Hochschild homology and cohomology for involutive A_∞-algebras

Ramsès Fernàndez-València

Abstract

We present a study of the homological algebra of bimodules over A_∞-algebras endowed with an involution. Furthermore we introduce a derived description of Hochschild homology and cohomology for involutive A_∞-algebras.

Contents

1 Introduction

Hochschild homology and cohomology are homology and cohomology theories developed for associative algebras which appears naturally when one studies its deformation theory. Furthermore, Hochschild homology plays a central role in topological field theory in order to describe the closed states part of a topological field theory.

An involutive version of Hochschild homology and cohomology was developed by Braun in [Bra14] by considering associative and A_∞-algebras endowed with an involution and morphisms which commute with the involution.

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Email: ramses.fernandez.valencia@gmail.com

This paper pretends to take a step further with regards to [FVG15]. Whilst in the latter paper we develop the homological algebra required to give a derived version of Braun's involutive Hochschild homology and cohomology for involutive associative algebras, this research is devoted to develop the machinery required to give a derived description of involutive Hochschild homology and cohomology for A_∞-algebras endowed with an involution.

As in [FVG15], this research has been driven by the author's research on Costello's classification of topological conformal field theories [Cos07], where he proves that an open 2-dimensional theory is equivalent to a Calabi-Yau A_∞-category. In [FV15], the author extends the picture to unoriented topological conformal field theories, where open theories now correspond to involutive Calabi-Yau A_∞-categories, and the closed state space of the universal open-closed extension turns out to be the involutive Hochschild chain complex of the open state algebra.

2 Basic concepts

2.1 Coalgebras and bicomodules

An *involutive graded coalgebra* over a field $\mathbb{K}$ is a graded $\mathbb{K}$-module C endowed with a coproduct $\Delta : C \to C \otimes_{\mathbb{K}} C$ of degree zero together with an involution $\star : C \to C$ such that:

1. The map Δ makes the following diagram commute

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes_{\mathbb{K}} C \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta\otimes \mathrm{Id}_C} \\ C \otimes_{\mathbb{K}} C & \xrightarrow[\mathrm{Id}_C \otimes \Delta]{} & C \otimes_{\mathbb{K}} C \otimes_{\mathbb{K}} C \end{array}$$

2. the involution and Δ are compatible: $\Delta(c^\star) = (\Delta(c))^\star$, for $c \in C$, where the involution on $C \otimes_{\mathbb{K}} C$ is defined as: $(c_1 \otimes c_2)^\star = c_2^\star \otimes c_1^\star$, for $c_1, c_2 \in C$.

An *involutive coderivation* on an involutive coalgebra C is a map $L : C \to C$ preserving involutions and making the following diagram commutative:

$$\begin{array}{ccc} C & \xrightarrow{L} & C \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta} \\ C \otimes_{\mathbb{K}} C & \xrightarrow[L\otimes \mathrm{Id}_C + \mathrm{Id}_C \otimes L]{} & C \otimes_{\mathbb{K}} C \end{array}$$

Denote with $\mathrm{iCoder}(-)$ the spaces of coderivations of involutive coalgebras. Observe that $\mathrm{iCoder}(-)$ are Lie subalgebras.

An *involutive differential graded coalgebra* is an involutive coalgebra C equipped with an involutive coderivation $b : C \to C$ of degree -1 such that $b^2 = b \circ b = 0$.

A morphism between two involutive coalgebras C and D is a graded map $C \xrightarrow{f} D$ compatible with the involutions which makes the following diagram commutative:

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ \downarrow{\scriptstyle \Delta_C} & & \downarrow{\scriptstyle \Delta_D} \\ C \otimes_{\mathbb{K}} C & \xrightarrow[f \otimes f]{} & D \otimes_{\mathbb{K}} D \end{array} \tag{1}$$

Example 2.1. *The cotensor coalgebra of an involutive graded $\mathbb{K}$-bimodule A is defined as $\widehat{T}A = \bigoplus_{n \geq 0} A^{\otimes_{\mathbb{K}} n}$. We define an involution in $A^{\otimes_{\mathbb{K}} n}$ by stating:*

$$(a_1 \otimes \cdots \otimes a_n)^\star := (a_n^\star \otimes \cdots \otimes a_1^\star).$$

The coproduct on $\widehat{T}A$ is given by:

$$\Delta(a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n} (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_n).$$

Observe that Δ commutes with the involution.

Proposition 2.2. *There is a canonical isomorphism of complexes:*

$$\text{iCoder}\,(TSA) \cong \text{Hom}_{A\text{-}\,i\mathcal{B}imod}\,(\text{Bar}(A), A)\,.$$

Proof. The proof follows the arguments in Proposition 4.1.1 [FVG15], where we show the result for the non-involutive setting in order to restrict to the involutive one.

