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Infinite Horizon Risk-Sensitive Control of Diffusions Without Any Blanket Stability Assumptions

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Abstract

We consider the infinite horizon risk-sensitive problem for nondegenerate diffusions with a compact action space, and controlled through the drift. We only impose a structural assumption on the running cost function, namely near-monotonicity, and show that there always exists a solution to the risk-sensitive Hamilton–Jacobi–Bellman (HJB) equation, and that any minimizer in the Hamiltonian is optimal in the class of stationary Markov controls. Under the additional hypothesis that the data of the diffusion is bounded, and satisfies a condition that limits (even though it still allows) transient behavior, we prove that the solution of the HJB is unique, establish the usual verification result, and show that there exists a stationary Markov control which is optimal in the class of all admissible controls. We also present some new results concerning the multiplicative Poisson equation.

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1. Introduction

Optimal control under a risk-sensitive criterion has been an active area of research for the past 30 years. It has found applications in finance [6, 23, 34], missile guidance [37], cognitive neuroscience [35], and many more. There are many situations which dictate the use of a risk-sensitive penalty. For example, if one considers the *risk parameter* to be small then it approximates the standard mean-variance type cost structure. Another reason that the risk-sensitive criterion is often desirable is because it captures the effects of higher order moments of the running cost in addition to its expectation. To the best of our knowledge, the risk-sensitive criterion was first considered in [27]. We also refer the reader to [39, 40] for an early account of risk-sensitive optimal controls. For discrete state space controlled Markov chains, the risk-sensitive optimal control problem is studied in [12–18, 28, 38]. For optimal control problems where the dynamics are modeled by controlled diffusions, we refer the reader to [3–5, 7–9, 11, 20–22, 29, 32, 33].

In this article we deal with nondegenerate diffusions, controlled through the drift, with the control taking values in a compact metric space (see (1.1) below). The goal is to minimize an infinite horizon average risk-sensitive penalty, where the running cost is assumed to satisfy a *near-monotonicity* hypothesis (Definition 1.1 below). We study the associated Hamilton-Jacobi-Bellman (HJB) equation and characterize the class of optimal stationary Markov controls. In [22] a similar control problem is studied under the

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assumption of asymptotic flatness, and existence of a unique solution to the HJB is established. This work is generalized in [33], where the authors impose some structural assumptions on the drift and cost (e.g., the cost necessarily grows to infinity, the action set is a Euclidean space, etc). Risk-sensitive control problems with periodic data are studied in [32]. Risk-sensitive control for a general class of controlled diffusions is considered in [7–9], under the assumption that all stationary Markov controls are stable. However, the studies in [7–9] neither establish uniqueness of the solution to the HJB, nor do they fully characterize the optimal stationary Markov controls. One of our main contributions in this article is the development of a basic theory that parallels existing results for optimal ergodic control problems. To this end, we remove the stability hypothesis on the drift, and replace it by a much weaker hypothesis (see Assumption 1.1). Under this hypothesis and the near-monotone structure of the running cost, we show that any optimal Markov control is necessarily stable.

The dynamics are modeled by a controlled diffusion process $X = \{X_t,\ t \ge 0\}$ which takes values in the d-dimensional Euclidean space $\mathbb{R}^d$, and is governed by the Itô stochastic differential equation

$$\mathrm{d}X_t \;=\; b(X_t, U_t)\,\mathrm{d}t + \upsigma(X_t)\,\mathrm{d}W_t\,. \tag{1.1}$$

All random processes in (1.1) live in a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. The process W is a d-dimensional standard Wiener process independent of the initial condition X_0. The control process U takes values in a compact, metrizable set $\mathbb{U}$, and $U_t(\omega)$ is jointly measurable in $(t, \omega) \in [0, \infty) \times \Omega$. The set $\mathfrak{U}$ of *admissible controls* consists of the control processes U that are *non-anticipative*: for $s < t$, $W_t - W_s$ is independent of

$$\mathfrak{F}_s \;:=\; \text{the completion of } \sigma\{X_0, U_r, W_r,\ r \le s\} \text{ relative to } (\mathfrak{F}, \mathbb{P})\,.$$

We impose the standard assumptions on the drift b and the diffusion matrix $\upsigma$ to guarantee existence and uniqueness of solutions. For more details on the model see Section 1.1.

Let $c\colon \mathbb{R}^d \times \mathbb{U} \to [1, \infty)$ be continuous, and locally Lipschitz in its first argument uniformly with respect to the second. For $U \in \mathfrak{U}$ we define the risk-sensitive penalty by

$$\mathcal{R}(U) = \mathcal{R}(U; c) \;:=\; \limsup_{T\to\infty}\, \frac{1}{T}\, \log \mathbb{E}_x^U\left[\mathrm{e}^{\int_0^T c(X_t, U_t)\,\mathrm{d}t}\right],$$

and the risk-sensitive optimal values by

$$\lambda^* \;:=\; \inf_{U \in \mathfrak{U}}\, \mathcal{R}(U)\,, \qquad \lambda^*_{\mathrm{m}} \;:=\; \inf_{U \in \mathfrak{U}_{\mathrm{SM}}}\, \mathcal{R}(U)\,,$$

where $\mathfrak{U}_{\mathrm{SM}}$ is the class of stationary Markov controls.

Unless λ^* is finite, the optimal control problem, is of course ill-posed. For nonlinear models as in the current paper, standard Foster-Lyapunov conditions are usually imposed to guarantee that $\lambda^* < \infty$. However, the objective of this paper is different. Rather, we impose a structural assumption on the running cost function c, and investigate whether this is sufficient for characterization of optimality via the risk-sensitive HJB equation. We need the following definition.

Definition 1.1 (near-monotone). A Borel measurable $f\colon \mathcal{X} \to \mathbb{R}$, where $\mathcal{X}$ is a locally compact topological space, is said to be *near-monotone relative to* $\lambda \in \mathbb{R}$, if a sublevel set $\bigl\{x \in \mathcal{X} \times \mathbb{U} : f(x) \le \gamma\bigr\}$ for some $\gamma > \lambda$, is nonempty and is contained in some compact subset of $\mathcal{X}$. We also say that f is *norm-like* (or *inf-compact*) if it is near-monotone relative to all $\lambda \in \mathbb{R}$.

Note that the notion of near-monotonicity in the literature is often stricter—a function f is sometimes called near-monotone if it is near-monotone relative to all $\lambda < \|f\|_\infty$ [2].

The main results of the paper are summarized in Propositions 1.1–1.2, and Theorem 1.3 in Section 1.3. Proposition 1.1 assumes that c is near-monotone with respect to λ^*, and imposes a mild condition that limits the transient behavior of the controlled process. Existence and uniqueness of a solution to the HJB equation, a characterization of the optimal stationary Markov controls, and a stochastic representation of the solution to the HJB are obtained. If the running cost is near-monotone relative to λ^*_{m}, then we show in Proposition 1.2 that there exists a pair $(V^*, \lambda^*_{\mathrm{m}}) \in \mathcal{C}^2(\mathbb{R}^d) \times \mathbb{R}$ solving the HJB equation and any measurable selector of the HJB is a stable, optimal Markov control. Under the same near-monotonicity hypothesis, together with the assumption that c is norm-like, the risk-sensitive problem for denumerable Markov decision processes is treated in [12], where a dynamic programming inequality is established.

Another interesting result proved in this article concerns uncontrolled diffusions and is stated in Theorem 1.3 below. We show that if the process is recurrent and the running-cost function f near-monotone relative to $\Lambda(f)$ (the risk-sensitive penalty with running cost f), then the process is stable and the multiplicative Poisson equation has a unique solution. These results are the diffusion counterpart of the results obtained in [2]. Let us also remark that unlike [2] our results do not assume any (geometric) Lyapunov type stability on the dynamics.

The notation used in the paper is summarized in Section 1.2. Section 2 contains various results on the multiplicative Poisson equation, which lead to the proof of Theorem 1.3. Section 3 is devoted to the proofs of Propositions 1.1–1.2.

1.1. The model

The following assumptions on the diffusion (1.1) are in effect throughout the paper unless otherwise mentioned.

(A1) *Local Lipschitz continuity:* The functions

$$b \;=\; \bigl[b^1,\dotsc,b^d\bigr]^{\mathsf{T}} : \mathbb{R}^d\times\mathbb{U}\to\mathbb{R}^d\,, \quad \text{and} \quad \upsigma \;=\; \bigl[\upsigma^{ij}\bigr] : \mathbb{R}^d\to\mathbb{R}^{d\times d}$$

are locally Lipschitz in x with a Lipschitz constant $C_R > 0$ depending on $R > 0$. In other words, for all $x, y \in B_R$ and $u \in \mathbb{U}$, we have

$$|b(x,u) - b(y,u)| + \|\upsigma(x) - \upsigma(y)\| \;\le\; C_R\,|x-y|\,.$$

We also assume that b is continuous in (x, u).

(A2) *Affine growth condition:* b and $\upsigma$ satisfy a global growth condition of the form

$$|b(x,u)|^2 + \|\upsigma(x)\|^2 \;\le\; C\bigl(1+|x|^2\bigr) \qquad \forall\,(x,u)\in\mathbb{R}^d\times\mathbb{U}\,,$$

where $\|\upsigma\|^2 \;:=\; \mathrm{trace}\,\bigl(\upsigma\upsigma^{\mathsf{T}}\bigr)$.

(A3) *Nondegeneracy:* For each $R > 0$, it holds that

$$\sum_{i,j=1}^{d} a^{ij}(x)\xi_i\xi_j \;\ge\; C_R^{-1}|\xi|^2 \qquad \forall\,x\in B_R\,,$$

and for all $\xi = (\xi_1, \dotsc, \xi_d)^{\mathsf{T}} \in \mathbb{R}^d$, where $a := \frac{1}{2}\upsigma\upsigma^{\mathsf{T}}$.

