SINGULARITY

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Abstract. This is a user guide for the first version of our developed Maple library, named `Singularity`. The first version here is designed for the qualitative study of local real zeros of scalar smooth maps. This library will be extended for symbolic bifurcation analysis and control of different singularities including autonomous differential singular systems and local real zeros of multidimensional smooth maps. Many tools and techniques from computational algebraic geometry have been used to develop `Singularity`. However, we here skip any reference on how this library is developed. This package is useful for both pedagogical and research purposes. `Singularity` will be updated as our research progresses and will be released for public access once our draft paper [4] is peer-reviewed in a refereed journal.

Contents

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Part 1. Introduction

This is a user guide on how to use the first version of our developed Maple library (named `Singularity`) for local bifurcation analysis of real zeros of scalar smooth maps; see [4] for the main ideas. We remark that the term *singularity theory* has been used in many different Mathematics disciplines with essentially different objectives and tools but yet sometimes with similar terminologies; for examples of these see [4]. For more detailed information, definitions, and related theorems in what we call here *singularity theory*, we refer the reader to [1–3, 5–9].

Verification and warning note. `Singularity` is able to check and verify all of its computations. However, this sometimes adds an extra computational cost. This happens mainly for finding out the correct and suitable truncation degree and computational ring. Therefore, it is beneficial to skip the extra computations when it is not necessary. For an instance of benefit, consider that you need to obtain certain results for a large family of problems arising from the same origin. Therefore, you might be able to only check a few problems and conclude about the suitable truncation degree and computational ring for the whole family. Thereby, the commands of `Singularity` check and verify the output results unless it requires extra computation. In this case, a warning note of not verified output or possible errors is given; in these cases, a recommendation is always provided on how to verify or circumvent the problem. Lack of warning notes indicates that the output results have been successfully verified.

1.1. List of commands

The following table shows a complete list of commands from singularity theory that have so far been implemented in `Singularity`.

Commands in Singularity theory

Verify	Normalform	UniversalUnfolding
RecognitionProblem	CheckUniversal	Transformation
TransitionSet	PersistentDiagram	NonPersistent
AlgObjects	RT	T
P	S	TangentPerp
SPerp	IntrinsicGen	Intrinsic

The following table enlists all implemented tools from computational algebraic geometry.

Tools from algebraic geometry

MultMatrix	Division	StandardBasis	ColonIdeal	Normalset

In this user guide we will explain the capabilities, options and how each of these commands work. After installing `Singularity` on Maple software running on your computer, the above list is accessible by using the command `with(Singularity)`. In fact, Maple enlists all the commands in the above two tables as its output.

Part 2. Singularity theory

The terminology "singularity theory" has been used to deal with many different problems in different mathematical disciplines. Singularity theory here refers the methodologies in dealing with the qualitative behavior of local zeros

$$g(x, \lambda) = 0, \tag{2.0.1}$$

where $x \in \mathbb{R}$ is a state variable and λ is a distinguished parameter. The cases of multi-dimensional parameters are dealt with through the notions of unfolding. `Singularity` will be soon enhanced to deal with the cases of multi-dimensional state variables.

In many real life problems at certain solutions, $g(x, \lambda)$ is *singular*, i.e.,

$$g(x, \lambda) = g_x(x, \lambda) = 0.$$

A singular germ $g(x, \lambda)$ subjected to smooth changes demonstrates surprising changes in the *qualitative properties* of the solutions, e.g., changes in the number of solutions. This phenomenon is called a *bifurcation*.

2.1. Qualitative properties

We define the *qualitative properties* as the invariance of an equivalence relation. The equivalence relation used in `Singularity` is *contact equivalence* and is defined by

$$f \sim g \Leftrightarrow f(x, \lambda) = S(x, \lambda) g(X(x, \lambda), \Lambda(\lambda)) \tag{2.1.1}$$

where $S(x, \lambda) > 0$ while (X, Λ) is locally a diffeomorphism such that $X_x(x, \lambda)$ and $\Lambda'(\lambda) > 0$.

2.2. Ring and truncation degree

In this section we describe how to determine the permissible computational ring and the truncation degree. In fact, each smooth germ with a nonzero-infinite Taylor series expansion must be truncated at certain degree. Further, there are four different options in `Singularity` for computational rings, i.e., polynomial, fractional, formal power series and smooth germ rings, that they each can be used by commands in `Singularity` for bifurcation analysis of each singular germ. The command `Verify(g)` derives the following information about the singular germ g for correct and efficient computations:

(1) the permissible computational rings for the germ g.
(2) the least permissible truncation degree k for computations involving the germ g. In other words, the computations modulo degrees of higher than (but not equal to) k does not lead to error.
(3) our recommended computational ring.

We are also interested in the above information for the following purposes:

- A list of germs G in two variables for either division, standard basis computations, multiplication matrix, or intrinsic part of an ideal.
- A parametric germ $H(x, \lambda, \alpha)$ with $\alpha \in \mathbb{R}^p$ for either transition set computation or persistent bifurcation diagram classification.

Command	**Description**
`Verify`(g)	derives the permissible computational rings and permissible truncation degree.
Default upper bound for truncation	a permissible truncation degree is computed as long as it is less than or equal to 20.

2.2.1. **Options.**

- `Ideal`; `Persistent`; the command `Verify`(G, `Ideal`, `Vars`) deals with an ideal I generated by G, i.e., $I := \langle G \rangle_{\mathscr{E}}$, where G is a list of germs. It returns the permissible computational ring and a permissible truncation degree when the ideal I is of finite codimension. Otherwise, it remarks that "the ideal is of infinite codimension." However, the command `Verify`(H, `Persistent`, `Vars`) determines the least permissible truncation degree k so that the computations associated with either persistent bifurcation diagram classification or transition sets would be correct.
- `Fractional`; `Formal`; `SmoothGerms`; `Polynomial`; the command uses either the rings of fractional germs, formal power series or ring of smooth germs.
- Upper bound for truncation degree N; this lets the user to change the default upper bound truncation degree from 20 to N.

Example 2.2.1. `Verify`($x^3 - \sin(\lambda)$) gives
The following rings are allowed as the means of computations:
Ring of smooth germs
Ring of formal power series
Ring of fractional germs
The truncation degree must be: 3

Example 2.2.2. `Verify`($x^3 - \sin(\lambda)$, 2) gives the following warning message:
"Increase the upper bound for the truncation degree!"

Example 2.2.3. `Verify`$([x^4-x\sin(\lambda), x^3\lambda-\lambda\sin(\lambda), 3x^4, 3x^2\lambda]$, `Ideal`, $[x, \lambda]$) gives
The following rings are allowed as means of computations:
Ring of smooth germs
Ring of formal power series
Ring of fractional germs
The truncated degree must be: 4

Example 2.2.4. Command `Verify`$(x^3-\sin(\lambda)$, `Persistent`, $[x, \lambda])$ gives the least permissible truncation degree to be 2.