Since $\text{Bar}(A) = A \otimes_{\mathbb{K}} TSA \otimes_{\mathbb{K}} A$, the degree $-n$ part of $\text{Hom}_{A\text{-}\,\mathcal{B}imod}(\text{Bar}(A), A)$ is the space of degree $-n$ linear maps $TSA \to A$, which is isomorphic to the space of degree $(-n-1)$ linear maps $TSA \to SA$. By the universal property of the tensor coalgebra, there is a bijection between degree $(-n-1)$ linear maps $TSA \to SA$ and degree $(-n-1)$ coderivations on TSA. Hence the degree n part of $\text{Hom}_{A\text{-}\,\mathcal{B}imod}(\text{Bar}(A), A)$ is isomorphic to the degree n part of $\text{Coder}\,(TSA)$. One checks directly that this isomorphism restricts to an isomorphism of graded vector spaces

$$\text{Hom}_{A\text{-}\,i\mathcal{B}imod}(\text{Bar}(A), A) \cong \text{iCoder}\,(TSA)\,.$$

Finally, one can check that the differentials coincide under the above isomorphism, cf. Section 12.2.4 [LV12]. □

Remark 2.3. *Proposition 2.2 allows us to think of a coderivation as a map $\widehat{T}A \to A$. Such a map $f : \widehat{T}A \to A$ can be described as a collection of maps $\{f_n : A^{\otimes n} \to A\}$ which will be called the components of f.*

If b is a coderivation of degree -1 on $\widehat{T}A$ with $b_n : A^{\otimes_{\mathbb{K}} n} \to A$, then b^2 becomes a linear map of degree -2 with

$$b_n^2 = \sum_{i+j=n+1} \sum_{k=0}^{n-1} b_i \circ \left(\mathrm{Id}^{\otimes k} \circ b_j \circ \mathrm{Id}^{\otimes (n-k-j)} \right).$$

The coderivation b will be a differential for $\widehat{T}A$ if, and only if, all the components b_n^2 vanish.

Given a (involutive) graded $\mathbb{K}$-bimodule A, we denote the suspension of A by SA and define it as the graded (involutive) $\mathbb{K}$-bimodule with $SA_i = A_{i-1}$. Given such a bimodule A, we define the following morphism of degree -1 induced by the identity $s : A \to SA$ by $s(a) = a$.

Lemma 2.4 (cf. Lemma 1.3 [GJ90]). *If $b_k : (SA)^{\otimes_{\mathbb{K}} k} \to SA$ is an involutive linear map of degree -1, we define $m_k : A^{\otimes_{\mathbb{K}} k} \to A$ as $m_k = s^{-1} \circ b_k \circ s^{\otimes_{\mathbb{K}} k}$. Under these conditions:*

$$b_k(sa_1 \otimes \cdots \otimes sa_k) = \sigma m_k(a_1 \otimes \cdots \otimes a_k),$$

where $\sigma := (-1)^{(k-1)|a_1|+(k-2)|a_2|+\cdots+2|a_{k-2}|+|a_{k-1}|+\frac{k(k-1)}{2}}$.

Proof. The proof follows the arguments of Lemma 1.3 [GJ90]. We only need to observe that the involutions are preserved as all the maps involved in the proof are assumed to be involutive. □

Let $\overline{m}_k := \sigma m_k$, then we have $b_k(sa_1 \otimes \cdots \otimes sa_k) = \overline{m}_k(a_1 \otimes \cdots \otimes a_k)$.

Proposition 2.5. *Given an involutive graded $\mathbb{K}$-bimodule A, let $\epsilon_i = |a_1| + \cdots + |a_i| - i$ for $a_i \in A$ and $1 \leq i \leq n$. A boundary map b on $\widehat{T}SA$ is given in terms of the maps $\overline{m}_k$ by the following formula:*

$$\begin{aligned} & b_n(sa_1 \otimes \cdots \otimes sa_n) \\ = & \sum_{k=0}^{n} \sum_{i=1}^{n-k+1} (-1)^{\epsilon_{i-1}} (sa_1 \otimes \cdots \otimes sa_{i-1} \otimes \overline{m}_k(a_i \otimes \cdots \otimes a_{i+k-1}) \otimes \cdots \otimes sa_n). \end{aligned}$$

Proof. This proof follows the arguments of Proposition 1.4 [GJ90]. The only detail that must be checked is that b_n preserves involutions:

$$\begin{aligned} & b_n((sa_1 \otimes \cdots \otimes sa_n)^\star) \\ & = \sum_{j,k} \pm (sa_n^\star \otimes \cdots \otimes sa_j^\star \otimes \overline{m}_k(a_{j-1}^\star \otimes \cdots \otimes a_{j-k+1}^\star) \otimes \cdots \otimes sa_1^\star) \\ & = \sum_{j,k} \pm (sa_1 \otimes \cdots \otimes \overline{m}_k(a_{j-k+1} \otimes \cdots \otimes a_{j-1}) \otimes sa_j \otimes \cdots \otimes sa_n)^\star \\ & = (b_n(sa_1 \otimes \cdots \otimes sa_n))^\star. \end{aligned}$$

□

Given an involutive coalgebra C with coproduct ρ and counit ε, for an involutive graded vector space P, a *left coaction* is a linear map $\Delta^L : P \to C \otimes_{\mathbb{K}} P$ such that

1. $(\mathrm{Id} \otimes \rho) \circ \Delta^L = (\rho \otimes \mathrm{Id}) \circ \Delta^L$;

2. $(\mathrm{Id} \otimes \varepsilon) \circ = \mathrm{Id}$.