In integral form, (1.1) is written as

$$X_t \;=\; X_0 + \int_0^t b(X_s,U_s)\,\mathrm{d}s + \int_0^t \upsigma(X_s)\,\mathrm{d}W_s\,. \tag{1.2}$$

The third term on the right hand side of (1.2) is an Itô stochastic integral. We say that a process $X = \{X_t(\omega)\}$ is a solution of (1.1), if it is $\mathfrak{F}_t$-adapted, continuous in t, defined for all $\omega \in \Omega$ and $t \in [0,\infty)$, and satisfies (1.2) for all $t \in [0,\infty)$ a.s. It is well known that under (A1)–(A3), for any admissible control there exists a unique solution of (1.1) [1, Theorem 2.2.4]. We define the family of operators $\mathcal{L}^u : \mathcal{C}^2(\mathbb{R}^d) \mapsto \mathcal{C}(\mathbb{R}^d)$, where $u \in \mathbb{U}$ plays the role of a parameter, by

$$\mathcal{L}^u f(x) \;=\; a^{ij}(x)\,\partial_{ij} f(x) + b^i(x,u)\,\partial_i f(x)\,, \quad u\in\mathbb{U}\,.$$

We refer to $\mathcal{L}^u$ as the *controlled extended generator* of the diffusion.

Let $\mathfrak{U}_{\rm SM}$ denote the set of stationary Markov controls. It is well known that under $v \in \mathfrak{U}_{\rm SM}$ (1.1) has a unique strong solution [26]. Moreover, under $v \in \mathfrak{U}_{\rm SM}$, the process X is strong Markov, and we denote its transition function by $P_v^t(x, \cdot)$. It also follows from the work in [10] that under $v \in \mathfrak{U}_{\rm SM}$, the transition probabilities of X have densities which are locally Hölder continuous. Thus $\mathcal{L}^v$ defined by

$$\mathcal{L}^v f(x) \;=\; a^{ij}(x)\,\partial_{ij} f(x) + b^i\bigl(x, v(x)\bigr)\,\partial_i f(x)\,, \quad v \in \mathfrak{U}_{\rm SM}\,,$$

for $f \in \mathcal{C}^2(\mathbb{R}^d)$, is the generator of a strongly-continuous semigroup on $\mathcal{C}_b(\mathbb{R}^d)$, which is strong Feller. We let $\mathbb{P}_x^v$ denote the probability measure and $\mathbb{E}_x^v$ the expectation operator on the canonical space of the process under the control $v \in \mathfrak{U}_{\rm SM}$, conditioned on the process X starting from $x \in \mathbb{R}^d$ at $t = 0$. We denote by $\mathfrak{U}_{\rm SSM}$ the subset of $\mathfrak{U}_{\rm SM}$ that consists of *stable controls*, i.e., under which the controlled process is positive recurrent, and by μ_v the invariant probability measure of the process under the control $v \in \mathfrak{U}_{\rm SSM}$.

In the next section, we summarize the notation used in the paper.

1.2. Notation

The standard Euclidean norm in $\mathbb{R}^d$ is denoted by $|\cdot|$, and $\langle\cdot\,,\cdot\rangle$ denotes the inner product. The set of nonnegative real numbers is denoted by $\mathbb{R}_+$, $\mathbb{N}$ stands for the set of natural numbers, and $\mathbb{1}$ denotes the indicator function. Given two real numbers a and b, the minimum (maximum) is denoted by $a \wedge b$ ($a \vee b$), respectively. The closure, boundary, and the complement of a set $A \subset \mathbb{R}^d$ are denoted by $\bar{A}$, ∂A, and A^c, respectively. We denote by $\uptau(A)$ the *first exit time* of the process $\{X_t\}$ from the set $A \subset \mathbb{R}^d$, defined by

$$\uptau(A) \;:=\; \inf\,\{t > 0\,:\, X_t \notin A\}\,.$$

The open ball of radius r in $\mathbb{R}^d$, centered at the origin, is denoted by B_r, and we let $\uptau_r := \uptau(B_r)$, and $\breve{\uptau}_r := \uptau(B_r^c)$.

The term *domain* in $\mathbb{R}^d$ refers to a nonempty, connected open subset of the Euclidean space $\mathbb{R}^d$. For a domain $D \subset \mathbb{R}^d$, the space $\mathcal{C}^k(D)$ ($\mathcal{C}^\infty(D)$) refers to the class of all real-valued functions on D whose partial derivatives up to order k (of any order) exist and are continuous, and $\mathcal{C}_b(D)$ denotes the set of all bounded continuous real-valued functions on D. By a slight abuse of notation, whenever the whole space $\mathbb{R}^d$ is concerned, we write $f \in \mathcal{C}^k(\mathbb{R}^d)$ whenever $f \in \mathcal{C}^k(D)$ for all bounded domains $D \subset \mathbb{R}^d$. The space $L^p(D)$, $p \in [1, \infty)$, stands for the Banach space of (equivalence classes of) measurable functions f satisfying $\int_D |f(x)|^p\,\mathrm{d}x < \infty$, and $L^\infty(D)$ is the Banach space of functions that are essentially bounded in D. The standard Sobolev space of functions on D whose generalized derivatives up to order k are in $L^p(D)$, equipped with its natural norm, is denoted by $\mathscr{W}^{k,p}(D)$, $k \geq 0$, $p \geq 1$.

In general, if $\mathcal{X}$ is a space of real-valued functions on Q, $\mathcal{X}_{\rm loc}$ consists of all functions f such that $f\varphi \in \mathcal{X}$ for every $\varphi \in \mathcal{C}_c^\infty(Q)$, the space of smooth functions on Q with compact support. In this manner we obtain for example the space $\mathscr{W}^{2,p}_{\rm loc}(Q)$.

For a continuous function $g\colon \mathbb{R}^d \to [1, \infty)$ we let L_g^∞ (or $\mathcal{O}(g)$) denote the space of Borel measurable functions $f\colon \mathbb{R}^d \to \mathbb{R}$ satisfying $\operatorname{ess\,sup}_{x \in \mathbb{R}^d} \frac{|f(x)|}{g(x)} < \infty$, and by $\mathfrak{o}(g)$ the subspace of functions $f \in L_g^\infty$ such that $\limsup_{R \to \infty}\, \operatorname{ess\,sup}_{x \in B_R^c} \frac{|f(x)|}{g(x)} = 0$. We also let $\mathcal{C}_g(\mathbb{R}^d)$ denote the Banach space of continuous functions under the norm

$$\|f\|_g \;:=\; \sup_{x \in \mathbb{R}^d} \frac{|f(x)|}{g(x)}\,.$$

We adopt the notation $\partial_i := \frac{\partial}{\partial x_i}$ and $\partial_{ij} := \frac{\partial^2}{\partial x_i \partial x_j}$ for $i, j \in \mathbb{N}$. We often use the standard summation rule that repeated subscripts and superscripts are summed from 1 through d. For example,

$$a^{ij}\partial_{ij}\varphi + b^i\partial_i\varphi \;:=\; \sum_{i,j=1}^d a^{ij}\frac{\partial^2\varphi}{\partial x_i \partial x_j} + \sum_{i=1}^d b^i\frac{\partial\varphi}{\partial x_i}\,.$$

1.3. Main results

Assumption 1.1. The data b and σ in (1.1) are bounded, and the constant C_R in (A1) and (A3) does not depend on R. Moreover,

$$\max_{u\in\mathbb{U}} \frac{\langle b(x,u),\, x\rangle^{+}}{|x|} \xrightarrow[|x|\to\infty]{} 0\,. \tag{1.3}$$

Proposition 1.1. *Let Assumption-1.1 hold, and suppose that c is near-monotone relative to λ^*. Then the HJB equation*

$$\min_{u\in\mathbb{U}} \bigl[\mathcal{L}^u V(x) + c(x,u)\,V(x)\bigr] \;=\; \lambda\, V(x) \qquad \forall\, x\in\mathbb{R}^d$$

has a solution $(V^*,\lambda^*)\in\mathcal{C}^2(\mathbb{R}^d)\times\mathbb{R}$ satisfying $\inf_{\mathbb{R}^d} V^*>0$.

Moreover, the following hold:

(*i*) *The solution (V^*,λ^*) is unique in the class*

$$\mathfrak{V}_\circ \;:=\; \bigl\{(V,\lambda)\in\mathcal{C}^2(\mathbb{R}^d)\times\mathbb{R}\colon\; V(0)=1\,,\; \inf_{\mathbb{R}^d} V>0\,,\; \lambda\le\lambda^*\bigr\}\,,$$

and provided that c is bounded, it is also unique in the larger class

$$\mathfrak{V} \;:=\; \bigl\{(V,\lambda)\in\mathcal{C}^2(\mathbb{R}^d)\times\mathbb{R}\colon\; V(0)=1\,,\; V>0\,,\; \lambda\le\lambda^*\bigr\}\,.$$

(*ii*) *Any $v^*\in\mathfrak{U}_{\mathrm{SM}}$ that satisfies*

$$\mathcal{L}^{v^*} V^*(x) + c\bigl(x,v^*(x)\bigr)\,V^*(x) \;=\; \min_{u\in\mathbb{U}} \bigl[\mathcal{L}^u V^*(x) + c(x,u)\,V^*(x)\bigr] \quad \text{a.e. } x\in\mathbb{R}^d \tag{1.4}$$

is stable, and is optimal, i.e., $\mathcal{R}(v^*)=\lambda^*$.

(*iii*) *A control $v^*\in\mathfrak{U}_{\mathrm{SM}}$ is optimal only if it satisfies* (1.4).