2.3. Normal form

A germ $f(x, \lambda)$ is called a normal form for the singular germ $g(x, \lambda)$ when f has a minimal set of monomial terms in its Taylor expansion among all contact-equivalent germs to g. Therefore, it is easier to analyze the solution set of f while it has the same qualitative behavior as zeros of g do.

Command/the default options	**Description**
`Normalform`(g)	This function derives a normal form for g.
The computational ring	The default ring is the ring of fractional germs.
Verify/Warning/Suggestion	It automatically verifies if fractional germ ring is sufficient for normal form computation of the input germ g. Otherwise, it writes a warning note along with a cognitive suggestion for the user.
Truncation degree	It, by default, detects the maximal degree k in which $\mathcal{M}^{k+1} \subseteq \mathscr{P}(g)$. Thus, normal forms are computed modulo degrees higher than or equal to $k+1$.
Input germ	It, by default, takes the input germ g as a polynomial or a smooth germ. It truncates the smooth germs modulo the degree $k+1$.

2.3.1. Options.

- k; specifies the degree k so that computations are performed modulo degree $k+1$. When the input degree k is too small for the input singular germ g, the computation is not reliable. Thus, an error warning note is returned to inform the user of the situation along with cognitive suggestions to circumvent the problem.
- `Fractional`, `Formal`, `SmoothGerms`; `Polynomial`; the command uses either the rings of fractional germs, formal power series or ring of smooth germs. When it is necessary, warning/cognitive suggestions are given accordingly.
- `list`; this generates a list of possible normal forms for the germ g. Different normal forms may only occur due to possible alternative eliminations in intermediate order terms.

Example 2.3.1. `Normalform`$(x^3 - \sin(\lambda)$, 10, `SmoothGerms`) generates

$$x^3 - \lambda.$$

while `Normalform`$(1-\frac{1}{1+x^4-\lambda^2}, 10,$ `Formal`$)$ gives rise to

$$x^4-\lambda^2.$$

Using `Normalform`$(x^5+x^3\lambda+\sin(\lambda^2), 10,$ `Polynomial`$)$ gives the following suggestion and warning note.

Warning: The polynomial germ ring is not suitable for normal form computations.

Suggestion: Use the command `Verify` to find the appropriate computational ring.

The following output might be wrong.

The germ is an infinite codimensional germ.

In fact the above statement is wrong since the high order term ideal contains

$$\mathcal{M}^6+\mathcal{M}^4\langle\lambda\rangle+\mathcal{M}\langle\lambda^2\rangle.$$

Now the command `Verify`$(x^5+x^3\lambda+\sin(\lambda^2))$ gives rise to

Fractional germ ring; Formal power series ring; Smooth germ ring.

Thus, we use `Normalform`$(x^5+x^3\lambda+\sin(\lambda^2), 10,$ `Fractional`$)$ to obtain

$$x^5+x^3\lambda+\lambda^2.$$

2.4. Universal unfolding

Generally in dealing with singular problems, extra complications are experienced in the laboratory data than what are predicted by the modeling theoretical analysis. The problem here is due to *modeling imperfections*; natural phenomena can not be perfectly modeled by a mathematical model. In fact one usually neglects the impact of many factors like friction, pressure, and/or temperature, etc., to get a manageable mathematical model. Otherwise one will end up with a mathematical modeling problem with too many or infinite number of parameters. The imperfections around singular points may cause dramatic qualitative changes in the solution set of the model. Universal unfolding gives us a natural way to circumvent the problem of imperfections.

Definition 2.4.1. *A parametric germ $G(x,\lambda,\alpha)$ is called an* unfolding *for $g(x,\lambda)$ when*

$$G(x,\lambda,0)=g(x,\lambda).$$

An unfolding $G(x,\lambda,\alpha)$ for g is called a versal unfolding *when for each unfolding $H(x,\lambda,\beta)$ of $g(x,\lambda)$ there is a smooth germ $\alpha(\beta)$ so that H is contact-equivalent to $G(x,\lambda,\alpha(\beta))$. Roughly speaking, a versal unfolding is a parametric germ that contains a contact-equivalent copy of all small perturbations of $g(x,\lambda)$. A versal unfolding with insignificant parameters is not suitable for the bifurcation analysis. So, we are interested in a versal unfolding that has a minimum possible number of parameters, that is called* universal unfolding. *In other words, universal unfolding has the minimum possible number of parameters so that they accommodate all possible qualitative types that small perturbations of $g(x,\lambda)$ may experience.*

Command/the default options	Description
`UniversalUnfolding`(g)	This function computes a universal unfolding for g.
The computational ring	By default, `Singularity` uses the ring of fractional germs.
Verify/Warning/Suggestion	This automatically derives the least sufficient degree for truncations and also verifies if fractional germ ring is sufficient for computation. Otherwise, it writes a warning note along with guidance on the suitable rings for computations and hints at other possible capabilities of `Singularity`.
Degree	It, by default, detects the maximal degree k in which terms of degree higher than or equal to $k+1$ can be ignored. Thus, the computations are performed modulo degree $k+1$.
Input germ	The default input germ g is a polynomial or a smooth germ. For an input smooth germ, the default procedure `UniversalUnfolding` truncates the smooth germs modulo $k+1$, i.e., modulo degrees higher than and equal to $k+1$.

2.4.1. **Options.**

- `normalform`; A universal unfolding for normal form of g is derived by this option.
- `list`; this function provides the list of possible universal unfoldings for g.
- k; the degree k determines the truncation degree so that all computations are performed modulo $k+1$. For low degrees of k, it may derive wrong results. Thus, it gives a warning error and a suggestion for the user when k must be a larger number for correct result.
- `Fractional`; `Formal`; `SmoothGerms`; `Polynomial`; this determines the computational ring. The command `UniversalUnfolding` gives a warning note when the user's choice of computational ring is not suitable for computations involving the input germ g and writes a suggestion to circumvent the problem.

Example 2.4.2. `UniversalUnfolding`$(x^4+4x^3-\lambda x,$ `normalform`, `list`$)$ gives rise to

$$x^3 - x\lambda + \alpha_1 + \alpha_2\lambda$$
$$x^3 - x\lambda + \alpha_1 + \alpha_2 x^2.$$

`UniversalUnfolding`$(6x - 6\sin(x) - \lambda x,$ `normalform`, `list`$, 6,$ `Formal`$)$ leads to

$$x^3 - \lambda x + \alpha_2 x^2 + \alpha_1$$
$$x^3 - \lambda x + \alpha_2\lambda + \alpha_1.$$

Now consider $g(x,\lambda) := x^6 + x^4\lambda + \lambda^2$.