Analagously we introduce the concept of *right coaction*.

Given an involutive coalgebra (C, ρ, ε) with involution $\star$ we define an *involutive C-bicomodule* as an involutive graded vector space P with involution $\dagger$, a left coaction $\Delta^L : P \to C \otimes_{\mathbb{K}} P$ and a right coaction $\Delta^R : P \to P \otimes_{\mathbb{K}} C$ which are compatible with the involutions, that is the diagrams below commute:

$$\begin{array}{ccc} P & \xrightarrow{(-)^\star} & P \\ \downarrow{\scriptstyle \Delta^L} & & \downarrow{\scriptstyle \Delta^R} \\ C \otimes_{\mathbb{K}} P & \xrightarrow{(-,-)^\star} & P \otimes_{\mathbb{K}} C \end{array} \qquad (2)$$

$$\begin{array}{ccc} P & \xrightarrow{\Delta^L} & C \otimes P \\ \downarrow{\scriptstyle \Delta^R} & & \downarrow{\scriptstyle \mathrm{Id}_C \otimes \Delta^R} \\ P \otimes_{\mathbb{K}} C & \xrightarrow{\Delta^L \otimes \mathrm{Id}_C} & C \otimes_{\mathbb{K}} P \otimes_{\mathbb{K}} C \end{array} \qquad (3)$$

Where

$$\begin{array}{rccc} (-,-)^\star : & C \otimes_{\mathbb{K}} P & \to & P \otimes_{\mathbb{K}} C \\ & c \otimes p & \mapsto & p^\dagger \otimes c^\star \end{array}$$

For two involutive C-bicomodules (P_1, Δ_1) and (P_2, Δ_2), a morphism $P_1 \xrightarrow{f} P_2$ is defined as an involutive morphism making diagrams below commute:

$$\begin{array}{ccc} P_1 & \xrightarrow{\Delta_1^L} & C \otimes_{\mathbb{K}} P_1 \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{Id}_C \otimes f} \\ P_2 & \xrightarrow{\Delta_2^L} & C \otimes_{\mathbb{K}} P_2 \end{array} \qquad (4)$$

$$\begin{array}{ccc} P_1 & \xrightarrow{\Delta_1^R} & P_1 \otimes_{\mathbb{K}} C \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{Id}_C \otimes f} \\ P_2 & \xrightarrow{\Delta_2^R} & P_2 \otimes_{\mathbb{K}} C \end{array} \qquad (5)$$

2.2 A_∞-algebras and A_∞-quasi-isomorphisms

An *involutive* $\mathbb{K}$*-algebra* is an algebra A over a field $\mathbb{K}$ endowed with a $\mathbb{K}$-linear map (an involution) $\star : A \to A$ satisfying:

1. $0^\star = 0$ and $1^\star = 1$;
2. $(a^\star)^\star = a$ for each $a \in A$;
3. $(a_1 a_2)^\star = a_2^\star a_1^\star$ for every $a_1, a_2 \in A$.

Example 2.6. *1. Any commutative algebra A becomes an involutive algebra if we endoew it with the identity as involution.*

2. Let V an involutive vector space. The tensor algebra $\bigoplus_n V^{\otimes n}$ becomes an involutive algebra if we endow it with the following involution: $(v_1, \ldots, v_n)^\star = (v_n^\star, \ldots, v_1^\star)$. This example is particularly important and we will come back to it later on.

3. For a discrete group G, the group ring $\mathbb{K}[G]$ is an involutive $\mathbb{K}$-algebra with involution given by inversion $g^\star = g^{-1}$.

Given an involutive algebra A, an *involutive A-bimodule M* is an A-bimodule endowed with an involution satisfying $(a_1 m a_2)^\star = a_2^\star m^\star a_1^\star$.

Given two involutive A-bimodules M and N, a *involutive morphism* between them is a morphism of A-bimodules $f : M \to N$ compatible with the involutions.

Lemma 2.7. *The composition of involutive morphisms is an involutive morphism.*

Proof. Given $f : M \to N$ and $g : N \to P$ two involutive morphisms:

$$(f \circ g)(m^\star) = f((g(m^\star))) = f((g(m))^\star) = (f(g(m)))^\star \qquad \square$$

Involutive A-bimodules and involutive morphisms form the category A- $i\mathcal{B}imod$.

Given a (involutive) graded $\mathbb{K}$-module A, we denote the suspension of A by SA and define it as the graded (involutive) $\mathbb{K}$-module with $SA_i = A_{i-1}$. An *involutive A_∞-algebra* is an involutive graded vector space A endowed with involutive morphisms

$$b_n : (SA)^{\otimes_{\mathbb{K}} n} \to SA,\ n \geq 1, \tag{6}$$

of degree $n-2$ such that the identity below holds:

$$\sum_{i+j+l=n} (-1)^{i+jl} b_{i+1+l} \circ (\mathrm{Id}^{\otimes i} \otimes b_j \otimes \mathrm{Id}^{\otimes l}) = 0,\ \forall n \geq 1. \tag{7}$$

Remark 2.8. *Condition (7) says, in particular, that $b_1^2 = 0$.*

Example 2.9.

