal{O}^{\leq}(P,-n). \tag{34}$$

Proof. Since $\operatorname{Hom}^{<}(P,[n])$ is a finite poset, the cardinality is equal to the Euler characteristic: $\#\operatorname{Hom}^{<}(P,[n]) = e(\operatorname{Hom}^{<}(P,[n]))$. We apply the Euler characteristic reciprocity (Theorem 3.1),

$$e(\operatorname{Hom}^{<}(P,[n])) = (-1)^{\#P} \cdot e(\operatorname{Hom}^{\leq}(P,[n] \times (0,1))).$$

Note that $[n] \times (0,1)$ is a semialgebraic totally ordered set (with the lexicographic ordering) with the Euler characteristic $e([n]\times(0,1)) = -n$. Applying Theorem 3.5, we have

$$e(\operatorname{Hom}^{\leq}(P,[n] \times (0,1))) = \mathcal{O}^{\leq}(P,-n),$$

which implies (34). □

4 Chromatic polynomials for finite graphs

4.1 Chromatic polynomials and Euler characteristics

Let $G = (V,E)$ be a finite simple graph with vertex set V and edge (unoriented) set E. The chromatic polynomial is a polynomial $\chi(G,t) \in \mathbb{Z}[t]$ which satisfies

$$\chi(G,n) = \#\{c : V \longrightarrow [n] \mid v_1v_2 \in E \Longrightarrow c(v_1) \neq c(v_2)\},$$

for all $n > 0$. The chromatic polynomial is also characterized by the following properties:

- if $E = \emptyset$ then $\chi(G,t) = t^{\#V}$;
- if $e \in E$, then $\chi(G,t) = \chi(G-e,t) - \chi(G/e,t)$, where $G-e$ and G/e are the deletion and the contraction with respect to the edge e, respectively.

Definition 4.1. For a set X, define the set of vertex coloring with X (or the graph configuration space) by

$$\underline{\chi}(G,X) = \{c : V \longrightarrow X \mid v_1v_2 \in E \Longrightarrow c(v_1) \neq c(v_2)\}. \tag{35}$$

The assignment $X \longmapsto \underline{\chi}(G, X)$ can be considered as a functor ([12]). The space $\underline{\chi}(G, X)$ is also called the graph (generalized) configuration space ([7]).

The chromatic polynomial $\chi(G, t) \in \mathbb{Z}[t]$ satisfies $\chi(G, n) = \#\underline{\chi}(G, [n])$ for all $n \in \mathbb{N}$.

In this section, we investigate the Euler characteristic aspects of the chromatic polynomial for a finite graph.

When X is a semialgebraic set, $\underline{\chi}(G, X)$ is also a semialgebraic set. The following result generalizes [7, Theorem 2], where the result is shown when X is a complex projective space.

Theorem 4.2. *Let $G = (V, E)$ be a finite graph and X be a semialgebraic set. Then*

$$e(\underline{\chi}(G, X)) = \chi(G, e(X)). \tag{36}$$

Proof. This result is proved by induction on $\#E$. When $E = \emptyset$, $e(\underline{\chi}(G, X)) = e(X^{\#V}) = e(X)^{\#V} = \chi(G, e(X))$. Suppose $e \in E$. Then we can prove

$$\underline{\chi}(G - e, X) \simeq \underline{\chi}(G, X) \sqcup \underline{\chi}(G/e, X).$$

Using the additivity of the Euler characteristic and the recursive relation for the chromatic polynomial, we obtain (36). □

Note that for the complete graph $G = K_n$, $\underline{\chi}(K_n, X)$ is identical to the configuration space $C_n(X)$ of n-points. Applying Theorem 4.2 to the complete graph K_n (which has the chromatic polynomial $\chi(K_n, t) = t(t - 1) \cdots (t - n + 1)$), we have the following.

Corollary 4.3. $e(C_n(X)) = e(X)(e(X) - 1) \cdots (e(X) - n + 1)$.

4.2 Acyclic orientations

To formulate the reciprocity for chromatic polynomials, we recall the notion of acyclic orientations on a graph G.

Let $G = (V, E)$ be a finite simple graph. The set of edges E can be considered as a subset of

$$(V \times V \smallsetminus \Delta)/\mathfrak{S}_2,$$

where $\Delta = \{(v, v) \mid v \in V\}$ is the diagonal subset and $\mathfrak{S}_2$ acts on $V \times V$ by transposition.