`UniversalUnfolding`$(g(x,\lambda),$ `normalform`, `list`, $6,$ `Polynomial`) gives the following warning error and suggestion:
Warning: The ring of polynomial germs is not suitable for normal form computations of g.
Suggestion: The permissible computational ring options are `Fractional`, `SmoothGerms` *and* `Formal`.

2.5. Recognition problem

We describe the command `RecognitionProblem` on how it answers the recognition problem, that is, what kind of germs have the same normal form or universal unfolding for a given germ g?

2.5.1. **Normal form.**

- **Low order terms**. Low order terms refer to the monomials in $\mathscr{S}(g)^{\perp}$ which do not appear in any contact-equivalent copy of g.
- **High order terms**. The ideal $\mathscr{P}(g)$ represents the space of negligible terms that are called high order terms. These terms are eliminated in normal form of g.
- **Intermediate order terms**. A monomial term are called an intermediate order term when it is neither low order nor high order term. Intermediate order terms may or may not be simplified in normal form computation of smooth germs.

The answer for the recognition problem for normal form of a germ g is a list of zero and nonzero conditions for certain derivatives of a hypothetical germ f. When these zero and nonzero conditions are satisfied for a given germ f, the germ f and g are contact-equivalent. Each germ with a minimal list of monomial terms in its Taylor expansion constitutes a normal form for g.

2.5.2. **Universal unfolding.** Consider a parametric germ $G(x,\lambda,\alpha)$ and a germ $g(x,\lambda)$. Then, G is usually a universal unfolding for g when $G(x,\lambda,0)=g(x,\lambda)$ and certain matrix associated with G has a nonzero determinant. Thus, the answer of the recognition problem for universal unfolding is actually a matrix whose components are derivatives of a hypothetical parametric germ G satisfying $G(x,\lambda,0)=g(x,\lambda)$.

Command/ default	Description
`RecognitionProblem`(g)	returns a list of zero and nonzero conditions on certain derivatives of a hypothetical germ f. A given germ f is contact-equivalent to g when those conditions are satisfied.
Computational ring	The default is fractional germ ring. The lack of warning notes is a confirmation that the fractional germ ring is suitable for computation.
Truncation degree	It automatically computes an optimal truncation degree k and performs the remaining computations modulo degrees of higher than (but not equal to) k.
Verification/Warning	A warning note of possible errors is given when the computational ring is not suitable for the germ g. The truncation degree is also checked and if it is not sufficiently large enough, a warning note is given. Warning notes are accompanied with cognitive suggestions to circumvent the problem.

2.5.2.1. *Options.*

- k; this number represents the truncation degree.
- Computational ring: `Fractional`, `Formal`, `SmoothGerms`; `Polynomial`; the command accordingly uses either the rings of fractional germs, formal power series or smooth germs.
- `Universalunfolding`; it returns a matrix. The matrix components consists of certain derivatives of a hypothetical parametric germ G. Then, a parametric germ $G(x, \lambda, \alpha)$ is a universal unfolding for $g(x, \lambda)$ when $G(x, \lambda, 0) = g(x, \lambda)$ and the associated matrix has a nonzero determinant.

Example 2.5.1. `RecognitionProblem`$(x^3 + \sin(\lambda),\ 6,$ `Formal`$)$ gives rise to

$$\text{"nonzero condition="},\ [\tfrac{\partial}{\partial\lambda}f \neq 0,\ \tfrac{\partial^3}{\partial x^3}f \neq 0]$$
$$\text{"zero condition="},\ [f = 0,\ \tfrac{\partial}{\partial x}f = 0,\ \tfrac{\partial^2}{\partial x^2}f = 0].$$

`RecognitionProblem`$(x^3 + \exp(\lambda^2) - 1,$ universalunfolding, $6,$ `SmoothGerms`$)$ gives rise to

$$\det \begin{pmatrix} 0 & 0 & 0 & g_{x,x,x}(0) & g_{x,x,\lambda}(0) \\ 0 & g_{\lambda,\lambda}(0) & 0 & g_{x,x,\lambda}(0) & g_{x,\lambda,\lambda}(0) \\ G_{\alpha_1}(0) & G_{\lambda,\alpha_1}(0) & G_{x,\alpha_1}(0) & G_{x,x,\alpha_1}(0) & G_{x,\lambda,\alpha_1}(0) \\ G_{\alpha_2}(0) & G_{\lambda,\alpha_2}(0) & G_{x,\alpha_2}(0) & G_{x,x,\alpha_2}(0) & G_{x,\lambda,\alpha_2}(0) \\ G_{\alpha_3}(0) & G_{\lambda,\alpha_3}(0) & G_{x,\alpha_3}(0) & G_{x,x,\alpha_3}(0) & G_{x,\lambda,\alpha_3}(0) \end{pmatrix} \neq 0.$$

Let $G(x, \lambda, \alpha)$ be a parametric germ where $\alpha \in \mathbb{R}^k$.

Command	Description
`CheckUniversal`(G)	This function checks if a parametric germ G is a universal unfolding for $G(x, \lambda, 0)$

Example 2.5.2. `CheckUniversal`$(x^5 - \lambda + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3)$ gives

"Yes"

2.6. Transformations

For each two contact-equivalent germs f and g, there are diffeomorphic germs $(X(x,\lambda),\Lambda(\lambda))$ and smooth germ $S(x,\lambda)>0$ such that $X_x(x,\lambda)$, $\Lambda'(\lambda)>0$ and $f(x,\lambda)=S(x,\lambda)g(X(x,\lambda),\Lambda(\lambda))$.

Command/option	**Description**
`Transformation`(g)	This function computes the smooth germs X,Λ,S transforming the germ g into its normal form modulo degree k, where terms of degree higher than or equal to k are high order terms.
Transformation(g,f)	This function computes suitable smooth maps X,Λ,S for transforming the germ g into f modulo high order terms.

2.6.1. **Options.**

- k; this number specifies a degree k so that computations are performed modulo degrees of higher or equal to k. When k is less than the degrees of high order terms a warning note is given.

Example 2.6.1. `Transformation`$(x^3+\sin(\lambda)+\exp(x^5)-1, x^3+\lambda, 4)$ gives rise to

$$\begin{aligned} X &= x+\lambda+x\lambda+\lambda^2, \qquad \Lambda(\lambda)=\lambda,\\ S &= 1-3x^2-3x\lambda-\frac{5}{6}\lambda^2-3x^3-9\lambda x^2-9x\lambda^2-3\lambda^3. \end{aligned}$$

2.7. Bifurcation diagrams

Bifurcation diagram analysis of a parametric system is performed by the notion of *persistent* and *non-persistent* bifurcation diagrams. Bifurcation diagram of (2.0.1) is defined by

$$\{(x,\lambda)\mid g(x,\lambda,\alpha)=0\}.$$

A bifurcation diagram is called *persistent* when the bifurcation diagrams subjected to small perturbations in parameter space remain self contact-equivalent.