1. *The concept A_∞-algebra is a generalization for that of a differential graded algebra. Indeed, if the maps $b_n = 0$ for $n \geq 3$ then A is a differential $\mathbb{Z}$-graded algebra and conversely an A_∞-algebra A yields a differential graded algebra if we require $b_n = 0$ for $n \geq 3$.*
2. *The definition of A_∞-algebra was introduced by Stasheff whose motivation was the study of the graded abelian group of singular chains on the based loop space of a topological space.*

For an involutive A_∞-algebra (A, b_n), the *involutive bar complex* is the involutive differential graded coalgebra $\mathrm{Bar}(A) = \widehat{T}SA$, where we endow $\mathrm{Bar}(A)$ with a coderivation defined by $b_i = s^{-1} \circ b_i \circ s^{\otimes_K i}$ (cf. Definition 1.2.2.3 [LH03]).

Given two involutive A_∞-algebras C and D, a *morphism of A_∞-algebras* $f : C \to D$ is an involutive morphism of degree 0 between the associated involutive differential graded coalgebras $\mathrm{Bar}(C) \to \mathrm{Bar}(D)$.

It follows from Proposition 2.2 that the definition of an involutive A_∞-algebra can be summarized by saying that it is an involutive graded $\mathbb{K}$-module A equipped with an involutive coderivation on $\mathrm{Bar}(A)$ of degree -1.

Remark 2.10. *From [Bra14], Definition 2.8, we have that a morphism of involutive A_∞-algebras $f : C \to D$ can be given by an involutive morphism of differential graded coalgebras* $\mathrm{Bar}(C) \to \mathrm{Bar}(D)$*, that is, a series of involutive homogeneous maps of degree zero*

$$f_n : (SC)^{\otimes_{\mathbb{K}} n} \to SD,\ n \geq 1,$$

such that

$$\sum_{i+j+l=n} f_{i+l+1} \circ \left(\mathrm{Id}_{SC}^{\otimes i} \otimes b_j \otimes \mathrm{Id}_{SC}^{\otimes l}\right) = \sum_{i_1+\cdots+i_s=n} b_s \circ (f_{i_1} \otimes \cdots \otimes f_{i_s}). \tag{8}$$

The composition $f \circ g$ of two morphisms of involutive A_∞-algebras is given by

$$(f \circ g)_n = \sum_{i_1+\cdots+i_s=n} f_s \circ (g_{i_1} \otimes \cdots \otimes g_{i_s});$$

the identity on SC is defined as $f_1 = \mathrm{Id}_{SC}$ and $f_n = 0$ for $n \geq 2$.

For an involutive A_∞-algebra A, we define its associated homology algebra $\mathrm{H}_\bullet(A)$ as the homology of the differential b_1 on A: $\mathrm{H}_\bullet(A) = H_\bullet(A, b_1)$.

Remark 2.11. *Endowed with b_2 as multiplication, the homology of an A_∞-algebra A is an associative graded algebra, whereas A is not usually associative.*

Let $f : A_1 \to A_2$ a morphism of involutive A_∞-algebras with components f_n; for $n = 1$, f_1 induces a morphism of algebras $\mathrm{H}_\bullet(A_1) \to \mathrm{H}_\bullet(A_2)$. We say that $f : A_1 \to A_2$ is an *A_∞-quasi-isomorphism* if f_1 is a quasi-isomorphism.

2.3 A_∞-bimodules

Let (A, b^A) be an involutive A_∞-algebra. An *involutive A_∞-bimodule* is a pair (M, b^M) where M is a graded involutive $\mathbb{K}$-module and b^M is an involutive differential on the $\mathrm{Bar}(A)$-bicomodule

$$\mathrm{Bar}(M) := \mathrm{Bar}(A) \otimes_{\mathbb{K}} SM \otimes_{\mathbb{K}} \mathrm{Bar}(A).$$

Let $\left(M, b^M\right)$ and $\left(N, b^N\right)$ be two involutive A_∞-bimodules. We define a *morphism of involutive A_∞-bimodules* $f : M \to N$ as a morphism of $\mathrm{Bar}(A)$-bicomodules

$$F : \mathrm{Bar}(M) \to \mathrm{Bar}(N)$$

such that $b^N \circ F = F \circ b^M$.

Proposition 2.12. *If $f : A_1 \to A_2$ is a morphism of involutive A_∞-algebras, then A_2 becomes an involutive bimodule over A_1.*

Proof. As we are assuming that both A_1 and A_2 are involutive A_∞-algerbas and that f is involutive, we do not need to care about involutions. When it comes to the bimodule structure, this result holds as $\mathrm{Bar}(A_2)$ is made into a bicomodule of $\mathrm{Bar}(A_1)$ by the homomorphism of involutive coalgebras $f : \mathrm{Bar}(A_1) \to \mathrm{Bar}(A_2)$, see Proposition 3.4 [GJ90]. □