(*iv*) *For any $v^*\in\mathfrak{U}_{\mathrm{SM}}$ satisfying* (1.4), *we have*

$$V^*(x) \;=\; \mu_{v^*}(V^*)\;\liminf_{T\to\infty}\; \mathbb{E}_x^{v^*}\Bigl[\mathrm{e}^{\int_0^T [c(X_t,v^*(X_t))-\lambda^*]\,\mathrm{d}t}\Bigr]\,,$$

where, as defined earlier, μ_{v^*} is the invariant probability measure of (1.1) *under the control v^*, and $\mu_{v^*}(V^*) := \int_{\mathbb{R}^d} V^*\,\mathrm{d}\mu_{v^*}$.*

Proof. This is contained in Theorems 3.3–3.4 in Section 3. □

Remark 1.1. The hypothesis in (1.3) of Assumption 1.1 may be replaced by the following. There exists a $\mathcal{C}^2$ function $\mathcal{V}_{\mathrm{o}}$, satisfying $\liminf_{|x|\to\infty} \frac{\mathcal{V}_{\mathrm{o}}(x)}{1+|x|^2}>0$, such that

$$\frac{\bigl[\mathcal{L}^u \mathcal{V}_{\mathrm{o}}(x)\bigr]^{+}}{\sqrt{\mathcal{V}_{\mathrm{o}}(x)}} \xrightarrow[|x|\to\infty]{} 0 \qquad \forall\, u\in\mathbb{U}\,.$$

It is clear from the proof that the result of Lemma 3.2 in Section 3 holds under this assumption. It is also evident that (1.3) may be replaced by the more general hypothesis that $\mathbb{E}_x^U\bigl[|X_t|\bigr]\in\mathfrak{o}(t)$ under any $U\in\mathfrak{U}$, which is the conclusion of Lemma 3.2 on which the proof of Theorem 3.3 is based. Note that when the data b and σ is bounded, it is always the case that $\mathbb{E}_x^U\bigl[|X_t|\bigr]\in\mathcal{O}(t)$.

Proposition 1.2. *Suppose that c is near-monotone relative to λ^*_{m}. Then the HJB equation*

$$\min_{u\in\mathbb{U}} \bigl[\mathcal{L}^u V(x) + c(x,u)\,V(x)\bigr] \;=\; \lambda\, V(x) \qquad \forall\, x\in\mathbb{R}^d$$

has a solution $(V^*,\lambda^*_{\mathrm{m}})\in\mathcal{C}^2(\mathbb{R}^d)\times\mathbb{R}$. Also, any $v^*\in\mathfrak{U}_{\mathrm{SM}}$ that satisfies (1.4) *is stable, and is optimal in the class $\mathfrak{U}_{\mathrm{SM}}$, i.e., $\mathcal{R}(v^*)=\lambda^*_{\mathrm{m}}$.*

The proof of Proposition 1.2 is in Section 3.

Remark 1.2. The linear growth condition in (A2) guarantees that trajectories do not suffer an explosion in finite time. This assumption is a quite standard, but may be restrictive for some applications. As far as the results of Proposition 1.2 are concerned, it may be replaced by the weaker condition

$$2\langle x, b(x,u)\rangle + \|\sigma(x)\|^2 \;\le\; C_1\bigl(1+|x|^2\bigr)\,,\quad \forall\,(x,u)\in\mathbb{R}^d\times\mathbb{U}\,.$$

The proofs of these results depend heavily on properties of the multiplicative Poisson equation (MPE). To summarize these we consider an uncontrolled diffusion

$$\mathrm{d}X_t \;=\; b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t\,, \tag{1.5}$$

where σ satisfies (A1)–(A3), while b is measurable and has affine growth. We let $\mathbb{E}_x$ denote the expectation operator induced by the strong Markov process with $X_0 = x$, governed by (1.5), and $\mathcal{L} := a^{ij}(x)\,\partial_{ij} + b^i(x)\,\partial_i$, with $a := \frac{1}{2}\sigma\sigma^{\mathsf{T}}$. Let $f:\mathbb{R}^d\to\mathbb{R}_+$ be measurable and locally bounded, and define

$$\Lambda(f) \;:=\; \limsup_{T\to\infty}\,\frac{1}{T}\,\log\,\mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^T f(X_t)\,\mathrm{d}t}\Bigr] \qquad \forall\,x\in\mathbb{R}^d\,.$$

We assume $\Lambda(f)<\infty$. We say that $(\Psi,\lambda)\in\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)\times\mathbb{R}$, $p>d$, $\Psi>0$, is a solution of the multiplicative Poisson equation (MPE) if it satisfies

$$\mathcal{L}\Psi(x) + f(x)\,\Psi(x) \;=\; \lambda\,\Psi(x) \qquad \text{a.e. } x\in\mathbb{R}^d\,. \tag{1.6}$$

We refer to λ as the *eigenvalue* of the MPE.

With respect to Theorem 1.3 stated below, note that we do not assume that the running cost is norm-like, or even near-monotone in the sense of [2, p. 126]. Nor do we assume that $\Lambda(\alpha f)<\infty$ for some $\alpha>1$, as is common in the literature. This should be compared with [2, Theorem 1.2], and [12, Theorem 2.2] for irreducible Markov chains, as well as the more general results in [30, 31]. Therefore, in the case of nondegenerate diffusions, Theorem 1.3 is an improvement of existing results on the MPE.

Theorem 1.3. *Suppose that the diffusion in* (1.5) *is recurrent, and that* $f\colon\mathbb{R}^d\to\mathbb{R}_+$ *is near-monotone relative to* $\Lambda(f)$*. Then, for* $\lambda\in[0,\Lambda(f)]$*, there exists a unique solution* $(\Psi,\lambda)\in\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)\times\mathbb{R}$*,* $p>d$*,* $\Psi>0$*, to the MPE in* (2.4)*, satisfying* $\Psi(0)=1$*, and the following hold.*

(*i*) *The diffusion in* (1.5) *is positive recurrent, and* $\lambda=\Lambda(f)$.

(*ii*) *For some positive constants* C_0 *and* β*, it holds that*

$$\Bigl|\Psi(x)-\mu(\Psi)\,\mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^t[f(X_s)-\Lambda(f)]\,\mathrm{d}s}\Bigr]\Bigr| \;\le\; C_0\,\mathrm{e}^{-\beta t}\,\Psi(x) \qquad \forall\,x\in\mathbb{R}^d\,,\quad \forall\,t>0\,,$$

where μ *is the invariant probability measure of the diffusion in* (1.5).

(*iii*) *The function* Ψ *satisfies*

$$\Psi(x) \;=\; \mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^T[f(X_t)-\Lambda(f)]\,\mathrm{d}t}\,\Psi(X_T)\Bigr] \qquad \forall\,T>0\,.$$

(*iv*) *There exists a bounded open ball* $\mathcal{B}_\circ$*, and* $\xi_\circ<\Lambda(f)$ *such that, with* $\tau_\circ:=\tau(\mathcal{B}_\circ^c)$*, we have*

$$\mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{\tau_\circ}[f(X_t)-\xi_\circ]\,\mathrm{d}t}\Bigr] \;<\; \infty \qquad \forall\,x\in\overline{\mathcal{B}}_\circ^c\,.$$

and

$$\Psi(x) \;=\; \mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{\tau_\circ}[f(X_t)-\Lambda(f)]\,\mathrm{d}t}\,\Psi(X_{\tau_\circ})\Bigr] \qquad \forall\,x\in\overline{\mathcal{B}}_\circ^c\,.$$

Proof. The proof of Theorem 1.3 is contained in Lemmas 2.4 and 2.7 and Corollary 2.8 of Section 2. □

2. Some results on the multiplicative Poisson equation

In this section we establish basic properties of the MPE through lemmas that lead to the proof of Theorem 1.3. Some well known properties of the MPE are summarized in the following lemma.

Lemma 2.1. *Let f be near-monotone relative to λ, and $(\Psi,\lambda)\in\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)\times\mathbb{R}$, $p>d$, $\Psi>0$ on $\mathbb{R}^d$, be a solution to* (1.6)*. Suppose that at least one of* (*a*) *or* (*b*) *are true:*

(*a*) *The diffusion* (1.5) *is recurrent.*

(*b*) $\inf_{\mathbb{R}^d}\,\Psi>0$.

Then the following hold:

(*i*) *The function Ψ is inf-compact. In particular,* $\inf_{\mathbb{R}^d}\,\Psi>0$.

(*ii*) *The diffusion* (1.5) *is geometrically ergodic, i.e., it is positive recurrent with invariant probability measure μ, and there exist positive constants κ and β, such that if $g:\mathbb{R}^d\to\mathbb{R}$ is any locally bounded measurable function satisfying $\|g\|_\Psi<\infty$, it holds that*

$$\bigl|\mathbb{E}_x[g(X_t)]-\mu(g)\bigr| \;\le\; \kappa\mathrm{e}^{-\beta t}\,\|g\|_\Psi\bigl(1+\Psi(x)\bigr) \qquad \forall\,t>0\,. \tag{2.1}$$

(*iii*) *It holds that $\Lambda(f)\le\lambda$.*

Proof. Let $\mathcal{B}$ be a bounded ball and $\delta>0$ a constant, such that $f-\lambda>\delta$ in $\mathcal{B}^c$. Then with $\breve{\uptau}\equiv\uptau(\mathcal{B}^c)$ we have

$$\Psi(x) \;\ge\; \mathbb{E}_x\Bigl[\mathrm{e}^{\delta\breve{\uptau}}\,\Psi(X_{\breve{\uptau}})\,\mathbb{1}\{\breve{\uptau}<\infty\}\Bigr] \qquad \forall\,x\in\overline{\mathcal{B}}^c\,. \tag{2.2}$$

If (a) holds, then since $\inf_{\mathcal{B}}\,\Psi>0$ by the Harnack inequality, part (i) follows. Therefore (a) implies (b), and we continue the proof by assuming (b).

Since $\mathcal{L}\Psi<-\delta\Psi$ on $\mathcal{B}^c$, and $\inf_{\mathbb{R}^d}\,\Psi>0$, it is well known that (2.1) holds (see [19, 24]). This proves part (ii).

By (1.6) and Fatou's lemma we have

$$\begin{aligned}\Psi(x) \;&\ge\; \mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^T[f(X_t)-\lambda]\,\mathrm{d}t}\,\Psi(X_T)\Bigr]\\ &\ge\; \Bigl(\inf_{\mathbb{R}^d}\;\Psi\Bigr)\,\mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^T[f(X_t)-\lambda]\,\mathrm{d}t}\Bigr]\,,\end{aligned}$$

and part (iii) follows by taking log and dividing by T. □

We quote a result from [36] on eigensolutions of the Dirichlet problem.