2.7.1. **Persistent bifurcation diagram classification and transition set.** The classification of persistent bifurcation diagrams are performed by the notion of transition sets. In fact, a subset of parameter space is called *transition set* when the associated bifurcation diagrams are non-persistent. *Transition set* is denoted by Σ and is usually a hypersurface of codimension one for germs of finite codimension. Then, one choice from each connected components of the complement of the transition set Σ makes a complete persistent bifurcation diagram classification of a given parametric germ. This provides a comprehensive insight into the persistent zero solutions of a parametric germ.

2.7.1.1. *Transition set.* The parameters associated with non-persistent bifurcation diagrams are split into three categories: *bifurcation*, *hysteresis*, and *double limit point*. These are defined and denoted by

$$\begin{aligned} \mathscr{B} &= \{\alpha\in\mathbb{R}^p \mid G=G_x=G_\lambda=0 \text{ at } (x,\lambda,\alpha) \text{ for some } (x,\lambda)\in\mathbb{R}\times\mathbb{R}\},\\ \mathscr{H} &= \{\alpha\in\mathbb{R}^p \mid G=G_x=G_{xx}=0 \text{ at } (x,\lambda,\alpha) \text{ for some } (x,\lambda)\in\mathbb{R}\times\mathbb{R}\},\\ \mathscr{D} &= \{\alpha\in\mathbb{R}^p \mid \exists(x_1,x_2,\lambda)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R} \text{ so that } G=G_x=0 \text{ at } (x_1,\lambda,\alpha) \text{ and } \\ &\qquad (x_1,\lambda,\alpha)\}. \end{aligned}$$

The transition set Σ is now given by $\Sigma := \mathscr{B} \cup \mathscr{H} \cup \mathscr{D}$. Suppose that H is a singular parametric germ.

Command/the default options	**Description**
`TransitionSet`$(H(x, \lambda, \alpha))$	This function estimates the transition set in terms of parameters of H. The default is to eliminate x and λ variables from the equations given by $\mathscr{B}, \mathscr{H}, \mathscr{D}$.
Truncation degree	For non-polynomial input germs, by default, it automatically computes a suitable truncation degree k and truncates the input germ at degree k, i.e., preserving degrees of less than and equal to k.

2.7.1.2. *Options.*

- $[\alpha_1, \alpha_2, \ldots, \alpha_p]$; this hints to derive the transition set in terms of these variables while the attempts are best made to eliminate the rest of variables from the equations (as many as possible).
- `plot`; this function plots/animates transition set in parameter space.
- α_i; When codimension is more than or equal to three 3, some parameters $\alpha_3, \alpha_4, \ldots$ will be taken as fixed by default. This option refines the `plot/animate` option by allowing to change the fixed parameters to α_i.
- k; determines the truncation degree. The user may use the command `Verify` to find an appropriate degree k.

Example 2.7.1. Here, we bring two examples from [6, Page 206].
`TransitionSet`$(x^4 - \lambda x + \alpha_1 + \alpha_2\lambda + \alpha_3 x^2, [x, \lambda, \alpha_1, \alpha_2, \alpha_3])$ gives rise to

$$\begin{aligned}
\mathscr{B} &:= \{(\alpha_1, \alpha_2, \alpha_3) \,|\, \alpha_2^4 + \alpha_2^2\alpha_3 + \alpha_1 = 0\},\\
\mathscr{H} &:= \{(\alpha_1, \alpha_2, \alpha_3) \,|\, 128\alpha_2^2\alpha_3^3 + 3\alpha_3^4 + 72\alpha_1\alpha_3^2 + 432\alpha_1^2 = 0\},\\
\mathscr{D} &:= \{(\alpha_1, \alpha_2, \alpha_3) \,|\, -\alpha_3^2 + 4\alpha_1 = 0,\ \alpha_3 \leq 0\}.
\end{aligned}$$

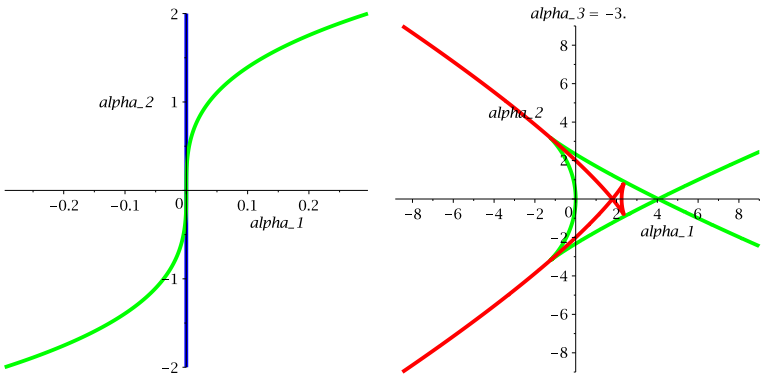
`TransitionSet`$(x^5 - \lambda + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3, [x, \lambda, \alpha_1, \alpha_2, \alpha_3])$ derives

$$\begin{aligned}
\mathscr{B} &:= \emptyset,\\
\mathscr{H} &:= \{(\alpha_1, \alpha_2, \alpha_3) \,|\, -81\alpha_1\alpha_3^4 + 27\alpha_2^2\alpha_3^3 + 360\alpha_1^2\alpha_3^2\\
&\quad -540\alpha_1\alpha_2^2\alpha_3 + 135\alpha_2^4 - 400\alpha_1^3 = 0\},\\
\mathscr{D} &:= \{(\alpha_1, \alpha_2, \alpha_3) \,|\, -16\alpha_3^6 + 224\alpha_1\alpha_3^4 - 88\alpha_2^2\alpha_3^3\\
&\quad -1040\alpha_1^2\alpha_3^2 + 360\alpha_1\alpha_2^2\alpha_3 + 135\alpha_2^4 + 1600\alpha_1^3 = 0\}.
\end{aligned}$$

Example 2.7.2. `TransitionSet`$(x^3 + \sin(\lambda x) + \alpha_1 + \alpha_2 x^2, \texttt{plot}, [\alpha_1, \alpha_2], [x, \lambda, \alpha_1, \alpha_2], 5)$ generates Figure 3(a).

`TransitionSet`$(x^5 - \lambda + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3, \texttt{plot}, [\alpha_1, \alpha_2, \alpha_3], [x, \lambda, \alpha_1, \alpha_2, \alpha_3])$ creates an animation ending at Figure 1(b).

2.7.1.3. *Persistent bifurcation diagrams.* The command `PersistentDiagram` follows the following table.



(a) (b)

FIGURE 1. Blue color is for bifurcation $\mathscr{B}$, green stands for hysteresis $\mathscr{H}$ while the red color depicts transition set associated with double limit point $\mathscr{D}$.