Remark 2.13 (Section 5.1 [KS09]). *Let $i\mathcal{V}ect$ be the category of involutive $\mathbb{Z}$-graded vector spaces and involutive morphisms. For an involutive A_∞-algebra A, involutive A-bimodules and their respective morphisms form a differential graded category. Indeed, following [KS09], Definition 5.1.5: let A be an* *involutive A_∞-algebra and let us define the category $\overline{A\text{- }i\mathcal{B}imod}$ whose class of objects are involutive A-bimodules and where $\mathrm{Hom}_{\overline{A\text{- }i\mathcal{B}imod}}(M,N)$ is:*

$$\underline{\mathrm{Hom}}^n_{i\mathcal{V}ect}(\mathrm{Bar}(A)\otimes_{\mathbb{K}} SM\otimes_{\mathbb{K}}\mathrm{Bar}(A),\mathrm{Bar}(A)\otimes_{\mathbb{K}} SN\otimes_{\mathbb{K}}\mathrm{Bar}(A)).$$

Let us recall that

$$\underline{\mathrm{Hom}}^n_{i\mathcal{V}ect}(\mathrm{Bar}(A)\otimes_{\mathbb{K}} SM\otimes_{\mathbb{K}}\mathrm{Bar}(A),\mathrm{Bar}(A)\otimes_{\mathbb{K}} SN\otimes_{\mathbb{K}}\mathrm{Bar}(A))$$

is by definition

$$\prod_{i\in\mathbb{Z}}\mathrm{Hom}_{i\mathcal{V}ect}((\mathrm{Bar}(A)\otimes_{\mathbb{K}} SM\otimes_{\mathbb{K}}\mathrm{Bar}(A))^i,(\mathrm{Bar}(A)\otimes_{\mathbb{K}} SN\otimes_{\mathbb{K}}\mathrm{Bar}(A))^{i+n}).$$

The morphism

$$\begin{aligned}&\underline{\mathrm{Hom}}^n_{i\mathcal{V}ect}(\mathrm{Bar}(A)\otimes_{\mathbb{K}} SM\otimes_{\mathbb{K}}\mathrm{Bar}(A),\mathrm{Bar}(A)\otimes_{\mathbb{K}} SN\otimes_{\mathbb{K}}\mathrm{Bar}(A))\to\\ &\qquad\qquad\underline{\mathrm{Hom}}^{n+1}_{i\mathcal{V}ect}(\mathrm{Bar}(A)\otimes_{\mathbb{K}} SM\otimes_{\mathbb{K}}\mathrm{Bar}(A),\mathrm{Bar}(A)\otimes_{\mathbb{K}} SN\otimes_{\mathbb{K}}\mathrm{Bar}(A))\end{aligned}$$

sends a family $\{f_i\}_{i\in\mathbb{Z}}$ to a family $\{b^N\circ f_i-(-1)^n f_{i+1}\circ b^M\}_{i\in\mathbb{Z}}$. Observe that the zero cycles in $\underline{\mathrm{Hom}}^\bullet_{i\mathcal{V}ect}(\mathrm{Bar}(A)\otimes_{\mathbb{K}} M\otimes_{\mathbb{K}}\mathrm{Bar}(A),\mathrm{Bar}(A)\otimes_{\mathbb{K}} N\otimes_{\mathbb{K}}\mathrm{Bar}(A))$ are precisely the morphisms of involutive A-bimodules. This morphism defines a differential, indeed: for fixed indices $i,n\in\mathbb{Z}$ we have

$$\begin{aligned}d^2(f_i) &= d\left(b^N f_i-(-1)^n f_{i+1}b^M\right)\\ &= b^N\left(b^N f_i-(-1)^n f_{i+1}b^M\right)-(-1)^{n+1}\left(b^N f_i-(-1)^n f_{i+1}b^M\right)b^M\\ &\overset{(!)}{=} -(-1)^n b^N f_{i+1}b^M-(-1)^{n+1}b^N f_{i+1}b^M=0,\end{aligned}$$

where (!) points out the fact that $b^N\circ b^N=0=b^M\circ b^M$.

For a morphism $\phi\in\underline{\mathrm{Hom}}^n_{i\mathcal{V}ect}(\mathrm{Bar}(A)\otimes_{\mathbb{K}} M\otimes_{\mathbb{K}}\mathrm{Bar}(A),\mathrm{Bar}(A)\otimes_{\mathbb{K}} N\otimes_{\mathbb{K}}\mathrm{Bar}(A))$ and an element $x\in\mathrm{Bar}(A)\otimes_{\mathbb{K}} M\otimes_{\mathbb{K}}\mathrm{Bar}(A)$, $\mathrm{Hom}_{\overline{A\text{- }i\mathcal{B}imod}}(M,N)$ becomes an involutive complex if we endowed it with the involution $\phi^\star(x)=\phi(x^\star)$.