There is a natural projection

$$\pi : V \times V \smallsetminus \Delta \longrightarrow (V \times V \smallsetminus \Delta)/\mathfrak{S}_2.$$

An edge orientation on G is a subset $\widetilde{E} \subset V \times V \setminus \Delta$ such that $\pi|_{\widetilde{E}} : \widetilde{E} \xrightarrow{\simeq} E$ is a bijection. An orientation $\widetilde{E}$ is said to contain an oriented cycle, if there exists a cyclic sequence $(v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n), (v_n, v_1) \in \widetilde{E}$ for some $n > 2$. The orientation $\widetilde{E}$ is called *acyclic* if it does not contain oriented cycles.

Definition 4.4. Let $G = (V, E)$ be a finite graph. Fix an acyclic orientation $\widetilde{E} \subset V \times V \setminus \Delta$. Let T be a totally ordered set.

(1) A map $c : V \longrightarrow T$ is said to be *compatible* with $\widetilde{E}$ if

$$(v, v') \in \widetilde{E} \Longrightarrow c(v) \leq c(v').$$

(2) A map $c : V \longrightarrow T$ is said to be *strictly compatible* with $\widetilde{E}$ if

$$(v, v') \in \widetilde{E} \Longrightarrow c(v) < c(v').$$

We denote the sets of all pairs of an acyclic orientation with a compatible map, and with a strictly compatible map, by

$$\mathcal{AOC}^{\leq}(G, T) := \left\{ (\widetilde{E}, c) \;\middle|\; \begin{array}{l} \widetilde{E} \text{ is an acyclic orientation, and } c : V \to T \\ \text{is a map compatible with } \widetilde{E} \end{array} \right\},$$

and

$$\mathcal{AOC}^{<}(G, T) := \left\{ (\widetilde{E}, c) \;\middle|\; \begin{array}{l} \widetilde{E} \text{ is an acyclic orientation, and } c : V \to T \\ \text{is a map strictly compatible with } \widetilde{E} \end{array} \right\},$$

respectively. If T is a semialgebraic totally ordered set, then these spaces possess the structures of semialgebraic sets. In the next section, we will see a reciprocity between these two spaces from which Stanley's reciprocity for chromatic polynomials is deduced.

4.3 Euler characteristic reciprocity for chromatic polynomials

We formulate a reciprocity for chromatic polynomials in terms of Euler characteristics.

Theorem 4.5. *Let $G = (V, E)$ be a finite simple graph and T be a semialgebraic totally ordered set. Then*

$$e(\mathcal{AOC}^{\leq}(G, T)) = (-1)^{\#V} \cdot e(\mathcal{AOC}^{<}(G, T \times (0, 1))), \tag{37}$$

$$e(\mathcal{AOC}^{<}(G, T)) = (-1)^{\#V} \cdot e(\mathcal{AOC}^{\leq}(G, T \times (0, 1))). \tag{38}$$

To prove Theorem 4.5, we give alternative descriptions of $\mathcal{AOC}^{\leq(<)}(G,T)$ in terms of poset homomorphisms and graph configuration spaces. Let $\widetilde{E}$ be an acyclic orientation of $G=(V,E)$. Then $\widetilde{E}$ determines an ordering on V, called the transitive closure of $\widetilde{E}$, defined by

$$v < v' \Longleftrightarrow \exists v_0,\dots,v_n \in V \text{ s.t. } \begin{cases} v=v_0, v'=v_n, \text{ and} \\ (v_{i-1},v_i)\in\widetilde{E} \text{ for } 1\leq i\leq n. \end{cases}$$

This ordering defines a poset which we denote by $P(V,\widetilde{E})$.