Lemma 2.2. *For each $n\in\mathbb{N}$, there exists a unique pair $(\widehat{\Psi}_n,\hat{\lambda}_n)\in\bigl(\mathcal{W}^{2,p}(B_n)\cap\mathcal{C}(\bar{B}_n)\bigr)\times\mathbb{R}$, for any $p>d$, satisfying $\widehat{\Psi}_n>0$ on B_n, $\widehat{\Psi}_n=0$ on ∂B_n, and $\widehat{\Psi}_n(0)=1$, which solves*

$$\mathcal{L}\widehat{\Psi}_n(x)+f(x)\,\widehat{\Psi}_n(x) \;=\; \hat{\lambda}_n\,\widehat{\Psi}_n(x) \qquad a.e.\ x\in B_n\,. \tag{2.3}$$

Moreover, $\hat{\lambda}_n\le\hat{\lambda}_{n+1}$ for all $n\in\mathbb{N}$.

We refer to $(\widehat{\Psi}_n,\hat{\lambda}_n)$ as the Dirichlet eigensolution of the MPE on B_n.

Lemma 2.3. *Let $(\widehat{\Psi}_n,\hat{\lambda}_n)$ be as in Lemma 2.2. Then*

(*i*) $\hat{\lambda}_n\le\Lambda(f)$ *for all $n\in\mathbb{N}$.*

(ii) *Any limit point* $(\widehat{\Psi}, \hat{\lambda}) \in \mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d) \times \mathbb{R}$ *of the Dirichlet eigensolutions* $(\widehat{\Psi}_n, \hat{\lambda}_n)$ *in* (2.3) *as* $n \to \infty$ *is a solution of the MPE* (1.6).

(iii) *If* f *is near-monotone relative to* $\Lambda(f)$, *then* $\hat{\lambda}_n \nearrow \Lambda(f)$ *as* $n \to \infty$.

(iv) *It holds that* $\hat{\lambda}_n < \hat{\lambda}_{n+1}$ *for all* $n \in \mathbb{N}$.

Proof. Parts (i) and (ii) are as in [8, Lemma 2.1], while part (iii) follows by part (i) and Lemma 2.1 (iii).

Part (iv) is a straightforward application of the strong maximum principle. Suppose $\hat{\lambda}_n = \hat{\lambda}_{n+1}$. Then we can find constant $\kappa > 0$ such that $\phi := \kappa\hat{\Psi}_{n+1} - \hat{\Psi}_n \geq 0$ in B_n and ϕ attains the value 0 at some point in B_n. By (2.3) we have

$$\mathcal{L}\phi - (f - \hat{\lambda}_n)^{-}\phi \;=\; -(f - \hat{\lambda}_n)^{+}\phi \;\leq\; 0 \quad \text{in } B_n\,.$$

Therefore by [25, Theorem 9.6] we must have $\phi = 0$ on B_n, which is a contradiction to the fact that $\phi > 0$ on ∂B_n. □

For the remaining of this section we assume that (1.5) is recurrent, and $f\colon \mathbb{R}^d \to \mathbb{R}_+$ is near-monotone relative to $\Lambda(f)$. This implies that (1.5) is positive recurrent. We let μ denote the invariant probability measure of the Markov process governed by (1.5). Lemma 2.3 shows that there exists a positive solution $\Psi \in \mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p > d$, to

$$\mathcal{L}\Psi(x) + f(x)\,\Psi(x) \;=\; \Lambda(f)\,\Psi(x) \qquad \text{a.e. } x \in \mathbb{R}^d\,. \tag{2.4}$$

Then, necessarily $\inf_{\mathbb{R}^d} \Psi > 0$ by Lemma 2.1.

Lemma 2.4. *The map*

$$\Psi(x) \;=\; \lim_{T\to\infty}\, \mathbb{E}_x\left[\mathrm{e}^{\int_0^T [f(X_t)-\Lambda(f)]\,\mathrm{d}t}\right]. \tag{2.5}$$

is in $\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p > d$, *and is a positive solution of* (2.4). *It also satisfies*

$$\Psi(x) \;=\; \mathbb{E}_x\left[\mathrm{e}^{\int_0^T [f(X_t)-\Lambda(f)]\,\mathrm{d}t}\,\Psi(X_T)\right] \qquad \forall\, T > 0\,. \tag{2.6}$$

In general, if a function $\Psi \in \mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p > d$, *such that* $\inf_{\mathbb{R}^d} \Psi > 0$, *solves* (2.4) *and satisfies* (2.6), *then for some positive constants* C_0 *and* β, *it holds that*

$$\left|\Psi(x) - \mu(\Psi)\,\mathbb{E}_x\left[\mathrm{e}^{\int_0^t [f(X_s)-\Lambda(f)]\,\mathrm{d}s}\right]\right| \;\leq\; C_0\,\mathrm{e}^{-\beta t}\,\Psi(x) \qquad \forall\, x \in \mathbb{R}^d, \quad \forall\, t > 0\,. \tag{2.7}$$

Proof. We first establish (2.7). Using the martingale property in (2.6) over a $2t$ horizon, and conditioning at $\mathfrak{F}^X_t := \sigma(X_s\,,\ 0 \leq s \leq t)$, we have

$$\begin{aligned}\Psi(x) \;&=\; \mathbb{E}_x\left[\mathrm{e}^{\int_0^{2t} [f(X_s)-\Lambda(f)]\,\mathrm{d}s}\,\Psi(X_{2t})\right]\\ &=\; \mathbb{E}_x\left[\mathrm{e}^{\int_0^{t} [f(X_s)-\Lambda(f)]\,\mathrm{d}s}\,\mathbb{E}_{X_t}[\Psi(X_t)]\right].\end{aligned}$$

Therefore, by (2.1), we obtain

$$\begin{aligned}\left|\Psi(x) - \mu(\Psi)\,\mathbb{E}_x\left[\mathrm{e}^{\int_0^t [f(X_s)-\Lambda(f)]\,\mathrm{d}s}\right]\right| \;&\leq\; \kappa\mathrm{e}^{-\beta t}\,\mathbb{E}_x\left[\mathrm{e}^{\int_0^t [f(X_s)-\Lambda(f)]\,\mathrm{d}s}\big(1 + \Psi(X_t)\big)\right]\\ &\leq\; 2\kappa\Big(\inf_{\mathbb{R}^d}\Psi \wedge 1\Big)^{-1}\mathrm{e}^{-\beta t}\,\Psi(x) \qquad \forall\, t > 0\,,\end{aligned} \tag{2.8}$$

and (2.7) follows.

If f is bounded then any positive solution of (2.4) satisfies (2.6). Indeed, by [1, Lemma 3.7.2], we obtain

$$\mathbb{E}_x\left[\mathrm{e}^{\int_0^{t\wedge\tau_n}[f(X_s)-\Lambda(f)]\,\mathrm{d}s}\,\Psi(X_{t\wedge\tau_n})\,\mathbb{1}\{t\ge\tau_n\}\right] \;\le\; \mathrm{e}^{\|f\|_\infty t}\,\mathbb{E}_x\bigl[\Psi(X_{\tau_n})\,\mathbb{1}\{t\ge\tau_n\}\bigr] \xrightarrow[n\to\infty]{} 0 \qquad \forall\, t>0\,.$$

Thus the claim follows by applying the monotone convergence theorem to

$$\mathbb{E}_x\left[\mathrm{e}^{\int_0^{t}[f(X_s)-\Lambda(f)]\,\mathrm{d}s}\,\Psi(X_t)\,\mathbb{1}\{t<\tau_n\}\right].$$

If f is not bounded, then the truncated function $f\wedge\ell$ is clearly near-monotone relative to $\Lambda(f\wedge\ell)$ for all large enough $\ell\in\mathbb{N}$. Let $(\Psi^{(\ell)},\Lambda(f\wedge\ell))$ be the limit of the Dirichlet eigensolutions corresponding to $f\wedge\ell$. Then $\Psi^{(\ell)}$ satisfies (2.6). Since any limit point of $\bar\Psi$ of $\Psi^{(\ell)}$ as $\ell\to\infty$ satisfies $\mathcal{L}\bar\Psi+f\bar\Psi=\bar\Lambda\bar\Psi$, with $\bar\Lambda=\lim_{\ell\to\infty}\Lambda(f\wedge\ell)\le\Lambda(f)$, it follows by Lemma 2.1 (iii) that $\Lambda(f)\le\bar\Lambda$ and therefore $\bar\Lambda=\Lambda(f)$. It is straightforward to show that for some bounded ball $\mathcal{B}$ and $\bar\ell\in\mathbb{N}$, we have $\mathcal{L}\Psi^{(\ell)}\le-\delta\Psi^{(\ell)}$ on $\mathcal{B}^c$ for all $\ell\ge\bar\ell$. This implies that $\inf_{\ell\ge\bar\ell}\inf_{\mathbb{R}^d}\Psi^{(\ell)}>0$, and also that (2.1) holds for $\Psi=\Psi^{(\ell)}$, for some constants κ and β that do not depend on $\ell\ge\bar\ell$.

We normalize $\Psi^{(\ell)}$ so that $\mu\bigl(\Psi^{(\ell)}\bigr)=1$, and with $C_0:=2\kappa\Bigl(\inf_{\ell\ge\bar\ell}\inf_{\mathbb{R}^d}\Psi^{(\ell)}\wedge 1\Bigr)^{-1}$ we use (2.8) to write

$$\left|\Psi^{(\ell)}(x)-\mathbb{E}_x\left[\mathrm{e}^{\int_0^{T}[(f\wedge\ell)(X_t)-\Lambda(f\wedge\ell)]\,\mathrm{d}t}\right]\right| \;\le\; C_0\mathrm{e}^{-\beta T}\Psi^{(\ell)}(x) \qquad \forall \ell\ge\bar\ell\,. \tag{2.9}$$

Taking limits as $\ell\to\infty$ we obtain

$$\left|\bar\Psi(x)-\mathbb{E}_x\left[\mathrm{e}^{\int_0^{T}[f(X_t)-\Lambda(f)]\,\mathrm{d}t}\right]\right| \;\le\; C_0\mathrm{e}^{-\beta T}\bar\Psi(x)\,. \tag{2.10}$$

This also shows that $\bar\Psi=\lim_{\ell\to\infty}\Psi^{(\ell)}$.