Command/the default options	**Description**
`PersistentDiagram`(H)	this function plots/animates bifurcation diagrams in $x-\lambda$ plane by passing through the parameter space. The default path is a circle around the singular point and it may include at most two parameters.

2.7.1.4. *Options.*

- α_i; for parameter space of dimension more than two, it chooses three values for α_i, i.e., a negative, zero and a positive value for α_i. Then, it plots/animates bifurcation diagrams in $x-\lambda$ plane by passing (circular path by default) through parameter space for fixed parameter α_i.
- $f(\zeta), g(\zeta)$; this function animates bifurcation diagrams in $x-\lambda$ plane by passing through the given path $(f(\zeta), g(\zeta))$.
- `ShortList`; `IntermediateList`; `CompleteList`; either of these generates a list of points associated persistent bifurcation diagrams.
- `plot`; this option plots the persistent bifurcation diagrams associated with parameter points output of the previous option, i.e., `ShortList`; `IntermediateList`; `CompleteList`.
- k; determines the truncation degree of the germ H.

Example 2.7.3. Now we present how the command `PersistentDiagram` works.

`PersistentDiagram`$(x^5-\lambda+\alpha_1 x+\alpha_2 x^2+\alpha_3 x^3$, `plot`, `CompleteList`, $[x,\lambda,\alpha_1,\alpha_2,\alpha_3])$ generates a list from which the list of inequivalent bifurcation diagrams are chosen in Figure 2.

2.7.2. **Singular boundary conditions.** Extra sources of non-persistent is caused by singular boundary conditions of a parametric scalar map restricted to a bounded domain. Let $W \subset \mathbb{R}^p$ be a closed disk and $U, L \subset \mathbb{R}$ be two closed intervals. Next, consider

$$F(x,\lambda,\alpha) = 0$$

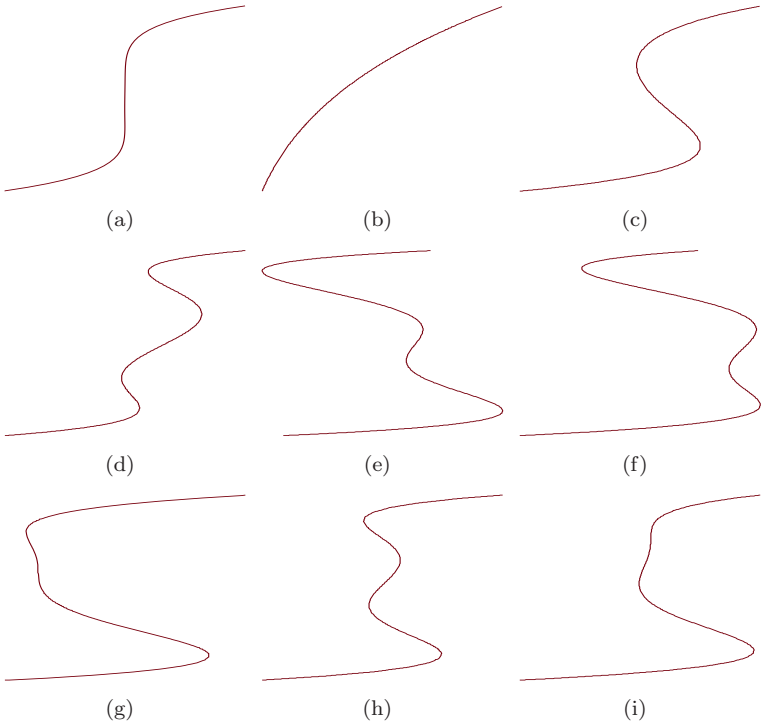


FIGURE 2. Persistent bifurcation diagrams associated with $x^5 - \lambda + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$.

where $\alpha \in W$ and $(x, \lambda) \in U \times L$; see [6, Pages 154-158]. The new non-persistent sources are defined by

$$\begin{aligned}
\mathscr{L}_C &:= \{\alpha \in W \mid F(x,\lambda,\alpha) = 0 \text{ for some } (x,\lambda) \in \partial U \times \partial L\},\\
\mathscr{L}_{SH} &:= \{\alpha \in W \mid F = F_x = 0 \text{ for some } (x,\lambda) \in \partial U \times L\},\\
\mathscr{L}_{SV} &:= \{\alpha \in W \mid F = F_x = 0 \text{ for some } (x,\lambda) \in U \times \partial L\},\\
\mathscr{L}_T &:= \{\alpha \in W \mid F = F_\lambda = 0 \text{ for some } (x,\lambda) \in \partial U \times L\},
\end{aligned}$$

$$\begin{aligned}
\mathscr{G}_1 &:= \{\alpha \in W \mid F = 0 \text{ at } (x_0,\lambda,\alpha) \text{ for some } (x_0,\lambda) \in \partial U \times L,\\
&\quad x_0 \neq x \text{ and } F = F_x = 0 \text{ at } (x,\lambda,\alpha) \text{ for some } (x,\lambda) \in U \times L\},\\
\mathscr{G}_2 &:= \{\alpha \in W \mid \exists (x_1,\lambda),(x_2,\lambda) \in \partial U \times L, x_1 \neq x_2 \text{ s.t } F = 0 \text{ at}\\
&\quad (x_i,\lambda,\alpha) \text{ for } i = 1,2\},
\end{aligned}$$

$$\begin{aligned}
\mathscr{L}_{\mathscr{B}} &:= \{\alpha \in W \mid F = F_x = F_\lambda = 0 \text{ at } (x,\lambda,\alpha) \text{ for some}\\
&\quad (x,\lambda) \in U \times L\},\\
\mathscr{L}_{\mathscr{H}} &:= \{\alpha \in W \mid F = F_x = F_{xx} = 0 \text{ at } (x,\lambda,\alpha) \text{ for some}\\
&\quad (x,\lambda) \in U \times L\},\\
\mathscr{G}_D &:= \{\alpha \in W \mid \exists (x_1,\lambda),(x_2,\lambda) \in U \times L, x_1 \neq x_2 \text{ s.t } F = F_x = 0 \text{ at}\\
&\quad (x_i,\lambda,\alpha) \text{ for } i = 1,2\}.
\end{aligned}$$

In this case, the transition set is given by $\Sigma := \mathscr{L} \cup \mathscr{G}$, here

$$\begin{aligned} \mathscr{L} &:= \mathscr{L}_{\mathscr{B}} \cup \mathscr{L}_{\mathscr{H}} \cup \mathscr{L}_C \cup \mathscr{L}_{SH} \cup \mathscr{L}_{SV} \cup \mathscr{L}_T, \\ \mathscr{G} &:= \mathscr{G}_D \cup \mathscr{G}_1 \cup \mathscr{G}_2. \end{aligned}$$

For a finite codimension singular germ, Σ is a hypersurface of codimension one and each two choices from a connected component in the complement of Σ are contact-equivalent. Therefore, we can classify the persistent bifurcation diagrams by merely choosing one representative parameter from each components of the complement set of Σ and plotting the associated bifurcation diagrams. The command `NonPersistent` is designed for this purpose.