A map $c: V \longrightarrow T$ is compatible with $\widetilde{E}$ if and only if c is an increasing map from $P(V,\widetilde{E})$ to T. Hence the set of maps compatible with $\widetilde{E}$ is identified with $\mathrm{Hom}^{\leq}(P(V,\widetilde{E}),T)$. We have the following decomposition.

$$\mathcal{AOC}^{\leq}(G,T) \simeq \bigsqcup_{\widetilde{E}\text{: acyclic ori.}} \mathrm{Hom}^{\leq}(P(V,\widetilde{E}),T). \tag{39}$$

Similarly, $\mathcal{AOC}^{<}(G,T)$ is decomposed as follows.

$$\mathcal{AOC}^{<}(G,T) \simeq \bigsqcup_{\widetilde{E}\text{: acyclic ori.}} \mathrm{Hom}^{<}(P(V,\widetilde{E}),T). \tag{40}$$

Proof of Theorem 4.5. We prove (37). Using the above decompositions (39) and (40) together with Theorem 3.1, we obtain

$$\begin{aligned} e(\mathcal{AOC}^{\leq}(G,T)) &= e\left(\bigsqcup_{\widetilde{E}\text{: acyclic ori.}} \mathrm{Hom}^{\leq}(P(V,\widetilde{E}),T)\right) \\ &= \sum_{\widetilde{E}\text{: acyclic ori.}} e\left(\mathrm{Hom}^{\leq}(P(V,\widetilde{E}),T)\right) \\ &= (-1)^{\#V}\cdot \sum_{\widetilde{E}\text{: acyclic ori.}} e\left(\mathrm{Hom}^{<}(P(V,\widetilde{E}),T\times(0,1))\right) \\ &= (-1)^{\#V}\cdot e\left(\bigsqcup_{\widetilde{E}\text{: acyclic ori.}} \mathrm{Hom}^{<}(P(V,\widetilde{E}),T\times(0,1))\right) \\ &= (-1)^{\#V}\cdot e(\mathcal{AOC}^{<}(G,T\times(0,1))). \end{aligned}$$

This completes the proof. The second formula (38) is proved similarly. ☐

To deduce Stanley's reciprocity on chromatic polynomials ([11]), we need the following.

Proposition 4.6. Let $G = (V, E)$ be a finite simple graph and T be a semialgebraic totally ordered set. Then we have an isomorphism of semialgebraic sets:

$$\mathcal{AOC}^{<}(G, T) \simeq \underline{\chi}(G, T).$$

Proof. The map from $\underline{\chi}(G, T)$ to $\mathcal{AOC}^{<}(G, T)$ is constructed as follows. Let $c : V \longrightarrow T$ be an element of $\underline{\chi}(G, T)$. Let $vv' \in E$ be an edge. Define $\widetilde{E}$ by

$$\widetilde{E} = \{(v, v') \mid vv' \in E \text{ and } c(v) < c(v') \text{ in } T\}.$$

Then $\widetilde{E}$ is the unique acyclic orientation of G with which c is strictly compatible. The assignment $c \longmapsto (\widetilde{E}, c)$ determines a map $\underline{\chi}(G, T) \longrightarrow \mathcal{AOC}^{<}(G, T)$. It is easy to see that $(\widetilde{E}, c) \longmapsto c$ gives the converse. Hence we have the isomorphism $\mathcal{AOC}^{<}(G, T) \simeq \underline{\chi}(G, T)$. □

Theorem 4.2 and Proposition 4.6 imply the following.

Corollary 4.7. *Let $G = (V, E)$ be a finite simple graph and T be a semialgebraic totally ordered set. Then*

$$e(\mathcal{AOC}^{<}(G, T)) = \chi(G, e(T)). \tag{41}$$

Applying Theorem 4.5 and Corollary 4.7, we can compute as follows (note that $T \times (0, 1)$ is also a semialgebraic totally ordered set).

$$\begin{aligned} e(\mathcal{AOC}^{\leq}(G, T)) &= (-1)^{\#V} \cdot e(\mathcal{AOC}^{<}(G, T \times (0, 1))) \\ &= (-1)^{\#V} \cdot \chi(G, e(T \times (0, 1))) \\ &= (-1)^{\#V} \cdot \chi(G, -e(T)). \end{aligned}$$

Putting $T = [n]$, we have the following Stanley's reciprocity.

Corollary 4.8. *Let $G = (V, E)$ be a finite graph and $n \in \mathbb{N}$. Then*

$$\# \mathcal{AOC}^{\leq}(G, [n]) = (-1)^{\#V} \cdot \chi(G, -n).$$

Acknowledgements. T. H. was supported by Grant-in-Aid for Young Scientists (B) 15K17549, JSPS and M. Y. was partially supported by Grant-in-Aid for Scientific Research (C) 25400060, JSPS.

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