It easily follows by (2.9)–(2.10) that there exist constants $T_0\in\mathbb{R}_+$ and $M=M(T)>0$ such that

$$\Psi^{(\ell)} \;\le\; M(T)\bar\Psi(x) \qquad \forall T>T_0\,,\ \ell\ge\bar\ell\,.$$

Therefore, using the identity

$$\Psi^{(\ell)}(x) \;=\; \mathrm{e}^{[\Lambda(f)-\Lambda(f\wedge\ell)]T}\,\mathbb{E}_x\left[\mathrm{e}^{\int_0^{T}[(f\wedge\ell)(X_t)-\Lambda(f)]\,\mathrm{d}t}\,\Psi^{(\ell)}(X_T)\right],$$

and passing to the limit as $\ell\to\infty$, employing dominated convergence for the integral, it follows that

$$\bar\Psi(x) \;=\; \mathbb{E}_x\left[\mathrm{e}^{\int_0^{T}[f(X_t)-\Lambda(f)]\,\mathrm{d}t}\,\bar\Psi(X_T)\right] \qquad \forall\, T>T_0\,. \tag{2.11}$$

By Fatou's lemma

$$\bar\Psi(x) \;\ge\; \mathbb{E}_x\left[\mathrm{e}^{\int_0^{T}[f(X_t)-\Lambda(f)]\,\mathrm{d}t}\,\bar\Psi(X_T)\right] \qquad \forall\, T>0\,. \tag{2.12}$$

Combining (2.11)–(2.12) we deduce that (2.11) holds for all $T>0$. This completes the proof. □

To simplify the notation define the Nisio semigroup $\mathcal{T}_t$, $t\ge0$, acting on nonnegative measurable functions by

$$\mathcal{T}_t\,g(x) \;:=\; \mathbb{E}_x\left[\mathrm{e}^{\int_0^{t}[f(X_s)-\Lambda(f)]\,\mathrm{d}s}\,g(X_t)\right].$$

Note that if Ψ is a positive solution of (2.4), then $\mathcal{T}_t\colon L^\infty_\Psi\to L^\infty_\Psi$.

Lemma 2.5. *Let Ψ be a positive solution of* (2.4) *satisfying* (2.6), *and let $\mathcal{B}_\circ$ be an open ball centered at the origin such that* $\inf_{\mathcal{B}_\circ^c}\Psi\ge2\mu(\Psi)$. *Then*

$$\varliminf_{t\to\infty}\,\mathcal{T}_t\,\mathbb{1}_{\mathcal{B}_\circ}(x) \;\ge\; \frac{\Psi(x)}{2\mu(\Psi)} \qquad \forall\, x\in\mathbb{R}^d\,. \tag{2.13}$$

Proof. By (2.6) we have

$$\Bigl(\inf_{\mathcal{B}_\circ^c} \Psi\Bigr) \mathcal{T}_t \mathbb{1}_{\mathcal{B}_\circ^c} \;\le\; \mathcal{T}_t\bigl(\Psi \mathbb{1}_{\mathcal{B}_\circ^c}\bigr) \;\le\; \Psi\,,$$

which after rearranging we write as

$$\mathcal{T}_t \mathbb{1}_{\mathcal{B}_\circ} \;\ge\; \mathcal{T}_t \mathbb{1}_{\mathbb{R}^d} - \Bigl(\inf_{\mathcal{B}_\circ^c} \Psi\Bigr)^{-1} \Psi\,.$$

The inequality in (2.13) follows by letting $t \to \infty$ and using Lemma 2.4. □

Lemma 2.6. *There exists a bounded open ball $\mathcal{B}_\circ$ and a constant $\delta > 0$, such that if $\tilde{f} := f - \delta \mathbb{1}_{\mathcal{B}_\circ}$, then $\Lambda(\tilde{f}) < \Lambda(f)$.*

Proof. Let $\delta > 0$ be small enough such that the sublevel set $\{x\colon f(x) \le \Lambda(f) + 2\delta\}$ is bounded, and let $\mathcal{B}$ be a bounded open ball that contains it. Let $\mathcal{G}$ be the class of measurable functions $g\colon \mathbb{R}^d \to [0,\delta]$, and for each $g \in \mathcal{G}$ let $\bigl(\Psi_g, \Lambda(f-g)\bigr) \in \mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d) \times \mathbb{R}$, $p > d$, $\Psi_g > 0$, be the solution to the multiplicative Poisson equation

$$\mathcal{L}\Psi_g(x) + (f-g)(x)\,\Psi_g(x) \;=\; \Lambda(f-g)\,\Psi_g(x) \qquad \text{a.e. } x \in \mathbb{R}^d\,,$$

as defined in Lemma 2.4. Since $\Lambda(f-g) \le \Lambda(f)$, then $\Lambda(f-g) - f + g + \delta < 0$ on $\mathcal{B}^c$, and combining this with Harnack's inequality, we deduce that there exists a finite constant κ_0 such that $(\Lambda(f-g) - f + g + \delta)\Psi_g \le \kappa_0$ for all $g \in \mathcal{G}$. With $\breve{\tau} \equiv \tau(\mathcal{B}^c)$ we have

$$\Psi_g(x) \;\ge\; \mathbb{E}_x\Bigl[\mathrm{e}^{\delta\breve{\tau}}\,\Psi_g(X_{\breve{\tau}})\,\mathbb{1}\{\breve{\tau} < \infty\}\Bigr] \qquad \forall\, x \in \overline{\mathcal{B}}^c\,,\ \forall\, g \in \mathcal{G}\,,$$

from which if follows, again by using Harnack's inequality, that there exists a bounded ball $\mathcal{B}_\circ$ such that $\inf_{\mathcal{B}_\circ^c} \Psi_g \ge \frac{2\kappa_0}{\delta}$ for all $g \in \mathcal{G}$. Since

$$\begin{aligned}\mathcal{L}\Psi_g \;&=\; \bigl(\Lambda(f-g) - f + g\bigr)\,\Psi_g\\ &\le\; \kappa_0 - \delta\,\Psi_g\,,\end{aligned}$$

it follows that $\mu(\Psi_g) \le \frac{\kappa_0}{\delta}$ for all $g \in \mathcal{G}$. Therefore, $\inf_{\mathcal{B}_\circ^c} \Psi_g \ge 2\mu(\Psi_g)$ for all $g \in \mathcal{G}$, and in particular for $g = \delta\mathbb{1}_{\mathcal{B}_\circ}$.

It is clear that $\Lambda(\tilde{f}) \le \Lambda(f)$. Suppose that $\Lambda(f) = \Lambda(\tilde{f})$. Write (2.4) as

$$\mathcal{L}\Psi(x) + \delta\Psi(x)\,\mathbb{1}_{\mathcal{B}_\circ}(x) \;=\; [\Lambda(\tilde{f}) - \tilde{f}(x)]\,\Psi(x)\,.$$

Let

$$\widetilde{\mathcal{T}}_t\, g(x) \;:=\; \mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^t [\tilde{f}(X_s) - \Lambda(\tilde{f})]\,\mathrm{d}s} g(X_t)\Bigr]\,.$$

By Itô's formula, we obtain

$$\Psi(x) \;\ge\; \delta \int_0^T \widetilde{\mathcal{T}}_t\bigl(\Psi\,\mathbb{1}_{\mathcal{B}_\circ}\bigr)(x)\,\mathrm{d}t + \widetilde{\mathcal{T}}_T\,\Psi(x)\,. \tag{2.14}$$

Since $\inf_{\mathcal{B}_\circ} \Psi > 0$, it follows by Lemma 2.5 that $\lim_{t\to\infty} \widetilde{\mathcal{T}}_t\bigl(\Psi\,\mathbb{1}_{\mathcal{B}_\circ}\bigr)(x) > 0$, for all $x \in \mathbb{R}^d$, and therefore the first term of (2.14) diverges as $T \to \infty$. This of course is a contradiction, and therefore, we must have $\Lambda(f) > \Lambda(\tilde{f})$. □

The following lemma plays a crucial role in obtaining the stochastic representations in Proposition 1.1 (iii).

Lemma 2.7. *Let $\mathcal{B}_\circ$ be as in Lemma 2.6, and define $\tau_\circ := \tau(\mathcal{B}_\circ^c)$. Then, the following hold*

(*i*) *There exists $\xi_\circ < \Lambda(f)$ such that*

$$\mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{\tau_\circ} [f(X_t) - \xi_\circ]\,\mathrm{d}t}\Bigr] \;<\; \infty \qquad \forall\, x \in \overline{\mathcal{B}}_\circ^c\,. \tag{2.15}$$

(*ii*) *The Dirichlet eigensolutions* $(\widehat{\Psi}_n, \hat{\lambda}_n)$ *in* (2.3) *have the stochastic representation*

$$\widehat{\Psi}_n(x) \;=\; \mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{\uptau_\circ}[f(X_t)-\hat{\lambda}_n]\,\mathrm{d}t}\,\widehat{\Psi}_n(X_{\uptau_\circ})\,\mathbb{1}\{\uptau_\circ<\uptau_n\}\Bigr] \qquad \forall\,x\in B_n\setminus\overline{\mathcal{B}}_\circ\,,$$

for all large enough $n\in\mathbb{N}$.

(*iii*) *If* Ψ *is any limit point, as* $n\to\infty$, *of the Dirichlet eigensolutions* $(\widehat{\Psi}_n, \hat{\lambda}_n)$ *in* (2.3), *then*

$$\Psi(x) \;=\; \mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{\uptau_\circ}[f(X_t)-\Lambda(f)]\,\mathrm{d}t}\,\Psi(X_{\uptau_\circ})\Bigr] \qquad \forall\,x\in\overline{\mathcal{B}}_\circ^c\,. \tag{2.16}$$

Proof. Let $\tilde{f} := f-\delta\mathbb{1}_{\mathcal{B}_\circ}$. Then (2.15) follows by selecting $\xi_\circ=\Lambda(\tilde{f})$, which by Lemma 2.6 is smaller than $\Lambda(f)$.