Command/default option	**Description**
`NonPersistent`(F, U, L)	This function computes transition set for F where bifurcation diagrams are limited on $U \times L$. Here, U and L are only taken as closed intervals. Further, it plots the transition set.
Box of figures	It plots transition set in $[-1, 1] \times [-1, 1]$ by default.

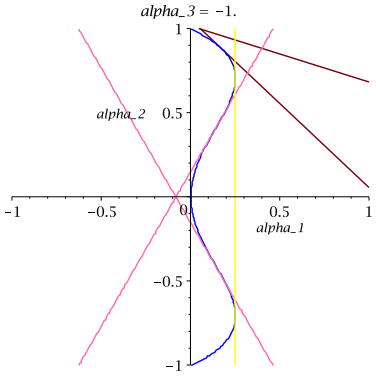
2.7.2.1. *Options.*

- V, W; this option enforces that the computed transition set is plotted in $V \times W$ instead of the default square $[-1, 1] \times [-1, 1]$.
- `Vertical` (`Horizontal` is also similar); this assumes that the boundary conditions is $U \times \mathbb{R}$, i.e., there is only singular boundary conditions on vertical boundary lines.

Example 2.7.4. `NonPersistent`$(x^4-\lambda x+\alpha_1 x+\alpha_2\lambda+\alpha_3 x^2,[-2,2],[1,3])$ gives rise to

$$\begin{aligned}
\mathscr{L}_C &= \{(\alpha_1,\alpha_2,\alpha_3)\,|\,\alpha_1+18+4\alpha_3+\alpha_2=0,\alpha_1+22+4\alpha_3+3\alpha_2=0,\\
&\quad \alpha_1+14+4\alpha_3+\alpha_2=0,\alpha_1+10+4\alpha_3+3\alpha_2=0\},\\
\mathscr{L}_{SH} &= \{(\alpha_1,\alpha_2,\alpha_3)\,|\,4\alpha_2\alpha_3-\alpha_1+32\alpha_2+4\alpha_3+48=0,4\alpha_2\alpha_3+\alpha_1\\
&\quad +32\alpha_2-4\alpha_3-48=0\},\\
\mathscr{L}_{SV} &= \{(\alpha_1,\alpha_2,\alpha_3)\,|\,16\alpha_1\alpha_3^4+16\alpha_2\alpha_3^4-128\alpha_1^2\alpha_3^2-256\alpha_1\alpha_2\alpha_3^2-128\alpha_2^2\alpha_3^2\\
&\quad +256\alpha_1^3+768\alpha_1^2\alpha_2+768\alpha_1\alpha_2^2+256\alpha_2^3-4\alpha_3^3+144\alpha_1\alpha_3+144\alpha_2\alpha_3-27=0,\\
&\quad 16\alpha_1\alpha_3^4+48\alpha_2\alpha_3^4-128\alpha_1^2\alpha_3^2-768\alpha_1\alpha_2\alpha_3^2-1152\alpha_2^2\alpha_3^2+256\alpha_1^3+2304\alpha_1^2\alpha_2\\
&\quad +6912\alpha_1\alpha_2^2+6912\alpha_2^3-36\alpha_3^3+1296\alpha_1\alpha_3+3888\alpha_2\alpha_3-2187=0\},\\
\mathscr{L}_T &= \{(\alpha_1,\alpha_2,\alpha_3)\,|\,\alpha_2+2=0,4\alpha_3+16+\alpha_1=0,\alpha_2-2=0,4\alpha_3+16+\alpha_1=0\},\\
\mathscr{G}_1 &= \{(\alpha_1,\alpha_2,\alpha_3)\,|\,4\alpha_2^2\alpha_3^3+80\alpha_2^2\alpha_3^2+16\alpha_2\alpha_3^3-72\alpha_1\alpha_2\alpha_3+512\alpha_2^2\alpha_3\\
&\quad +32\alpha_2\alpha_3^2+16\alpha_3^3+27\alpha_1^2-320\alpha_1\alpha_2+72\alpha_1\alpha_3+1024\alpha_2^2-384\alpha_2\alpha_3\\
&\quad +176\alpha_3^2+224\alpha_1-1024\alpha_2+640\alpha_3+768,4\alpha_2^2\alpha_3^3+80\alpha_2^2\alpha_3^2-16\alpha_2\alpha_3^3\\
&\quad +72\alpha_1\alpha_2\alpha_3+512\alpha_2^2\alpha_3-32\alpha_2\alpha_3^2+16\alpha_3^3+27\alpha_1^2+320\alpha_1\alpha_2+72\alpha_1\alpha_3\\
&\quad +1024\alpha_2^2+384\alpha_2\alpha_3+176\alpha_3^2+224\alpha_1+1024\alpha_2+640\alpha_3+768=0\},\\
\mathscr{G}_2 &= \{(\alpha_1,\alpha_2,\alpha_3)\,|\,4\alpha_3+16+\alpha_1=0\},\\
\mathscr{L}_{\mathscr{B}} &= \{(\alpha_1,\alpha_2,\alpha_3)\,|\,\alpha_2^4+\alpha_2^2\alpha_3+\alpha_1=0\},\\
\mathscr{L}_{\mathscr{H}} &= \{(\alpha_1,\alpha_2,\alpha_3)\,|\,128\alpha_2^2\alpha_3^3+3\alpha_3^4+72\alpha_1\alpha_3^2+432\alpha_1^2=0\},\\
\mathscr{G}_D &= \{(\alpha_1,\alpha_2,\alpha_3)\,|\,\alpha_3^2-4\alpha_1=0\}.
\end{aligned}$$

generates Figure 3.



(a)

FIGURE 3. Transition set

Part 3. Tools from algebraic geometry

In this section we describe how to compute some tools from computational algebraic geometry; see [4] for more information.

3.1. Multiplication Matrix

Let $\mathcal{R}$ be either of the rings of germs $\mathscr{E}$, $\mathbb{R}[[x,\lambda]]$, $\mathbb{R}[x,\lambda]_{\langle x,\lambda\rangle}$ or $R[x,\lambda]$; see [4]. Now we describe how to compute the multiplication Matrix defined by

$$\varphi_{u,J} : \frac{\mathcal{R}}{J} \to \frac{\mathcal{R}}{J}, \quad \varphi_{u,J}(f+J) := uf + J, \tag{3.1.1}$$

where J is an ideal generated by a finite set $A \subset \mathcal{R}$, i.e., $J := \langle A\rangle_{\mathcal{R}}$, and u is a monomial; also see [4, Equation 3.4].