The functor $\mathrm{Hom}_{\overline{A\text{- }i\mathcal{B}imod}}(M,-)$ pairs an involutive A-bimodule F with the involutive $\mathbb{K}$-vector space $\mathrm{Hom}_{\overline{A\text{- }i\mathcal{B}imod}}(M,F)$ of involutive homomorphisms. Given a homomorphism $f:F\to G$, for $F,G\in\mathrm{Obj}\left(\overline{A\text{- }i\mathcal{B}imod}\right)$, $\mathrm{Hom}_{\overline{A\text{- }i\mathcal{B}imod}}(M,-)$ pairs f with the involutive map:

$$\begin{array}{rccc}f_\star: & \mathrm{Hom}_{\overline{A\text{- }i\mathcal{B}imod}}(M,F) & \to & \mathrm{Hom}_{\overline{A\text{- }i\mathcal{B}imod}}(M,G)\\ & \phi & \mapsto & f\circ\phi\end{array}.$$

We prove that $f_\star$ preserves involutions:

$$(f_\star\phi^\star)(x)=(f\circ\phi^\star)(x)=f(\phi(x^\star))=f((\phi(x))^\star)=(f(\phi(x)))^\star=(f_\star\phi(x))^\star.$$

We define the functor $\mathrm{Hom}_{\overline{A\text{- }i\mathcal{B}imod}}(-,M)$, which sends an involutive homomorphism $f:F\to G$, for $F,G\in\mathrm{Obj}\left(\overline{A\text{- }i\mathcal{B}imod}\right)$, to

$$\begin{array}{rccc}\varphi: & \mathrm{Hom}_{\overline{A\text{- }i\mathcal{B}imod}}(G,M) & \to & \mathrm{Hom}_{\overline{A\text{- }i\mathcal{B}imod}}(F,M)\\ & \phi & \mapsto & \phi\circ f\end{array}$$

Let us check that the involution is preserved:

$$\varphi(\phi^\star)(x) = (\phi^\star \circ f)(x) = \phi(f(x)^\star) = \phi(f(x^\star)) = \varphi(\phi)(x^\star) = (\varphi(\phi))^\star(x)$$

Let A be an involutive A_∞-algebra and let $\left(M, b^M\right)$ and $\left(N, b^N\right)$ be involutive A-bimodules. For $f, g : M \to N$ morphisms of A-bimodules, an A_∞-*homotopy* between f and g is a morphism $h : M \to N$ of A-bimodules satisfying

$$f - g = b^N \circ h + h \circ b^M.$$

We say that two morphisms $u : M \to N$ and $v : N \to M$ of involutive A-bimodules are *homotopy equivalent* if $u \circ v \sim \mathrm{Id}_N$ and $v \circ u \sim \mathrm{Id}_M$.

3 The involutive tensor product

For an involutive A_∞-algebra A and involutive A-bimodules M and N, the involutive tensor product $M \widetilde{\boxtimes}_\infty N$ is the following object in $i\mathcal{V}ect_{\mathbb{K}}$:

$$M \widetilde{\boxtimes}_\infty N := \frac{M \otimes_{\mathbb{K}} \mathrm{Bar}(A) \otimes_{\mathbb{K}} N}{(m^\star \otimes a_1 \otimes \cdots \otimes a_k \otimes n - m \otimes a_1 \otimes \cdots \otimes a_k \otimes n^\star)}.$$

Observe that, for an element of $M \widetilde{\boxtimes}_\infty N$ of the form $m \otimes a_1 \otimes \cdots \otimes a_k \otimes n$, we have: $(m \otimes a_1 \otimes \cdots \otimes a_k \otimes n)^\star = m^\star \otimes a_1 \otimes \cdots \otimes a_k \otimes n = m \otimes a_1 \otimes \cdots \otimes a_k \otimes n^\star$.

Proposition 3.1. *For an involutive A_∞-algebra A and involutive A-bimodules M, N and L, the following isomorphism holds:*

$$\tau : \mathrm{Hom}_{i\mathcal{V}ect}\left(M \widetilde{\boxtimes}_\infty N, L\right) \xrightarrow{\cong} \mathrm{Hom}_{i\mathcal{V}ect}\left(\frac{M \otimes_{\mathbb{K}} \mathrm{Bar}(A)}{\sim}, \mathrm{Hom}_{\overline{A\text{-}i\mathcal{B}imod}}(N, L)\right),$$

where in $M \otimes_{\mathbb{K}} \mathrm{Bar}(A)$: $(m \otimes a_1 \otimes \cdots \otimes a_k)^\star = m^\star \otimes a_1 \otimes \cdots \otimes a_k$, $\sim$ denotes the relation $m \otimes a_1 \otimes \cdots \otimes a_k = m^\star \otimes a_1 \otimes \cdots \otimes a_k$ and $\frac{M \otimes_{\mathbb{K}} \mathrm{Bar}(A)}{\sim}$ has the identity map as involution.

Proof. Let $f : M \widetilde{\boxtimes}_\infty N \to L$ be an involutive map. We define:

$$\tau(f) := \tau_f \in \mathrm{Hom}_{i\mathcal{V}ect}\left(\frac{M \otimes_{\mathbb{K}} \mathrm{Bar}(A)}{\sim}, \mathrm{Hom}_{\overline{A\text{-}i\mathcal{B}imod}}(N, L)\right),$$

where $\tau_f(m \otimes a_1 \otimes \cdots \otimes a_k) := \tau_f[m \otimes a_1 \otimes \cdots \otimes a_k] \in \mathrm{Hom}_{\overline{A\text{-}i\mathcal{B}imod}}(N, L)$. Finally, for $n \in N$ we define:

$$\tau_f[m \otimes a_1 \otimes \cdots \otimes a_k](n) := f\left(m \otimes a_1 \otimes \cdots \otimes a_k \otimes n\right).$$