Let $n_0\in\mathbb{N}$ be such that $\hat{\lambda}_n>\xi_\circ$ for all $n\ge n_0$. Also, without loss of generality we may assume that $f-\xi_\circ>0$ on $\mathcal{B}_\circ^c$. With $M(x)$ denoting a bound for (2.15), we obtain

$$\begin{aligned}\limsup_{T\to\infty}\;\mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{T}[f(X_t)-\hat{\lambda}_n]\,\mathrm{d}t}\,\widehat{\Psi}_n(X_T)\,\mathbb{1}\{T<\uptau_\circ\wedge\uptau_n\}\Bigr] \;&\le\; M(x)\Bigl(\sup_{B_n\setminus B_\circ}\;\widehat{\Psi}_n\Bigr)\;\limsup_{T\to\infty}\;\mathrm{e}^{-(\hat{\lambda}_n-\xi_\circ)T}\\ &=\;0 \qquad \forall\,x\in B_n\setminus\overline{\mathcal{B}}_\circ^c\,.\end{aligned}\tag{2.17}$$

Therefore, using (2.17) and monotone convergence, and since $\widehat{\Psi}_n=0$ on ∂B_n, we obtain

$$\begin{aligned}\widehat{\Psi}_n(x) \;&=\; \lim_{T\to\infty}\;\mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{T}[f(X_t)-\hat{\lambda}_n]\,\mathrm{d}t}\,\widehat{\Psi}_n(X_T)\,\mathbb{1}\{T<\uptau_\circ\wedge\uptau_n\}\Bigr]\\ &\qquad\qquad + \lim_{T\to\infty}\;\mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{\uptau_\circ}[f(X_t)-\hat{\lambda}_n]\,\mathrm{d}t}\,\widehat{\Psi}_n(X_{\uptau_\circ})\,\mathbb{1}\{\uptau_\circ\le T\wedge\uptau_n\}\Bigr]\\ &=\;\mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{\uptau_\circ}[f(X_t)-\hat{\lambda}_n]\,\mathrm{d}t}\,\widehat{\Psi}_n(X_{\uptau_\circ})\,\mathbb{1}\{\uptau_\circ<\uptau_n\}\Bigr]\,.\end{aligned}\tag{2.18}$$

This proves part (ii) of the lemma.

To prove (2.16) we write (2.18) as

$$\begin{aligned}\widehat{\Psi}_n(x) \;&=\; \mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{\uptau_\circ}[f(X_t)-\hat{\lambda}_n]\,\mathrm{d}t}\,\Psi(X_{\uptau_\circ})\,\mathbb{1}\{\uptau_\circ<\uptau_n\}\Bigr]\\ &\qquad\qquad + \Bigl(\sup_{\mathcal{B}_\circ}\bigl|\Psi-\widehat{\Psi}_n\bigr|\Bigr)\;\mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{\uptau_\circ}[f(X_t)-\hat{\lambda}_n]\,\mathrm{d}t}\,\mathbb{1}\{\uptau_\circ<\uptau_n\}\Bigr]\\ &\le\;\mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{\uptau_\circ}[f(X_t)-\hat{\lambda}_n]\,\mathrm{d}t}\,\Psi(X_{\uptau_\circ})\Bigr] + \Bigl(\sup_{\mathcal{B}_\circ}\bigl|\Psi-\widehat{\Psi}_n\bigr|\Bigr)\;\mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{\uptau_\circ}[f(X_t)-\hat{\lambda}_n]\,\mathrm{d}t}\Bigr]\,.\end{aligned}\tag{2.19}$$

Note that the terms on the right hand side of (2.19) are finite by (2.15). Since $\hat{\lambda}_n$ is nondecreasing in n, and $\hat{\lambda}_n\nearrow\Lambda(f)$, we have

$$\mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{\uptau_\circ}[f(X_t)-\hat{\lambda}_n]\,\mathrm{d}t}\,\Psi(X_{\uptau_\circ})\Bigr] \;\xrightarrow[n\to\infty]{}\; \mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{\uptau_\circ}[f(X_t)-\Lambda(f)]\,\mathrm{d}t}\,\Psi(X_{\uptau_\circ})\Bigr] \;\le\; \Psi(x) \tag{2.20}$$

by monotone convergence and (2.2). Since $\widehat{\Psi}_n\to\Psi$ as $n\to\infty$, uniformly on compact sets, it follows that the second term of the right hand side of (2.19) converges to 0 as $n\to\infty$. Thus taking limits in (2.19) as $n\to\infty$, and using (2.20) we obtain (2.16). This completes the proof. □

Corollary 2.8. *Suppose that the diffusion in* (1.5) *is recurrent, and* $f\colon\mathbb{R}^d\to\mathbb{R}_+$ *is near-monotone relative to* $\Lambda(f)$. *Then there exists a unique positive solution* $\Psi\in\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p>d$, *satisfying* $\Psi(0)=1$, *to the MPE in* (2.4). *In particular,* Ψ *satisfies* (2.5)–(2.7).

Proof. Let Ψ be as in Lemma 2.7 (iii), $\mathcal{B}_\circ$ as in Lemma 2.6, and $\tilde{\Psi}$ some other solution to (2.4). By Itô's formula and Fatou's lemma we have

$$\tilde{\Psi}(x) \;\ge\; \mathbb{E}_x\Bigl[\mathrm{e}^{\int_0^{\uptau_\circ}[f(X_t)-\Lambda(f)]\,\mathrm{d}t}\,\tilde{\Psi}(X_{\uptau_\circ})\Bigr] \qquad \forall\, x \in \overline{\mathcal{B}}_\circ^c\,,$$

while Ψ satisfies (2.16). Therefore, if $\Psi < \tilde{\Psi}$ on $\mathcal{B}_\circ$ then $\Psi < \tilde{\Psi}$ on $\mathbb{R}^d$. This implies that if we scale Ψ until it touches $\tilde{\Psi}$ in at least one point from below, then it has to touch it at a point $\tilde{x} \in \overline{\mathcal{B}}_\circ$. Thus $\tilde{\Psi} = \Psi$ on $\mathbb{R}^d$ by the strong maximum principle. □

3. Proofs of Propositions 1.1–1.2

For the proof of Proposition 1.1 we need some auxiliary lemmas. The lemma which follows is the nonlinear Dirichlet eigenvalue problem studied in [36], combined with a result from [8, Lemma 2.1].

Lemma 3.1. *For each $n \in \mathbb{N}$, there exists a unique pair $(\widehat{V}_n, \hat{\lambda}_n) \in \bigl(C^2(B_n)\cap\mathcal{C}(\bar{B}_n)\bigr)\times\mathbb{R}$, satisfying $\widehat{V}_n > 0$ on B_n, $\widehat{V}_n = 0$ on ∂B_n, and $\widehat{V}_n(0) = 1$, which solves*

$$\min_{u\in\mathbb{U}} \bigl[\mathcal{L}^u \widehat{V}_n(x) + c(x,u)\,\widehat{V}_n(x)\bigr] \;=\; \hat{\lambda}_n\,\widehat{V}_n(x)\,, \qquad x \in B_n\,. \tag{3.1}$$

Moreover, $\hat{\lambda}_n \le \lambda^*$.

Lemma 3.2. *Suppose that $\upsigma$ is bounded and*

$$\max_{u\in\mathbb{U}} \frac{\bigl\langle b(x,u),\, x\bigr\rangle^+}{|x|} \xrightarrow[|x|\to\infty]{} 0\,.$$

Then,

$$\limsup_{t\to\infty}\; \frac{1}{t}\; \mathbb{E}_x^U\bigl[|X_t|\bigr] \;=\; 0 \qquad \forall\, U \in \mathfrak{U}\,.$$

Proof. We claim that for each $\varepsilon > 0$ there exists a positive constant C_ε such that $\varepsilon C_\varepsilon \to 0$, as $\varepsilon \searrow 0$, and

$$\max_{u\in\mathbb{U}} \bigl\langle b(x,u),\, x\bigr\rangle^+ \;\le\; C_\varepsilon\bigl(1+\varepsilon\,|x|\bigr) \qquad \forall\, x \in \mathbb{R}^d\,.$$

Indeed, if f is nonnegative and $f(x) \in \mathfrak{o}(|x|)$, we write

$$\begin{aligned} f(x) &\le \sup_{|x|<R} f(x) + \Bigl(\sup_{|x|\ge R} \tfrac{f(x)}{|x|}\Bigr)|x| \\ &= M_R + \varepsilon_R|x| \\ &< 1 + M_R + \varepsilon_R|x| \\ &= (1+M_R)\Bigl(1 + \tfrac{\varepsilon_R}{1+M_R}|x|\Bigr)\,, \end{aligned}$$

which proves the claim since $\varepsilon_R \searrow 0$ as $R \nearrow \infty$.

By Itô's formula, under any control $U \in \mathfrak{U}$, we have

$$\begin{aligned} \mathbb{E}_x^U\bigl[|X_t|^2\bigr] &\le\; |x|^2 + \int_0^t \mathbb{E}_x^U\bigl[2\,\langle b(X_s,U_s)\,,X_s\rangle^+ + \operatorname{trace}\bigl(a(X_s)\bigr)\bigr]\,\mathrm{d}s \\ &\le\; |x|^2 + C'_\varepsilon \int_0^t \bigl(1+\varepsilon\,\mathbb{E}_x^U[|X_s|]\bigr)\,\mathrm{d}s\,, \end{aligned} \tag{3.2}$$

where C'_ε also satisfies $\varepsilon C'_\varepsilon \to 0$, as $\varepsilon \searrow 0$. Let $\varphi(t)$ denote the right hand side of (3.2). Then

$$\dot\varphi(t) \;\le\; C'_\varepsilon\bigl(1+\varepsilon\sqrt{\varphi(t)}\bigr)\,,$$

which implies that

$$\frac{\dot\varphi(t)}{\sqrt{\varepsilon^{-2}+\varphi(t)}} \;\le\; 2\,\varepsilon C'_\varepsilon\,. \tag{3.3}$$

Integrating (3.3), and using (3.2), we obtain

$$\mathbb{E}^U_x\bigl[|X_t|\bigr] \;\le\; \sqrt{\varepsilon^{-2}+\varphi(t)} \;\le\; \varepsilon C'_\varepsilon\, t + \frac{1}{\varepsilon^2}\,. \tag{3.4}$$

Since (3.4) holds for all $\varepsilon>0$, the result follows. □

Theorem 3.3. *Let Assumption-1.1 hold, and suppose that c is near-monotone relative to λ^*. Then the following hold*