Command/option	**Description**
`MultMatrix`(A, u)	This function derives $\varphi_{u,J}$ where u is a monomial, $\varphi_{u,J}$ is defined by Equation (3.1.1).
Default computational ring	the fractional germ ring.
Truncation degree	When the input set of germs A only includes polynomials, `MultMatrix`(A, u) does not need truncation degree. However, for non-polynomial input germs, a truncation degree k needs to be included.

3.1.1. **Options.**

- k; determines the truncation degree. The user is advised to use the command `Verify` to find an appropriate truncation degree k.
- Computational ring: `Fractional`, `Formal`, `SmoothGerms`; `Polynomial`; the command uses either the rings of fractional germs, formal power series or ring of smooth germs. The command `Verify` is an appropriate tool to find/verify the appropriate computational ring.

Example 3.1.1. The command `MultMatrix`($[x^6 + \frac{12}{27}x^10\lambda^9, \frac{5}{3}x^5 + \lambda\sin(x^3), \lambda^2 - \frac{2}{3}(1-\exp(x^5))], x, 6,$ `Formal`) gives rise to

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -\frac{5}{3} \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.$$

3.2. Divisions of germs

The following table describes how to use the command `Division` to divide a germ g by a set of germs $G := \{f_i | i = 1, \dots n\}$ where all these germs are in terms of the variables in `Vars`. Note that the ordering in the list of variables `Vars` is important and determines how anti-lexicographical ordering is defined.

Command	**Description**
`Division`(g, G, `Vars`)	This divides the germ g by the germs in G using anti-lexicographical ordering.

3.2.1. **Options.**

- `Formal`; `SmoothGerms`; `Polynomial`; `Fractional`; this determines the computational germ ring. In order to verify/check the computational ring for the division, one is advised to use the command `Verify`(G, `Ideal`, `Vars`) and find the permissible computational ring.
- k; this option enforces the computations modulo degrees higher than or equal to $k+1$. The command `Verify`(G, `Ideal`, `Vars`) also suggests an optimal permissible truncation degree k.

Example 3.2.1. For an example we use `Division`($\sin(x^7)-1, [x^5+x^6\exp(\lambda), xy^3-\frac{2}{7}x\lambda^6-x^7, \lambda\cos(x^7)]$, $[x,\lambda], 8,$ `SmoothGerms`). This returns

$$-1$$

as the remainder of

$$\sin(x^7)-1$$

divided by

$$\{x^5+x^6\exp(\lambda), xy^3-\frac{2}{7}x\lambda^6-x^7, \lambda\cos(x^7)\}$$

using the anti-lexicographical ordering with $x \succ \lambda$.

3.3. Standard basis

Now we describe how to compute a standard basis for a set of germs in either of the following local rings: fractional germs $\mathbb{R}[x,\lambda]_{\langle x,\lambda\rangle}$, formal power series $\mathbb{R}[[x,\lambda]]$, and ring of smooth germs $\mathscr{E}$; see [4]. The command `StandardBasis` follows the table and options described below. Note that `Vars` denotes an order list of variables and the germs in G are in terms of the variables in `Vars`. The anti-lexicographical ordering is here used while it is determined by the ordering of variables in `Vars`.

Command	**Description**
`StandardBasis`(G, `Vars`)	computes the standard basis of the polynomial germs in G in the fractional germ ring.
Default computational ring	fractional germ ring.
Default input	the default germs in G must be polynomial germs. For the cases of non-polynomials, it needs a truncation degree k.

3.3.1. **Options.**

- k; this determines the truncation degree.
- Consider the cases that the option k is used. A warning note is given when the truncation degree k is not sufficiently high to guarantee that the output is correct. A warning note is given like "The truncation degree is not sufficiently high and thus, the following results might be wrong." In this case, for an appropriate truncation degree l is given.
- `Formal`; `SmoothGerms`; `Fractional`; this determines the computational germ ring and computes the standard basis accordingly.

Example 3.3.1. For an example, the command `StandardBasis`($[x^5+x^2\sin(\lambda+x)+\lambda^2, x^3\lambda^2+\cos(\lambda)x, \lambda^6+x^4-\lambda x]$, $[x,\lambda], 7,$ `SmoothGerms`) computes the standard basis of the set of germs

$$\{x^5+x^2\sin(\lambda+x)+\lambda^2, x^3\lambda^2+\cos(\lambda)x, \lambda^6+x^4-\lambda x\}$$

as $\{x, \lambda^2\}$.

3.4. Colon ideals

The colon ideal $I : \langle g\rangle$ refers to the ideal defined by

$$I : \langle g\rangle_{\mathscr{E}} = \{f \in \mathscr{E} : f\langle g\rangle_{\mathscr{E}} \subseteq I\}.$$

Using the arguments on [4, Page 22], we have $I : \langle g\rangle_{\mathscr{E}} = \langle \frac{h_i}{g} | i = 1, \ldots, n\rangle_{\mathscr{E}}$, where $h_1, h_2, \ldots, h_n$ is a standard basis for the ideal $I \cap \langle g\rangle_{\mathbb{R}[x,\lambda]}$. The command `ColonIdeal` follows this.

Command	Description
`ColonIdeal`(I, g)	computes the colon ideal $I : g$ for g in $\mathscr{E}$.

Example 3.4.1. `ColonIdeal`$([x^7 + \lambda x^3 - \lambda^2 x, \lambda x^6 + \lambda^2 x^2 - \lambda^3, x^3\lambda + x], \lambda)$ leads to the ideal generated by

$$[x(\lambda x^2 + 1), \lambda^2 x^2 - x^4 - \lambda^3, \lambda^4 + \lambda^2 - x^2, x(x^4 + \lambda^3 + \lambda)].$$

3.5. Complement spaces

For computing universal unfolding of a singular germ g, we need to compute a basis for a complement vector space for the tangent space $T(g)$ associated with g. This is equivalent to computing a basis for the quotient space $\mathscr{E}/T(g)$. More generally, the command `Normalset`(I) computes a monomial basis for $\mathscr{E}/I$, when I is either an ideal or a vector space with finite codimension in the local ring $\mathscr{E}$.

Command	Description
`Normalset`(A)	computes a monomial basis for $\mathscr{E}/I$, when A is a list of germs generating an ideal $I := \langle A\rangle$.

Example 3.5.1. For example the command `Normalset`$([x^6 + \lambda x^4 + \lambda^2 x, \lambda x^5 + \lambda^2 x^3 + \lambda^3, 5x^6 + 3\lambda x^4, 5\lambda x^4 + 3\lambda^2 x^2, -3x^7 - 3\lambda x^5 - \frac{25}{3}x^6])$ returns

$$[1, \lambda, x, \lambda^2, x^2, x^3, x^4, x^5, \lambda x, \lambda x^3, x^2\lambda]$$

as a list of monomials for the complement of I in $\mathscr{E}$.