We need to check that τ preserves the involutions, indeed:

$$\begin{aligned}\tau_{f^\star}[m \otimes a_1 \otimes \cdots \otimes a_k](n) &= f^\star\left(m \otimes a_1 \otimes \cdots \otimes a_k \otimes n\right) = \\ &= \left(f\left(m \otimes a_1 \otimes \cdots \otimes a_k \otimes n\right)\right)^\star = (\tau_f)^\star[m \otimes a_1 \otimes \cdots \otimes a_k](n).\end{aligned}$$

In order to see that τ is an isomorphism, we build an inverse. Let us consider an involutive map

$$\begin{aligned} g_1 : \quad \frac{M \otimes_{\mathbb{K}} \mathrm{Bar}(A)}{\sim} &\to \mathrm{Hom}_{\overline{A\text{-}i\mathcal{B}imod}}(N, L) \\ m \otimes a_1 \otimes \cdots \otimes a_k &\mapsto g_1[m \otimes a_1 \otimes \cdots \otimes a_k] \end{aligned}$$

and define a map

$$\begin{array}{rccc} g_2: & M\widetilde{\boxtimes}_\infty N & \to & L \\ & m\otimes a_1\otimes\cdots\otimes a_k\otimes n & \mapsto & g_1[m\otimes a_1\otimes\cdots\otimes a_k](n) \end{array}$$

We check that g_2 is involutive:

$$\begin{aligned} g_2((m\otimes a_1\otimes\cdots\otimes a_k\otimes n)^\star) &= g_2(m^\star\otimes a_1\otimes\cdots\otimes a_k\otimes n) = \\ &= g_1[m^\star\otimes a_1\otimes\cdots\otimes a_k](n) = (g_1[m\otimes a_1\otimes\cdots\otimes a_k])^\star(n) \\ &= (g_1[m\otimes a_1\otimes\cdots\otimes a_k](n))^\star = (g_2(m\otimes a_1\otimes\cdots\otimes a_k\otimes n))^\star. \end{aligned}$$

The rest of the proof is standard and follows the steps of Theorem 2.75 [Rot09] or Proposition 2.6.3 [Wei94]. □

For an A-bimodule M, let us define $(-)\widetilde{\boxtimes}_\infty M$ as the covariant functor

$$\begin{array}{ccc} \overline{A\text{-}i\mathcal{B}imod} & \xrightarrow{(-)\widetilde{\boxtimes}_\infty M} & \overline{A\text{-}i\mathcal{B}imod} \\ B & \rightsquigarrow & B\widetilde{\boxtimes}_\infty M \end{array}.$$

This functor sends a map $B_1 \xrightarrow{f} B_2$ to $B_1\widetilde{\boxtimes}_\infty M \xrightarrow{f\widetilde{\boxtimes}_\infty \mathrm{Id}_M} B_2\widetilde{\boxtimes}_\infty M$.

The functor $(-)\widetilde{\boxtimes}_\infty M$ is involutive: let us consider an involutive map $f: B_1 \to B_2$ and its image under the tensor product functor, $g = f\widetilde{\boxtimes}_\infty \mathrm{Id}_M$. Hence:

$$g((b,a)^\star) = g(b^\star, a) = (f(b^\star), a) = (f(b), a)^\star = (g(b,a))^\star.$$

Given an involutive A_∞-algebra A, we say that an involutive A-bimodule F is *flat* if the tensor product functor $(-)\widetilde{\boxtimes}_\infty F : \overline{A\text{-}i\mathcal{B}imod} \to \overline{A\text{-}i\mathcal{B}imod}$ is exact, that is: it takes quasi-isomorphisms to quasi-isomorphisms. From now on, we will assume that all the involutive A-bimodules are flat.

Lemma 3.2. *If P and Q are homotopy equivalent as involutive A_∞-bimodules then, for every involutive A_∞-bimodule M, the following quasi-isomorphism in the category of involutive A_∞-bimodules holds:*

$$P\widetilde{\boxtimes}_\infty M \simeq Q\widetilde{\boxtimes}_\infty M.$$

Proof. Let $f: P \leftrightarrows Q : g$ be a homotopy equivalence. It is clear that

$$h \sim k \Rightarrow h\widetilde{\boxtimes}_\infty \mathrm{Id}_M \sim k\widetilde{\boxtimes}_\infty \mathrm{Id}_M\,.$$

Therefore, we have:

$$\begin{array}{ccccc} P\widetilde{\boxtimes}_\infty M & \to & Q\widetilde{\boxtimes}_\infty M & \to & P\widetilde{\boxtimes}_\infty M \\ p\widetilde{\boxtimes}a & \mapsto & f(p)\widetilde{\boxtimes}a & \mapsto & g(f(p))\widetilde{\boxtimes}a \end{array}$$

and

$$\begin{array}{ccccc} Q\widetilde{\boxtimes}_\infty M & \to & P\widetilde{\boxtimes}_\infty M & \to & Q\widetilde{\boxtimes}_\infty M \\ q\widetilde{\boxtimes}a & \mapsto & g(q)\widetilde{\boxtimes}a & \mapsto & f(g(q))\widetilde{\boxtimes}a \end{array}$$

the result follows since $f\circ g \sim \mathrm{Id}_Q$ and $g\circ f \sim \mathrm{Id}_P$. □