(*i*) *There exists a solution $(V^*,\lambda^*)\in\mathcal{C}^2(\mathbb{R}^d)\times\mathbb{R}$, satisfying $V^*(0)=1$ and $\inf_{\mathbb{R}^d} V^*>0$, to the HJB equation*

$$\min_{u\in\mathbb{U}}\,\bigl[\mathcal{L}^u V(x) + c(x,u)\,V(x)\bigr] \;=\; \lambda\, V(x) \qquad \forall\, x\in\mathbb{R}^d\,. \tag{3.5}$$

(*ii*) *If c is bounded, and $(V,\lambda)\in\mathfrak{V}$ satisfies* (3.5)*, then $\lambda=\lambda^*$, and $\inf_{\mathbb{R}^d} V>0$. In general, if $(V,\lambda)\in\mathfrak{V}_\circ$ satisfies* (3.5)*, then any measurable selector from the minimizer of* (3.5) *belongs to $\mathfrak{U}_{\mathrm{SSM}}$ and is optimal, i.e., $\mathcal{R}(v)=\lambda^*$.*

Proof. Truncate c by letting $c^{(\ell)} := c\wedge\ell$, $\ell\in\mathbb{N}$. As shown in [8, Lemma 2.1], any limit point, $(V_\ell,\lambda_\ell)\in\mathcal{C}^2(\mathbb{R}^d)\times\mathbb{R}$ of the eigensolutions $\bigl(\widehat V_{n,\ell},\hat\lambda_{n,\ell}\bigr)$ of the Dirichlet problem on B_n in Lemma 3.1, as $n\to\infty$, satisfies

$$\min_{u\in\mathbb{U}}\,\bigl[\mathcal{L}^u V_\ell(x) + c^{(\ell)}(x,u)\,V_\ell(x)\bigr] \;=\; \lambda_\ell\, V_\ell(x) \qquad \forall\, x\in\mathbb{R}^d\,. \tag{3.6}$$

Clearly then, $V_\ell>0$ on $\mathbb{R}^d$, $V_\ell(0)=1$, and $\lambda_\ell\le\lambda^*$ by Lemma 3.1. Since b, $\upsigma$, and $c^{(\ell)}$ are bounded, the uniform Harnack property implies that there exists a constant $\kappa=\kappa(\ell)>0$ such that

$$\mathrm{e}^{-\kappa(1+|x|)} \;\le\; V_\ell(x) \;\le\; \mathrm{e}^{\kappa(1+|x|)} \qquad \forall\, x\in\mathbb{R}^d\,. \tag{3.7}$$

Let v_ℓ be a measurable selector from the minimizer of (3.6). A straightforward application of Fatou's lemma on the stochastic representation of the solution V_ℓ of (3.6) shows that

$$V_\ell(x) \;\ge\; \mathbb{E}^{v_\ell}_x\Bigl[\mathrm{e}^{\int_0^T [c^{(\ell)}(X_t,v(X_t))-\lambda_\ell]\,\mathrm{d}t}\, V_\ell(X_T)\Bigr] \qquad \forall\, T>0\,. \tag{3.8}$$

Evaluating (3.8) at $x=0$, taking the logarithm on both sides, applying Jensen's inequality, dividing by T, and rearranging terms, we obtain

$$\frac{1}{T}\;\mathbb{E}^{v_\ell}_x\biggl[\int_0^T c^{(\ell)}\bigl(X_t,v(X_t)\bigr)\,\mathrm{d}t\biggr] + \frac{1}{T}\;\mathbb{E}^{v_\ell}_x\bigl[\log V_\ell(X_T)\bigr] \;\le\; \lambda_\ell + \frac{1}{T}\log V_\ell(x)\,. \tag{3.9}$$

Hence, since $\bigl|\log V_\ell(X_T)\bigr| \le \kappa\,\bigl(1+|X_T|\bigr)$ by (3.7), it follows by Lemma 3.2 that

$$\limsup_{T\to\infty}\;\frac{1}{T}\;\mathbb{E}^{v_\ell}_x\bigl[\bigl|\log V_\ell(X_T)\bigr|\bigr] \;=\; 0\,.$$

Therefore, by (3.9) we obtain

$$\limsup_{T\to\infty}\;\frac{1}{T}\;\mathbb{E}^{v_\ell}_x\biggl[\int_0^T c^{(\ell)}\bigl(X_t,v_\ell(X_t)\bigr)\,\mathrm{d}t\biggr] \;\le\; \lambda_\ell\,.$$

Since $\lambda_\ell \le \lambda^*$, it follows that v_ℓ is stable for all large enough ℓ (by [1, Theorem 2.6.10 (e)]). Therefore V_ℓ is inf-compact, and $\mathcal{R}(v_\ell; c^{(\ell)}) \le \lambda_\ell$ by Lemma 2.1. By the same argument as in the proof of Lemma 2.5 it follows that $\liminf_{\ell\to\infty} \inf_{\mathbb{R}^d} V_\ell > 0$. Hence, (V_ℓ, λ_ℓ) converges as $\ell \to \infty$, along a subsequence, to some $(V^*, \bar\lambda)$ satisfying (3.5), and such that $V^*(0) = 1$ and $\inf_{\mathbb{R}^d} V^* > 0$. Let v be a measurable selector from the minimizer of (3.5). Then, by Lemma 2.1, v is stable and $\mathcal{R}(v; c) \le \bar\lambda$. This of course implies that $\bar\lambda = \lambda^*$. This proves part (i).

The proof of part (ii) follows by repeating the argument in the preceding paragraph. □

Theorem 3.4. *Let the assumptions of Theorem 3.3 hold. Then:*

- *(i) The solution $(V^*, \lambda^*) \in \mathcal{C}^2(\mathbb{R}^d) \times \mathbb{R}$ of (3.5) is unique in the class $\mathfrak{V}_\circ$, and provided c is bounded, it is also unique in $\mathfrak{V}$.*
- *(ii) Any $\check v \in \mathfrak{U}_{\mathrm{SM}}$ that is optimal satisfies*

$$\mathcal{L}^{\check v} V^*(x) + c\bigl(x, \check v(x)\bigr)\, V^*(x) \;=\; \min_{u\in\mathbb{U}} \bigl[\mathcal{L}^u V^*(x) + c(x,u)\, V^*(x)\bigr] \quad \text{a.e. } x \in \mathbb{R}^d\,. \tag{3.10}$$

- *(iii) Part (iv) of Proposition 1.1 holds.*

Proof. We first show uniqueness. By Theorem 3.3 (ii) it suffices to prove uniqueness in the class $\mathfrak{V}_\circ$. Suppose that a pair $(\check V, \check\lambda) \in \mathfrak{V}_\circ$ solves

$$\min_{u\in\mathbb{U}} \bigl[\mathcal{L}^u \check V(x) + c(x,u)\, \check V(x)\bigr] \;=\; \check\lambda\, \check V(x)\,, \qquad x \in \mathbb{R}^d\,. \tag{3.11}$$

Let $\check v$ be a measurable selector from the minimizer of (3.11). It then follows by Theorem 3.3 that $\mathcal{R}(\check v) = \check\lambda = \lambda^*$, and $\inf_{\mathbb{R}^d} \check V > 0$.

We simplify the notation, by defining $\bar c_v(x) := c\bigl(x, v(x)\bigr)$, for $v \in \mathfrak{U}_{\mathrm{SM}}$. If c is bounded, we let V^* be some limit point of $\widehat V_n$ in Lemma 3.1 as $n \to \infty$. Otherwise, we let V^* be some limit point of V_ℓ in the proof of Theorem 3.3 as $\ell \to \infty$. By (3.11) and (3.5) we have

$$\mathcal{L}^{\check v} \check V(x) + \bar c_{\check v}(x)\, \check V(x) \;=\; \lambda^*\, \check V(x)\,, \tag{3.12}$$

$$\mathcal{L}^{\check v} V^*(x) + \bar c_{\check v}(x)\, V^*(x) \;\ge\; \lambda^*\, V^*(x) \tag{3.13}$$

for all $x \in \mathbb{R}^d$. Let $\mathcal{B}_\circ$ be defined as in Lemma 2.6 with Ψ replaced by $\check V$ and $\mu \equiv \mu_{\check v}$. Recall that $\uptau_\circ := \uptau(\mathcal{B}_\circ^c)$. By (3.12), Lemma 2.7 and Corollary 2.8, we obtain

$$\check V(x) \;=\; \mathbb{E}^{\check v}_x\Bigl[\mathrm{e}^{\int_0^{\uptau_\circ} [\bar c_{\check v}(X_t) - \lambda^*]\,\mathrm{d}t}\, \check V(X_{\uptau_\circ})\Bigr] \qquad \forall\, x \in \overline{\mathcal{B}}_\circ^c\,. \tag{3.14}$$

By Lemma 2.7 (i), there exists $\xi_\circ < \lambda^*$ such that

$$\mathbb{E}^{\check v}_x\Bigl[\mathrm{e}^{\int_0^{\uptau_\circ} [\bar c_{\check v}(X_t) - \xi_\circ]\,\mathrm{d}t}\Bigr] \;<\; \infty \qquad \forall\, x \in \overline{\mathcal{B}}_\circ^c\,. \tag{3.15}$$

First suppose c is bounded. Let $\hat\lambda_n$ be as in Lemma 3.1, and $n_0 \in \mathbb{N}$ be large enough so that $\hat\lambda_n > \xi_\circ$, and $B_{n_0} \Supset \mathcal{B}_\circ$. Since $\check v$ is in general suboptimal for (3.1), by Itô's formula, and also using (3.15) and the fact that $\widehat V_n = 0$ on ∂B_n, following the argument in (2.17)–(2.18), we obtain

$$\widehat V_n(x) \;\le\; \mathbb{E}^{\check v}_x\Bigl[\mathrm{e}^{\int_0^{\uptau_\circ} [\bar c_{\check v}(X_t) - \hat\lambda_n]\,\mathrm{d}t}\, \widehat V_n(X_{\uptau_\circ})\, \mathbb{1}\{\uptau_\circ < \uptau_n\}\Bigr] \qquad \forall\, x \in B_n \setminus \overline{\mathcal{B}}_\circ\,,$$

and using a triangle inequality as in (2.19), and taking limits as $n \to \infty$, we obtain

$$V^*(x) \;\le\; \mathbb{E}^{\check v}_x\Bigl[\mathrm{e}^{\int_0^{\uptau_\circ} [\bar c_{\check v}(X_t) - \lambda^*]\,\mathrm{d}t}\, V^*(X_{\uptau_\circ})\Bigr] \qquad \forall\, x \in \overline{\mathcal{B}}_\circ^c\,. \tag{3.16}$$