Part 4. Technical objects in singularity theory

A user who is not interested in technical details and their commands may simply skip this section.

4.1. Intrinsic ideals

In this section we describe how to compute the intrinsic part of an ideal or a vector space. Let A and B be two lists indicating the generators of an ideal and a vector space, respectively. We intend to compute the maximal intrinsic ideal contained in $\langle A\rangle_{\mathscr{E}} + \langle A\rangle_{\mathbb{R}}$.

Command/option	Description
`Intrinsic`(A)	computes intrinsic part of the ideal generated by A. It remarks when the ideal is of infinite codimension.
`Intrinsic`(A, B)	computes intrinsic part of a vector space given by $\langle A\rangle_{\mathscr{E}} + \langle B\rangle_{\mathbb{R}}$. It remarks when the vector space is of infinite codimension.
Computational ring	The default computational ring is the ring of fractional germs.
Verify/Warning/Suggestion	By default, `Intrinsic` checks and verifies whether the fractional germ ring is sufficient for computation of the intrinsic part of the ideal or vector space spanned by A (and B). If fractional germ ring is not sufficient, it gives a warning note of possible errors and a suggestion to circumvent the problem. `Intrinsic` remarks when the problem might be of infinite codimension. Despite possible warning errors, `Intrinsic` computes the intrinsic part using the fractional germ ring.

4.1.1. **Options.**

- k; this option enforces that the computations are performed modulo degree k.
- `Formal`; `SmoothGerms`; `Polynomial`; `Fractional`; this determines the computational germ ring. It checks and verifies the computations. It gives warning notes of possible errors and alternative suggestions when it finds them necessary.

Example 4.1.1. For an example `Intrinsic`$([x^3\lambda+\lambda^2, 3x^3\lambda, 3x^2\lambda^2]$, `Fractional`, `InfCodim`) leads to

$$M^3\langle\lambda\rangle + \langle\lambda^2\rangle.$$

As for a second example `Intrinsic`$([x^5+\lambda x^3+\lambda^2, 5x^5+3x^3\lambda, 5x^4\lambda+3x^2\lambda^2], [\lambda x^3+2\lambda^2, x^3+2\lambda, x^4+\frac{3}{5}\lambda x^2, \lambda^2, x^5])$ results in

$$M^5 + M^3\langle\lambda\rangle + \langle\lambda^2\rangle.$$

4.2. Algebraic objects

Singularity theory defines and uses many algebraic objects in the bifurcation analysis of zeros of smooth germs. These include restricted tangent space RT, tangent space T, high order term ideal $\mathscr{P}$, smallest intrinsic ideal associated with a singular germ $\mathscr{S}$, a basis for complement of the tangent space $\mathscr{E}/T$, and low order terms $\mathscr{S}^{\perp}$; see [4, 6]. These can be computed in `Singularity` using the command `AlgObjects` as well as the individual commands `RT`, `T`, `P`, and `TangentPerp`. The individual commands `RT`, `T`, and `P` have the same default and non-default options as `AlgObjects` has as follows.

Command/option	Description
`AlgObjects`(g)	This function computes RT, T, $\mathscr{P}$, $\mathscr{E}/T$, $\mathscr{S}$, $\mathscr{S}^\perp$ and intrinsic generators of $\mathscr{S}$ for given g.
`RT`(g)	This derives the restricted tangent space associated with a scalar smooth germ g.
`T`(g)	This command provides a nice representation of the tangent space associated with the singular smooth germ g. The representation uses intrinsic ideal representation as for the intrinsic part of $T(g)$.
`p`(g)	This computes the high order term ideal associated with the germ g.
`TangentPerp`(g)	This first computes $T(g)$, i.e., the tangent space of the germ g, and then returns a monomial basis for the complement space of $T(g)$.
`S`(g)	Computes the smallest intrinsic ideal containing the germ g.
`SPerp`(g)	This derives a set of monomials of low order terms for the germ g.
`IntrinsicGen`(g)	This derives the intrinsic generators of $\mathscr{S}(g)$ that determine the nonzero conditions for recognition problem for normal forms.
Computational ring	The default computational ring is the ring of fractional germs.
Default degree k	For non-polynomial input germs, it computes the least degree k so that truncations at degree k is permissible. Next, the germ g is truncated and all algebraic objects are computed modulo degrees higher than or equal to $k+1$.

4.2.1. **Options.**

- k; this option enforces the computations modulo degree $k+1$.
- `Formal`; `SmoothGerms`; `Polynomial`; `Fractional`; this determines the computational germ ring. It checks and verifies/gives warning notes of possible errors.

Example 4.2.1. Now we present three examples of singular germs of high codimension; see and compare these examples with the examples in [7, Page 4]. For example we consider a codimension 10 singularity and use `AlgObjects`($x^5+x^3\lambda^2+\lambda^3$,`Fractional`). It gives

$$\begin{aligned}
\mathscr{P} &= M^6 + M\langle\lambda^3\rangle,\\
RT &= M^6 + M\langle\lambda^3\rangle + \langle x^4\lambda, 3\lambda^2x^3+5x^5, \lambda^2x^3+x^5+\lambda^3\rangle,\\
T &= M^5 + \langle\lambda^3\rangle + \{\frac{3}{5}\lambda^2x^2 + x^4, x^3\lambda + \frac{3}{2}\lambda^2\},\\
\mathscr{E}/T &= \{1,\lambda,x,\lambda^2,x^2,x^3,\lambda x^2,\lambda^2x,\lambda^2x^2,x\lambda\},\\
\mathscr{S} &= M^5 + \langle\lambda^3\rangle,\\
\mathscr{S}^\perp &= \{1,\lambda,x,\lambda^2,x^2,x^3,x^4,\lambda x^2,\lambda^2x,\lambda^2x^2,x\lambda,x^3\lambda\},\\
\text{intrinsic generators} &= \{x^5,\lambda^3\}.
\end{aligned}$$

A codimension 20 singularity: `TangentPerp`$(x^8+\sin(\lambda^3))$ derives the following

$$\begin{aligned} T &= M^9+\langle\lambda^3\rangle+\{x^7,x^8,\lambda x^7,-2\lambda^2\},\\ \text{TangentPerp} &= \{1,\lambda,x,x^2,x^3,x^4,x^5,x^6,\lambda x,\lambda x^2,\lambda x^3,\lambda x^6,\\ &\quad \lambda^2x,\lambda^2x^2,\lambda^2x^5,x^3\lambda^2,x^4\lambda,x^4\lambda^2,x^5\lambda,x^6\lambda^2\}. \end{aligned}$$

Now we use the command `IntrinsicGen` for an example of codimension 13. `IntrinsicGen`$(\lambda x^8+x^7-\lambda^3x^2-\lambda^2x)$ leads to

$$\text{intrinsic generators} = \{x^7, x\lambda^2\}.$$

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