Lemma 3.3. *Let A be an involutive A_∞-algebra. If P and Q are homotopy equivalent as involutive A-bimodules then, for every involutive A-bimodule M, the following quasi-isomorphism holds:*

$$\mathrm{Hom}_{\overline{A\text{-}i\mathcal{B}imod}}(P,M) \simeq \mathrm{Hom}_{\overline{A\text{-}i\mathcal{B}imod}}(Q,M).$$

Proof. Consider $f : P \to Q$ a homotopy equivalence and let $g : Q \to P$ be its homotopy inverse. If $[-,-]$ denotes the homotopy classes of morphisms, then both f and g induce the following maps:

$$\begin{array}{rccc} f_\star : & [P,M] & \to & [Q,M] \\ & \alpha & \mapsto & \alpha \circ g \end{array}$$

$$\begin{array}{rccc} g_\star : & [Q,M] & \to & [P,M] \\ & \beta & \mapsto & \beta \circ f \end{array}$$

Now we have:

$$f_\star \circ g_\star \circ \beta = f_\star \circ \beta \circ f = \beta \circ g \circ f \sim \beta;$$
$$g_\star \circ f_\star \circ \alpha = g_\star \circ \alpha \circ g = \alpha \circ f \circ g \sim \alpha.$$

□

4 Involutive Hochschild homology and cohomology

4.1 Hochschild homology for involutive A_∞-algebras

We define the *involutive Hochschild chain complex* of an involutive A_∞-algebra A with coefficients in an A-bimodule M as follows:

$$C_\bullet^{\mathrm{inv}}(M,A) = M \widetilde{\boxtimes}_\infty \mathrm{Bar}(A).$$

The differential is the same given in Section 7.2.4 [KS09]. The involutive Hochschild homology of A with coefficients in M is

$$\mathrm{HH}_n(M,A) = \mathrm{H}\, C_n^{\mathrm{inv}}(M,A).$$

Lemma 4.1. *For an involutive A_∞-algebra A and a flat A-bimodule M, the following quasi-isomorphism holds:*

$$C_\bullet^{inv}(M,A) \simeq M \widetilde{\boxtimes}_\infty A.$$

Proof. The result follows from:

$$M \widetilde{\boxtimes}_\infty A \simeq M \widetilde{\boxtimes}_\infty \mathrm{Bar}(A) = C_\bullet^{\mathrm{inv}}(M,A).$$

Observe that we are using that M is flat and that there is a quasi-isomorphism between $\mathrm{Bar}(A)$ and A (Proposition 2, Section 2.3.1 [Fer12]). □

4.2 Hochschild cohomology for involutive A_∞-algebras

The *involutive Hochschild cochain complex* of an involutive A_∞-algebra A with coefficients on an A-bimodule M is defined as the $\mathbb{K}$-vector space

$$\mathrm{C}^\bullet_{\mathrm{inv}}(A, M) := \mathrm{Hom}_{\overline{A\text{- }i\mathcal{B}imod}}(\mathrm{Bar}(A), M),$$

with the differential defined in section 7.1 of [KS09].

Proposition 4.2. *For an involutive A_∞-algebra A and an A-bimodule M, we have the following quasi-isomorphism: $C^\bullet_{inv}(A, M) \simeq \mathrm{Hom}_{\overline{A\text{- }i\mathcal{B}imod}}(A, M)$.*

Proof. The result follows from:

$$\begin{aligned}
\mathrm{C}^\bullet_{\mathrm{inv}}(A, M) = \mathrm{Hom}_{\overline{A\text{- }i\mathcal{B}imod}}(\mathrm{Bar}(A), M) :=& \\
\mathrm{Hom}^n_{i\mathcal{V}ect}(\mathrm{Bar}(A) \otimes_{\mathbb{K}} S\,\mathrm{Bar}(A) \otimes_{\mathbb{K}} \mathrm{Bar}(A), \mathrm{Bar}(A) \otimes_{\mathbb{K}} SM \otimes_{\mathbb{K}} \mathrm{Bar}(A)) &\overset{(!)}{\simeq} \\
\mathrm{Hom}^n_{i\mathcal{V}ect}(\mathrm{Bar}(A) \otimes_{\mathbb{K}} SA \otimes_{\mathbb{K}} \mathrm{Bar}(A), \mathrm{Bar}(A) \otimes_{\mathbb{K}} SM \otimes_{\mathbb{K}} \mathrm{Bar}(A)) &=: \\
\mathrm{Hom}_{\overline{A\text{- }i\mathcal{B}imod}}(A, M)&.
\end{aligned}$$

Here (!) points out the fact that $S\,\mathrm{Bar}(A)$ is a projective resolution of SA in $i\mathcal{V}ect$ and hence we have the quasi-isomorphism $S\,\mathrm{Bar}(A) \simeq SA$. Observe that $S\,\mathrm{Bar}(A)$ is projective in $i\mathcal{V}ect$, therefore the involved functors in the proof are exact and preserve quasi-isomorphisms. □

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