If c is not bounded, following the argument in the previous paragraph, we obtain

$$V_\ell(x) \;\le\; \mathbb{E}_x^{\check v}\Bigl[\mathrm{e}^{\int_0^{\uptau_\circ}[\bar c_{\check v}^{(\ell)}(X_t)-\lambda_\ell]\,\mathrm{d}t}\,V_\ell(X_{\uptau_\circ})\Bigr] \qquad \forall\,x\in\overline{\mathcal{B}}_\circ^c\,, \tag{3.17}$$

for all ℓ sufficiently large. Let $\ell_0\in\mathbb{N}$ be such that $\lambda_\ell>\xi_\circ$ for all $\ell\ge\ell_0$. By monotone convergence, we obtain from (3.17) that

$$V_\ell(x) \;\le\; \mathbb{E}_x^{\check v}\Bigl[\mathrm{e}^{\int_0^{\uptau_\circ}[\bar c_{\check v}(X_t)-\lambda_\ell]\,\mathrm{d}t}\,V_\ell(X_{\uptau_\circ})\Bigr] \qquad \forall\,x\in\overline{\mathcal{B}}_\circ^c\,, \tag{3.18}$$

and that the right hand side of (3.18) is finite for all $\ell\ge\ell_0$. Hence, using (3.18) and the triangle inequality in (2.19), and letting $\ell\to\infty$ along a subsequence over which V_ℓ converges, we obtain (3.16).

Hence, in either case, following the proof of Corollary 2.8, and using (3.12)–(3.14) and (3.16), we deduce that $\check V=V^*$.

Next we show that every optimal control satisfies (3.10). Let $\check v\in\mathfrak{U}_{\mathrm{SM}}$ such that $\mathcal{R}(\check v)=\lambda^*$. Since c is near-monotone relative to λ^*, it follows that $\check v\in\mathfrak{U}_{\mathrm{SSM}}$. Thus, by Theorem 1.3 there exists a unique positive $\check V\in\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p>d$, with $\check V(0)=1$, which solves the MPE in (3.12) a.e. in $\mathbb{R}^d$, and clearly (3.13) holds. Using the argument in the preceding paragraph we obtain (3.14) and (3.16). Thus, again by the argument in the proof of Corollary 2.8, we must have $\check V=V^*$. This proves part (ii).

Part (iii) follows by Theorem 1.3. □

Proof of Proposition 1.2. Let $\varepsilon_0>0$ be small enough so that c is near-monotone relative to $\lambda^*_{\mathrm{m}}+\varepsilon_0$, and for $\varepsilon\in(0,\varepsilon_0)$, let $v_\varepsilon\in\mathfrak{U}_{\mathrm{SM}}$ be a ε-optimal control relative to λ^*_{m}. In other words, v_ε satisfies $\mathcal{R}(v_\varepsilon)\le\lambda^*_{\mathrm{m}}+\varepsilon$. Define

$$b_n^{v_\varepsilon}(x,u) \;:=\; \begin{cases} b(x,u) & \text{for } x\in B_n \quad\text{and}\quad u\in\mathbb{U}\,,\\ b\bigl(x,v_\varepsilon(x)\bigr) & \text{for } x\in B_n^c\,,\end{cases}$$

and $c_n^{v_\varepsilon}(x,u)$ in an exactly analogous manner. Let $\mathfrak{U}_{\mathrm{SM}}^{n,v_\varepsilon}$ denote the class of stationary Markov controls that agree with v_ε on B_n^c. By [36, Theorem 1.1], there exists a unique pair $(V_{n,k}^{v_\varepsilon},\lambda_{n,k}^{v_\varepsilon})\in\bigl(\mathcal{W}^{2,p}(B_k)\cap\mathcal{C}(\overline{B}_k)\bigr)\times\mathbb{R}$, for any $p>d$, satisfying $V_{n,k}^{v_\varepsilon}>0$ on B_k and $V_{n,k}^{v_\varepsilon}(0)=1$, which solves

$$\min_{u\in\mathbb{U}}\;\bigl[\mathcal{L}^u V_{n,k}^{v_\varepsilon}(x)+c_n^{v_\varepsilon}(x,u)\,V_{n,k}^{v_\varepsilon}(x)\bigr] \;=\; \lambda_{n,k}^{v_\varepsilon}\,V_{n,k}^{v_\varepsilon}(x) \qquad \text{a.e. } x\in B_k\,, \tag{3.19}$$

and $V_{n,k}^{v_\varepsilon}=0$ on ∂B_k (compare with Lemma 2.2). Following the proof of [8, Lemma 2.1], we deduce that $\lambda_{n,k}^{v_\varepsilon}\le\lambda^*_{\mathrm{m}}+\varepsilon$. Taking limits as $k\to\infty$ along some subsequence, we obtain by (3.19) a pair $(V_n^{v_\varepsilon},\lambda_n^{v_\varepsilon})\in\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)\times\mathbb{R}$, for any $p>d$, satisfying $\lambda_n^{v_\varepsilon}\le\lambda^*_{\mathrm{m}}+\varepsilon$, $V_n^{v_\varepsilon}>0$ on $\mathbb{R}^d$, and $V_n^{v_\varepsilon}=1$, which solves

$$\min_{u\in\mathbb{U}}\;\bigl[\mathcal{L}^u V_n^{v_\varepsilon}(x)+c_n^{v_\varepsilon}(x,u)\,V_n^{v_\varepsilon}(x)\bigr] \;=\; \lambda_n^{v_\varepsilon}\,V_n^{v_\varepsilon}(x) \qquad \text{a.e. } x\in\mathbb{R}^d\,. \tag{3.20}$$

It also follows by elliptic regularity that the restriction of $V_n^{v_\varepsilon}$ in B_n is in $\mathcal{C}^2(\mathbb{R}^d)$.

Let $\hat v_n$ be a measurable selector from the minimizer of (3.20). Since $\hat v_n\in\mathfrak{U}_{\mathrm{SM}}^{n,v_\varepsilon}$, it follows that $\hat v_n\in\mathfrak{U}_{\mathrm{SSM}}$. Let $\mathcal{B}(\varepsilon_0)$ be a bounded open ball such that $c(x,u)>\lambda^*_{\mathrm{m}}+\varepsilon_0$ for all $(x,u)\in\mathcal{B}^c(\varepsilon_0)\times\mathbb{U}$. Applying Itô's formula to (3.20), and using Fatou's lemma, we obtain, with $\check\uptau\equiv\uptau\bigl(\mathcal{B}^c(\varepsilon_0)\bigr)$, that

$$\begin{aligned} V_n^{v_\varepsilon}(x) \;&\ge\; \mathbb{E}_x^{\hat v_n}\Bigl[\mathrm{e}^{\int_0^{\check\uptau}[c_n^{v_\varepsilon}(X_t,\hat v_n(X_t))-\lambda_n^{v_\varepsilon}]\,\mathrm{d}t}\,V_n^{v_\varepsilon}(X_{\check\uptau})\Bigr]\\ &\ge\; \Bigl(\min_{\partial\mathcal{B}(\varepsilon_0)}\;V_n^{v_\varepsilon}\Bigr)\,\mathbb{E}_x^{\hat v_n}\Bigl[\exp\Bigl(\tfrac{1}{2}(\varepsilon_0-\varepsilon)\,\check\uptau\Bigr)\Bigr] \qquad \forall\,x\in\mathcal{B}^c(\varepsilon_0)\,. \end{aligned} \tag{3.21}$$

It follows by (3.21) and Harnack's inequality that

$$\inf_{n\in\mathbb{N}}\;\inf_{\mathbb{R}^d}\;V_n^{v_\varepsilon} \;>\; 0\,. \tag{3.22}$$

Thus, taking limits in (3.21) as $n \to \infty$, along some subsequence, we obtain a pair $(V^{v_\varepsilon}, \lambda^{v_\varepsilon}) \in \mathcal{C}^2(\mathbb{R}^d) \times \mathbb{R}$, satisfying $\lambda^{v_\varepsilon} \le \lambda^*_{\mathrm{m}} + \varepsilon$, $\inf_{\mathbb{R}^d} V^{v_\varepsilon} > 0$, and $V^{v_\varepsilon} = 1$, which solves

$$\min_{u\in\mathbb{U}} \bigl[\mathcal{L}^u V^{v_\varepsilon}(x) + c(x,u)\, V^{v_\varepsilon}(x)\bigr] \;=\; \lambda^{v_\varepsilon}\, V^{v_\varepsilon}(x) \qquad \text{a.e. } x \in \mathbb{R}^d\,. \tag{3.23}$$

By Lemma 2.1 (iii), we have $\lambda^{v_\varepsilon} = \mathcal{R}(v_\varepsilon)$. Taking any limit of (3.23) as $\varepsilon \searrow 0$ along some subsequence, we obtain a function $V^* \in \mathcal{C}^2(\mathbb{R}^d)$, satisfying

$$\min_{u\in\mathbb{U}} \bigl[\mathcal{L}^u V^*(x) + c(x,u)\, V^*(x)\bigr] \;=\; \lambda^*_{\mathrm{m}}\, V^*(x)\,, \qquad x \in \mathbb{R}^d\,. \tag{3.24}$$

It holds that $\inf_{\mathbb{R}^d} V^* > 0$ by (3.22), and $V^*(0) = 1$ by construction.

Let v^* be a measurable selector from the minimizer of (3.24). Then $v^* \in \mathfrak{U}_{\mathrm{SSM}}$, and $\mathcal{R}(v^*) = \lambda^*_{\mathrm{m}}$ by Lemma 2.1. □

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